Entire Problem Solutions

Mathematics 3–4 · 907 worked exercises

Start with the printed question and work it on paper. Each numbered step names an idea and shows how it changes the calculation or the diagram. An amber mark on a drawing picks out the object discussed beside it. Each exercise ends with a check you can perform yourself.

Open the interactive studio for movable diagrams and guided playback.

Find an exercise

The bookmarks list exercises individually. Each exercise heading also opens its original printed page in the Phillips Exeter Academy book.

Exercises 1–100
Exercises 101–200
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Appendix exercises

Read the omission note for appendix exercise 391-1.

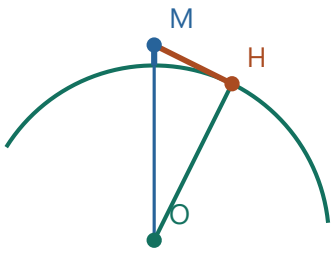
Exercise 1 · Mountain horizon

Printed page 1 · The Pythagorean theorem / Tangent lines / Unit conversion

From the top of Mt Washington, which is 6288 feet above sea level, how far is it to the horizon? Assume that the earth has a 3960-mile radius (one mile is 5280 feet), and give your answer to the nearest mile.

1. Locate the horizon

Call the center of Earth O, the observer M, and the last visible point H. The sight line MH just grazes the spherical surface. A line passing through the surface would hide part of itself behind Earth.



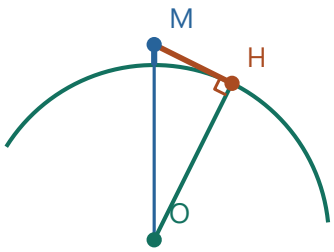
Altitude exaggerated

O is Earth's center; H is the contact point.

The sight line from the observer M just touches the curved surface.

2. Put the lengths in miles

The radius OH is 3960 mi. The observer is h=6288/5280≈1.1909 mi above the surface, so OM=3960+h. The altitude alone is only the outermost part of OM.



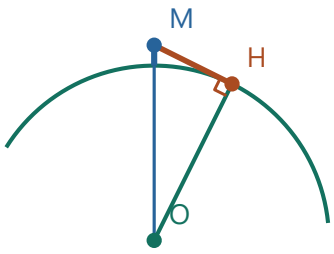
Altitude exaggerated

OH is the 3,960-mi radius; OM adds 6,288 ft above the surface.

The sight line is perpendicular to OH at H.

3. Identify the right angle

A radius to a point of tangency is perpendicular to the tangent there. In triangle OHM the right angle is at H, OM is the hypotenuse, and MH=d is the length sought.



Altitude exaggerated

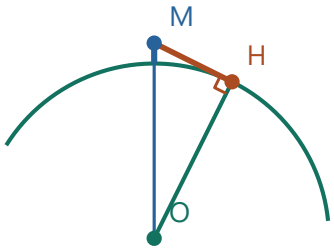
$OH^2 + HM^2 = OM^2$.

Subtract the squared radius before taking the square root.

4. Solve without rounding early

Pythagoras gives $d^2=(3960+h)^2-3960^2=h(7920+h)$. Taking the positive square root gives 97.126 mi, or 97 mi to the nearest mile.

Observer altitude · 6,288 ft



Altitude exaggerated

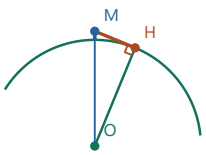
Altitude 6,288 ft = 1.191 mi.

Tangent distance √[h(7920+h)] = 97.126 mi; h is in miles.

1. $d^2+R^2=(R+h)^2$	The tangent and radius are the legs of the right triangle.
2. $d^2=R^2+2Rh+h^2-R^2$	Expand the squared hypotenuse and subtract the squared radius.
3. $d=\sqrt{h(2R+h)}$	Cancel R^2, factor out h, and choose the nonnegative length.

Compare settings · Observer altitude

1,000 ft



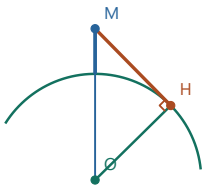
Altitude exaggerated

Altitude 1,000 ft = 0.189 mi.

Tangent distance √[h(7920+h)] = 38.73 mi; h is in miles.

Altitude 1,000 ft = 0.189 mi. Tangent distance √[h(7920+h)] = 38.73 mi; h is in miles.

52,000 ft



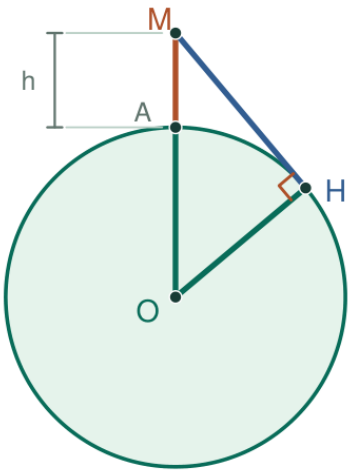
Altitude exaggerated

Altitude 52,000 ft = 9.848 mi.

Tangent distance √[h(7920+h)] = 279.458 mi; h is in miles.

Altitude 52,000 ft = 9.848 mi. Tangent distance √[h(7920+h)] = 279.458 mi; h is in miles.

A radius meets the horizon sight line



O · Earth's center

A · Surface beneath the observer

M · Observer on the mountain

H · Horizon point

OH = OA = 3960 mi

OM = 3960 mi + altitude h

MH = distance along the tangent

Altitude is exaggerated to make the tangent triangle visible.

Check. The small-height estimate √(2Rh) also rounds to 97 mi.

Exercise 2 · Stacking equal slices

Printed page 1 · Prism volume / Cross sections / Area of a circle

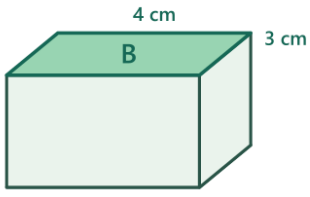
In mathematical discussion, a *right prism* is defined to be a solid figure that has two parallel, congruent polygonal bases, and rectangular *lateral faces*. How would you find the volume of such a figure? Explain your method. What about the volume of a cylinder?

1. Find the two bases

Look at the two matching end faces of the prism. Those are its bases. Congruent means that one fits exactly on the other. Parallel means that the distance between their planes is constant. The rectangular side faces connect corresponding edges. The model uses a rectangle as its base so you can count its area. The argument will work with any polygonal base.

2. Measure the whole shaded face

Call the area of one base B. The green shading covers the entire face. Its outline measures a perimeter, which would be a length. B measures how many square units cover the inside. In the starting example the rectangle is 4 cm by 3 cm, so B = 4 × 3 = 12 cm². Try changing its width and watch the shaded area change.



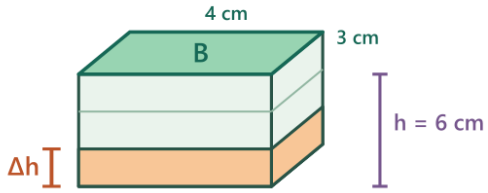
B = 4 × 3 = 12 cm²

3. Measure straight between the bases

The purple bracket measures h, the perpendicular distance from one base plane to the other. In a right prism the side edges run in this same direction. At any height, a cut parallel to the base produces a translated copy of the base. Each such cross section therefore has area B.

4. Give one layer a thickness

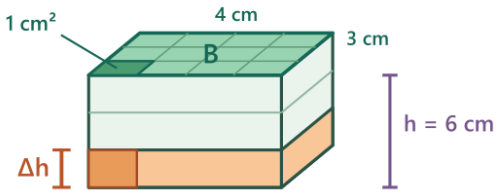
Imagine making N equal layers with cuts parallel to the base. Δ is the Greek letter delta. Here Δh means the change in height from the bottom to the top of one layer. The orange bracket spans exactly that gap. All N gaps together make the full height, so Δh = h/N. At h = 6 cm and N = 3, one layer is 2 cm thick.



$B = 4 \times 3 = 12\ \text{cm}^2$ $\Delta h = 6 / 3 = 2\ \text{cm}$

5. Measure what one layer holds

Cover the rectangular base with 1 cm by 1 cm squares. The dark green tile is one of these squares. Its area is 1 cm^2. The darker orange face belongs to its column in the bottom layer. Carried straight through a thickness Δh, it sweeps out a column of volume 1 $\text{cm}^2 \times \Delta h$. Adding across the whole base gives $B \times \Delta h$. With $B = 12\ \text{cm}^2$ and $\Delta h = 2$ cm, that is 24 cm^3 for one layer. For a polygon with angled edges, use a finer grid. The partial squares along its boundary occupy an ever smaller total area, so the same rule holds.

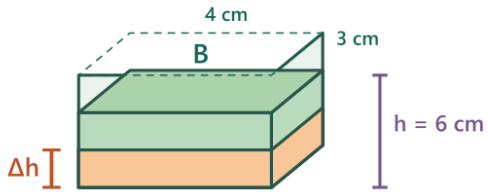


$B = 4 \times 3 = 12\ \text{cm}^2$ $\Delta h = 6 / 3 = 2\ \text{cm}$

6. Put all the layers back

The layers meet at faces and fill the prism without gaps. Add their volumes: V = N × B × Δh. Since N × Δh = h, this becomes V = Bh. Try doubling N with the height fixed. Each layer becomes half as thick, so the total stays the same. For the starting prism, 3 × 24 cm^3 = 12 cm^2 × 6 cm = 72 cm^3. Build the stack one layer at a time. Each new layer adds the same volume.

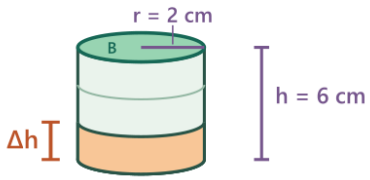
2 of 3 layers assembled



B = 4 × 3 = 12 cm^2 Δh = 6 / 3 = 2 cm

7. Trade the polygon for a circle

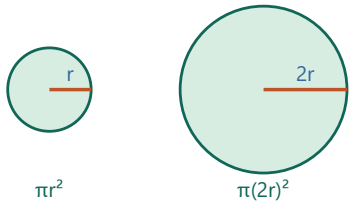
Look at the right circular cylinder. A cut parallel to its circular base is a disk of the same radius r at every height. The radius runs from the center to the rim. The disk area is B = πr^2. The same stack has volume V = Bh = $\pi r^2 h$. One way to justify the disk step is to fit regular polygons just inside and just outside the circle. Their prism volumes approach the same value as their base areas approach πr^2.



B = π × 2^2 ≈ 12.57 cm^2 Δh = 6 / 3 = 2 cm

8. Check what changes the answer

At r = 2 cm and h = 6 cm the cylinder holds 24π cm^3, about 75.40 cm^3. Raise h from 6 to 12 with N = 3. The same three layers become twice as thick, and the total volume doubles. Now change N to 6. Each layer is back to 2 cm thick and you have twice as many as at the start. Reset the dimensions and double r instead. Each disk has four times its original area, so volume quadruples. In both shapes, square centimeters times centimeters give cubic centimeters.



Top views

Same cylinder height h

1. $V(2r)=\pi(2r)^2h$
Change only the radius in the cylinder formula.

2. $V(2r)=4\pi r^2h=4V(r)$
Squaring the factor of two multiplies the circular base area by four.

Compare the two circular bases while holding the cylinder height fixed.

Check. Doubling height doubles the number of equal-thickness slices and doubles volume.

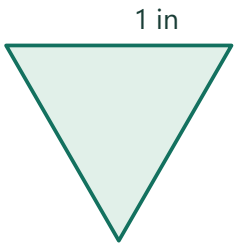
Exercise 3 · Triangular chocolates

Printed page 1 · Prism volume / Tessellations / Packing

A chocolate company has a new candy bar in the shape of a prism whose base is a 1-inch equilateral triangle and whose sides are rectangles that measure 1 inch by 2 inches. These prisms will be packed in a box that has a regular hexagonal base with 2-inch edges, and rectangular sides that are 6 inches tall. How many candy bars fit in such a box?

1. Look at a candy bar end-on

Stand the bars with their 2-inch edges vertical. Each bar then covers a unit equilateral triangle in the box's horizontal base and fills 2 inches of height.

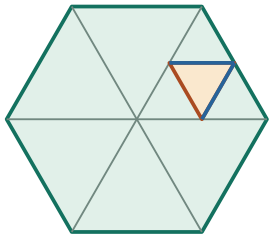


One triangular candy end.

Each bar rises 2 in above this end.

2. Tile one sixth of the base

Join the hexagon’s center to its six vertices. Each resulting triangle has sides of length 2. Joining the midpoints of its three sides cuts it into four equilateral triangles of side 1, including one pointed the other way.

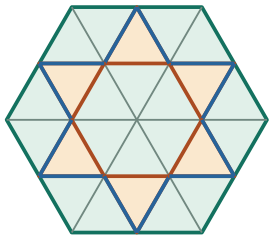


One of six sectors

Its three midsegments divide the 2-in triangle into four 1-in triangles.

3. Count one layer

Six large triangles each hold four candy-bar ends. That gives $6 \cdot 4 = 24$ bars in one layer, with the small triangular ends sharing edges and leaving no gaps.

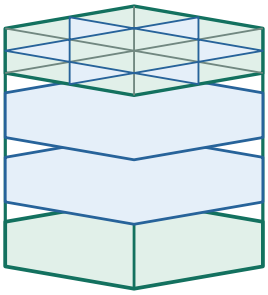


6 sectors × 4 ends = 24 bars

No gap remains across a layer.

4. Stack the layers

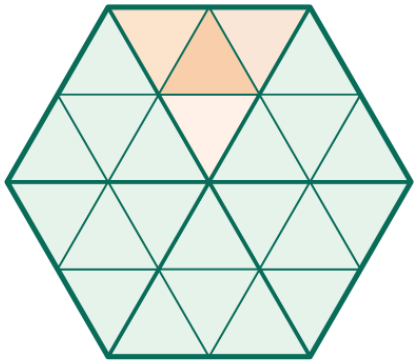
The 6-inch box height accommodates $6/2 = 3$ layers. The construction fills the entire box with $3 \cdot 24 = 72$ bars. A further bar would require extra volume.



Each 2-in tier contains 24 triangular bars.

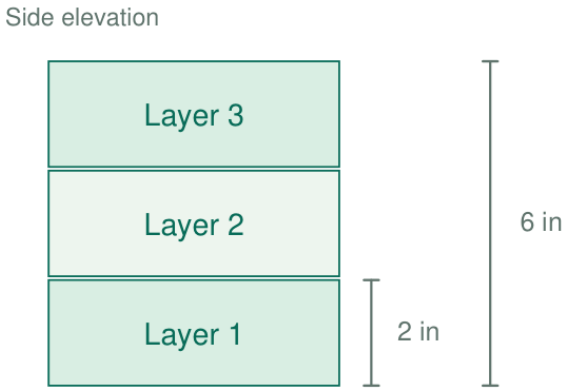
The full 6-in hexagonal prism takes $3 \times 24 = 72$.

One layer, seen from above



6 sectors × 4 unit triangles = 24 ends

Stack through the height



Each bar extends 2 in into one layer.

Check. The box-to-bar volume ratio is 24×3=72, so no packing can beat it.

Exercise 4 · Packing rectangular bars

Printed page 1 · Packing / Upper and lower bounds / Coloring arguments

(Continuation) The same company also markets a rectangular chocolate bar that measures 1 cm by 2 cm by 4 cm. How many of these bars can be packed in a rectangular box that measures 8 cm by 12 cm by 12 cm? How many of these bars can be packed in a rectangular box that measures 8 cm by 5 cm by 5 cm? How would you pack them?

1. Use the bar as a unit of volume

Each bar occupies 1·2·4=8 cm³. Division by 8 gives an upper bound, but demonstrating that many bars fit requires a physical arrangement.

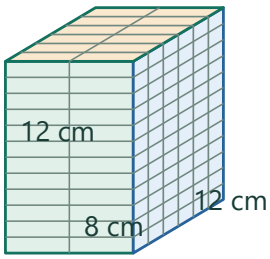


A bar has three perpendicular edges.

Its 4 × 2 × 1 cm³ volume gives 8 cm³ per bar.

2. Fill the larger box

Align the 4-cm edge along the 8-cm direction. Along the other two directions use the 2-cm and 1-cm edges. There are (8/4)(12/2)(12/1)=144 bars. Their total volume equals the box volume.

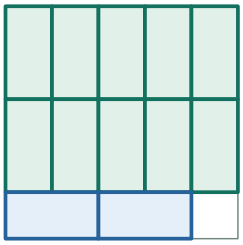


Follow the divisions along all three edges.

There are 2 × 6 × 12 = 144 bars.

3. Build the smaller arrangement

Look at the 5-by-5 end. Place ten 1-by-2 bar ends in the first four rows and two in the last row. The resulting 12 bars extend 4 cm into the box. Repeat once along its 8-cm length to fit 24 bars.

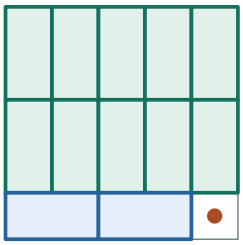


5 cm × 5 cm end

Ten vertical dominoes and two horizontal dominoes fit per 4 cm of depth.

4. Find the only possible improvement

This box holds 8·5·5=200 cm^3, exactly 25 bar volumes. If 25 could fit, no space could remain uncovered, including anywhere on the square end.

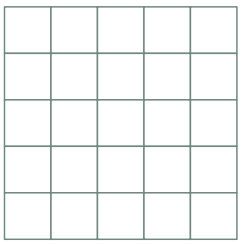


One square remains per 5 × 5 end

25 bars would have to fill the box completely.

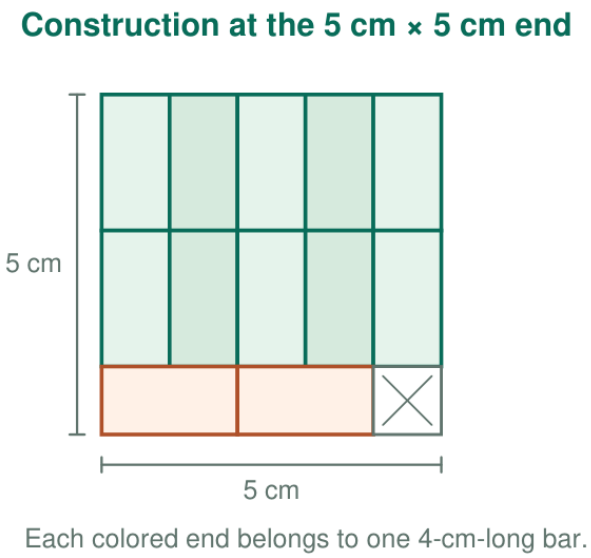
5. Test whether that end can be tiled

A bar touching the end in positive area must have one whole face in its plane. An edge or corner contributes zero area. Each possible face area is 2, 4, or 8 cm². These even areas cannot sum to 25 cm², so 24 is the maximum even with tilted bars.



25 one-square-centimeter cells

An end tiled by complete faces of areas 2, 4 or 8 must have even area.



Check. The bound allows arbitrary orientations. A bar inside the box can meet a supporting end plane only in a full face, an edge or a point. Edges and points contribute zero area.

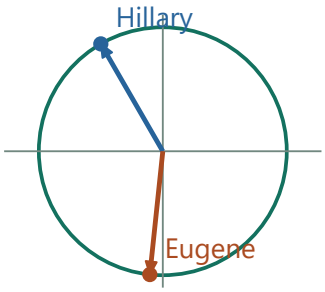
Exercise 5 · Runners on a track

Printed page 1 · Angular and linear speed / Arc length / Chords and inscribed angles

Starting at the same spot on a circular track that is 80 meters in diameter, Hillary and Eugene run in opposite directions, at 300 meters per minute and 240 meters per minute, respectively. They run for 50 minutes. What distance separates Hillary and Eugene when they finish? There is more than one way to interpret the word distance in this question.

1. Track one runner from the other

In one minute a 300 m advance in one direction and a 240 m advance in the opposite direction change their separation along the track by 540 m. After 50 min the relative travel is 27,000 m.

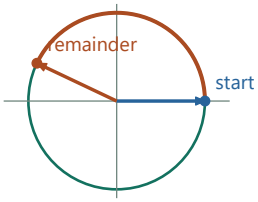


Opposite directions, schematic positions.

Relative travel each minute is 300 + 240 = 540 m.

2. Remove whole laps

A lap is π times the diameter, or 80π m. After subtracting 107 whole laps, the remaining arc measures $27{,}000 - 107 \cdot 80\pi \approx 107.967$ m.



Remove 107 laps of 80π m.

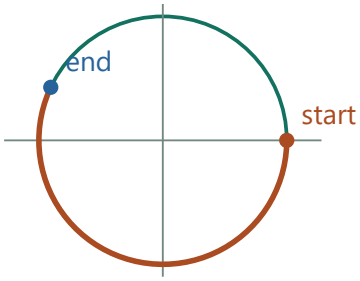
The remaining arc measures about 107.967 m.

1. $27{,}000/(80\pi) \approx 107.43$
Divide relative travel by the length of one lap.

2. $27{,}000 - 107(80\pi) \approx 107.97$ m
Subtract only completed laps to keep the remaining arc.

3. Compare the two curved routes

The complementary arc is $80\pi - 107.967 \approx 143.361$ m. The smaller number is the shortest distance if travel must stay on the track.



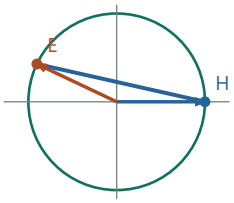
Two arcs join the endpoints.

The other way around is 143.361 m.

4. Cut across the circle

For a straight-line distance use the chord. The radius is 40, so the short arc s subtends s/40 radians. Halving the central triangle gives chord=2·40 sin(s/80)≈78.051 m.

Angular separation · 154.7°



For the 40-m track radius, arc = 107.967 m.

The chord spans 78.051 m.

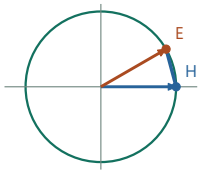
1. θ=s/40
Divide the short arc by the radius to measure its angle in radians.

2. (c/2)/40=sin(θ/2)
The radius perpendicular to the chord bisects both the chord and its central angle.

3. c=80 sin(s/80)
Multiply by 80 and substitute θ=s/40.

Compare settings · Angular separation

30.0°

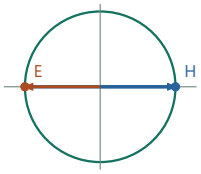


For the 40-m track radius, arc = 20.944 m.

The chord spans 20.706 m.

For the 40-m track radius, arc = 20.944 m. The chord spans 20.706 m.

180.0°



For the 40-m track radius, arc = 125.664 m.

The chord spans 80 m.

For the 40-m track radius, arc = 125.664 m. The chord spans 80 m.

Check. The two arc lengths add to 80π. The chord is shorter than either arc and smaller than the diameter.

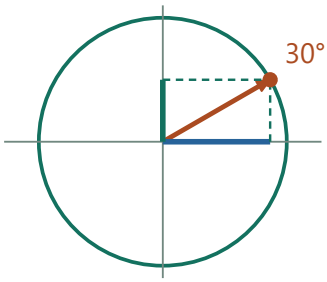
Exercise 6 · A trigonometric invariant

Printed page 1 · Trigonometric identities / The unit circle

Choose a positive number θ (*Greek* "theta") less than 90.0 and use a calculator to find $\sin\theta$ and $\cos\theta$. Square these numbers and add them. Could you have predicted the sum?

1. Pick an angle and a unit

Use 30°, which is inside the specified interval. Put the calculator in degree mode. Its two readings are sin 30°=1/2 and cos 30°=√3/2≈0.8660254.



Unit radius at 30°

Horizontal = √3/2; vertical = 1/2.

2. Square each reading separately

The square of the sine is 1/4. The square of the cosine is 3/4. Their sum is 1. A rounded decimal cosine can make the calculator's sum appear slightly off.



Squared projections

The cosine contributes 3/4 and the sine contributes 1/4.

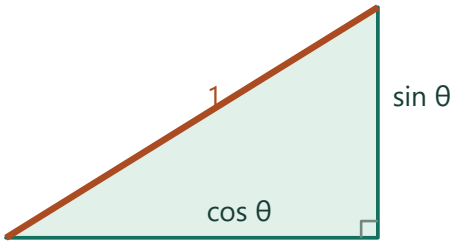
1. $\sin^2 30° = (1/2)^2 = 1^2/2^2 = 1/4$
Square both parts of the exact fraction.

2. $\cos^2 30° = (\sqrt{3}/2)^2 = 3/4$
The square undoes the numerator's square root.

3. $1/4 + 3/4 = (1+3)/4 = 1$
Equal denominators let us add the two numerators.

3. Explain why the number was predictable

On the unit circle the point at angle θ has horizontal coordinate cos θ and vertical coordinate sin θ. These are the legs of a right triangle whose hypotenuse is the unit radius. Pythagoras gives $\cos^2\theta + \sin^2\theta = 1$.



Schematic section

Pythagoras

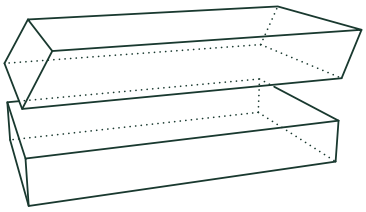
The same two perpendicular legs give $\cos^2\theta + \sin^2\theta = 1$.

Check. On a right triangle, dividing the Pythagorean equation by the squared hypotenuse gives the same identity.

Exercise 7 · Shearing a deck

Printed page 1 · Cavalieri's principle / Volume

Playing cards measure 2.25 inches by 3.5 inches. A full deck of fifty-two cards is 0.75 inches high. What is the volume of a deck of cards? If the cards were uniformly shifted (turning the bottom illustration into the top illustration), would this volume be affected? How might you generalize this?



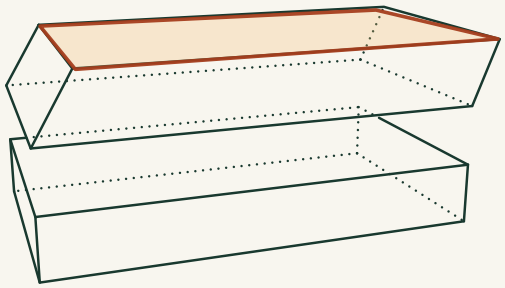
1. Find the area under one card

One flat card covers $2.25 \cdot 3.5 = 7.875$ in^2. The stack occupies this same horizontal area throughout its 0.75-inch height.



2.25 in × 3.5 in

The area of a single face is 7.875 in^2.

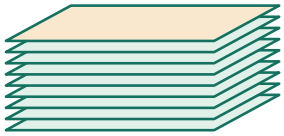


Top face of a shifted card

The outline on top is a card viewed in perspective. Sliding that card preserves its face area.

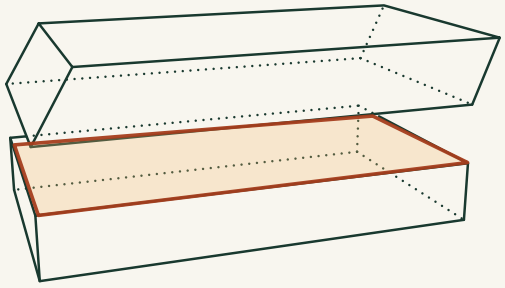
2. Compute the original volume

Area times perpendicular height gives $7.875 \cdot 0.75 = 5.90625$ in^3. The full stack height already counts the thicknesses of all 52 cards.



52 cards span 0.75 in altogether.

The displayed layers represent the full deck.



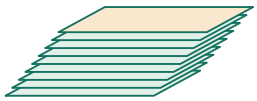
Aligned deck before sliding

In the lower drawing the stacked faces sit directly above one another. Compare their footprint with the translated layers above.

3. Slide the cards in the picture

Follow one card as it shifts sideways. Neither its face area nor its thickness changes. Each card still contributes exactly its original volume, so adding the 52 contributions gives 5.90625 in^3 again.

Sideways shift per displayed layer · 5 drawing units



Shift each displayed slice by 5 drawing units.

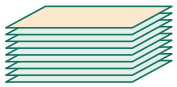
The physical 52-card deck still contains 5.90625 in^3: each card keeps its size and thickness.

1. $t_{card} = 0.75/52$ in
The cards have equal thickness, so divide the total height by their number.

2. $52V_{card} = 52 \cdot 7.875 \cdot (0.75/52) = 7.875 \cdot 0.75$ in^3
Horizontal displacement preserves each card's dimensions. Summing their volumes cancels the factor of 52.

Compare settings · Sideways shift per displayed layer

0 drawing units

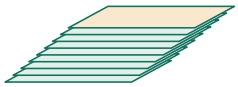


Shift each displayed slice by 0 drawing units.

The physical 52-card deck still contains 5.90625 in^3: each card keeps its size and thickness.

Shift each displayed slice by 0 drawing units. The physical 52-card deck still contains 5.90625 in^3: each card keeps its size and thickness.

7 drawing units



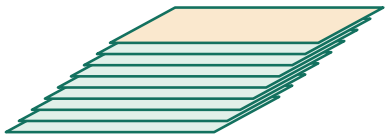
Shift each displayed slice by 7 drawing units.

The physical 52-card deck still contains 5.90625 in^3: each card keeps its size and thickness.

Shift each displayed slice by 7 drawing units. The physical 52-card deck still contains 5.90625 in^3: each card keeps its size and thickness.

4. Extend the argument

Two solids of equal height have equal volume when their sections at every corresponding height have equal area. Moving sections sideways changes the outline while preserving this volume comparison.



At every height the horizontal area is unchanged.

Consequently the total remains 5.90625 in^3.

Check. Equal horizontal cross-sectional areas at equal heights give equal volumes by Cavalieri's principle.

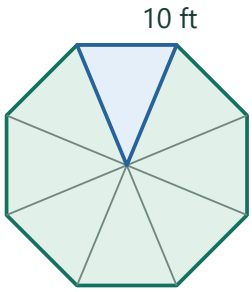
Exercise 8 · An octagonal silo

Printed page 1 · Regular polygons / Prism volume / Right-triangle trigonometry

In the middle of the nineteenth century, octagonal barns and silos (and even some houses) became popular. How many cubic feet of grain would an octagonal silo hold if it were 12 feet tall and had a regular base with 10-foot edges?

1. Use the center of the octagon

Connecting the center to its eight vertices gives eight congruent triangles. Each has central angle 360°/8=45° and base 10 ft.



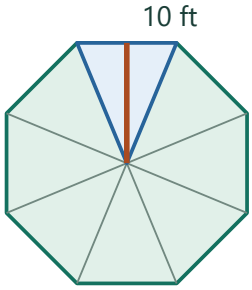
Eight center triangles

Each spans 360°/8 = 45°.

2. Find one triangle's height

Drop a perpendicular to the midpoint of a 10-ft side. The resulting right triangle has half-angle 22.5°, opposite side 5 and adjacent side a, the apothem. Hence tan 22.5°=5/a.

Regular polygon side count · 8 sides

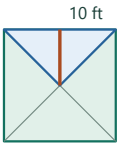


Keep the side 10 ft.

Half the central angle is 22.5°. Apothem = 5/tan(22.5°) = 12.071 ft.

Compare settings · Regular polygon side count

4 sides

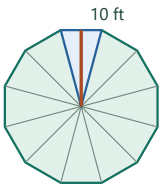


Keep the side 10 ft.

Half the central angle is 45°. Apothem = 5/tan(45°) = 5 ft.

Keep the side 10 ft. Half the central angle is 45°. Apothem = 5/tan(45°) = 5 ft.

12 sides



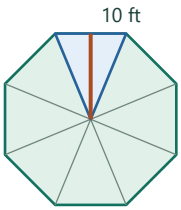
Keep the side 10 ft.

Half the central angle is 15°. Apothem = 5/tan(15°) = 18.66 ft.

Keep the side 10 ft. Half the central angle is 15°. Apothem = 5/tan(15°) = 18.66 ft.

3. Assemble the floor area

One triangle has area 10a/2. Eight give B=40a=200 cot 22.5°=200(1+√2) ft².



B = 8(10a/2) = 40a

The apothem a is each triangle's altitude.

1. a=5/tan(22.5°)
Multiply tan(22.5°)=5/a by a and divide by the nonzero tangent.

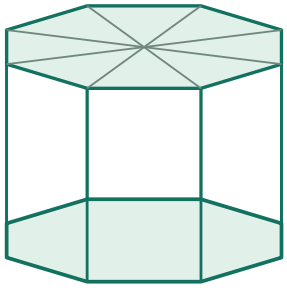
2. cot(22.5°)=(1+cos45°)/sin45°
Use the half-angle identity for the apothem ratio.

3. (1+√2/2)/(√2/2)=1+√2
Multiply numerator and denominator by √2.

4. B=8(10a/2)=200(1+√2) ft²
Add the areas of eight base triangles.

4. Fill the silo

Each horizontal layer covers B. Multiplying by the 12-ft height gives 2400(1+√2)≈5794.1 ft³ of grain.



Stack the octagonal floor vertically for 12 ft.

Volume = 12 × 200(1+√2) ≈ 5,794.1 ft³.

Check. Multiplying half the perimeter by the apothem gives the same base area.

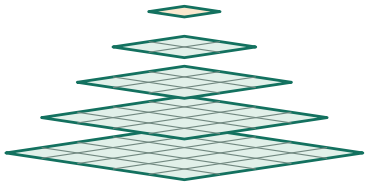
Exercise 9 · Five tiers of cubes

Printed page 1 · Sums of squares / Area and volume scaling

Build a sugar-cube pyramid as follows: First make a $5 \times 5 \times 1$ bottom layer. Then center a $4 \times 4 \times 1$ layer on the first layer, center a $3 \times 3 \times 1$ layer on the second layer, and center a $2 \times 2 \times 1$ layer on the third layer. The fifth layer is a single $1 \times 1 \times 1$ cube. Express the volume of this pyramid as a percentage of the volume of a $5 \times 5 \times 5$ cube.

1. Count by floors

All cubes have equal volume, so a ratio of cube counts is also a ratio of volumes. The bottom floor is a five-by-five square containing 25 cubes.



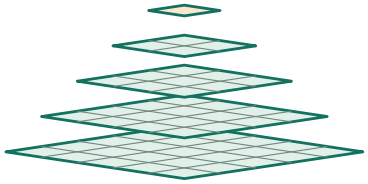
Five square floors

The bottom floor is 5×5.

2. Work upward

Each next floor loses one cube from each side length. The remaining counts are $4^2=16$, $3^2=9$, $2^2=4$, and $1^2=1$. Centering them changes their positions but preserves these counts.

Number of floors · 5 floors



Square floors from 1^2 to 5^2 give 55 cubes.

The containing cube holds 125.

Compare settings · Number of floors

2 floors

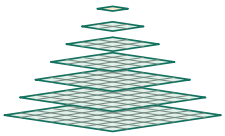


Square floors from 1^2 to 2^2 give 5 cubes.

The containing cube holds 8.

Square floors from 1^2 to 2^2 give 5 cubes. The containing cube holds 8.

7 floors



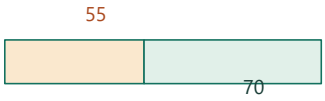
Square floors from 1^2 to 7^2 give 140 cubes.

The containing cube holds 343.

Square floors from 1^2 to 7^2 give 140 cubes. The containing cube holds 343.

3. Compare with a full cube

The tiered solid contains 25+16+9+4+1=55 cubes. A 5-by-5-by-5 cube contains 125. The percentage is (55/125)·100=44%.



55 of 125 cubes are present.

The occupied fraction is 44%.

1. 55/125=(55÷5)/(125÷5)=11/25
Reduce the ratio of occupied cubes to available cubes.

2. 11/25=(11·4)/(25·4)=44/100
A denominator of 100 gives the volume percentage directly.

Check. Pairing outer tiers gives (25+1)+(16+4)+9=55.

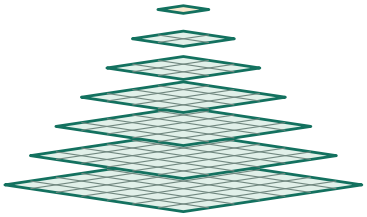
Exercise 10 · Square layers in the limit

Printed page 1 · Sums of squares / Limits and convergence

(Continuation) Repeat the sugar-cube construction, starting with a $10 \times 10 \times 1$ base, the dimensions of each square decreasing by one unit per layer. Using a calculator, express the volume of the pyramid as a percentage of the volume of a $10 \times 10 \times 10$ cube. Repeat, using $20 \times 20 \times 1$, and $30 \times 30 \times 1$. Do you see a trend?

1. Identify what changes from tier to tier

Starting with n cubes on a side gives n layers. Reading from the top down, their counts are 1^2, 2^2, …, n^2. Their sum S is the number used.

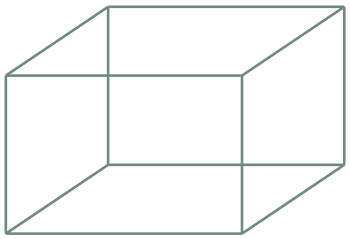


The top contributes 1^2 and the bottom n^2.

A square of each possible side length occurs.

2. Choose the comparison for each base

A complete cube of side n contains n^3 unit cubes. Compare S with n^3, so the denominator changes each time n changes.



Bounding cube

Its width, depth and height all equal n.

3. Compute the three percentages

The sums for n=10, 20, 30 are 385, 2870 and 9455. Divide respectively by 1000, 8000 and 27000, then multiply by 100. The results are 38.5%, 35.875%, and about 35.0185%.

tiers	cubes	occupied
10	385	38.5%
20	2870	35.875%
30	9455	35.019%

Divide the step-pyramid count by n^3.

The three base sizes use 38.5%, 35.875%, and about 35.0185% of their cubes.

1. $S_{10}=10\cdot11\cdot21/6=385$, $S_{20}=20\cdot21\cdot41/6=2870$
Substitute each base length into the sum of the first n squares.

2. $S_{30}=30\cdot31\cdot61/6=5\cdot31\cdot61=9455$
The factor of six cancels against thirty before multiplication.

3. $100\cdot9455/30^3=945500/27000\approx35.0185\%$
Compare against the cube with the same 30-unit edge.

4. Explain the observed trend

Using $S=n(n+1)(2n+1)/6$, the ratio is $1/3+1/(2n)+1/(6n^2)$. Both positive extra terms shrink as n increases, so the percentages decrease toward 33⅓%.



Limit of the ratio

$1/3 + 1/(2n) + 1/(6n^2)$ approaches 1/3.

1. $S/n^3=(n+1)(2n+1)/(6n^2)$
Divide the square sum by n^3 and cancel one positive factor n.

2. $=(2n^2+3n+1)/(6n^2)$
Distribute each term of n+1 across 2n+1.

3. $=1/3+1/(2n)+1/(6n^2)$
Divide each numerator term by the same denominator.

Check. Expanding yields $1/3+1/(2n)+1/(6n^2)$, so the excess above one-third tends to zero.

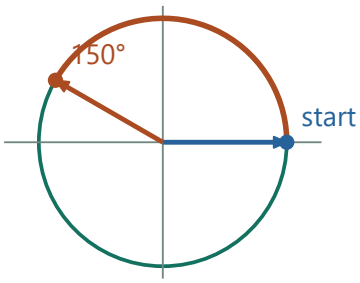
Exercise 11 · Components at 150 degrees

Printed page 2 · Vectors and components / Trigonometric ratios

A vector $\mathbf{v}$ of length 6 makes a 150-degree angle with the vector $[1, 0]$, when they are placed *tail-to-tail*. Find the components of $\mathbf{v}$.

1. Start from the reference vector

The vector (1,0) points along the positive horizontal axis. A unit arrow making directed angle θ with that axis ends at (cos θ,sin θ).

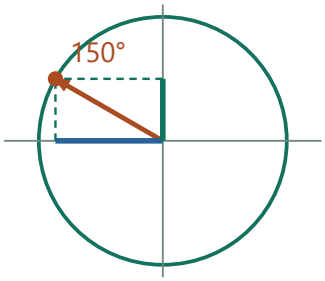


Turn from the positive horizontal axis.

The radius length is 6.

2. Locate the 150° ray

A counterclockwise rotation of 150° stops 30° above the negative x-axis. Its horizontal projection is negative and its vertical projection positive. The unit direction is (−√3/2,1/2).



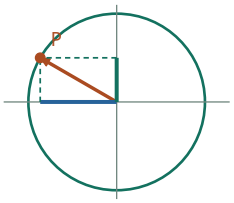
Horizontal projection = 6 cos 150°.

Vertical projection = 6 sin 150°.

3. Scale to the stated length

Multiply both projections by 6. The components are (−3√3,3), whose squares sum to 36. If “angle” here is unsigned, the reflected vector (−3√3,−3) makes the same tail-to-tail angle.

Radius direction · 150°



Radius 6; angle 150°.

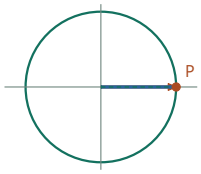
Projection onto x = −5.196. Projection onto y = 3.

1. $v = 6(-\sqrt{3}/2, 1/2) = (-3\sqrt{3}, 3)$
Scale each unit-vector component by the vector length.

2. $\|v\|^2 = (-3\sqrt{3})^2 + 3^2 = 27 + 9 = 36$
The squared projections add to the squared length.

Compare settings · Radius direction

0°

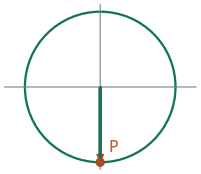


Radius 6; angle 0°.

Projection onto x = 6. Projection onto y = 0.

Radius 6; angle 0°. Projection onto x = 6. Projection onto y = 0. The drawing uses a counterclockwise direction. For an unsigned angle, the reflected point has the opposite vertical coordinate.

270°



Radius 6; angle 270°.

Projection onto x = −0. Projection onto y = −6.

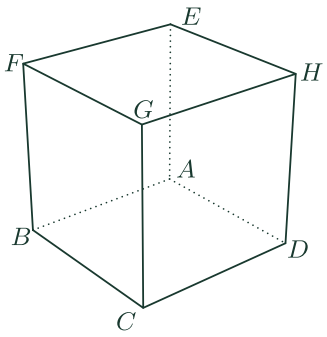
Radius 6; angle 270°. Projection onto x = −0. Projection onto y = −6. The drawing uses a counterclockwise direction. For an unsigned angle, the reflected point has the opposite vertical coordinate.

Check. Both have length √(27+9)=6 and x component divided by length equals cos150°.

Exercise 12 · Three pyramids in a cube

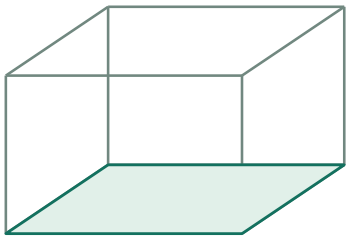
Printed page 2 · Pyramid volume / Symmetry

Given that $ABCDEFGH$ is a cube (shown at right), what is significant about the square pyramids $ADHEG$, $ABCDG$, and $ABFEG$?



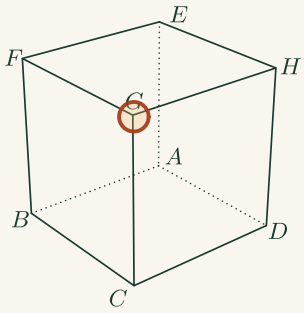
1. Identify the common tip

The final letter G names the apex of each pyramid. Their bases are the square faces ADHE, ABCD, and ABFE, the three faces opposite G.



Choose one base face opposite G.

The cube edge is s.

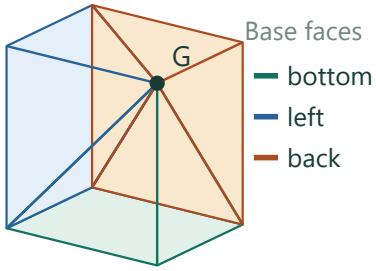


Common apex G

All three pyramid names end in G. Follow their edges back to this same cube vertex.

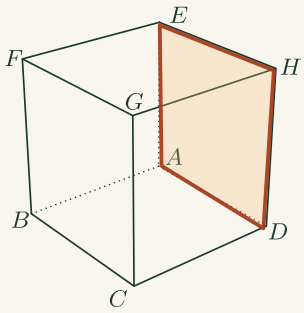
2. Account for the whole cube

Trace a ray from G through an interior point until it leaves the cube. Its exit belongs to one of these three opposite faces, placing that point inside the corresponding pyramid. The dividing faces between pyramids have zero volume.



Each of the three faces opposite G serves as one base.

Join each base to G. The three pyramids meet along common triangular boundaries.

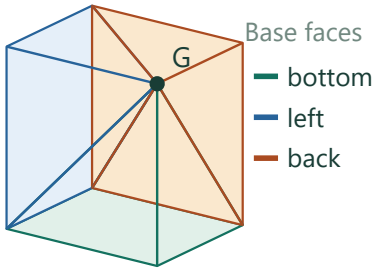


Base ADHE

The third square meets the first two along cube edges. The pyramids can share boundary points while dividing the full volume.

3. Use cube symmetry

A symmetry about the body diagonal through G permutes the three square bases. The three pyramids are congruent and have equal volumes. Together they make the cube, so each has volume $s^3/3$ for cube edge s.



Each pyramid uses an entire square face and the same perpendicular height s.

A symmetry permutes the three pyramids, so each occupies $s^3/3$.

Check. The pyramid formula gives $(1/3)s^2 \cdot s$ for each.

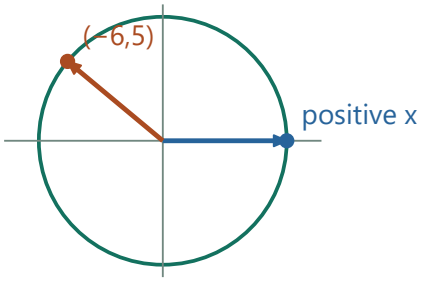
Exercise 13 · Angle from vector components

Printed page 2 · Vectors and components / Inverse trigonometric functions

To the nearest tenth of a degree, find the size of the angle formed by placing the vectors $[4, 0]$ and $[-6, 5]$ tail-to-tail at the origin. It is understood in questions such as this that the answer is smaller than 180 degrees.

1. Place both tails at the origin

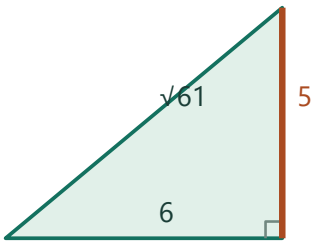
The first vector (4,0) sets the positive horizontal reference. The other endpoint has x=−6 and y=5, above and to the left of the origin.



The negative x and positive y components place the arrow in quadrant II.

2. Read the small reference triangle

Between the second vector and the negative x-axis, the acute reference angle α has opposite side 5 and adjacent side 6, so α=arctan(5/6).

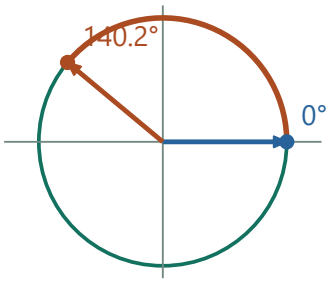


The reference angle has tan α = 5/6.

This is the acute angle above the negative x-axis.

3. Measure from the positive axis

A straight angle from the positive to the negative x-axis is 180°. The desired angle stops α short of it. Hence θ=180°−arctan(5/6)≈140.2°.



Measure from positive x.

The requested angle is 180° − arctan(5/6).

Check. The cosine from the dot product is −6/√61.

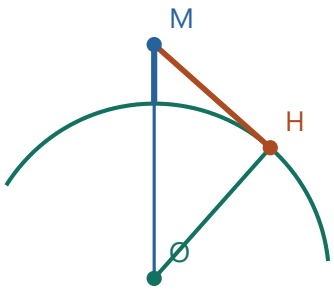
Exercise 14 · Airplane horizon

Printed page 2 · The Pythagorean theorem / Tangent lines / Unit conversion

Flying at an altitude of 39 000 feet one clear day, Cameron looked out the window of the airplane and wondered how far it was to the horizon. Rounding your answer to the nearest mile, answer Cameron's question.

1. Locate the tangent sight line

Let O be Earth's center, M the airplane and H the horizon. OH=3960 mi and MH is tangent at H, so OH is perpendicular to MH. The sought distance is MH.



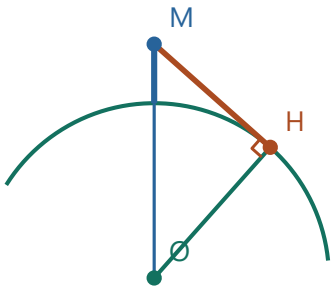
Altitude exaggerated

O is Earth's center; H is the contact point.

The sight line from the observer M just touches the curved surface.

2. Match units and lengths

The plane is h=39000/5280≈7.3864 mi above the surface. From Earth's center to the plane the distance is 3960+h; this is the hypotenuse OM.



Altitude exaggerated

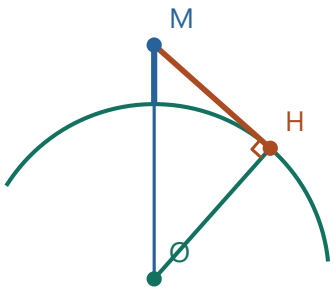
OH is the 3,960-mi radius; OM adds 39,000 ft above the surface.

The sight line is perpendicular to OH at H.

3. Calculate and round once

Pythagoras yields $MH^2=(3960+h)^2-3960^2=h(7920+h)$. The positive root is 241.980 mi, giving 242 mi to the nearest mile.

Observer altitude · 39,000 ft



Altitude exaggerated

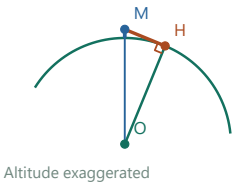
Altitude 39,000 ft = 7.386 mi.

Tangent distance √[h(7920+h)] = 241.98 mi; h is in miles.

1. $MH^2=(R+h)^2-R^2$	Move the radius term across the Pythagorean equation.
2. $=(R+h-R)(R+h+R)=h(2R+h)$	Factor the difference of squares before evaluating.
3. $MH=\sqrt{[h(7920+h)]}$	Use R=3960 miles and the positive length.

Compare settings · Observer altitude

1,000 ft

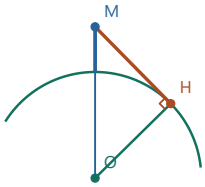


Altitude exaggerated

Altitude 1,000 ft = 0.189 mi.

Tangent distance √[h(7920+h)] = 38.73 mi; h is in miles.

52,000 ft



Altitude exaggerated

Altitude 52,000 ft = 9.848 mi.

Tangent distance √[h(7920+h)] = 279.458 mi; h is in miles.

Altitude 1,000 ft = 0.189 mi. Tangent distance √[h(7920+h)] = 38.73 mi; h is in miles.

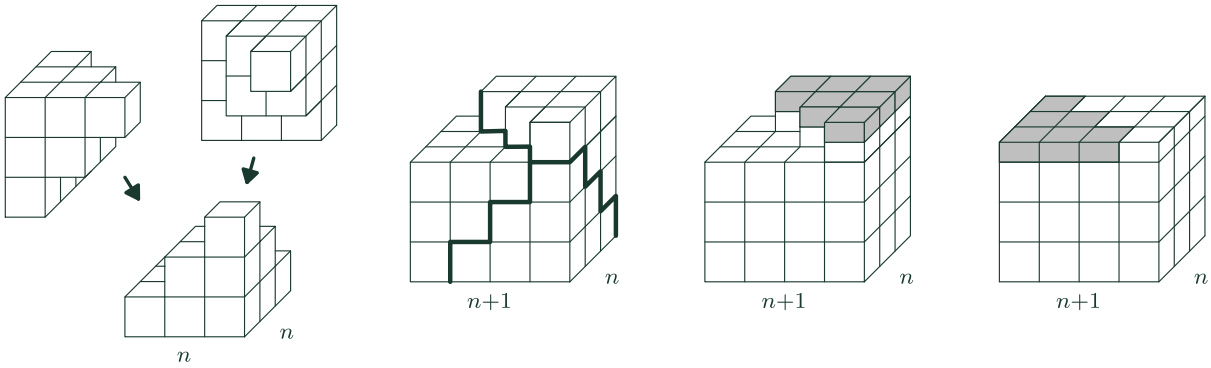
Altitude 52,000 ft = 9.848 mi. Tangent distance √[h(7920+h)] = 279.458 mi; h is in miles.

Check. √(2Rh)≈241.9 mi independently confirms the rounding.

Exercise 15 · Sum of square layers

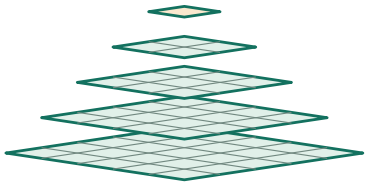
Printed page 2 · Sums of squares / Area and volume scaling / Limits and convergence

Explain how the diagrams lead to a simplified expression for $1^2 + 2^2 + 3^2 + \cdots + n^2$. Does this explain the trend in #10? Why or why not?



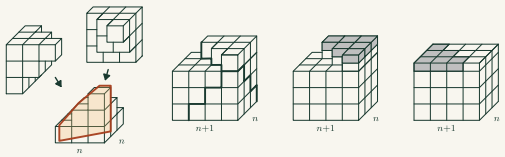
1. Read the stepped solid as a sum

One solid has floors of area 1^2, 2^2, …, n^2 and unit thickness. Its volume is S, the sum we want. Rearranging three identical copies preserves their total volume 3S.



$S = 1^2 + 2^2 + \ldots + n^2$

Each tier adds a new square.

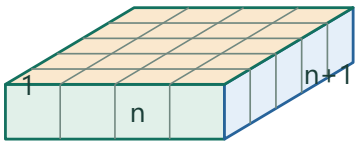


Stepped stack of square layers

Read the horizontal tiers from a small square at the top to increasingly large squares below.

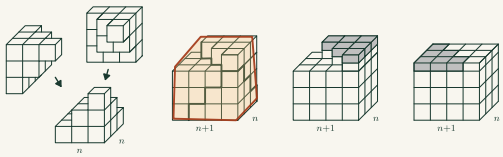
2. Read the footprint of the combined block

The base marked n by n+1 has area n(n+1). Above it are n full layers and an extra amount equal to half a layer. In the last pictures the shaded half-height pieces redistribute that extra material across the top.



Example footprint for n=4.

One complete layer of the rearranged box has n(n+1)=20 square positions.

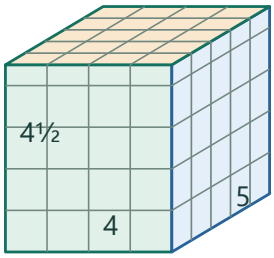


Combining the stepped pieces

The middle assembly joins copies of the same stack. A rearrangement changes how the pieces meet, not the number of unit cubes.

3. Turn the reassembly into algebra

The block's total effective height is n+½. Thus 3S=n(n+1)(n+½). Divide by three and write n+½=(2n+1)/2 to obtain S=n(n+1)(2n+1)/6.



Three 4-tier staircases have 3(1+4+9+16)=90 unit cubes.

Their volume equals 4 × 5 × 4½; the last layer is half the usual thickness in this equivalent-volume box.

4. Return to the sugar-cube percentages

A full n-sided cube has volume n^3. The ratio $S/n^3=1/3+1/(2n)+1/(6n^2)$. Increasing n reduces both positive correction terms, explaining the decrease in exercise 10 toward one third.



Divide by n^3.

The extra positive terms approach zero.

1. $S/n^3=(n+1)(2n+1)/(6n^2)$
Cancel one common n from the volume comparison; n counts layers and is positive.

2. $=(2n^2+3n+1)/(6n^2)$
Expand the two factors in the numerator.

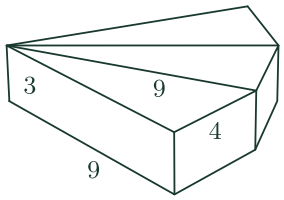
3. $=1/3+1/(2n)+1/(6n^2)$
Separate the numerator terms so their limiting sizes are visible.

Check. Subtracting the formula for n−1 from that for n gives n^2, with starting value 1 at n=1.

Exercise 16 · Cheese wedges

Printed page 2 · Packing / Isosceles triangles / Inverse trigonometric functions

A triangular prism of cheese is measured and found to be 3 inches tall. The edges of its base are 9, 9, and 4 inches long. Several congruent prisms are to be arranged around a common 3-inch segment, as shown. How many prisms can be accommodated? To the nearest cubic inch, what is their total volume?



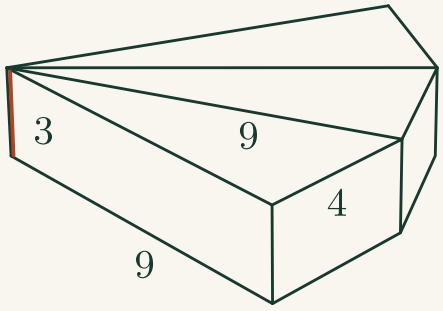
1. Look down the shared vertical edge

The 3-inch prism edges point toward the viewer in this end view. Each cheese wedge occupies the angle of its triangular base at the shared corner. Use the narrowest available corner to fit the largest number.



Look along the 3-in prism edge.

The two 9-in sides meet at the narrow corner.



Shared vertical edge

The short upright at the left is the common 3-inch edge. Looking down it turns the packing question into one about angles in the base.

2. Find that angle

The shortest side, length 4, is opposite the smallest angle. Bisect the isosceles 9–9–4 triangle. Each half has hypotenuse 9 and side 2 opposite half the apex angle. Thus $\theta = 2\arcsin(2/9) \approx 25.68°$.



The half-apex angle has sine 2/9.

The whole wedge angle is ≈ 25.68°.

1. $\sin(\theta/2) = 2/9$
The bisector makes the opposite side half the four-inch edge.

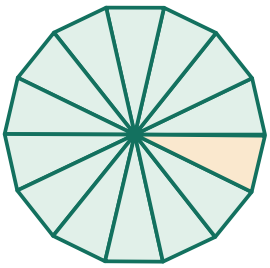
2. $\theta/2 = \arcsin(2/9)$
Use the acute angle in one half of the wedge.

3. $\theta = 2\arcsin(2/9)$
Double the half-angle to recover the full opening.

3. Fit only complete wedges

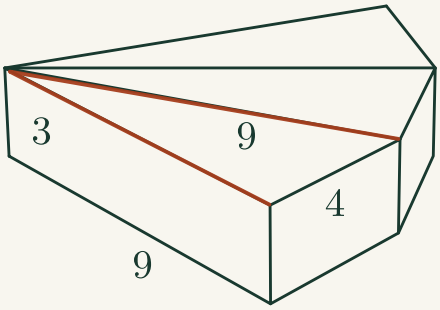
Fourteen such angles total about 359.51°, so 14 fit. Fifteen would need at least 385.19°. Choosing a wider corner would use still more angle.

Complete wedges · 14 wedges



Each wedge uses 25.679°.

14 wedges use 359.508°; 0.492° remains.



Narrow corner between equal sides

From that left-hand apex the two long sides of one triangular end extend rightward. The angle between them consumes part of the 360-degree turn.

Compare settings · Complete wedges

5 wedges

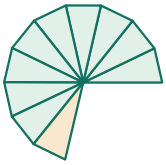


Each wedge uses 25.679°.

5 wedges use 128.396°; 231.604° remains.

Each wedge uses 25.679°. 5 wedges use 128.396°; 231.604° remains.

10 wedges



Each wedge uses 25.679°.

10 wedges use 256.792°; 103.208° remains.

Each wedge uses 25.679°. 10 wedges use 256.792°; 103.208° remains.

4. Calculate the occupied volume

The perpendicular height of each triangular base is $\sqrt{(9^2-2^2)}=\sqrt{77}$. Its area is $4\sqrt{77}/2=2\sqrt{77}$. Multiply by prism height 3 and by 14 wedges: $84\sqrt{77}\approx737$ in^3.



Each prism contains $6\sqrt{77}$ in^3.

Fourteen contain $84\sqrt{77}$ in^3.

Check. The law of cosines gives $\cos(\text{apex})=(81+81-16)/162=73/81$.

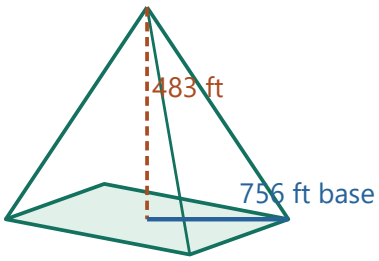
Exercise 17 · Stone in a pyramid

Printed page 2 · Pyramid volume / Density

The Great Pyramid at Gizeh was originally 483 feet tall, and it had a square base that was 756 feet on a side. It was built from rectangular stone blocks measuring 7 feet by 7 feet by 14 feet. Such a block weighs seventy tons. Approximately how many tons of stone were used to build the Great Pyramid? The volume of a pyramid is one third the base area times the height. What do you think the volume of a cone is?

1. Calculate the whole pyramid's volume

The square base area is 756^2 ft^2. Use perpendicular height 483 ft in the supplied pyramid formula to get $756^2 \cdot 483/3$ ft^3.

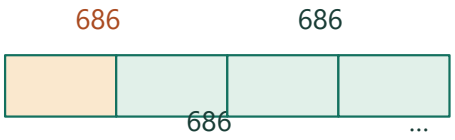


Square base and perpendicular altitude

$V = 756^2 \times 483 / 3$ cubic feet.

2. Estimate the block count

One block occupies $7\cdot7\cdot14=686$ ft³. Dividing the whole pyramid's volume by this block volume estimates how many equivalent solid blocks it contains.



One representative block occupies 686 ft³.

Divide total volume by volume per block.

3. Convert volume into weight

Each block weighs 70 tons, so the estimate is $(756^2\cdot483/3)\cdot70/686\approx9{,}389{,}520$ tons. The estimate treats the interior as solid stone of the same density.

blocks	×70 tons
one	70

Multiply the estimated block count by 70 tons.

The estimated total is about 9.39 million tons.

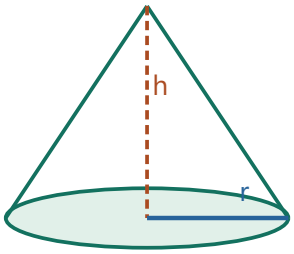
1. $V=756^2\cdot483/3=92{,}017{,}296$ ft³
Compute the square pyramid's volume first.

2. $N=92{,}017{,}296/686$
Divide by cubic feet per block.

3. $\text{mass}=N\cdot70=9{,}389{,}520$ tons
Multiply the equivalent block count by tons per block.

4. Apply the same base-area rule to a cone

Parallel sections of a cone shrink by the same length factor as sections of a pyramid at the same relative height. Thus equal base area and height give equal volumes. A cone with base area πr^2 has volume $\pi r^2 h/3$.



Replace the square base by a disk.

$V = \pi r^2 h/3$.

Check. Both solids have cross-sectional areas proportional to the square of distance from their tips.

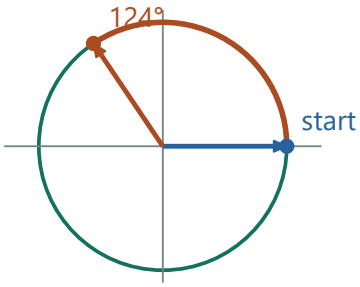
Exercise 18 · A vector of length twelve

Printed page 2 · Vectors and components / Trigonometric ratios

The angle formed by placing the vectors $[4, 0]$ and $[a, b]$ tail-to-tail at the origin is 124 degrees. The length of $[a, b]$ is 12. Find a and b.

1. Use the reference direction

The vector (4,0) points due right, so it sets the positive x direction regardless of its length. The new arrow has length 12.

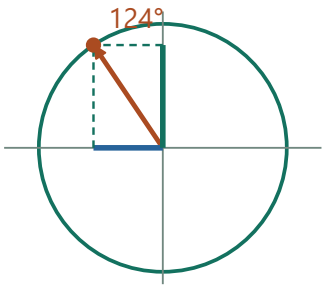


Turn from the positive horizontal axis.

The radius length is 12.

2. Project the arrow onto the axes

At counterclockwise angle 124°, the arrow ends in quadrant II. The horizontal projection is $a=12\cos 124°$ and the vertical projection is $b=12\sin 124°$.



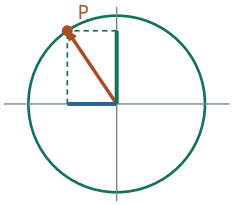
Horizontal projection = $12\cos 124°$.

Vertical projection = $12\sin 124°$.

3. Evaluate and interpret the sign

This gives $a\approx -6.71$ and $b\approx 9.95$. If the problem specifies only the unsigned angle between arrows, reflecting the endpoint below the x-axis preserves that angle, giving $b\approx -9.95$ as well.

Radius direction · 124°



Radius 12; angle 124°.

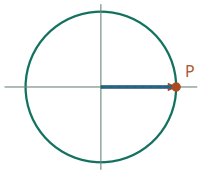
Projection onto x = −6.71. Projection onto y = 9.948.

1. $a=12\cos(-124°)=12\cos 124°$
Reflecting the direction across the horizontal axis preserves its horizontal projection.

2. $b=12\sin(-124°)=-12\sin 124°$
The vertical projection reverses, accounting for the second signed answer.

Compare settings · Radius direction

0°

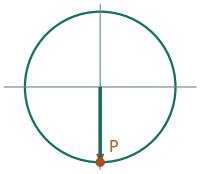


Radius 12; angle 0°.

Projection onto x = 12. Projection onto y = 0.

Radius 12; angle 0°. Projection onto x = 12. Projection onto y = 0. The drawing uses a counterclockwise direction. For an unsigned angle, the reflected point has the opposite vertical coordinate.

270°



Radius 12; angle 270°.

Projection onto x = −0. Projection onto y = −12.

Radius 12; angle 270°. Projection onto x = −0. Projection onto y = −12. The drawing uses a counterclockwise direction. For an unsigned angle, the reflected point has the opposite vertical coordinate.

Check. The x component is negative and its ratio to 12 is cos124°. Squaring the components and adding gives 144.

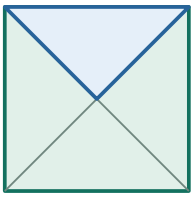
Exercise 19 · A square pyramid

Printed page 3 · Pyramid volume / The Pythagorean theorem

Pyramid $TABCD$ has a 20-cm square base $ABCD$. The edges that meet at T are 27 cm long. Make a diagram of $TABCD$, showing F, the point of $ABCD$ closest to T. Find the height TF, to the nearest cm. Find the volume of $TABCD$, to the nearest cc.

1. Locate the foot F

A perpendicular from T to the base gives the closest base point. The right triangles from T through F to each corner have equal hypotenuses of 27 cm and the same leg TF. All four corners are therefore equally far from F, the square's center.

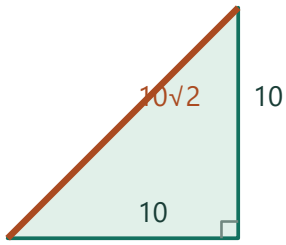


F is at the center of the square.

Four equal distances connect F to the corners.

2. Read the horizontal leg

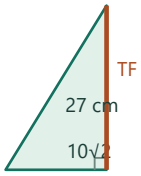
A base diagonal has length $\sqrt{(20^2+20^2)}=20\sqrt{2}$ cm. F is its midpoint, so the horizontal leg from F to a vertex is $10\sqrt{2}$ cm.



Half the square diagonal is $10\sqrt{2}$ cm.

3. Find the vertical height

In the triangle with tip T and a base vertex, $27^2=TF^2+(10\sqrt{2})^2$. Subtract 200 from 729 and take the positive square root to get TF=23 cm.



$TF^2 = 27^2 - (10\sqrt{2})^2 = 529$.

The perpendicular height is 23 cm.

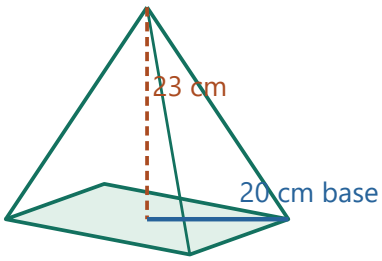
1. $TF^2=27^2-(10\sqrt{2})^2$
The 27-cm lateral edge is opposite the right angle.

2. $TF^2=729-10^2\cdot2=729-200=529$
Square each factor of the horizontal leg.

3. $TF=\sqrt{529}=23$ cm
Use the positive root since TF is a length.

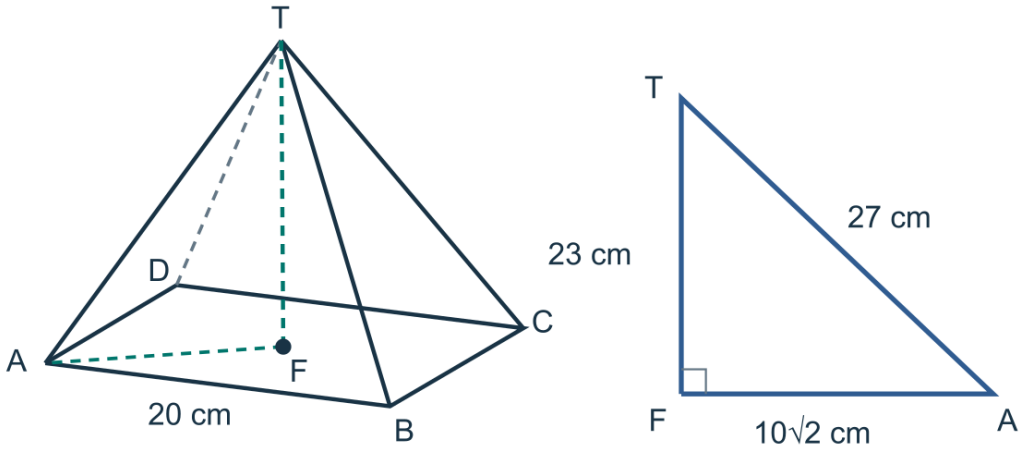
4. Use height, not slant edge, for volume

The base area is $20^2=400$ cm^2. The volume is $(1/3)\cdot400\cdot23=9200/3$ cm^3, or 3067 cm^3 to the nearest cubic centimeter.



$V = 400 \times 23 / 3$

The base area uses the 20 cm edge.



Check. $23^2+(10\sqrt{2})^2=27^2$.

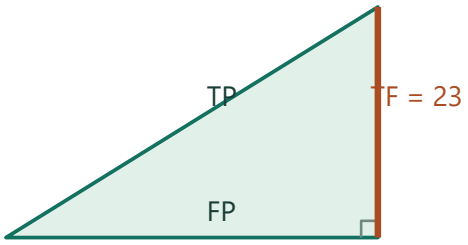
Exercise 20 · Smallest elevation angle

Printed page 3 · Optimization / Right-triangle trigonometry

(Continuation) Let P be a point on edge AB, and consider the possible sizes of angle TPF. What position for P makes this angle as small as it can be? How do you know?

1. Use the triangle through T, F and P

The fixed side TF=23 cm is perpendicular to the base. As P slides along AB, FP changes and the right angle remains at F.



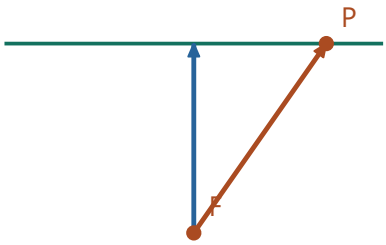
Schematic section

Elevation angle at P

tan(angle) = 23/FP.

2. Translate the angle into a ratio

At P the opposite side is TF and the adjacent side is FP. Thus $\tan\angle TPF = 23/FP$. Tangent increases on acute angles, so the smallest angle occurs when FP is largest.



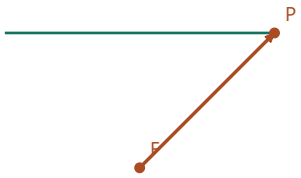
F stays fixed while P slides along AB.

A longer horizontal run makes a smaller angle.

3. Compare all points on the edge

Let M be the midpoint of AB. Then $FP^2 = FM^2 + MP^2$ with FM=10. The distance MP is at most 10 and reaches 10 at A and B only. Thus both endpoints give the minimum angle $\arctan(23/(10\sqrt{2}))$.

Position on the 20 cm edge · 10 cm from midpoint



$FP = \sqrt{(10^2+10^2)} = 14.142$ cm.

Elevation = $\arctan(23/FP) \approx 58.414°$.

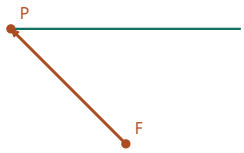
1. $FP^2 = 10^2 + MP^2 \le 100 + 100 = 200$
P is on the side, at most 10 cm from its midpoint.

2. $FP \le \sqrt{200} = 10\sqrt{2}$
Take the nonnegative square root. Equality holds at either endpoint.

3. $\tan(\angle TPF) \ge 23/(10\sqrt{2})$
A larger positive denominator makes this fixed-height ratio smaller.

Compare settings · Position on the 20 cm edge

-10 cm from midpoint

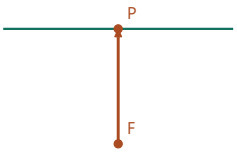


$FP = \sqrt{(10^2+10^2)} = 14.142$ cm.

Elevation = $\arctan(23/FP) \approx 58.414°$.

$FP = \sqrt{(10^2+10^2)} = 14.142$ cm. Elevation = $\arctan(23/FP) \approx 58.414°$.

0 cm from midpoint



$FP = \sqrt{(10^2+0^2)} = 10$ cm.

Elevation = $\arctan(23/FP) \approx 66.501°$.

$FP = \sqrt{(10^2+0^2)} = 10$ cm. Elevation = $\arctan(23/FP) \approx 66.501°$.

Check. Writing P=(x,10) relative to the center gives $FP^2 = x^2 + 100$, maximal at $x = \pm 10$.

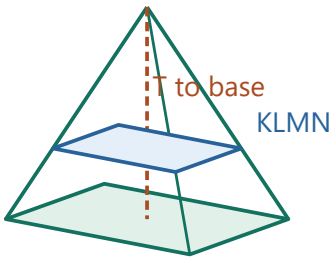
Exercise 21 · A slice parallel to the base

Printed page 3 · Similarity and scaling / Cross sections

(Continuation) Let K, L, M, and N be the points on TA, TB, TC, and TD, respectively, that are 18 cm from T. What can be said about polygon $KLMN$? Explain.

1. Mark corresponding points on the four edges

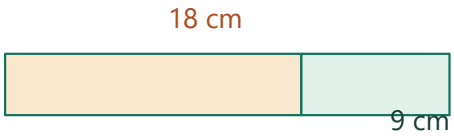
Each full edge from T to the base has length 27 cm. Points K,L,M,N lie 18 cm from T on these same rays. Their distance along every ray is 18/27=2/3 of the original.



Measure 18 cm down each 27-cm lateral edge.

2. Recognize a single dilation

A dilation about T by 2/3 sends A to K, B to L, C to M, and D to N. It scales all segments by 2/3 and preserves angles.



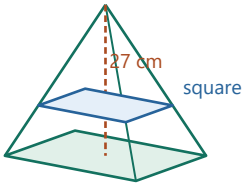
Fraction from T = 18/27 = 2/3.

One dilation moves all four vertices.

3. Describe the section

Corresponding sides of KLMN and ABCD are parallel. The section is a square in a plane parallel to the base. Each side has length (2/3)·20=40/3 cm.

Distance down the lateral edge · 18 cm



The cut is 18/27 of the way from the tip.

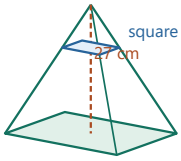
Its side is (20 cm)·18/27 = 13.333 cm.

1. KL=(18/27)AB=(2/3)20
Every side of the section is the image of the matching base edge under the same dilation.

2. KL=40/3 cm
The same computation applies to LM, MN and NK.

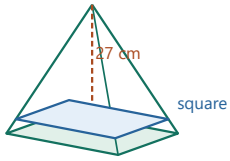
Compare settings · Distance down the lateral edge

9 cm



The cut is 9/27 of the way from the tip.
Its side is (20 cm)·9/27 = 6.667 cm.

24 cm



The cut is 24/27 of the way from the tip.
Its side is (20 cm)·24/27 = 17.778 cm.

The cut is 9/27 of the way from the tip. Its side is (20 cm)·9/27 = 6.667 cm.

The cut is 24/27 of the way from the tip. Its side is (20 cm)·24/27 = 17.778 cm.

Check. Each joining segment is parallel to a base side by similar triangles.

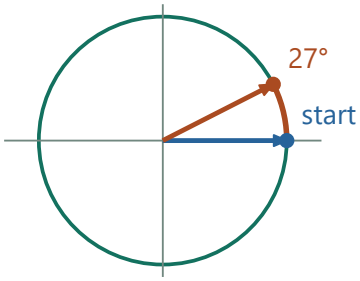
Exercise 22 · Paint on a wheel

Printed page 3 · Rotations / The unit circle

A wheel of radius one foot is placed so that its center is at the origin, and a paint spot on the rim is at $(1, 0)$. The wheel is spun 27 degrees in a counterclockwise direction. What are the coordinates of the paint spot?

1. Locate the starting radius

Draw the radius from (0,0) to (1,0). A counterclockwise turn of 27° takes the spot into quadrant I while keeping its distance from the center at 1 foot.

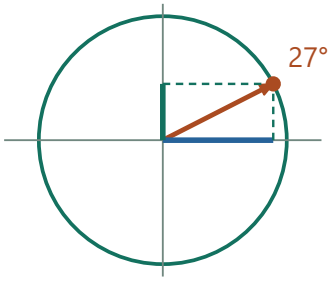


Turn from the positive horizontal axis.

The radius length is 1.

2. Read the projections

Drop a perpendicular from the new spot to the x-axis. In the resulting right triangle, the horizontal leg is 1·cos 27° and the vertical leg is 1·sin 27°.



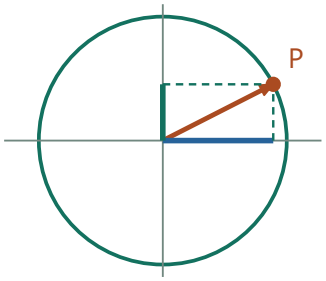
Horizontal projection = 1 cos 27°.

Vertical projection = 1 sin 27°.

3. Write the point in coordinate order

Horizontal comes first. The new spot is approximately (0.8910,0.4540) ft. Both coordinates are positive and their squared lengths sum to one apart from rounding.

Radius direction · 27°

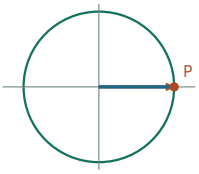


Radius 1; angle 27°.

Projection onto x = 0.891. Projection onto y = 0.454.

Compare settings · Radius direction

0°

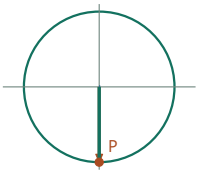


Radius 1; angle 0°.

Projection onto x = 1. Projection onto y = 0.

Radius 1; angle 0°. Projection onto x = 1. Projection onto y = 0.

270°



Radius 1; angle 270°.

Projection onto x = −0. Projection onto y = −1.

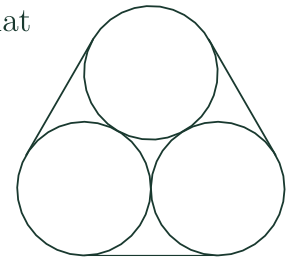
Radius 1; angle 270°. Projection onto x = −0. Projection onto y = −1.

Check. Both coordinates are positive and their squares sum to one.

Exercise 23 · Band around three pipes

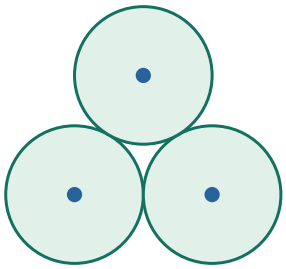
Printed page 3 · Circle tangency / Arc length

The figure shows three circular pipes, all with 12-inch diameters, that are strapped together by a metal band. How long is the band?



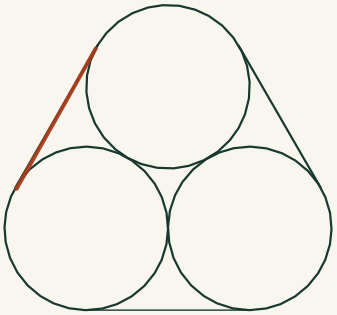
1. Separate straight belt from curved belt

Look down on the three pipes. Each radius is 6 in, and touching pipes have centers separated by 12 in. The three centers form an equilateral triangle.



Touching centers are 12 in apart.

Each pipe radius is 6 in.

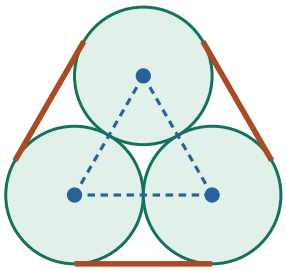


Upper-left straight run

Between its two contact points this strip of band is straight and parallel to the line joining the two pipe centers.

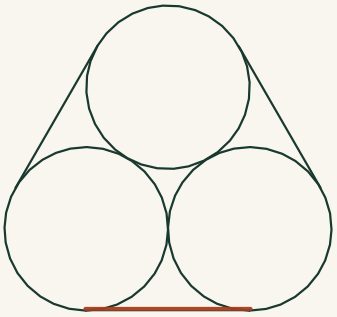
2. Measure the straight pieces

Between two tangent points, the band runs parallel to the segment joining the centers. The two radii to the tangent points are parallel, so each straight piece has the same length as the center separation, 12 in. Together the three pieces contribute 36 in.



The three external straight runs match the center triangle.

Straight total = 36 in.

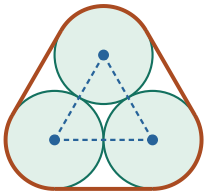


Bottom straight run

The lower two contact points lie directly below their centers. The joining band segment spans one center-to-center spacing.

3. Close the curves

The three rounded corners turn the band through one complete revolution in total. At radius 6 their combined arc length is $2\pi\cdot 6 = 12\pi$. The total is $36 + 12\pi \approx 73.70$ in.



The three rounded turns add to 360°.

Curved total = 12π in.

1. turn per corner $= 180° - 60° = 120°$
Change from the angle inside the equilateral center triangle to its exterior turn.

2. rounded length $= 3(120/360)(2\pi\cdot 6) = 12\pi$ in
Each pipe contributes a third of its circumference.

Check. Equivalently each pipe supplies a 120° arc, one-third of its circumference.

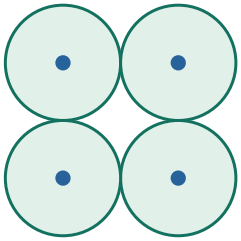
Exercise 24 · Four-pipe bundle

Printed page 3 · Circle tangency / Arc length

(Continuation) Suppose that four such pipes are strapped together with a snugly-fitting band. How long is the band?

1. Arrange the four centers in a square

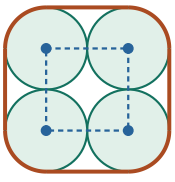
For a compact two-by-two arrangement, the four centers form a square of side 12 in. A snug band follows the outside of that square at a distance of 6 in.



The four centers form a 12-in square.

2. Add the straight and rounded parts

Each straight tangent is as long as the corresponding center-to-center side. The straight lengths total $4\cdot 12 = 48$ in. The four corner arcs are quarter circles of radius 6; together they contribute 12π in.



Four straight runs plus four quarter circles

48 + 12π in.

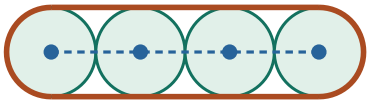
1. $s_1 = (90°/360°)\cdot 2\pi\cdot 6 = 3\pi$ in
One end arc sweeps a quarter of the circle about its pipe center.

2. $s_{all} = 4(3\pi) = 12\pi$ in
The four equal turns close the band.

3. State the result and its condition

The compact bundle needs $48+12\pi \approx 85.70$ in. A single row would have two straight runs of 36 in and two semicircles, requiring $72+12\pi$ in. Placement affects the answer.

Four-pipe placement · Single row

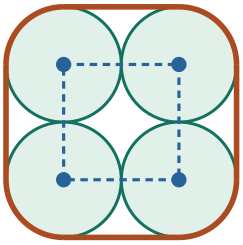


Two straight runs of 36 in plus two semicircles.

Band ≈ 109.70 in.

Compare settings · Four-pipe placement

Square bundle

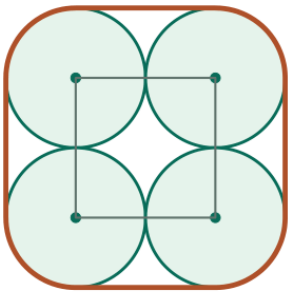


Square center perimeter 48 in plus 12π in of rounded contact.

Band ≈ 85.70 in.

Square center perimeter 48 in plus 12π in of rounded contact. Band ≈ 85.70 in.

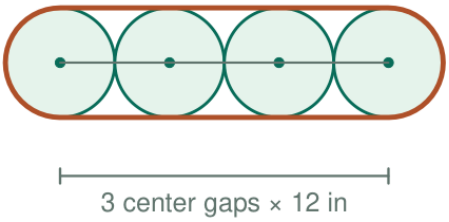
Square arrangement



4 straight runs, each 12 in

4 quarter circles at radius 6 in

Straight-row arrangement



2 straight runs, each 36 in

2 semicircles at radius 6 in

In both arrangements the curved portions turn through a full circle in total.

Check. Both answers have the same curved contribution. The straight segments equal the perimeter of the centers' convex hull (twice its length for a collinear row).

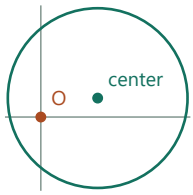
Exercise 25 · Extremal points on a circle

Printed page 3 · Standard form of a circle / Distance and loci / Optimization

Which point on the circle $x^2 + y^2 - 12x - 4y = 50$ is closest to the origin? Which point is farthest from the origin? Explain.

1. Identify the circle

Complete each square by adding 36 and 4: $(x-6)^2+(y-2)^2=90$. Its center C is (6,2) and its radius r is $3\sqrt{10}$. The origin O is inside because $OC=2\sqrt{10}$.



Center C = (6,2), radius = $3\sqrt{10}$.

O is inside the circle.

1. $x^2-12x+y^2-4y=50$
Group terms by variable.

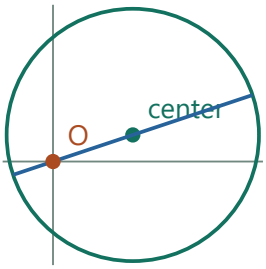
2. $x^2-12x+36+y^2-4y+4=50+36+4$
Half each linear coefficient and square it on both sides.

3. $(x-6)^2+(y-2)^2=90$
Replace the trinomials by squares.

4. $r=\sqrt{90}=3\sqrt{10}$
Take the positive radius and extract the square factor 9.

2. Locate the extreme distances

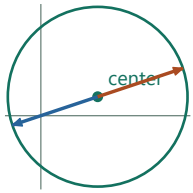
For any point P on the rim, the triangle inequality and its reverse give $r-OC\le OP\le r+OC$. The limiting cases occur when O, C and P lie on the same straight line.



Distance extremes occur along the center line.

3. Move a radius in each direction

The vector from O to C is (6,2). Divide it by OC to obtain the unit vector $(3,1)/\sqrt{10}$. Multiplying by r gives the radius displacement (9,3).



The center line has unit direction $(3,1)/\sqrt{10}$.

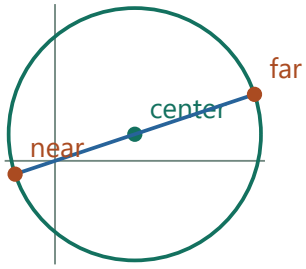
Multiplying by radius $3\sqrt{10}$ gives displacement (9,3) in either direction.

1. $OC=\sqrt{(6^2+2^2)}=\sqrt{40}=2\sqrt{10}$
The origin and center determine the length of the direction vector.

2. $(3\sqrt{10})(6,2)/(2\sqrt{10})=(9,3)$
Scale that vector to the circle radius before adding or subtracting it.

4. Read the two endpoints

Subtract the displacement from C for (−3,−1), the nearer point. Add it for (15,5), the farther point. Their distances to O are √10 and 5√10, matching the bounds.



Subtract or add (9,3) to the center.

Nearest (−3,−1); farthest (15,5).

Check. Distances are √10 and 5√10, equal to the difference and sum of the circle radius and OC.

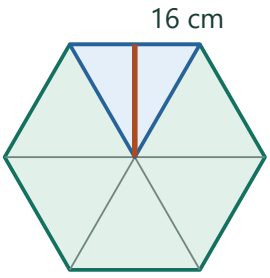
Exercise 26 · Hexagonal pyramid

Printed page 3 · Pyramid volume / Surface area / Regular polygons

The lateral edges of a regular hexagonal pyramid are all 20 cm long, and the base edges are all 16 cm long. To the nearest cc, what is the volume of this pyramid? To the nearest square cm, what is the combined area of the base and six lateral faces?

1. Separate the two needed heights

The height of the solid ends at the center of the hexagon. The height of a triangular face ends at the midpoint of a base edge. Both use a 20-cm lateral edge as hypotenuse, but they have different horizontal legs.

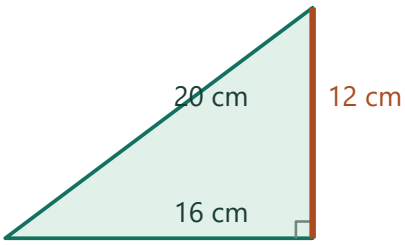


Center-to-vertex radius = 16 cm.

The face altitude and solid altitude end at different points.

2. Compute the solid's height

A regular hexagon splits into six equilateral triangles, so its center-to-vertex radius is its side length 16. Pythagoras gives $h=\sqrt{(20^2-16^2)}=12$ cm.

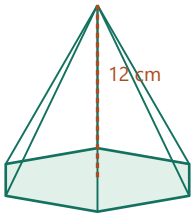


Vertical section through a vertex

$h^2 = 20^2 - 16^2$.

3. Find the volume

Each equilateral base triangle has area $(\sqrt{3}/4)\cdot 16^2$. Six give $384\sqrt{3}$ cm^2. The pyramid volume is $(1/3)\cdot 384\sqrt{3}\cdot 12=1536\sqrt{3}\approx 2660$ cm^3.



The hexagonal base has six 16-cm equilateral triangles.

V = ⅓ × 384√3 cm^2 × 12 cm = 1,536√3 cm^3.

1. $B=6\cdot(1/2)\cdot 16\cdot(8\sqrt{3})=384\sqrt{3}$
The altitude of each equilateral base triangle is $\sqrt{(16^2-8^2)}=8\sqrt{3}$.

2. $V=384\sqrt{3}\cdot 12/3=1536\sqrt{3}$ cm^3
Use the solid's vertical height, not the face altitude.

4. Find the surface area

In one face the half-base is 8, so the face altitude is $\sqrt{(20^2-8^2)}=\sqrt{336}$. Add six triangle areas to the hexagon: $384\sqrt{3}+6\cdot(16\sqrt{336}/2)=384\sqrt{3}+48\sqrt{336}\approx 1545$ cm^2.



Half of one lateral face

Its altitude uses half a side, 8 cm, as the horizontal leg.

1. face altitude=$\sqrt{(400-64)}=\sqrt{336}$
Halving the 16-cm edge leaves an 8-cm leg in the lateral face.

2. lateral area=$6(16\sqrt{336}/2)=48\sqrt{336}$
Use that slant altitude in each of six face areas.

Check. The base apothem is $8\sqrt{3}$; slant height squared equals $12^2+(8\sqrt{3})^2=336$.

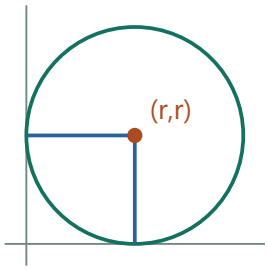
Exercise 27 · Two tangent circles

Printed page 3 · Circle tangency / Standard form of a circle

There are two circles that go through $(9, 2)$. Each one is tangent to both coordinate axes. Find the center and the radius for each circle. Start by drawing a clear diagram.

1. Draw radii to the two axes

A tangent radius meets its axis at right angles. A circle of radius r touching both axes must therefore have its center a distance r from each. A circle through the first-quadrant point (9,2) lies in that quadrant, so its center is (r,r).

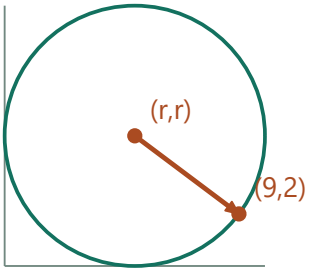


Tangency requires equal distances r to both axes.

The given point is in quadrant I.

2. Use the given point

The distance from (r,r) to (9,2) equals r. Squaring removes the root: $(9-r)^2+(2-r)^2=r^2$.

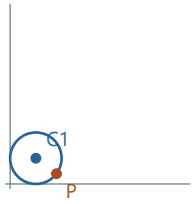


Use the squared distance from the center to the given point.

$(9-r)^2+(2-r)^2=r^2$. The candidate drawing illustrates this constraint.

3. Solve for both possible radii

Expand to get $81-18r+r^2+4-4r+r^2=r^2$, hence $r^2-22r+85=(r-5)(r-17)=0$. Each positive root gives a circle.



One radius solves the distance equation at r = 5.

1. $81-18r+r^2+4-4r+r^2=r^2$
Expand both squared displacements from the center.

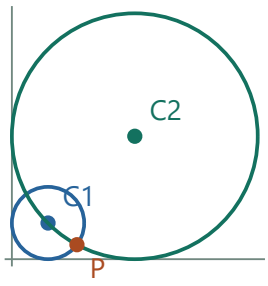
2. $r^2-22r+85=0$
Subtract the right-hand r^2 and collect terms.

3. $(r-5)(r-17)=0$
The two constants have product 85 and sum −22.

4. r=5 or r=17
At least one factor is zero; both radii are positive.

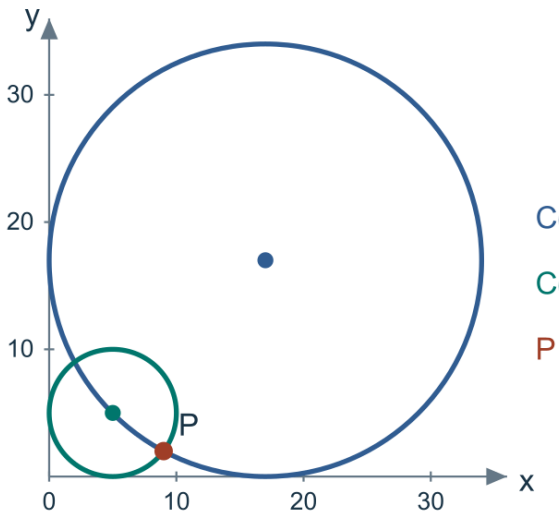
4. Place both centers

For r=5 the center is (5,5); for r=17 it is (17,17). Their distances to (9,2) are respectively $\sqrt{(4^2+3^2)}=5$ and $\sqrt{(8^2+15^2)}=17$.



Both circles pass through P=(9,2).

The centers are (5,5) and (17,17).



Center (17, 17), radius 17

Center (5, 5), radius 5

P = (9, 2)

Check. Distances from (9,2) to the centers are 5 and 17 by the 3-4-5 and 8-15-17 triangles.

Exercise 28 · Pizza share

Printed page 3 · Circular sectors / Percentage change

Some Exonians bought a circular pizza for \$10.80. Kyle's share was \$2.25. What was the central angle of Kyle's slice?

1. Connect cost to area

Assume the pizza is uniform and Kyle pays in proportion to the area received. His area fraction equals his fraction of the price.



The cost of the whole pizza is $10.80.

2. Compute the fraction without rounding

In cents that fraction is 225/1080=5/24. For a sector of a fixed circle, area is proportional to the central angle.



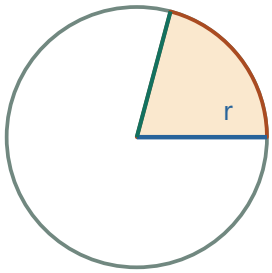
Kyle buys 5/24 of the area.

1. f=225/1080=(225÷45)/(1080÷45)=5/24
Convert both prices to cents and divide numerator and denominator by their common factor.

2. θ/360°=5/24
The area and price share are the same fraction of the pizza as the central angle is of a turn.

3. Convert the fraction to an angle

A whole pizza subtends 360°. Kyle's angle is (5/24)·360°=75°. This is between one sixth and one quarter of a turn, consistent with the fraction.



The same fraction of a turn is 75°.

Check. A 75° sector represents 20⅚% of the full disk.

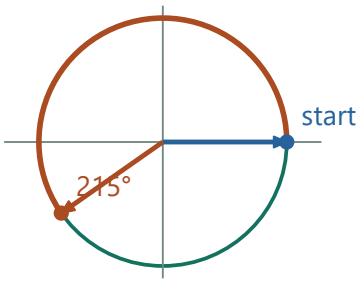
Exercise 29 · Third-quadrant coordinates

Printed page 3 · The unit circle / Reference angles and symmetry

Representing one unit by at least five squares on your graph paper, draw the *unit circle*, which is centered at the origin and goes through point $A = (1, 0)$. Use a protractor to mark the third-quadrant point P on the circle for which arc AP has angular size 215 degrees. Estimate the coordinates of P, reading from your graph paper. Notice that both are negative numbers. Use a calculator to find the cosine and sine values of a 215-degree angle. Explore further to explain why sine and cosine are known as *circular functions*.

1. Lay out the angular turn

Use the origin as center and at least five grid squares per unit. From A=(1,0), turn 180° counterclockwise to the negative x-axis and then another 35° to reach the 215° ray.

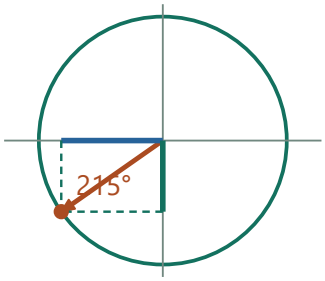


Turn from the positive horizontal axis.

The radius length is 1.

2. Read point P from the grid

Where the ray meets the unit circle, move horizontally and vertically back to the axes. P is left of the vertical axis and below the horizontal axis; a five-square scale gives roughly (−0.8,−0.6).



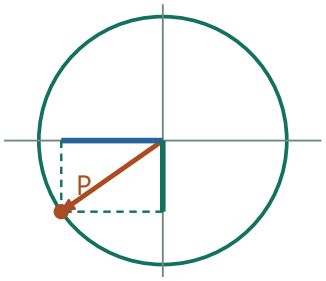
Horizontal projection = 1 cos 215°.

Vertical projection = 1 sin 215°.

3. Refine the estimate

The acute reference angle is 35°. The exact projections are $(-\cos 35°, -\sin 35°) = (\cos 215°, \sin 215°)$, approximately $(-0.8192, -0.5736)$.

Radius direction · 215°

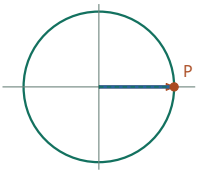


Radius 1; angle 215°.

Projection onto x = −0.819. Projection onto y = −0.574.

Compare settings · Radius direction

0°

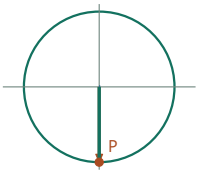


Radius 1; angle 0°.

Projection onto x = 1. Projection onto y = 0.

Radius 1; angle 0°. Projection onto x = 1. Projection onto y = 0.

270°



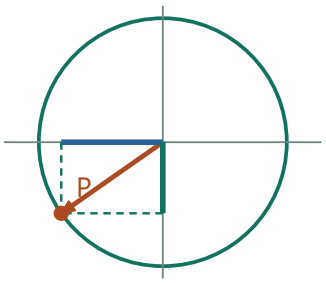
Radius 1; angle 270°.

Projection onto x = −0. Projection onto y = −1.

Radius 1; angle 270°. Projection onto x = −0. Projection onto y = −1.

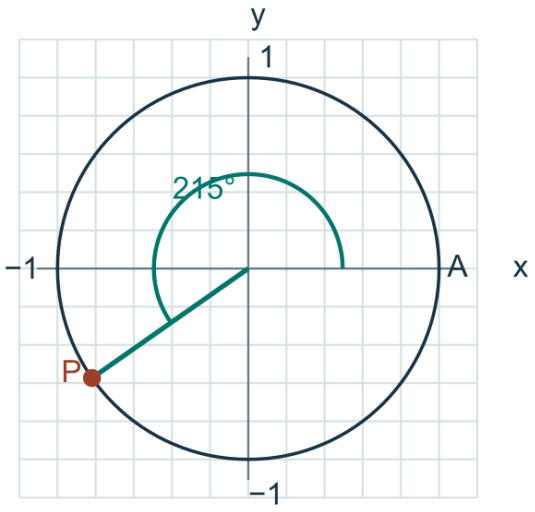
4. Connect this to circular functions

Every directed angle gives an endpoint on the unit circle. Cosine tracks its horizontal position and sine its vertical position. A full extra revolution restores both values.



In quadrant III both coordinates are negative.

Cosine reads x and sine reads y on the same unit point.



One small square = 0.2 unit

Grid estimate: (−0.8, −0.6)

Calculator: (−0.8192, −0.5736)

Check. The coordinates are both negative and their squared sum is 1.

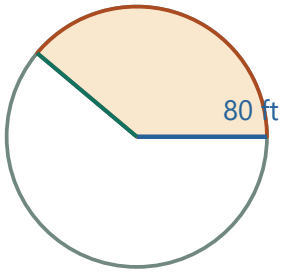
Exercise 30 · Fenced sector

Printed page 3 · Circular sectors / Arc length

A plot of land is bounded by a 140-degree circular arc and two 80-foot radii of the same circle. Find the *perimeter* of the plot, as well as its area.

1. Determine the fraction of a disk

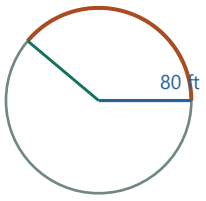
The radius is 80 ft, and the two radii enclose 140°. The sector therefore takes 140/360=7/18 of a complete circle.



140° of 360° gives 7/18 of the disk.

2. Find the curved boundary

A full circumference would be 2π·80=160π ft. Take 7/18 of this, giving an arc of 560π/9 ft. The perimeter must also include both straight radii, for 160+560π/9 ft.



Only the arc curves.

Its length is (7/18)·160π ft; add two 80-ft radii for the perimeter.

1. s=(140/360)·2π·80=(7/18)160π
The boundary arc occupies the sector's fraction of a circumference.

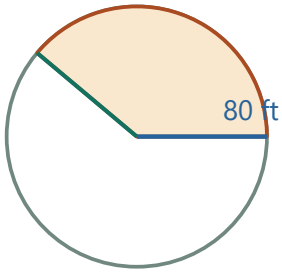
2. s=560π/9 ft
Cancel a factor of two in 1120/18.

3. P=80+80+s
Add the two straight radii as well as the arc.

3. Find the enclosed area

A full disk has area $\pi \cdot 80^2 = 6400\pi$ ft^2. Its 7/18 sector has area $22400\pi/9$ ft^2. The radii contribute length to the boundary but do not add separate regions of area.

Sector angle · 140°

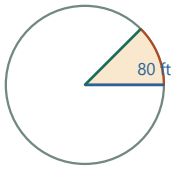


At radius 80 ft, arc = 195.477 ft.

Sector area = ½rs = 7819.075 square feet.

Compare settings · Sector angle

45°

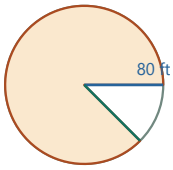


At radius 80 ft, arc = 62.832 ft.

Sector area = ½rs = 2513.274 square feet.

At radius 80 ft, arc = 62.832 ft. Sector area = ½rs = 2513.274 square feet.

315°



At radius 80 ft, arc = 439.823 ft.

Sector area = ½rs = 17592.919 square feet.

At radius 80 ft, arc = 439.823 ft. Sector area = ½rs = 17592.919 square feet.

Check. Area also equals half the product of radius and arc length.

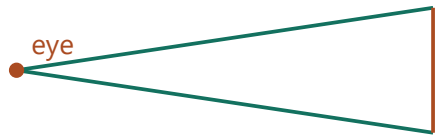
Exercise 31 · Angular diameter of the Sun

Printed page 3 · Angular diameter / Right-triangle trigonometry

Deniz notices that the sun can barely be covered by closing one eye and holding an aspirin tablet, whose diameter is 7 mm, at arm's length, which means 80 cm from Deniz's eye. Find the *apparent size* of the sun, which is the size of the angle *subtended* by the sun.

1. Draw what the eye sees

The two rays from the eye to opposite edges of the tablet define the same angular width as the Sun when the tablet just covers it. The measure sought is this angle at the eye.

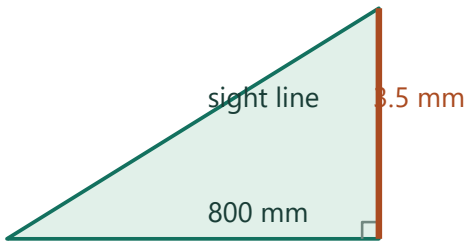


The eye sees two rim directions.

Angle exaggerated so both rays are visible.

2. Use the midpoint

With the tablet perpendicular to the viewing direction, its midpoint splits the triangle in half. The adjacent leg is 80 cm=800 mm. The opposite leg is half the 7-mm diameter, or 3.5 mm.



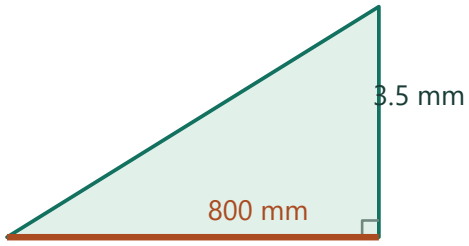
Schematic section

Work with the half-angle.

Half of the 7 mm tablet spans 3.5 mm.

3. Double the half-angle

If α is the half-angle, tan α=3.5/800. Thus the apparent angular diameter is 2 arctan(3.5/800)≈0.501°.



Schematic section

Full angle = twice arctan(3.5/800).

The angular diameter is about 0.501°.

1. α=arctan[(7/2)/(80·10)]	Use millimeters for both triangle legs before taking the inverse tangent.
2. 2α≈2(0.250667°)≈0.50133°	The eye sees equal half-angles on either side of the sight line.

Check. The small-angle estimate 7/800 radians converts to almost the same half-degree.

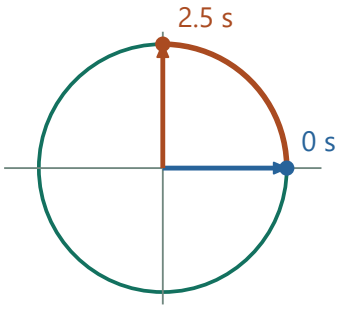
Exercise 32 · One revolution per ten seconds

Printed page 4 · Circular motion / Parametric equations

At constant speed, a wheel makes a full rotation once counterclockwise every 10 seconds. The center of the wheel is fixed at $(0, 0)$ and its radius is 1 foot. A paint spot is initially at $(1, 0)$; where is it t seconds later?

1. Turn elapsed time into an angle

One counterclockwise revolution is 2π radians and lasts 10 s. Constant speed gives angular speed 2π/10=π/5 radians per second. At time t the accumulated angle is πt/5.

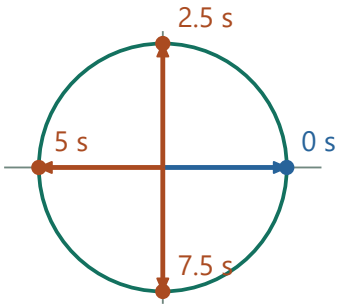


One turn takes 10 s.

2.5 seconds is a quarter-turn.

2. Carry the radius around the circle

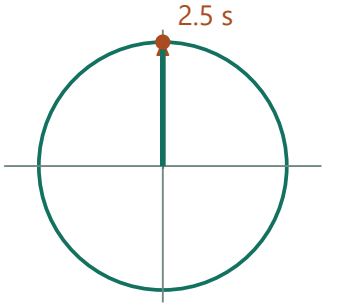
With center at the origin and radius 1 ft, the horizontal and vertical projections at angle θ are cos θ and sin θ.



At time t the radius has turned πt/5 radians.

3. Write and test the position

The spot is (cos(πt/5),sin(πt/5)) ft. At t=0 this is (1,0), and at t=2.5 s the quarter-turn gives (0,1), confirming the direction.



At 2.5 s the position is (0,1) ft.

General position: (cos(πt/5), sin(πt/5)).

Check. At 0, 2.5, 5, and 10 seconds the spot passes right, top, left, and back to right.

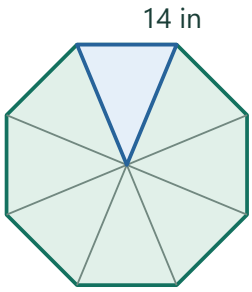
Exercise 33 · Height over an octagon

Printed page 4 · Pyramid volume / Regular polygons / The Pythagorean theorem

The base of a pyramid is the regular polygon $ABCDEFGH$, which has 14-inch sides. All eight of the pyramid's lateral edges, VA, VB, …, VH, are 25 inches long. To the nearest tenth of an inch, calculate the height of pyramid $VABCDEFGH$.

1. Draw the triangle inside the octagon

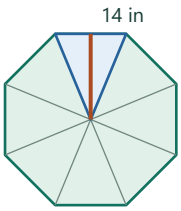
Connect the center O to adjacent vertices A and B. The eight equal central angles are 45°. The perpendicular from O to the midpoint of AB bisects both that angle and the 14-inch side.



The central angle is 45°.

2. Find the center-to-vertex radius

Let R=OA. In the half-triangle the side opposite 22.5° is 7. Thus sin 22.5°=7/R and R=7/sin 22.5°.



Half the side = 7 in.

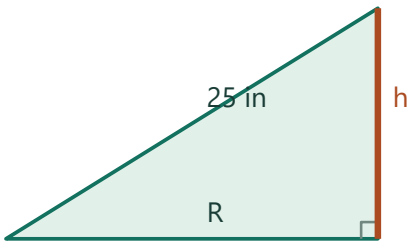
Center-to-vertex radius = 7/sin 22.5°.

1. sin(22.5°)=7/R
Half the octagon side is opposite half the central angle.

2. R sin(22.5°)=7 ⇒ R=7/sin(22.5°)
Multiply by the positive radius and isolate it.

3. Lift the triangle to the apex

The equal 25-in lateral edges put the apex V above the center O. Triangle VOA is right at O with hypotenuse 25 and base leg R. Its height VO=√(25²−R²)≈17.0 in.



Schematic section

Lift the apex above the center.

$h^2 = 25^2 - R^2$.

Check. The base apothem is 7 cot22.5° and its square plus 7^2 equals R^2.

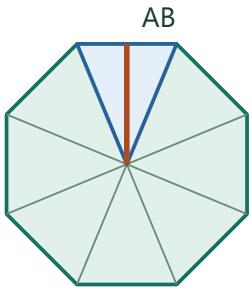
Exercise 34 · Angle of the octagonal face

Printed page 4 · Planes in space / Right-triangle trigonometry

(Continuation) To the nearest tenth of a degree, calculate the size of the *dihedral* angle formed by the octagonal base and the triangular face VAB.

1. Select a cut that measures the angle between planes

Let M be the midpoint of AB and O the base center. OM in the base and VM in triangular face VAB are both perpendicular to the common edge AB. The angle between those two segments at M is the dihedral angle.

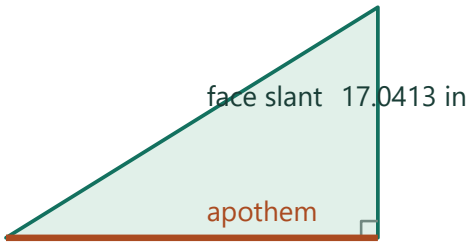


Use the midpoint of AB.

The base apothem runs perpendicular to AB.

2. Measure the horizontal run

In the base's half-central triangle, tan 22.5°=7/OM. Thus OM=7 cot 22.5°. The vertical rise VO≈17.0413 in comes from exercise 33.



Schematic section

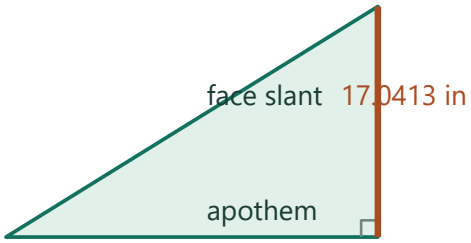
Apothem = 7 cot 22.5°.

Both traces meet at the base edge midpoint.

1. tan(22.5°)=7/a	In half a base triangle the apothem a is the adjacent leg.
2. a=7/tan(22.5°)=7 cot(22.5°)	Divide by the nonzero tangent to find the horizontal run.

3. Take the angle in the vertical section

In right triangle VOM, tan∠VMO=VO/OM. Keeping the unrounded height gives ∠VMO≈45.2°.



Schematic section

The dihedral angle has tangent height/apothem.

The result is about 45.2°.

Check. The face slant height is √(25²−7²)=24, giving the same angle from its cosine.

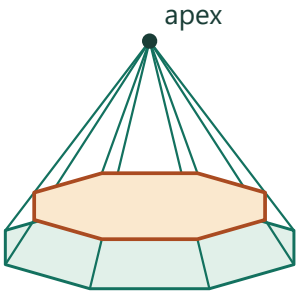
Exercise 35 · Pyramid and frustum

Printed page 4 · Area and volume scaling / Pyramid volume

(Continuation) Points A', B', C', D', E', F', G', and H' are marked on edges VA, VB, VC, VD, VE, VF, VG, and VH, respectively, so that segments VA', VB', $\ldots$, VH' are all 20 inches long. Find the volume ratio of pyramid $VA'B'C'D'E'F'G'H'$ to pyramid $VABCDEFGH$. Find the volume ratio of *frustum* $A'B'C'D'E'F'G'H'ABCDEFGH$ to pyramid $VABCDEFGH$.

1. Compare the rays from V

Each primed vertex is 20 inches along a 25-inch edge. A dilation centered at V with scale 20/25=4/5 takes every original vertex to its primed counterpart.

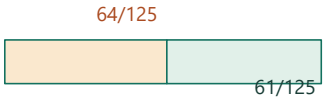


The source base is an octagon. Join points at equal fraction 20/25 on its eight lateral edges.

The parallel cut is another octagon.

2. Scale the volume in three dimensions

Each length shrinks by 4/5, so base area shrinks by $(4/5)^2$ and height by another 4/5. The upper pyramid's volume fraction is $(4/5)^3=64/125$.



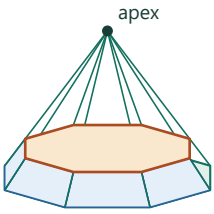
Upper volume fraction = $(4/5)^3$.

1. $B'/B=(4/5)^2=16/25$, $h'/h=4/5$
Two independent base dimensions change, while the vertical height changes once.

2. $V'/V=[(B'h')/3]/[(Bh)/3]=(16/25)(4/5)=64/125$
Cancel the common pyramid factor and multiply the area and height ratios.

3. Recover the frustum

The frustum is the piece left underneath the small pyramid. Together these pieces fill the original, so the frustum fraction is $1-64/125=61/125$.



The lower band is the octagonal frustum.

It is everything outside the 64/125-sized upper pyramid, or 61/125 of the volume.

1. $V_{frustum}/V=(V-V')/V=1-V'/V$
Distribute division by the full volume over the difference.

2. $1-64/125=(125-64)/125=61/125$
Use equal denominators for the subtraction.

Check. The two pieces together fill the original pyramid.

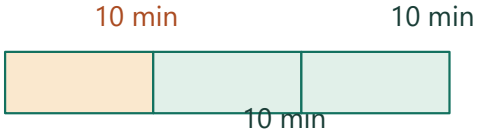
Exercise 36 · Running in radians

Printed page 4 · Radian measure / Circular motion

Quinn is running around the circular track $x^2+y^2=10\,000$, whose radius is 100 meters, at 4 meters per second. Quinn starts at the point $(100,0)$ and runs in the counterclockwise direction. After 30 minutes of running, what are Quinn's coordinates?

1. Turn time into arc length

Thirty minutes is 1800 s. At 4 m/s, Quinn travels $s=4\cdot1800=7200$ m along the circular track.

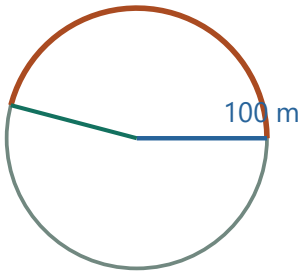


30 min = 1,800 s.

At 4 m/s the length traveled is 7,200 m.

2. Convert arc to angle

A radian measures arc divided by radius. Here θ=s/r=7200/100=72 radians. The counterclockwise direction from (100,0) gives a positive angle.



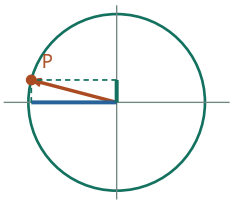
72 radians counts every whole revolution.

The highlighted remainder ends at the same direction.

3. Project the moving radius

Use radian mode to evaluate (100 cos 72,100 sin 72)≈(−96.725,25.382) m. The endpoint lies on the 100-m-radius circle even after many complete laps.

Radius direction · 165.296°



Radius 100; angle 165.296°.

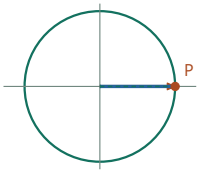
Projection onto x = −96.725. Projection onto y = 25.382.

1. 72−11·2π≈2.88496 radians
Remove eleven complete counterclockwise laps to locate the terminal radius in quadrant II.

2. x=100 cos(72−22π), y=100 sin(72−22π)
Full turns preserve both coordinates, explaining their signs without evaluating 72 as degrees.

Compare settings · Radius direction

0°

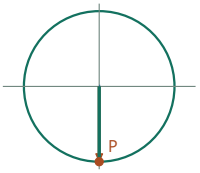


Radius 100; angle 0°.

Projection onto x = 100. Projection onto y = 0.

Radius 100; angle 0°. Projection onto x = 100. Projection onto y = 0.

270°



Radius 100; angle 270°.

Projection onto x = −0. Projection onto y = −100.

Radius 100; angle 270°. Projection onto x = −0. Projection onto y = −100.

Check. Subtracting eleven full revolutions leaves 72−22π radians, in quadrant II.

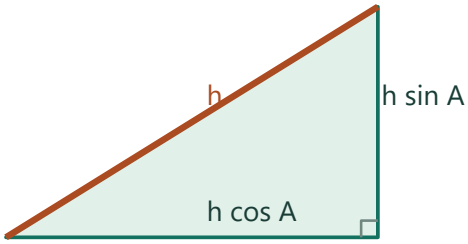
Exercise 37 · Right triangle area

Printed page 4 · Right-triangle trigonometry / Triangle area

The hypotenuse of a right triangle is 1000, and one of its angles is 87 degrees.
(a) Find the legs and the area of the triangle, correct to three decimal places.
(b) Write a formula for the area of a right triangle in which h is the length of the hypotenuse and A is the size of one of the acute angles.
(c) Apply your formula (b) to redo part (a). Did you get the same answer? Explain.

1. Resolve the hypotenuse into two legs

Relative to the 87° angle, adjacent leg/hypotenuse=cos 87° and opposite leg/hypotenuse=sin 87°. The legs are 1000 cos 87°≈52.336 and 1000 sin 87°≈998.630.



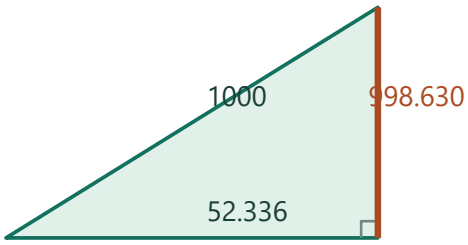
Schematic section

The 1,000-unit hypotenuse makes angle 87°.

Diagram is schematic.

2. Compute the area from the legs

These two sides are perpendicular, so either is a base and the other its height. One half of their product gives 26132.116 square units, using unrounded leg values.



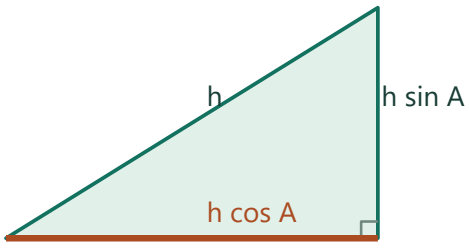
Schematic section

Use perpendicular legs for base and height.

A = ½(1000 cos 87°)(1000 sin 87°).

3. Replace the numbers by symbols

For hypotenuse h and acute angle A the same reasoning gives area=$(h^2/2)\sin A\cos A=(h^2/4)\sin(2A)$. Substituting h=1000 and A=87° reproduces the same area because the formula encodes the two leg calculations.



Schematic section

For any right triangle, multiply the two legs and divide by 2.

1. $A=(1/2)(h\cos A_0)(h\sin A_0)$	Here A_0 is the acute angle and A is the area.
2. $A=(h^2/2)\sin A_0\cos A_0$	Multiply the two copies of h.
3. $A=(h^2/4)\sin(2A_0)$	Replace twice the sine-cosine product with the sine of the double angle.

Check. The two area formulas agree by the double-angle identity, and the legs satisfy the Pythagorean theorem.

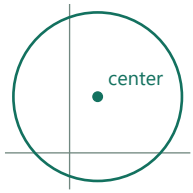
Exercise 38 · When a circle degenerates

Printed page 4 · Standard form of a circle / Completing the square

Find the center and the radius for each of the circles $x^2-2x+y^2-4y-4=0$ and $x^2-2x+y^2-4y+5=0$. How many points fit the equation $x^2-2x+y^2-4y+9=0$?

1. Complete the same squares each time

Because $(x-1)^2=x^2-2x+1$ and $(y-2)^2=y^2-4y+4$, making both squares adds 5 to the variable portion. Transfer the constant in each original equation before adding 5 to both sides.



Squared distance fixes the possible locations.

The center is the point where the two squares vanish.

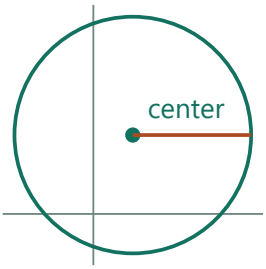
1. $x^2-2x=(x-1)^2-1$
Adding 1 inside the square requires subtracting 1 outside.

2. $y^2-4y=(y-2)^2-4$
Complete the other square using half of −4.

3. $x^2-2x+y^2-4y=(x-1)^2+(y-2)^2-5$
Combine the constant corrections; the right side increases by 5 when they are moved across.

2. Analyze the first equation

The squared-distance expression $(x-1)^2+(y-2)^2$ is 9, so the set is a circle centered at (1,2) with radius $\sqrt{9}=3$.

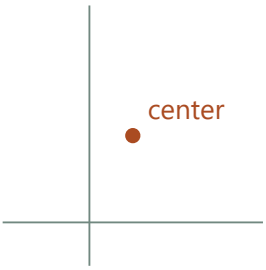


Positive squared radius

The points lie at one fixed distance from the center.

3. Interpret the other two right sides

The second gives zero. Both squares must vanish, so only (1,2) fits, a radius-zero circle. The third gives −4; a sum of two real squares cannot be negative, so there are no points.



A zero radius leaves a single point. A negative squared radius has no real points.

Check. A sum of real squares is nonnegative and is zero only when both terms vanish.

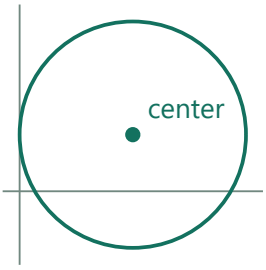
Exercise 39 · Circle in standard form

Printed page 4 · Standard form of a circle / Distance and loci

What is the result of graphing the equation $(x-h)^2+(y-k)^2=r^2$?

1. Read the horizontal and vertical displacements

For a test point (x,y), the differences x−h and y−k measure its offsets from (h,k). Their squared sum is its squared distance to that fixed point.

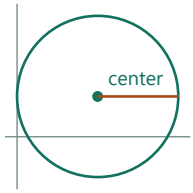


Squared distance fixes the possible locations.

The center is the point where the two squares vanish.

2. Interpret a positive radius

Setting the squared distance equal to r^2 requires every solution to lie a distance r from the center when r>0. Conversely every point that far away satisfies the equation. This is exactly the circle centered at (h,k).



Positive squared radius

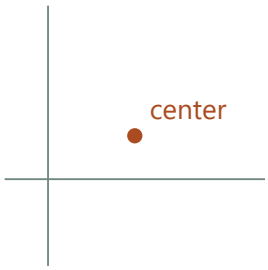
The points lie at one fixed distance from the center.

1. $d=\sqrt{[(x-h)^2+(y-k)^2]}$
The right triangle of coordinate differences has hypotenuse d.

2. $d^2=r^2 \Rightarrow d=\sqrt{(r^2)}=\text{abs}(r)$
A distance is always nonnegative, so taking its square root gives the absolute radius.

3. Include the special values

When r=0 both offsets must be zero, leaving the center alone. If r is negative, its square is unchanged and the geometric radius is |r|.



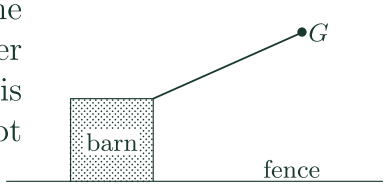
A zero radius leaves a single point. Squaring either sign of r gives the same radius magnitude.

Check. Its horizontal extreme points are (h±abs(r),k).

Exercise 40 · Goat at a fenced barn

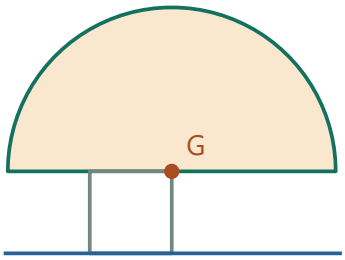
Printed page 4 · Circular sectors / Arc length

Find the total grazing area of the goat G represented in the figure (a top view) shown at right. The animal is tied to a corner of a $40' \times 40'$ barn, by an $80'$ rope. One of the sides of the barn is extended by a fence. Assume that there is grass everywhere except inside the barn.



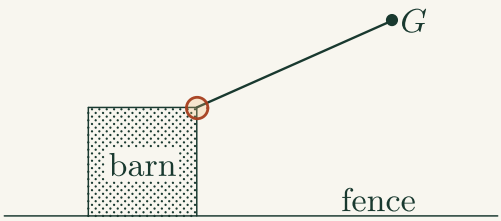
1. Trace sectors centered on the tie point

The 80-ft rope can swing freely through the upper half-plane. That reaches a semicircle of area $\frac{1}{2}\pi\cdot 80^2 = 3200\pi$ ft^2. Below the horizontal through the tie corner the barn blocks the left side and the fence cuts across the right-hand quarter disk.



The upper half-disk is within 80 ft of G.

The barn touches G below and to the left.

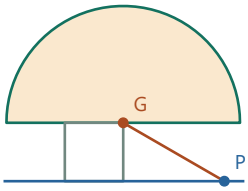


Rope attachment at the upper-right barn corner

The line from the goat G leads back to this corner. Use it as the center for the regions reached before the rope wraps.

2. Locate the fence intersection

The fence is 40 ft below G. At the right-hand intersection with the 80-ft circle, the horizontal displacement is $\sqrt{(80^2 - 40^2)} = 40\sqrt{3}$. The radius to that point is 30° below horizontal because $\sin 30° = 40/80$.



P lies on the fence and the 80-ft circle.

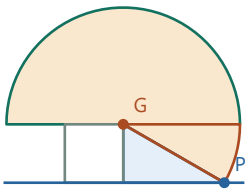
Its drop from G is 40 ft, giving a 30° angle below horizontal.

1. $x^2 + 40^2 = 80^2$
At the fence, the vertical displacement is 40 and the rope length is 80.

2. $x = \sqrt{(6400 - 1600)} = \sqrt{4800} = 40\sqrt{3}$
Use the positive horizontal displacement at the right boundary.

3. Measure the allowed lower-right strip

Separate the strip into a 30° sector and a right triangle with legs 40 and $40\sqrt{3}$. Their areas are $\frac{1}{2}\cdot 80^2\cdot(\pi/6) = 1600\pi/3$ and $\frac{1}{2}\cdot 40\cdot 40\sqrt{3} = 800\sqrt{3}$ ft^2.



The lower-right region splits along GP.

One piece is a 30° circular sector; the other is a right triangle.

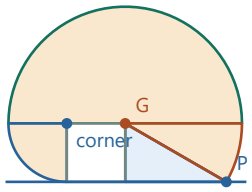
1. $\sin\theta = 40/80 = 1/2 \Rightarrow \theta = \pi/6$
Read the acute angle between the radius and the horizontal.

2. $A_{sector} = 80^2(\pi/6)/2 = 1600\pi/3$
Use the sector angle in radians.

3. $A_{triangle} = (40)(40\sqrt{3})/2 = 800\sqrt{3}$
The two perpendicular dimensions enclose the remaining triangular part.

4. Allow for wrapping around the barn

Going across the top edge to the upper-left corner uses 40 ft of rope. The remaining 40 ft sweeps a quarter disk on the other side, adding 400π ft^2. These regions share only boundaries. Their total is $12400\pi/3+800\sqrt{3}$ ft^2.



A further 40 ft remains after wrapping around the left barn corner.

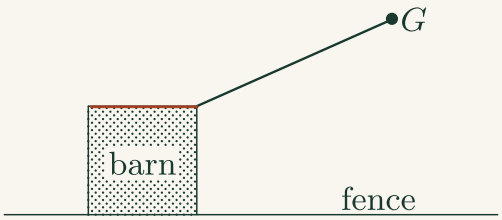
Its quarter disk adds 400π ft^2.

1. $3200\pi+1600\pi/3+400\pi=(9600+1600+1200)\pi/3$

Put the semicircle, strip sector and wrapped quarter circle over a common denominator.

2. $A=12400\pi/3+800\sqrt{3}$ ft^2

Include the triangular part of the strip once.



Upper barn edge used in wrapping

When the rope wraps to the upper-left corner, the length along this edge has already been used. Only the rest is free to swing.

Check. The right-hand strip also equals the integral of $\sqrt{(80^2-y^2)}$ from $y=0$ to 40. The three regions overlap only on their edges.

Exercise 41 · Half-turn around a point

Printed page 4 · Rotations / Coordinate geometry

A *half-turn* is a 180-degree rotation. Apply the half-turn centered at $(3, 2)$ to the point $(7, 1)$. Find coordinates of the image point. Find coordinates for the image of (x, y).

1. Move the origin to the turning point

Relative to (3,2), the point (7,1) has displacement $(7-3,1-2)=(4,-1)$. This arrow begins at the center of rotation.



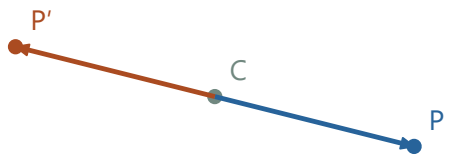
Begin at center C = (3, 2).

Measure the start relative to C.

2. Make a half-turn

Turning 180° reverses both components while preserving length. The new displacement is (−4,1). Starting at the center gives (3−4,2+1)=(−1,3).

Counterclockwise turn · 180°



Rotate by 180° about (3, 2).

New point (−1, 3); distance to center remains 4.123.

Compare settings · Counterclockwise turn

0°

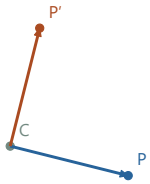


Rotate by 0° about (3, 2).

New point (7, 1); distance to center remains 4.123.

Rotate by 0° about (3, 2). New point (7, 1); distance to center remains 4.123.

90°



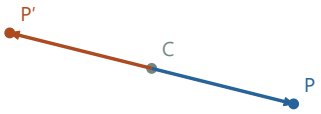
Rotate by 90° about (3, 2).

New point (4, 6); distance to center remains 4.123.

Rotate by 90° about (3, 2). New point (4, 6); distance to center remains 4.123.

3. Apply the same operation to (x,y)

Its displacement (x−3,y−2) becomes (3−x,2−y). Add (3,2) to obtain the general image (6−x,4−y). The midpoint of the pair is always (3,2).



The new arrow has the original length.

Blue shows the original displacement; orange shows its image.

1. P−C=(x−3,y−2)
Translate the center to the origin.

2. P′=C−(P−C)=(3−x+3,2−y+2)
A half-turn reverses each displacement before translating back.

3. P′=(6−x,4−y)
Collect the two copies of each center coordinate.

Check. Applying the same formula twice returns (x,y).

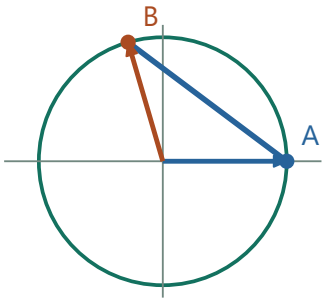
Exercise 42 · Arc cut off by a chord

Printed page 4 · Chords and inscribed angles / Arc length / Inverse trigonometric functions

A 16.0-inch chord is drawn in a circle whose radius is 10.0 inches. What is the angular size of the minor arc of this chord? What is the length of the arc, to the nearest tenth of an inch?

1. Bisect the chord

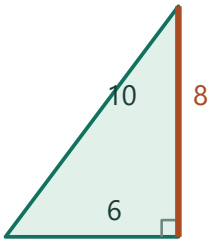
Draw both radii to the chord's endpoints, making an isosceles triangle. The perpendicular from the center to the chord halves both the chord and the central angle θ.



Two radii and a 16-in chord form an isosceles triangle.

2. Calculate the half-angle

Each half-triangle has hypotenuse 10 and opposite leg 16/2=8. Thus sin(θ/2)=8/10. Doubling the inverse sine gives the smaller angle θ≈106.26°.

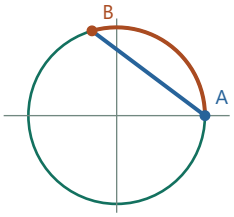


Half-chord = 8 in.

sin(θ/2) = 8/10.

3. Follow the curved path

Arc length equals radius times angle in radians. Here θ≈1.85459 radians, so s=10θ≈18.5 in. The arc is longer than its 16-inch chord.



θ ≈ 1.85459 radians.

Arc length = 10θ ≈ 18.5 in.

1. θrad=2 arcsin(8/10)≈1.85459
Evaluate the full central angle with the inverse sine in radians.

2. s=10θrad≈18.55 in
Multiply angular sweep by radius to get arc length.

Check. The chord reconstructed as 20 sin(θ/2) is 16 in.

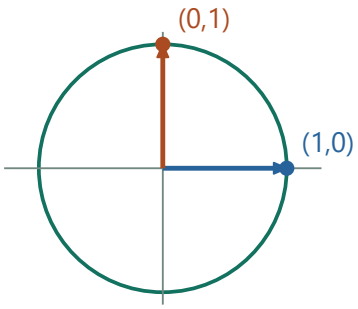
Exercise 43 · Parametrizing the circle

Printed page 4 · Parametric equations / The unit circle

What graph is traced by the parametric equation $(x, y) = (\cos t, \sin t)$?

1. Read the coordinates at a particular parameter

At t=0, cosine is 1 and sine is 0, giving (1,0). As t increases through a quarter-turn, the point moves upward to (0,1).

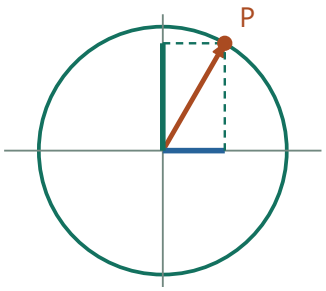


At t=0 the parameter gives (1,0).

One quarter-turn later its point is (0,1).

2. Determine the distance to the center

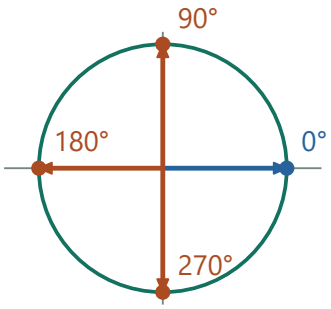
Substitute x=cos t and y=sin t. Then $x^2+y^2=\cos^2 t+\sin^2 t=1$, so the distance to the origin is always 1.



Every pair obeys $x^2+y^2=1$.

3. Identify the whole curve

Every radial direction occurs during one full turn of t. Thus the parametrization reaches every point of the unit circle and traces it counterclockwise.



Let the parameter span a full turn.

Every unit-circle direction appears.

Check. Quarter-turn parameter values give its four axis intercepts.

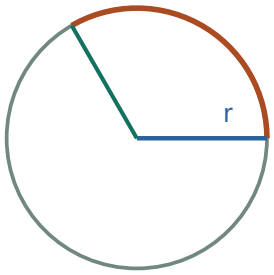
Exercise 44 · Sector from arc length

Printed page 5 · Circular sectors / Arc length

What is the area enclosed by a circular *sector* whose radius is r and arc length is s?

1. Compare the arc with a full turn

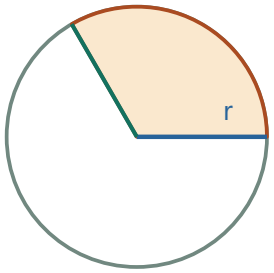
A whole radius-r circle has circumference 2πr. An arc of length s occupies the fraction s/(2πr) of that circumference.



The arc s is a fraction s/(2πr) of a full circumference.

2. Use the same fraction for area

The wedge bounded by the two radii subtends that fraction of a complete turn. Its share of the disk's area is therefore the same fraction.

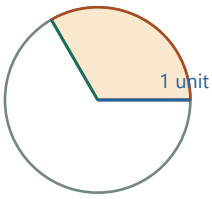


The sector occupies that same fraction of πr^2.

3. Cancel the common factors

Multiply the full area πr^2 by s/(2πr). Cancel π and one power of r to obtain A=rs/2. The product of two lengths supplies the correct area units.

Sector angle · 120°



At radius 1 unit, arc = 2.094 unit.

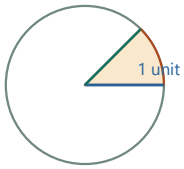
Sector area = ½rs = 1.047 square units.

1. $A=(s/(2\pi r))\pi r^2$
Use the arc's circumference fraction for the disk area.

2. $A=s(\pi r)r/(2\pi r)=sr/2$
Cancel the positive common factor πr.

Compare settings · Sector angle

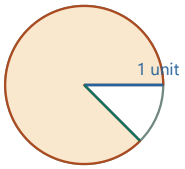
45°



At radius 1 unit, arc = 0.785 unit.
Sector area = ½rs = 0.393 square units.

At radius 1 unit, arc = 0.785 unit. Sector area = ½rs = 0.393 square units.

315°



At radius 1 unit, arc = 5.498 unit.
Sector area = ½rs = 2.749 square units.

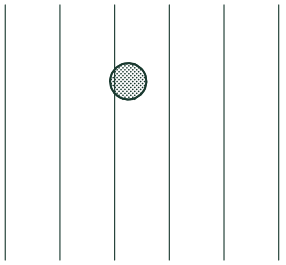
At radius 1 unit, arc = 5.498 unit. Sector area = ½rs = 2.749 square units.

Check. For a full circle $s=2\pi r$ recovers πr^2.

Exercise 45 · Coin on ruled paper

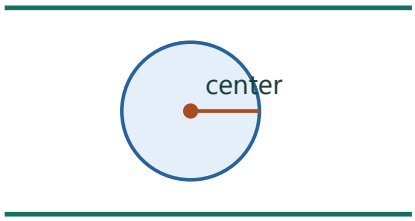
Printed page 5 · Geometric probability

A coin with a 2-cm diameter is dropped onto a sheet of paper ruled by parallel lines that are 3 cm apart. Which is more likely, that the coin will land on a line, or that it will not?



1. Follow the center of the coin

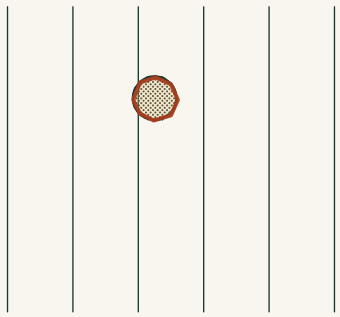
The coin's diameter is 2 cm, so any point on its rim is 1 cm from the center. Horizontal position along a line cannot affect the result; only distance across the 3-cm gap matters.



Line spacing 3 cm

A radius-1-cm coin lies between two rulings 3 cm apart.

Its center determines whether it reaches a line.

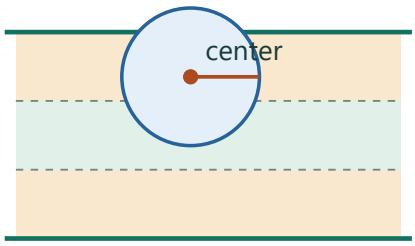


Coin and a nearby boundary

The coin extends one radius in every direction from its center. A boundary passing through the disk illustrates a touching or crossing outcome.

2. Mark the crossing bands

A center within 1 cm of either boundary lets the coin touch or cross that line. These two 1-cm bands occupy two of the three centimeters of width. The middle centimeter is safe.

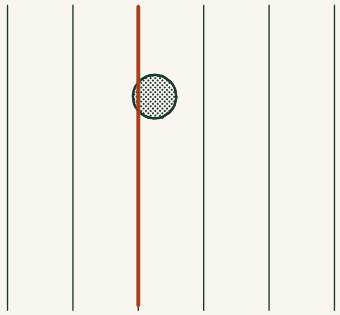


Line spacing 3 cm

The center lies in the shaded outer centimeter.

A 1-cm radius extends across the upper ruling.

1. safe width=3−1−1=1 cm	The center must clear the lower and upper boundaries by a coin radius.
2. hit width=3−1=2 cm	All remaining center positions in the period hit one of the ruled lines.



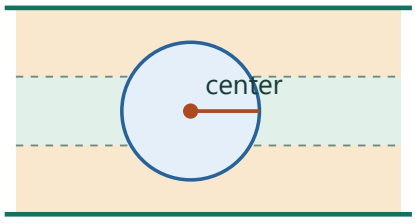
Nearest vertical line

It is the perpendicular horizontal distance to this line, not the distance moved up or down along it, that determines contact.

3. Compare probabilities

Assuming the center is uniform across a gap, width fractions give probabilities: 2/3 for touching a line and 1/3 for missing. Exact tangencies have probability zero under this continuous model.

Center distance from upper ruling · 1.5 cm



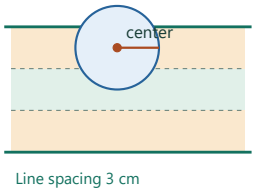
Line spacing 3 cm

The entire disk lies strictly between the rulings.

The nearer line is 1.5 cm from the center; the coin radius is 1 cm.

Compare settings · Center distance from upper ruling

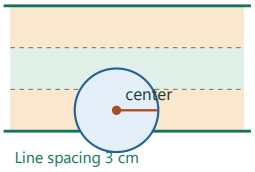
0.5 cm



The disk meets or touches a ruling.
The nearer line is 0.5 cm from the center; the coin radius is 1 cm.

The disk meets or touches a ruling. The nearer line is 0.5 cm from the center; the coin radius is 1 cm.

2.5 cm



The disk meets or touches a ruling.
The nearer line is 0.5 cm from the center; the coin radius is 1 cm.

The disk meets or touches a ruling. The nearer line is 0.5 cm from the center; the coin radius is 1 cm.

Check. The center may miss only inside the central 1-cm band.

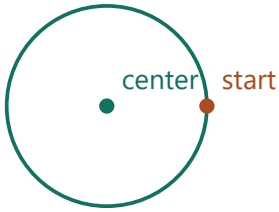
Exercise 46 · Shifted wheel

Printed page 5 · Rotations / Circular motion

A wheel whose radius is 1 is placed so that its center is at $(3, 2)$. A paint spot on the rim is found at $(4, 2)$. The wheel is spun θ degrees in the counterclockwise direction. Now what are the coordinates of that paint spot?

1. Measure the initial radius locally

From center (3,2) to paint spot (4,2) the displacement is (1,0). This is the familiar starting radius on the unit circle.

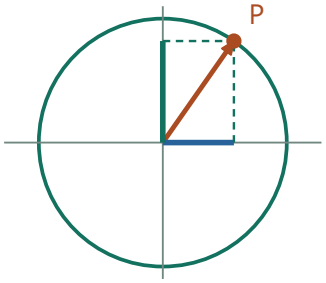


The wheel starts one foot right of its center (3,2).

2. Rotate that displacement

A counterclockwise turn through θ degrees changes its horizontal projection to cos θ and its vertical projection to sin θ.

Radius direction · 55°

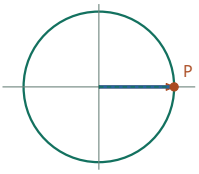


Radius 1; angle 55°.

Projection onto x = 0.574. Projection onto y = 0.819.

Compare settings · Radius direction

0°

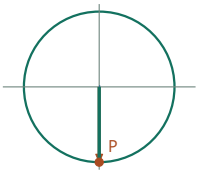


Radius 1; angle 0°.

Projection onto x = 1. Projection onto y = 0.

Radius 1; angle 0°. Projection onto x = 1. Projection onto y = 0.

270°



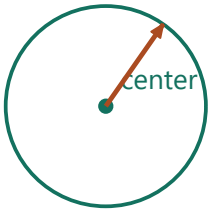
Radius 1; angle 270°.

Projection onto x = −0. Projection onto y = −1.

Radius 1; angle 270°. Projection onto x = −0. Projection onto y = −1.

3. Restore the location of the center

Add these signed projections to the center coordinates. The final spot is (3+cos θ,2+sin θ). At θ=90° the result is directly above the center at (3,3).



Add the hub coordinates.

Absolute position = (3+cos θ,2+sin θ).

Check. At θ=0 the formula returns (4,2).

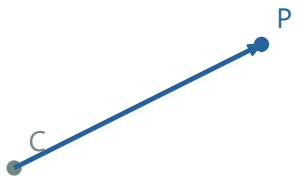
Exercise 47 · Rotating a point

Printed page 5 · Rotations / Vectors and components

A 36-degree counterclockwise rotation centered at the origin sends the point $A = (6, 3)$ to the image point A'. To three decimal places, find coordinates for A'.

1. Break the original arrow into two moves

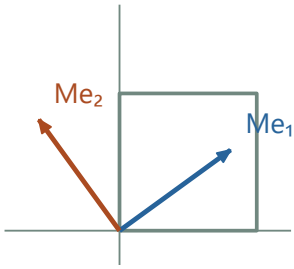
The point (6,3) represents six units along the horizontal axis and three along the vertical. Rotate both axis directions by 36° and then add their contributions.



The initial displacement is six right and three up.

2. Rotate the two basis directions

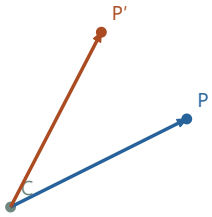
The unit horizontal vector becomes (cos 36°,sin 36°). The unit vertical direction stays a quarter-turn ahead and becomes (−sin 36°,cos 36°).



Both coordinate directions turn through 36°.

3. Combine the projections

The image is (6 cos 36°−3 sin 36°,6 sin 36°+3 cos 36°)≈(3.091,5.954). Compared to (6,3), its direction has turned upward and its distance to the origin is unchanged.



Add six rotated horizontal units and three rotated vertical units.

Image ≈ (3.091,5.954).

1. (6,3)=6(1,0)+3(0,1)
Split the arrow into horizontal and vertical basis directions.

2. P′=6(cos36°,sin36°)+3(−sin36°,cos36°)
Rotate the two basis arrows by the same turn.

3. P′=(6cos36°−3sin36°,6sin36°+3cos36°)
Multiply each pair by its coefficient and add like coordinates.

Check. Length is preserved at √45; the direction increases by 36°.

Exercise 48 · One second at the equator

Printed page 5 · Arc length / Unit conversion

In navigational terms, a *minute* is one sixtieth of a *degree*, and a *second* is one sixtieth of a minute. To the nearest foot, what is the length of a one-second arc on the equator? The radius of the earth is 3960 miles.

1. Decode an arcsecond

Each degree contains 60 arcminutes, each minute 60 arcseconds. One full turn therefore contains 360·60·60=1,296,000 arcseconds.

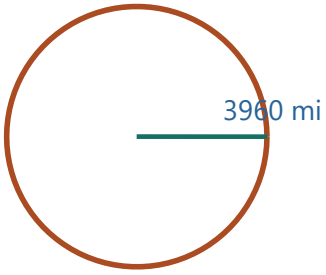


1° = 60′ = 3,600″.

A full turn contains 1,296,000 arcseconds.

2. Find a full equatorial circumference

The problem's spherical Earth has radius 3960 mi. Its equator has length 2π·3960 miles. Multiply by 5280 to measure that length in feet.



One equatorial revolution travels 2π·3960 miles.

3. Take the part for one arcsecond

Divide the circumference in feet by 1,296,000, obtaining about 101.369 ft. The requested whole-foot estimate is 101 ft.



One arcsecond is one of 1,296,000 equal pieces.

Its arc length is about 101 ft; schematic division.

1. $s=[2\pi(3960)\cdot5280]/[360\cdot60\cdot60]$ ft
Divide the full equatorial length in feet by the number of arcseconds in a full revolution.

2. $s=(3960/360)(5280/1800)\pi=(484/15)\pi$ ft
Cancel 2 against 3600 before reducing the distance per arcsecond.

Check. Multiplying 3960 miles by π/(180·3600) radians and converting gives the same result.

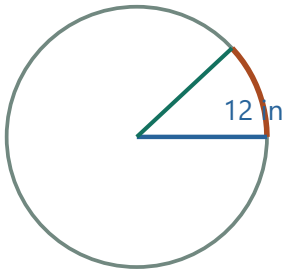
Exercise 49 · Sector from its dimensions

Printed page 5 · Circular sectors / Radian measure

A sector of a circle is enclosed by two 12.0-inch radii and a 9.0-inch arc. Its perimeter is therefore 33.0 inches. What is the area of this sector, to the nearest tenth of a square inch? What is the central angle of the sector, to the nearest tenth of a degree?

1. Connect arc length and angle

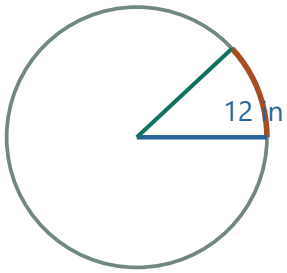
At radius 12 in the 9-in arc amounts to 9/12 of a radius-length of travel. By definition that is θ=0.75 radians.



The arc is 9 inches long on radius 12 inches.

2. Convert angular units

One radian is 180°/π. The angle is 0.75·180°/π≈43.0°.

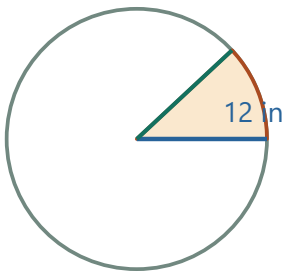


9/12 = 0.75 radians ≈ 43.0°.

3. Use the sector area

From the sector's proportion of a disk, A=rs/2. Insert r=12.0 and s=9.0 to get 54.0 in².

Sector angle · 42.972°

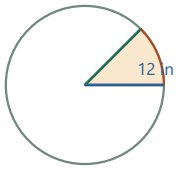


At radius 12 in, arc = 9 in.

Sector area = ½rs = 54 square inches.

Compare settings · Sector angle

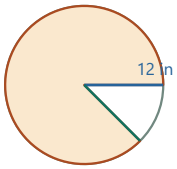
45°



At radius 12 in, arc = 9.425 in.
Sector area = ½rs = 56.549 square inches.

At radius 12 in, arc = 9.425 in. Sector area = ½rs = 56.549 square inches.

315°



At radius 12 in, arc = 65.973 in.
Sector area = ½rs = 395.841 square inches.

At radius 12 in, arc = 65.973 in. Sector area = ½rs = 395.841 square inches.

Check. The area fraction 54/(144π) equals the arc fraction 9/(24π).

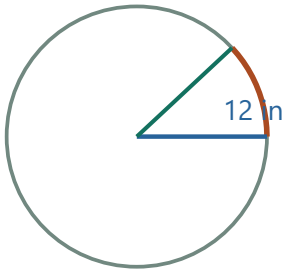
Exercise 50 · A second sector

Printed page 5 · Circular sectors / Quadratic equations

(Continuation) There is another circular sector — part of a circle of a different size — that has the same 33-inch perimeter and that encloses the same area. Find its central angle, radius, and arc length, rounding to the nearest tenth.

1. Write the two constraints

The boundary includes two radii and the arc, so 2r+s=33. The sector's area is rs/2=54. A second sector must satisfy both equalities.



First sector: arc = 9 in, radius = 12 in.
Boundary length = 33 in.

2. Eliminate the arc length

Substitute s=33−2r into the area equation and multiply by 2: r(33−2r)=108. Rearranging gives $2r^2-33r+108=(r-12)(2r-9)=0$.

P	2r+s
A	rs/2

With P=33, replace s by 33−2r in the area equation.
This leaves one quadratic for r.

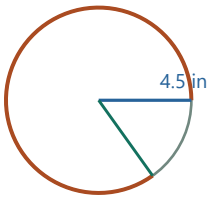
1. $54=r(33-2r)/2 \Rightarrow 108=33r-2r^2$
Multiply by 2 and distribute r.

2. $2r^2-33r+108=0$
Gather all terms on one side.

3. $(2r-9)(r-12)=0$
Factor the quadratic; the cross terms total −33r.

3. Interpret both radii

Radius 12 reproduces the first sector. The other root is r=4.5 in, and its arc is s=33−9=24 in.



The second radius is 4.5 in.

Its partner arc is 33 − 9 = 24 in.

1. 2r−9=0 or r−12=0
A product vanishes when at least one factor vanishes.

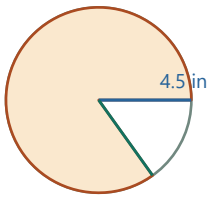
2. r=4.5 or 12 in
Both are positive; 12 is the original radius.

3. s=33−2(4.5)=24 in
Use the other radius in the same perimeter constraint.

4. Decide which part of the circle it uses

Its central angle is s/r=24/4.5 radians, about 305.6°. That is over half a circle, which explains how a smaller radius can enclose the same area.

Sector angle · 305.577°



At radius 4.5 in, arc = 24 in.

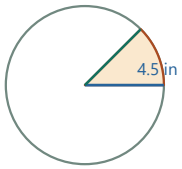
Sector area = ½rs = 54 square inches.

1. θ=s/r=24/(9/2)=24·(2/9)=16/3 rad
Divide the arc length by the radius in matching length units.

2. θdeg=(16/3)(180°/π)=960°/π
Multiply the radian measure by the degrees in one radian before rounding.

Compare settings · Sector angle

45°

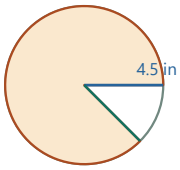


At radius 4.5 in, arc = 3.534 in.

Sector area = ½rs = 7.952 square inches.

At radius 4.5 in, arc = 3.534 in. Sector area = ½rs = 7.952 square inches.

315°



At radius 4.5 in, arc = 24.74 in.

Sector area = ½rs = 55.665 square inches.

At radius 4.5 in, arc = 24.74 in. Sector area = ½rs = 55.665 square inches.

Check. Two radii plus arc give 33, and 4.5·24/2=54.

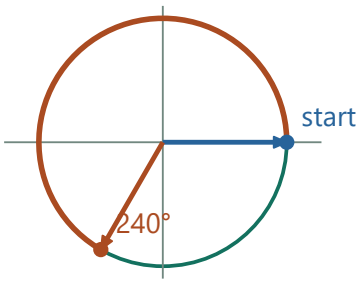
Exercise 51 · Special angle in quadrant III

Printed page 5 · The unit circle / Reference angles and symmetry

Use the unit circle and your knowledge of special triangles to find exact values for $\sin 240$ and $\cos 240$. Then use a calculator to check your answers. Notice that calculators use parentheses around the 240 because sin and cos are functions. In text, except where the parentheses are required for clarity, they are often left out.

1. Reduce to a familiar acute angle

The terminal ray at 240° is 60° beyond the negative x-axis. Its reference triangle therefore uses a 60° angle in quadrant III.

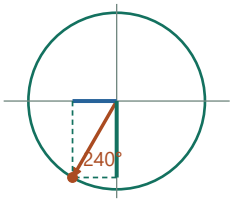


Turn from the positive horizontal axis.

The radius length is 1.

2. Get the exact leg lengths

Halving an equilateral triangle makes a 30–60–90 triangle. For hypotenuse 1, the side adjacent to 60° is 1/2 and the opposite side is √3/2.



Horizontal projection = 1 cos 240°.

Vertical projection = 1 sin 240°.

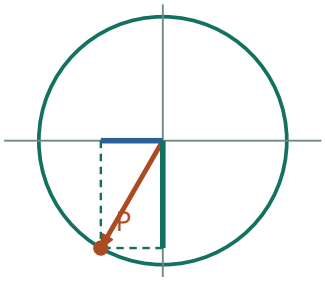
1. $a^2 + (1/2)^2 = 1^2$
Bisect a unit equilateral triangle to form a right triangle.

2. $a = \sqrt{(1 - 1/4)} = \sqrt{3}/2$
A side length uses the nonnegative square root.

3. Attach coordinate signs

In quadrant III both x and y are negative. Since sine is y and cosine is x on the unit circle, $\sin 240° = -\sqrt{3}/2 \approx -0.866$ and $\cos 240° = -1/2$.

Radius direction · 240°

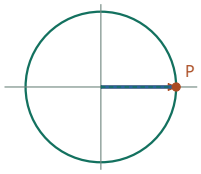


Radius 1; angle 240°.

Projection onto x = −0.5. Projection onto y = −0.866.

Compare settings · Radius direction

0°

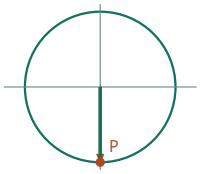


Radius 1; angle 0°.

Projection onto x = 1. Projection onto y = 0.

Radius 1; angle 0°. Projection onto x = 1. Projection onto y = 0.

270°



Radius 1; angle 270°.

Projection onto x = −0. Projection onto y = −1.

Radius 1; angle 270°. Projection onto x = −0. Projection onto y = −1.

Check. Adding 180° to 60° negates each coordinate of its unit-circle point.

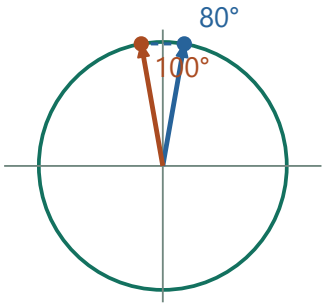
Exercise 52 · Related angles

Printed page 5 · Reference angles and symmetry / Trigonometric identities

Given that $\cos 80 = 0.173648\ldots$, explain how to find $\cos 100$, $\cos 260$, $\cos 280$, and $\sin 190$ without using a calculator.

1. Start with the known horizontal projection

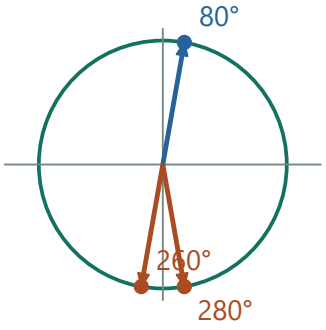
At 80° the unit-circle x-coordinate is 0.173648. Reflecting across the y-axis reverses x; reflecting across the x-axis preserves x.



A reflection across y reverses the horizontal coordinate.

2. Use the quadrant symmetries

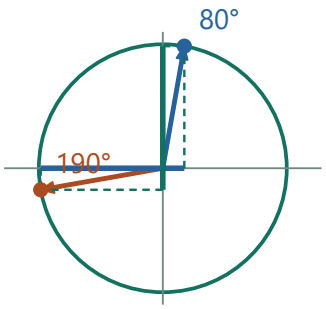
The 100° ray is the y-axis reflection of 80°, so cos 100°≈−0.173648. At 260° one half-turn from 80° also reverses x. At 280° the x-axis reflection preserves x, giving +0.173648.



The half-turn reverses x; reflection across x preserves x.

3. Recover a sine from a cosine

The 190° ray is 10° below the negative x-axis, so sin 190°=−sin 10°. The complementary angle identity sin 10°=cos 80° gives −0.173648.



sin 190° = −sin 10° = −cos 80°.

Check. Their signs agree with the corresponding quadrants.

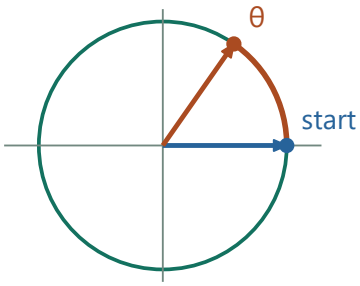
Exercise 53 · Circular definitions

Printed page 5 · The unit circle / Trigonometric ratios

Use the unit circle to define $\cos\theta$ and $\sin\theta$ for any number θ between 0 and 360, inclusive. Then explain how to use $\cos\theta$ and $\sin\theta$ to define $\tan\theta$.

1. Make a directed angle

Draw a circle of radius 1 centered at the origin. Starting with the radius to (1,0), rotate counterclockwise by the specified positive angle θ to meet the circle at P.

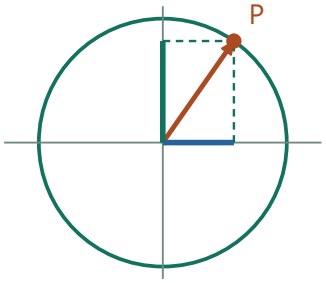


Measure the directed turn from (1,0).

2. Read two coordinates

The signed horizontal displacement to P defines cos θ; the signed vertical displacement defines sin θ. Thus P=(cos θ,sin θ), even in quadrants where one or both numbers are negative.

Radius direction · 55°

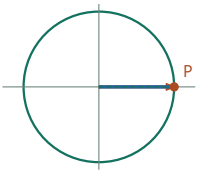


Radius 1; angle 55°.

Projection onto x = 0.574. Projection onto y = 0.819.

Compare settings · Radius direction

0°

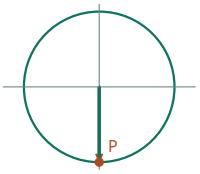


Radius 1; angle 0°.

Projection onto x = 1. Projection onto y = 0.

Radius 1; angle 0°. Projection onto x = 1. Projection onto y = 0.

270°



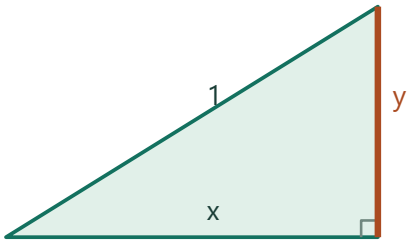
Radius 1; angle 270°.

Projection onto x = −0. Projection onto y = −1.

Radius 1; angle 270°. Projection onto x = −0. Projection onto y = −1.

3. Read the ray's slope

If P=(x,y) and x≠0, rise/run=y/x defines tan θ=sin θ/cos θ. At 90° and 270° the ray is vertical and the quotient is undefined.



Schematic section

Slope = y/x = tan θ wherever x ≠ 0.

Check. At 90° and 270°, x=0; a vertical ray has undefined slope and tangent.

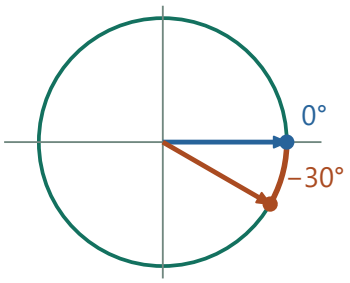
Exercise 54 · Angles of any size

Printed page 5 · The unit circle / Angle measure

Show that your method in the previous question allows you to define $\cos\theta$, $\sin\theta$, and $\tan\theta$ for numbers θ greater than 360 and also for numbers θ less than 0. What do you suppose it means for an angle to be negative?

1. Give angle a direction

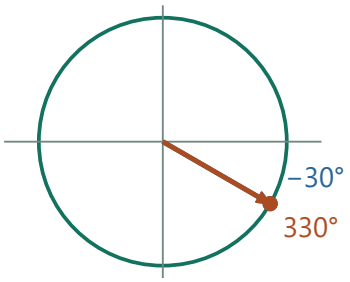
Starting at the positive x-axis, turn counterclockwise for a positive angle and clockwise for a negative one. The point on the unit circle still determines cosine and sine.



Clockwise is the negative direction.

2. Remove whole turns

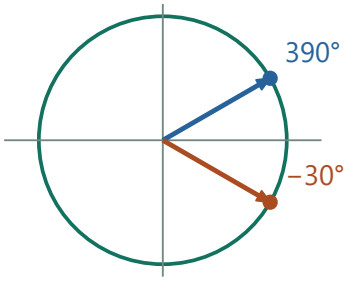
Adding or subtracting 360° changes the number of revolutions but leaves the endpoint unchanged. Such angles are called coterminal and give the same coordinates.



Add a full revolution and the endpoint stays the same.

3. Test both extensions

A 30° clockwise turn equals 330° counterclockwise, giving (√3/2,−1/2). A 390° turn ends at the same point as 30°, giving (√3/2,1/2). The tangent y/x is likewise unchanged where defined.



The first direction equals +30°; the second equals 330°.

Cosines match while the sines have opposite signs.

Check. For instance −30° and 330° give (√3/2,−1/2); 390° and 30° share (√3/2,1/2).

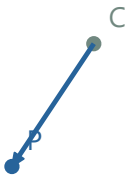
Exercise 55 · Half-turn in general

Printed page 5 · Rotations / Coordinate geometry

A half-turn centered at $(-3, 4)$ is applied to $(-5, 1)$. Find coordinates for the image point. What are the coordinates when the half-turn centered at (a, b) is applied to (x, y)?

1. Track the arrow from the rotation center

From (−3,4) to (−5,1) the displacement is (−2,−3). A 180° rotation reverses it to (2,3).

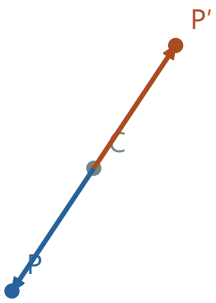


Begin at center C = (-3, 4).

Measure the start relative to C.

2. Locate the image

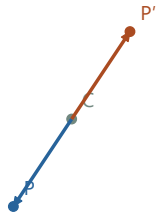
Add the new displacement to the center, obtaining $(-3+2,4+3)=(-1,7)$. The center is halfway between the original and image.



The midpoint between P and P′ is C. The two displacements from C have opposite signs.

3. Name the fixed center (a,b)

For an original (x,y) the displacement is $(x-a,y-b)$. Subtract it from (a,b) to reach the opposite endpoint, giving $(2a-x,2b-y)$.



The new arrow has the original length.

P′ = (−1,7).

1. (x+x′)/2=a, (y+y′)/2=b
The center is the midpoint of the original and rotated points.

2. x+x′=2a, y+y′=2b
Double each midpoint equation.

3. x′=2a−x, y′=2b−y
Subtract the starting coordinates to isolate the image.

Check. The midpoint of (−5,1) and (−1,7) is (−3,4).

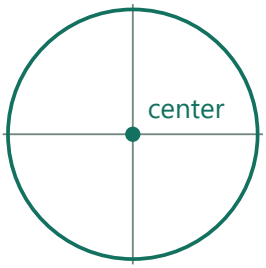
Exercise 56 · Translate a circle

Printed page 5 · Translations / Circle geometry

Translate the circle $x^2 + y^2 = 49$ by the vector $[3, -5]$. Write an equation for the image circle.

1. Locate the original circle

A circle of radius 7 centered at the origin consists of points at distance 7 from (0,0). The translation adds 3 to every x-coordinate and subtracts 5 from every y-coordinate.

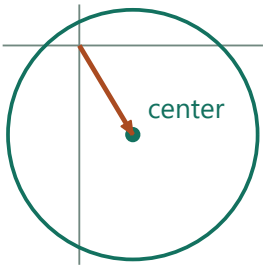


Original radius is 7.

A translation preserves all distances.

2. Move the center with the circle

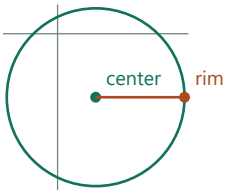
The center becomes (3,−5). Translation preserves all distances, including the 7-unit center-to-rim distance.



The new center is (3,−5).

3. Write the translated condition

The squared distance from a new point (x,y) to (3,−5) is $(x-3)^2+(y+5)^2$. Equating it to 7^2 gives $(x-3)^2+(y+5)^2=49$.



From the new center to the rim the squared distance is 49.

$(x-3)^2 + (y+5)^2 = 49$.

1. $x_{old}=x-3$, $y_{old}=y+5$
Undo the translation to express the starting location using the new coordinates.

2. $x_{old}^2+y_{old}^2=49 \Rightarrow (x-3)^2+(y+5)^2=49$
Every starting point satisfies the original equation, so replace both old coordinates.

Check. The original rightmost point (7,0) translates to (10,−5), which satisfies the equation.

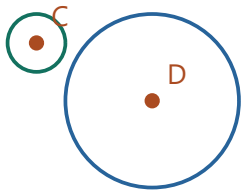
Exercise 57 · Dilation between circles

Printed page 6 · Similarity and scaling / Circle geometry

Point by point, a dilation transforms the circle $x^2-6x+y^2-8y=-24$ onto the circle $x^2-14x+y^2-4y=-44$. Find the center and the magnification factor of this transformation.

1. Read centers and radii first

Move the constant in each circle equation and complete both squares. The first becomes $(x-3)^2+(y-4)^2=1$ and the second $(x-7)^2+(y-2)^2=9$, so C=(3,4) and D=(7,2).



The radii are 1 and 3.

Their centers are (3,4) and (7,2).

2. Determine the scale

A magnification sends each radius to the corresponding new radius. Going from radius 1 to radius 3 requires positive scale factor 3.

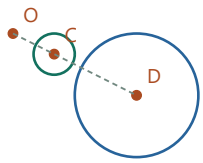


The new radius contains exactly three copies of the old radius.

The positive dilation factor is 3.

3. Solve for the fixed point

If O is the center of magnification, the displacement OD equals 3 times OC. In vector notation $D=O+3(C-O)=3C-2O$, hence $O=(3C-D)/2=(1,5)$.



D − O is 3(C − O).

The fixed center is O=(1,5).

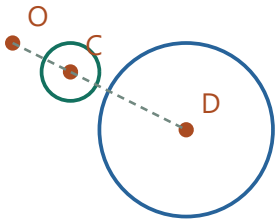
1. $D=O+3(C-O)=3C-2O$
Distribute the dilation factor on the displacement from O.

2. $2O=3C-D$
Move the center term to the other side.

3. $O=((9,12)-(7,2))/2=(1,5)$
Subtract corresponding coordinates and divide both by 2.

4. Check direction as well as length

From O the center displacements are (2,−1) and (6,−3), exactly a factor of 3. If signed factors are allowed, factor −3 about (4,3.5) gives another transformation.



O,C,D lie on one line with C between O and D.

The dilation sends C to D and radius 1 to radius 3.

Check. The displacement from O to D is three times that from O to C. If signed homotheties are allowed, factor −3 also works about (4,3.5).

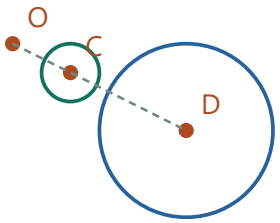
Exercise 58 · Common external tangents

Printed page 6 · Circle tangency / Right-triangle trigonometry

(Continuation) The circles have two common external tangent lines, which meet at the dilation center. Find the size of the angle formed by these lines, and write an equation for each line.

1. Use the previous magnification

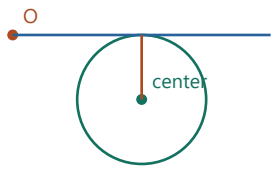
Both external common tangents pass through the positive magnification center (1,5). A line through that point maps onto itself. If tangent to the small circle, its image is tangent to the large one.



Both external common tangents pass through O.

2. Impose tangency to the small circle

For a nonvertical line $y-5=m(x-1)$, the distance from $(3,4)$ to the line is $|2m+1|/\sqrt{(1+m^2)}$. Set this equal to its radius 1 and square to obtain $4m^2+4m+1=1+m^2$.



The perpendicular distance from C to a tangent is the radius 1.

The horizontal tangent illustrates the condition before solving for every slope.

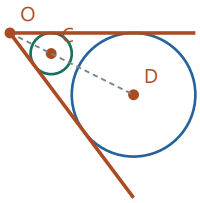
1. $mx-y+5-m=0$
Rearrange the line through the fixed point for the distance formula.

2. distance=abs(3m−4+5−m)/√(m²+1)=1
Insert center (3,4); tangency makes the distance one radius.

3. $(2m+1)^2=m^2+1$
Both distances are nonnegative, so square the equality.

3. Recover both lines

Then $m(3m+4)=0$. The lines are $y=5$ and $y-5=-(4/3)(x-1)$. The vertical line $x=1$ is 2 units from the small center, so it supplies no extra tangent.



One tangent is horizontal; the other has slope $-4/3$.

Tangency requires distance 1 to C.

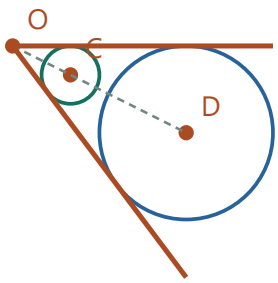
1. $4m^2+4m+1=m^2+1$
Expand the squared numerator.

2. $3m^2+4m=0 \Rightarrow m(3m+4)=0$
Subtract m^2+1 and factor out m.

3. $m=0$ or $m=-4/3$
Solve each factor separately; $x=1$ is two units from the small center and is not tangent.

4. Measure the angle

The first line is horizontal, and the second has slope magnitude 4/3. The smaller angle is $\arctan(4/3)\approx 53.13°$; the other is about 126.87°.



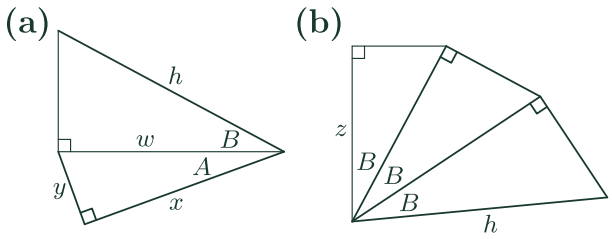
Comparing the two tangent directions gives an acute angle of 53.13°.

Check. Distances from (7,2) to both lines equal 3; the vertical line through (1,5) has distance 2 to the first center so none was lost.

Exercise 59 · Chained right triangles

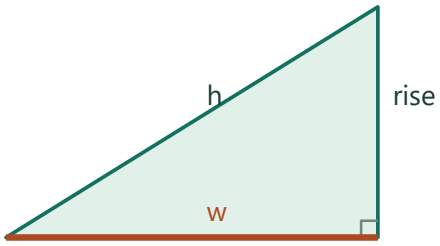
Printed page 6 · Right-triangle trigonometry / Trigonometric ratios

Using the figures at right, express the lengths w, x, y, and z in terms of length h and angles A and B.



1. Project h onto the horizontal side in (a)

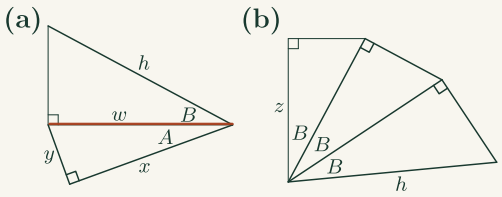
For angle B, h is the hypotenuse and the shared segment w is adjacent. Thus w/h=cos B, giving w=h cos B.



Schematic section

First projection

w = h cos B.

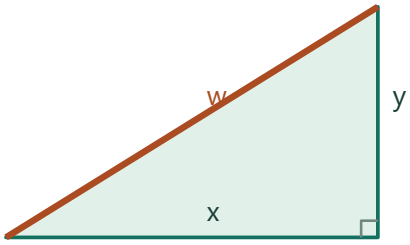


Shared side w in the first construction

This horizontal segment is adjacent to B in the upper triangle and serves as the hypotenuse of the lower triangle.

2. Use w as the next hypotenuse

In the lower triangle, x is adjacent to A and y is opposite. Therefore x=w cos A=h cos B cos A and y=w sin A=h cos B sin A.



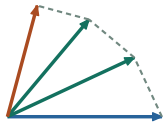
Schematic section

Now w is a hypotenuse.

x = w cos A; y = w sin A.

3. Track the repeated projection in (b)

Each new segment is adjacent to angle B relative to the preceding hypotenuse. The successive lengths are h cos B, h $\cos^2 B$, and finally z=h $\cos^3 B$.

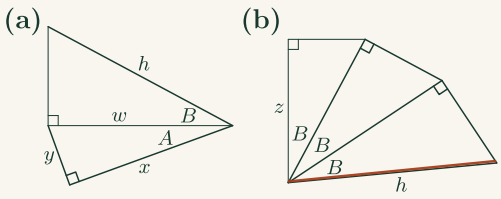


Three consecutive projections by angle B

Each multiplies the length by cos B, giving z = h $\cos^3 B$.

1. $\ell_1 = h \cos B$, $\ell_2 = \ell_1 \cos B$
Each new projection uses the preceding length as its hypotenuse.

2. $z = \ell_2 \cos B = (h \cos B) \cos B \cos B = h \cos^3 B$
Substitute backward through the three projections.



First direction in the repeated construction

In the right-hand drawing, successive rays share one tail. Start with the lower side marked h and project from ray to ray.

Check. The lower components satisfy $x^2+y^2=w^2$. The ratio z/h is the product of the three adjacent-to-hypotenuse ratios.

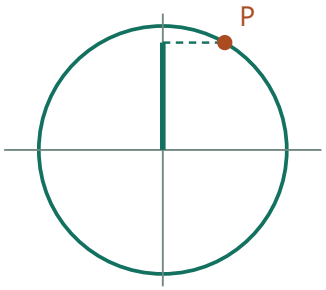
Exercise 60 · Solutions of a sine equation

Printed page 6 · Trigonometric equations / The unit circle

Find at least two values for θ that fit the equation $\sin\theta = \frac{1}{2}\sqrt{3}$. How many such values are there?

1. Read sine as a height

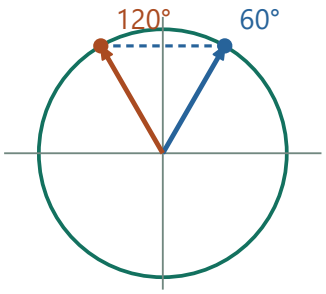
Draw the horizontal line y=√3/2 across the unit circle. Its positive height places both intersections above the x-axis.



Fix the unit-circle height at √3/2.

2. Locate both directions in one turn

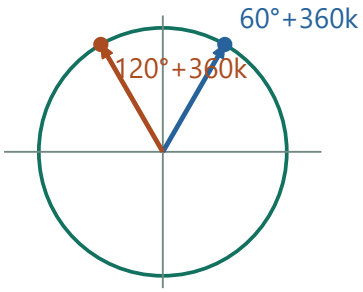
A 60° reference triangle has sine √3/2. One intersection occurs at 60° and its y-axis reflection occurs at 180°−60°=120°.



The same height occurs on either side of the y-axis.

3. Include all rotations

Every full turn leaves the point unchanged. All solutions are 60°+360°k and 120°+360°k, with k any integer. Thus there are infinitely many angles.



Whole turns preserve both endpoints.

Every integer k gives solutions.

Check. Both terminal points have y coordinate √3/2.

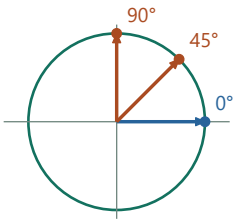
Exercise 61 · Testing the square identity

Printed page 6 · Trigonometric identities / The unit circle

Choose an angle θ and calculate $(\cos\theta)^2 + (\sin\theta)^2$. Repeat with several other values of θ. Explain the results. It is customary to write $\cos^2\theta + \sin^2\theta$ instead of $(\cos\theta)^2 + (\sin\theta)^2$.

1. Try angles with known coordinates

At 0° the unit point is (1,0), giving square sum 1+0. At 45° the coordinates are both √2/2, giving 1/2+1/2. At 90° the square sum is 0+1.



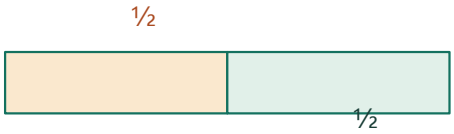
Read the horizontal and vertical components for three angles.

1. $(\sqrt{2}/2)^2=(\sqrt{2})^2/2^2=2/4=1/2$
Both coordinates at 45° have this squared magnitude.

2. 1/2+1/2=1
Add the horizontal and vertical squared distances from the origin.

2. Identify what these calculations measure

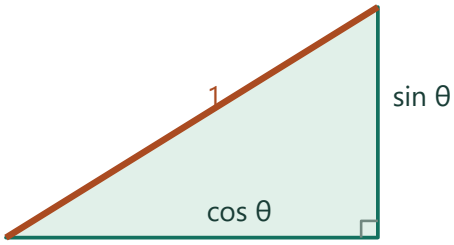
For a point (x,y), x^2+y^2 is its squared distance from the origin. Every point used here lies one unit from the origin.



At 45°, the two squared coordinates are each 1/2.

3. Generalize the calculation

For any angle θ the point is (cos θ,sin θ), so $\cos^2\theta+\sin^2\theta=1$. The rule holds also where either coordinate is negative, since the distance formula squares them.



Schematic section

The radius of the unit circle has length one.

The squared legs always sum to one.

Check. This is the unit circle's Cartesian equation $x^2+y^2=1$.

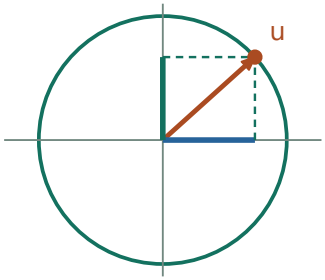
Exercise 62 · Translate a parametric curve

Printed page 6 · Parametric equations / Translations / Circle geometry

What graph is traced by the parametric equation $(x, y) = (2 + \cos t, 1 + \sin t)$?

1. Subtract the constant shifts

The point has coordinates x=2+cos t and y=1+sin t. Measured from (2,1), its displacement is (cos t,sin t).

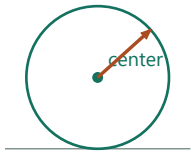


Subtract the center first.

(x−2,y−1) is a unit vector.

2. Measure the local radius

Squaring those local coordinates and adding gives $(x-2)^2+(y-1)^2=1$. Every point of the curve is one unit from (2,1).



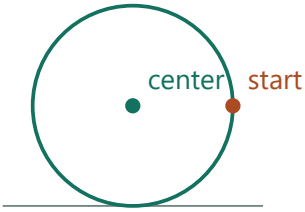
The local unit radius is unchanged by translation.

1. $x-2=\cos t$, $y-1=\sin t$
Subtract the center coordinates to isolate the unit displacement.

2. $(x-2)^2+(y-1)^2=\cos^2 t+\sin^2 t=1$
Square and add; the squared unit coordinates sum to one.

3. Describe the tracing

At t=0 it starts at (3,1), one unit right of the center. As t increases through one turn it passes all directions counterclockwise, drawing the whole circle.



Let the angle run through a turn.

The path is the complete circle centered at (2,1).

Check. At t=0 the curve begins at (3,1), the rightmost point.

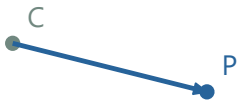
Exercise 63 · Quarter-turn about an offset center

Printed page 6 · Rotations / Coordinate geometry

A *quarter-turn* is a 90-degree rotation. If the counterclockwise quarter-turn centered at $(3, 2)$ is applied to $(7, 1)$, what are the coordinates of the image? What are the image coordinates when this transformation is applied to a general point (x, y)?

1. Measure relative to (3,2)

For the test point (7,1) the displacement is (4,−1). For a general point it is (u,v)=(x−3,y−2).

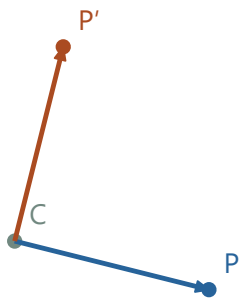


Begin at center C = (3, 2).

Measure the start relative to C.

2. Rotate the local arrow

A counterclockwise quarter-turn sends right to up and up to left. It therefore maps (u,v) to (−v,u); the test displacement becomes (1,4).



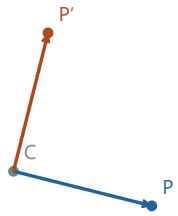
Both displacement arrows use the same center and have equal length.

The orange arrow is the rotated displacement.

3. Translate back

Add (3,2) to the new displacement. The test point becomes (4,6). In general the result is (3−(y−2),2+x−3)=(5−y,x−1).

Counterclockwise turn · 90°



Rotate by 90° about (3, 2).

New point (4, 6); distance to center remains 4.123.

1. P′=(3,2)+(−(y−2),x−3)
Add the center after a quarter-turn of the local coordinates.

2. P′=(3−y+2,2+x−3)=(5−y,x−1)
Distribute the minus sign and collect constants.

Compare settings · Counterclockwise turn

0°

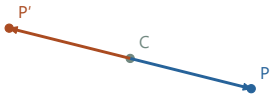


Rotate by 0° about (3, 2).

New point (7, 1); distance to center remains 4.123.

Rotate by 0° about (3, 2). New point (7, 1); distance to center remains 4.123.

180°



Rotate by 180° about (3, 2).

New point (−1, 3); distance to center remains 4.123.

Rotate by 180° about (3, 2). New point (−1, 3); distance to center remains 4.123.

Check. Before and after displacements (4,−1) and (1,4) are perpendicular, have equal length, and turn counterclockwise.

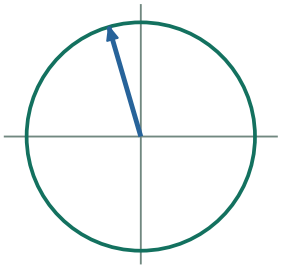
Exercise 64 · Tangent point from a normal

Printed page 6 · Circle tangency / Vector projections

A circle centered at the origin meets the line $-7x + 24y = 625$ tangentially. Find coordinates for the point of tangency.

1. Identify the contact radius

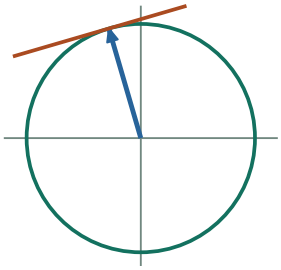
A radius to a tangency is perpendicular to the tangent. The line $-7x+24y=625$ has normal vector $(-7,24)$, so the radius from the origin to contact must follow that direction or its reverse.



The vector $(-7,24)$ is perpendicular to the line.

2. Put the possible contact on that line

Write the point as $(-7t,24t)$. This represents all points on the perpendicular through the origin; exactly one also belongs to the tangent.

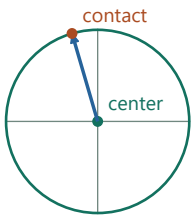


A tangent is perpendicular to the contact radius.

Its contact point must be $t(-7,24)$.

3. Solve the intersection

Substitution gives $49t+576t=625$ and hence $t=1$. The contact point is $(-7,24)$ and the radius is $\sqrt{625}=25$.



$625t = 625$, hence $t = 1$.

Contact point $(-7,24)$.

1. $-7(-7t)+24(24t)=625$
Substitute the normal-ray coordinates in the tangent line.

2. $(49+576)t=625 \Rightarrow t=1$
The common coefficient is 625.

3. $P=(-7,24)$
Insert the solved scale into the normal vector.

Check. Its distance from the origin is 25, matching the line's distance 625/25.

Exercise 65 · Expanding squares

Printed page 6 · Polynomial identities / Algebraic manipulation

Write without parentheses: **(a)** $(xy)^2$ **(b)** $(x+y)^2$ **(c)** $(a\sin B)^2$ **(d)** $(a+\sin B)^2$

1. Square a product in (a)

The exponent 2 calls for two identical copies of the entire expression. Write $(xy)^2$ as $(xy)(xy)$. These factors are numbers, so regroup $x{\cdot}y{\cdot}x{\cdot}y$ as $(x{\cdot}x)(y{\cdot}y)$. Removing the parentheses gives x^2y^2.

x	y
x	y

Two identical factors xy

Regroup their factors to obtain x^2y^2.

2. Distribute each term in (b)

For $(x+y)^2$ the two copies are sums. Multiply x from the first copy by each term of the second to get x^2+xy. Multiply y by each to get $yx+y^2$. The middle products are equal, so the answer is $x^2+2xy+y^2$.

x^2	xy
yx	y^2

In $(x+y)^2$ each term multiplies both terms.

Both cross-products contribute xy.

1. $(x+y)^2=(x+y)(x+y)$
A square contributes two identical factors.

2. $=x^2+xy+yx+y^2=x^2+2xy+y^2$
Distribute both terms and combine the equal cross products.

3. Keep the trigonometric factor together in (c)

For a given angle B, sin B is one number. Multiplying a sin B by itself gives $a{\cdot}a$ times sin B·sin B. The customary notation $\sin^2 B$ means the square of the sine value, so the result is $a^2\sin^2 B$.

a	sin B
a	sin B

The number sin B is one factor in each copy.

Reordering the four factors gives $a^2\sin^2 B$.

4. Expand the binomial in (d)

Multiply (a+sin B)(a+sin B) one term at a time. The products are a^2, a sin B, a sin B and $\sin^2 B$. Combining the two equal middle terms gives $a^2+2a \sin B+\sin^2 B$.

a^2	a sin B
a sin B	$\sin^2 B$

Use sin B as a single numerical factor.

The two equal cross-products give 2a sin B.

1. $(a+\sin B)(a+\sin B)=a^2+a \sin B+(\sin B)a+\sin^2 B$
Distribute each term of the first factor across the second.

2. $=a^2+2a \sin B+\sin^2 B$
The two middle products are equal.

Check. Distributing each factor against a second identical copy gives these terms.

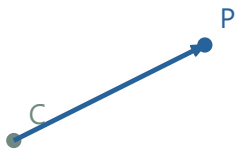
Exercise 66 · Recognize a half-turn

Printed page 6 · Rotations / Midpoints

Transformation $\mathcal{T}$ is defined by $\mathcal{T}(x, y) = (2, 7) + [2 - x, 7 - y]$. An equivalent definition is $\mathcal{T}(x, y) = (4 - x, 14 - y)$. Use the first definition to help you explain what kind of transformation $\mathcal{T}$ is.

1. Read the first definition as a point plus a displacement

Let C=(2,7) and P=(x,y). The vector from P to C has components (2−x,7−y). The printed rule starts at C and adds this very vector a second time, landing at the image P′.

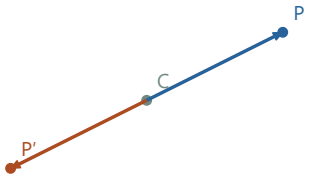


Begin at center C = (2, 7).

Measure the start relative to C.

2. Locate the center between P and P′

The displacement from P to C equals the displacement from C to P′. Thus P, C and P′ lie on the same line, and C is the midpoint of PP′. Equivalently, if P=(2+u,7+v), then P′=(2−u,7−v).



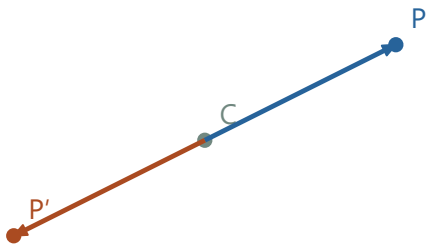
The midpoint between P and P′ is C.
The two displacements from C have opposite signs.

1. P′=(4−(2+u),14−(7+v))=(2−u,7−v)
Substitute the coordinates of P and subtract the complete parenthesized sums.

2. (P+P′)/2=((2+u+2−u)/2,(7+v+7−v)/2)=(2,7)
Opposite local displacements cancel, proving that C bisects every image segment.

3. Recognize and check the half-turn

Viewed from C, the radius to P′ points in the opposite direction from the radius to P, with the same length. That is a 180° rotation about C. Substituting C into the rule returns C itself, as a rotation center should.



The new arrow has the original length.

Blue shows the original displacement; orange shows its image.

Check. The midpoint of any point and its image is (2,7).

Exercise 67 · Rotate about (2,1)

Printed page 6 · Rotations / Vectors and components

A 15-degree counterclockwise rotation centered at $(2, 1)$ sends $(4, 6)$ to another point (x, y). Find x and y, correct to three decimal places.

1. Measure the radius from the point that stays fixed

Let C=(2,1) be the center and P=(4,6) the starting point. Traveling from C to P requires 4−2=2 units right and 6−1=5 units up. The displacement to rotate is d=(2,5).

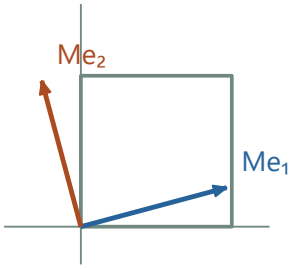


Begin at center C = (2, 1).

Measure the start relative to C.

2. Turn the two basic directions

Write c=cos 15° and s=sin 15°. A counterclockwise 15° turn sends a rightward unit arrow to (c,s). An upward unit arrow starts 90° farther around the circle, so it turns to (−s,c).



A rotated horizontal unit is (cos15°,sin15°).

The upward unit becomes (−sin15°,cos15°).

3. Rebuild the rotated displacement

The original displacement used two rightward units and five upward units. After rotation the same combination is 2(c,s)+5(−s,c)=(2c−5s,2s+5c). In degree mode this is approximately (0.637756,5.347267).



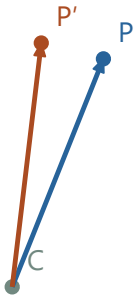
Combine two rotated rightward units and five rotated upward units.

The result is still relative to the center.

4. Return to the given coordinate system

The preceding numbers are measured from C. Adding C gives (2+0.637756,1+5.347267), so x≈2.638 and y≈6.347 to three decimal places. As a check, the squared radius stays $2^2+5^2=29$ because the rotated unit directions are still perpendicular.

Counterclockwise turn · 15°



Rotate by 15° about (2, 1).

New point (2.638, 6.347); distance to center remains 5.385.

Compare settings · Counterclockwise turn

0°



Rotate by 0° about (2, 1).
New point (4, 6); distance to center remains 5.385.

Rotate by 0° about (2, 1). New point (4, 6); distance to center remains 5.385.

180°



Rotate by 180° about (2, 1).
New point (−0, −4); distance to center remains 5.385.

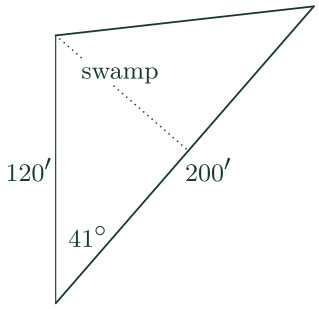
Rotate by 180° about (2, 1). New point (−0, −4); distance to center remains 5.385.

Check. Squared radius is 29 both before and after.

Exercise 68 · Survey across a swamp

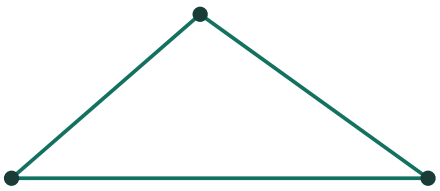
Printed page 6 · Right-triangle trigonometry / Triangle area

A triangular plot of land has the SAS description indicated in the figure shown at right. Although a swamp in the middle of the plot makes it awkward to measure the altitude that is dotted in the diagram, it can at least be calculated. Show how. Then use your answer to find the area of the triangle, to the nearest square foot.

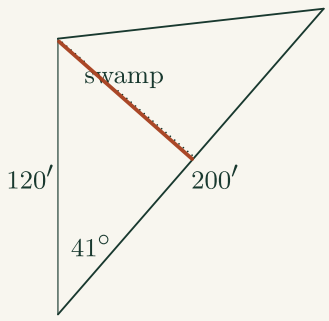


1. Construct the needed height

Use the 200-ft side as base. From the end of the 120-ft side, drop a perpendicular to that base. The resulting right triangle has the 120-ft side as hypotenuse.



Known sides 200 and 120 meet at 41°.

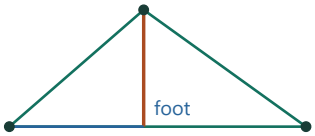


Perpendicular altitude to the base

The dotted segment from the upper-left vertex meets the 200-ft side at a right angle. This is the height needed with that base.

2. Find the perpendicular component

The altitude h lies opposite the included angle 41°, so sin 41°=h/120. This gives h≈78.73 ft.



Drop the perpendicular from the 120-unit side.

Its height is 120 sin 41°; its signed projection is 120 cos 41°.

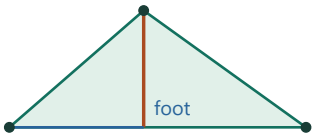
1. h/120=sin41° ⇒ 120(h/120)=120 sin41°
Multiply by the right triangle's hypotenuse.

2. h=120 sin41°≈78.7271 ft
Keep the unrounded value for the area.

3. Calculate the triangular area

A triangle occupies half a parallelogram of the same base and height. Its area is ½·200·120 sin 41°≈7873 ft².

Included angle · 41°

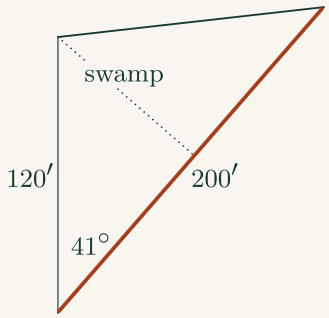


Base 200 ft, sloping side 120 ft.

Altitude 78.727 ft; triangle area 7872.708 ft²; opposite triangle side 134.811 ft.

1. A=(200/2)(120 sin41°)
Substitute the exact altitude for h in the triangle area formula.

2. A=12,000 sin41°≈7872.71 ft²
Multiply the constants before evaluating and rounding to 7873.



Base of length 200 ft

The sloping segment labeled 200′ is the base chosen in this calculation. A base need not be horizontal on the page.

Compare settings · Included angle

139°

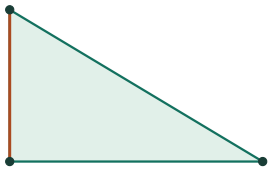


Base 200 ft, sloping side 120 ft.

Altitude 78.727 ft; triangle area 7872.708 ft²; opposite triangle side 301.042 ft.

Base 200 ft, sloping side 120 ft. Altitude 78.727 ft; triangle area 7872.708 ft²; opposite triangle side 301.042 ft.

90°



Base 200 ft, sloping side 120 ft.

Altitude 120 ft; triangle area 12000 ft²; opposite triangle side 233.238 ft.

Base 200 ft, sloping side 120 ft. Altitude 120 ft; triangle area 12000 ft²; opposite triangle side 233.238 ft.

Check. Taking 120 as base gives altitude 200 sin41°, hence identical area.

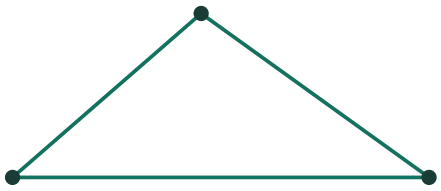
Exercise 69 · Third side of the plot

Printed page 6 · The cosine rule

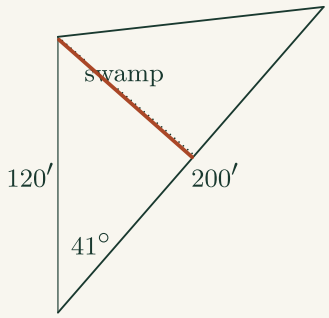
(Continuation) Find the length of the third side of the triangle, to the nearest foot.

1. Reuse the altitude from the field

Project the 120-ft side onto the 200-ft base. Its horizontal projection is 120 cos 41° and its vertical projection is 120 sin 41°.



Known sides 200 and 120 meet at 41°.

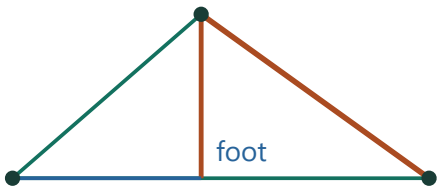


Perpendicular altitude to the base · Figure from exercise 68

The dotted segment from the upper-left vertex meets the 200-ft side at a right angle. This is the height needed with that base.

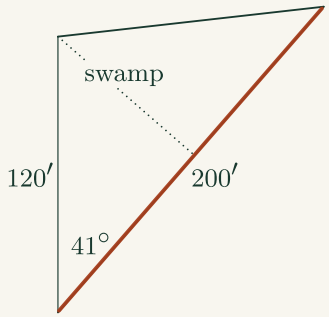
2. Locate the unknown side

From the perpendicular foot to the far end of the base, the remaining horizontal length is 200−120 cos 41°. The unknown side is the hypotenuse over this run and the altitude.



From the foot to the far end there are 200−120 cos 41° feet.

The opposite side is the hypotenuse over this remaining run and the altitude.



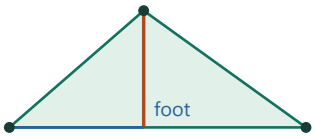
Base of length 200 ft · Figure from exercise 68

The sloping segment labeled 200′ is the base chosen in this calculation. A base need not be horizontal on the page.

3. Apply Pythagoras

Squaring and adding gives $c^2=(200-120\cos 41°)^2+(120\sin 41°)^2=200^2+120^2-2\cdot200\cdot120\cos 41°$. Therefore $c\approx134.811$ ft, or 135 ft.

Included angle · 41°



Base 200 ft, sloping side 120 ft.

Altitude 78.727 ft; triangle area 7872.708 ft²; opposite triangle side 134.811 ft.

1. $c^2=(200-120\cos41°)^2+(120\sin41°)^2$
Use the horizontal and vertical legs shown by the altitude.

2. $c^2=200^2-2\cdot200\cdot120\cos41°+120^2(\cos^2 41°+\sin^2 41°)$
Expand and group the squares belonging to the 120-ft side.

3. $c=\sqrt{(200^2+120^2-48{,}000\cos41°)}$
Use the unit-circle identity and select a positive length.

Compare settings · Included angle

139°

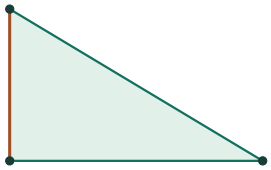


Base 200 ft, sloping side 120 ft.

Altitude 78.727 ft; triangle area 7872.708 ft²; opposite triangle side 301.042 ft.

Base 200 ft, sloping side 120 ft. Altitude 78.727 ft; triangle area 7872.708 ft²; opposite triangle side 301.042 ft.

90°



Base 200 ft, sloping side 120 ft.

Altitude 120 ft; triangle area 12000 ft²; opposite triangle side 233.238 ft.

Base 200 ft, sloping side 120 ft. Altitude 120 ft; triangle area 12000 ft²; opposite triangle side 233.238 ft.

Check. The projection of 120 along the 200-ft side is 120 cos41°; combining the leftover base with the altitude gives the same length.

Exercise 70 · All angles in one turn

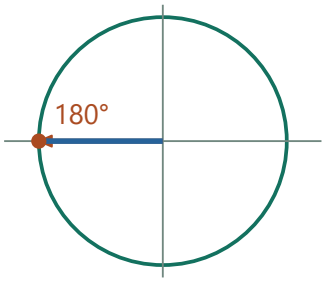
Printed page 6 · Trigonometric equations / Inverse trigonometric functions

Use the unit circle and, if necessary, a calculator to find all solutions θ for $0 \le \theta \le 360$:

(a) $\cos\theta = -1$ **(b)** $\cos\theta = 0.3420$ **(c)** $\sin\theta = -\frac{1}{2}\sqrt{2}$ **(d)** $\tan\theta = 6.3138$

1. Find the cosine directions in (a)

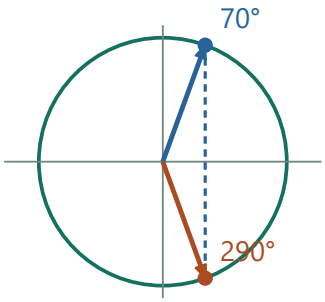
Cosine is the x-coordinate on the unit circle. The leftmost point is the only one with x=−1, at θ=180°.



cos θ = −1 only at the leftmost point.

2. Find both intersections in (b)

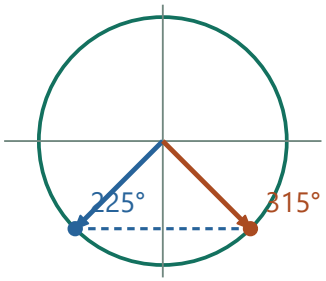
The vertical line x=0.3420 intersects above and below the x-axis. In degree mode the first angle is arccos(0.3420)≈70°. Its reflection gives 360°−70°≈290°.



Both points have x ≈ 0.3420.

3. Use the negative height in (c)

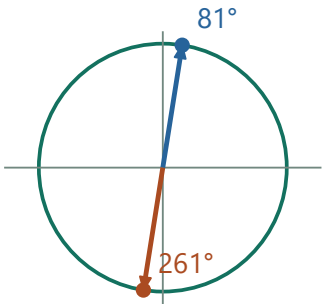
Sine $-\sqrt{2}/2$ has a 45° reference angle. The two lower quadrants give $180°+45°=225°$ and $360°-45°=315°$.



Both points have $y = -\sqrt{2}/2$.

4. Use the slope in (d)

For $\tan\theta=6.3138$ the ray has positive slope. Its first direction is $\arctan(6.3138)\approx 81°$. The opposite ray has the same slope at about 261°. These are all intersections within the given turn.



Opposite rays have the same positive slope, about 6.3138.

Check. The quadrant signs and periodicity produce every root in the specified interval.

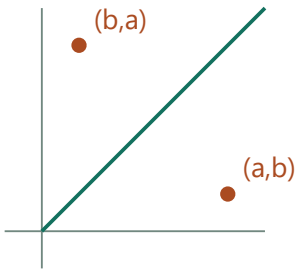
Exercise 71 · Reflection and nearest point

Printed page 7 · Reflections / Vector projections

Using the line $y = x$ as a mirror, find the reflected image of the point (a, b). What are the coordinates of the point on the line that is closest to (a, b)?

1. Reflect across the diagonal

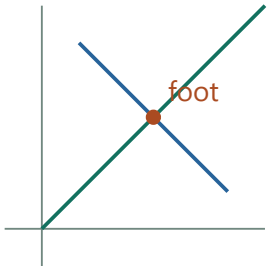
The mirror y=x exchanges the horizontal and vertical coordinate directions. The reflection of (a,b) is consequently (b,a).



Reflection in y=x interchanges the coordinate entries.

2. Locate the foot on the mirror

A mirror is the perpendicular bisector of the segment from a point to its reflected image. The foot of the perpendicular is therefore the midpoint.



The mirror perpendicularly bisects the segment joining the two reflected points.

3. Compute that midpoint

Average x-coordinates and then y-coordinates, giving ((a+b)/2,(a+b)/2). Its equal coordinates place it on the mirror, and the joining displacement has zero dot product with (1,1).

a	b
b	a

Average each coordinate separately.

Both coordinates of the foot equal (a+b)/2.

1. H=((a+b)/2,(b+a)/2)
Average each coordinate of P=(a,b) and its mirror P′=(b,a).

2. H=((a+b)/2,(a+b)/2)
Commutativity gives equal coordinates, as required on the line y=x.

Check. The segment joining the pair has direction (b−a,a−b), perpendicular to (1,1); if a=b the point is already on the line.

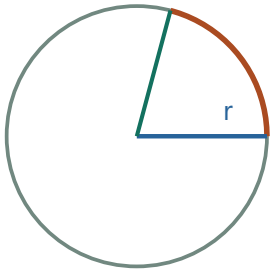
Exercise 72 · Sector length and boundary

Printed page 7 · Circular sectors / Arc length

The radius of a circular sector is r. The central angle of the sector is θ. Write formulas for the arc length and the perimeter of the sector.

1. Identify the three boundary pieces

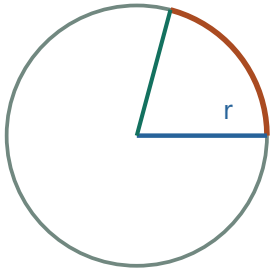
An ordinary sector consists of two straight segments of length r plus a curved arc of length s. Thus its perimeter is 2r+s.



The two straight sides contribute 2r; the curved side contributes s.

2. Use radian measure

By definition θ=s/r when θ is in radians. Solve for s=rθ, giving P=2r+rθ.



For a radian measure θ, s = rθ and P = 2r+rθ.

3. Use degrees if specified

For θ degrees, the arc is the fraction θ/360 of a full circumference. Thus s=2πrθ/360 and P=2r+2πrθ/360. At 180° this includes both a semicircular arc and a diameter.



For a degree measure the arc occupies θ/360 of the full circumference.

Check. At 180°, the boundary is the semicircular arc plus its diameter.

Exercise 73 · Projecting a flight path

Printed page 7 · Projection onto a plane / Parametric lines

A bird flies linearly, according to the equation $(x, y, z) = (5, 6, 7) + t[2, 3, 1]$. Assume that the sun is directly overhead, making the sun's rays perpendicular to the xy-plane which represents the ground. The bird's shadow is said to be *projected perpendicularly* onto the (level) ground. Find an equation that describes the motion of the shadow.

1. Separate the bird's three coordinates

At time t they are 5+2t, 6+3t and 7+t. The third coordinate measures altitude above the horizontal ground plane.



At t=0 the bird is at (5,6,7).

Its x and y locate it over the ground and z is its height.

2. Follow a vertical sunray

With the Sun overhead, light travels vertically through the bird. Its shadow occurs at the same x and y where altitude becomes zero.



A vertical sunray keeps x=5 and y=6 at the initial point while replacing z=7 by zero.

3. Write the moving shadow

The shadow is (5+2t,6+3t,0)=(5,6,0)+t(2,3,0) while the bird is above ground. The horizontal velocity components are unchanged and the vertical component is zero.



Ground path (5,6,0)+t(2,3,0).

For each unit of time the shadow moves 2 across and 3 forward.

1. 7+t≥0 ⇒ t≥−7
The light projects the bird down to ground only while its altitude is nonnegative.

2. (5+2t,6+3t,7+t)−(0,0,7+t)=(5+2t,6+3t,0)
The vertical ray removes the entire altitude and leaves horizontal position fixed.

Check. The difference between bird and shadow is always (0,0,7+t), purely vertical.

Exercise 74 · Coin inside a tile

Printed page 7 · Geometric probability

A coin of radius 1 cm is tossed onto a plane surface that has been tesselated (tiled) by rectangles whose measurements are all 8 cm by 15 cm. What is the probability that the coin lands within one of the rectangles?

1. Replace the falling coin by its center

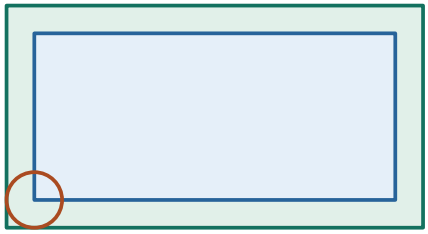
The coin's radius is 1 cm. For the whole disk to stay in an 8-by-15 tile, the center must stay at least 1 cm from each edge.



If the center is 1 cm from a tile edge, the coin just touches it.

2. Measure the safe rectangle

Removing a 1-cm strip from both ends of each dimension leaves safe dimensions $8-2=6$ and $15-2=13$ cm. Its area is 78 cm^2.

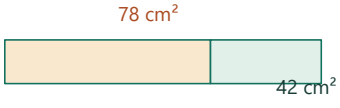


Subtract a 1-cm margin on both sides.

The safe center rectangle is 13 by 6 cm.

3. Use a uniform drop location

Because every tile repeats the same pattern, use one tile as the sample area, $8\cdot15=120$ cm^2. The probability is $78/120=13/20$; exact boundary events have zero probability.



Probability = safe area / tile area = 78/120.

1. $A_{safe}=(8-2)(15-2)=6\cdot13=78$ cm^2
The two margins cost one centimeter from each end of each dimension.

2. $P=(6\cdot13)/(8\cdot15)=78/120=13/20$
Divide both areas by six to simplify the probability.

Check. Subtracting the border region from 120 leaves 78 cm^2.

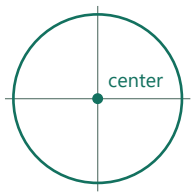
Exercise 75 · Two radius-three paths

Printed page 7 · Parametric equations / Circle geometry / Translations

What graph is traced by the parametric equation $(x, y) = (3\cos t, 3\sin t)$? What about the equation $(x, y) = (7 + 3\cos t, -2 + 3\sin t)$?

1. Track the first pair of coordinates

Multiplying both unit-circle coordinates by 3 scales every radius to 3. Thus (3 cos t,3 sin t) traces the circle $x^2+y^2=9$.



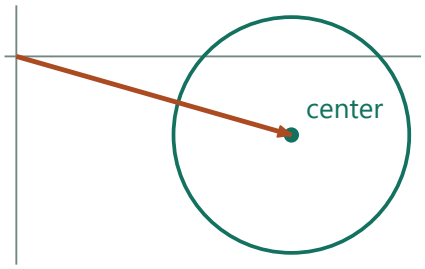
The first circle has radius 3 about the origin.

1. $x^2+y^2=(3\cos t)^2+(3\sin t)^2$
Substitute both parameter equations into squared distance.

2. $=9(\cos^2 t+\sin^2 t)=9$
The common factor is 3^2 and the bracket equals one.

2. Separate the second pair into a center and a displacement

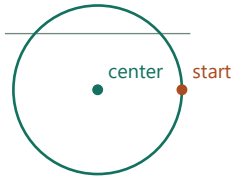
The second point equals (7,−2)+(3 cos t,3 sin t). Every point of the first circle has moved seven units right and two down.



Add the same vector (7,−2) at every parameter value.

3. Describe the translated curve

The radius remains 3, and the center becomes (7,−2). Its equation is $(x-7)^2+(y+2)^2=9$. In both examples a full turn of t reaches the whole circumference.



The radius is still 3.

The shifted path satisfies $(x-7)^2+(y+2)^2=9$.

1. $x-7=3\cos t$, $y+2=3\sin t$
Remove the translation to recover displacements from the center.

2. $(x-7)^2+(y+2)^2=9$
Squaring and adding gives the same squared radius as before.

Check. In each case subtracting the center gives a vector whose squared length is 9.

Exercise 76 · Rotate about (4,6)

Printed page 7 · Rotations / Vectors and components

A 15-degree counterclockwise rotation about $(4, 6)$ transforms $(2, 1)$ onto another point (x, y). Find x and y, correct to three decimal places.

1. Notice that the moving point and center have switched

From (4,6) to (2,1) the displacement is (−2,−5), exactly the reverse of the displacement in exercise 67.

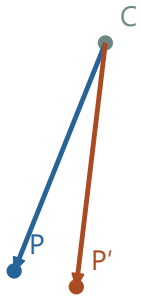


Begin at center C = (4, 6).

Measure the start relative to C.

2. Turn the reversed arrow

A rotation preserves negatives: rotating −v gives the negative of the rotated v. A 15° counterclockwise turn gives (−0.637756,−5.347267) for the local displacement.



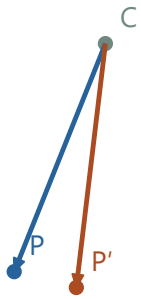
Both displacement arrows use the same center and have equal length.

The orange arrow is the rotated displacement.

3. Recover the absolute coordinates

Add the center (4,6) to obtain approximately (3.362,0.653). The distance from this point to (4,6) is still √29.

Counterclockwise turn · 15°



Rotate by 15° about (4, 6).

New point (3.362, 0.653); distance to center remains 5.385.

Compare settings · Counterclockwise turn

0°



Rotate by 0° about (4, 6).
New point (2, 1); distance to center remains 5.385.

Rotate by 0° about (4, 6). New point (2, 1); distance to center remains 5.385.

180°



Rotate by 180° about (4, 6).
New point (6, 11); distance to center remains 5.385.

Rotate by 180° about (4, 6). New point (6, 11); distance to center remains 5.385.

Check. Distances to the center remain √29.

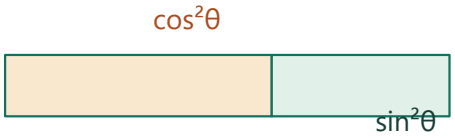
Exercise 77 · Factoring trigonometric identities

Printed page 7 · Trigonometric identities / Algebraic manipulation

Simplify: (a) $x\cos^2\theta + x\sin^2\theta$ (b) $x\cos^2\theta + x\cos^2\theta + 2x\sin^2\theta$

1. Identify the common factors in (a)

Both terms contain x, so distributivity lets us write x cos²θ+x sin²θ=x(cos²θ+sin²θ). The expression in parentheses is the squared radius of a unit-circle point.

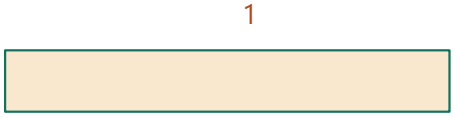


Schematic partition of one whole.

Multiplying both parts by x multiplies their total by x.

2. Apply the unit-circle identity

Because cos²θ+sin²θ=1, part (a) simplifies to x, for every θ.

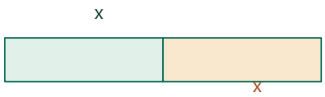


cos²θ + sin²θ = 1.

After factoring x, its coefficient is exactly 1.

3. Collect the matching terms in (b)

After combining the cosine-squared terms, the coefficients of $\cos^2\theta$ and $\sin^2\theta$ are both 2x. Factor 2x to leave the same identity in parentheses. The result is 2x.



For the second expression two matching copies remain.

Its value is 2x.

1. $x\cos^2\theta+x\cos^2\theta+2x\sin^2\theta=2x\cos^2\theta+2x\sin^2\theta$
Add the two identical cosine terms, leaving the sine term intact.

2. $=2x(\cos^2\theta+\sin^2\theta)=2x$
Distribute in reverse and use the unit-circle identity.

Check. Substitution $\theta=0$ or $90°$ gives the same results at each extreme.

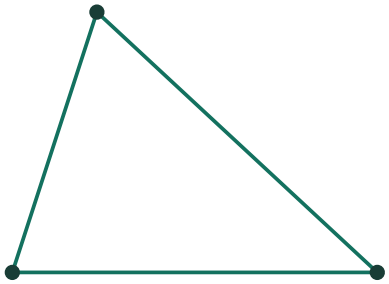
Exercise 78 · An acute SAS triangle

Printed page 7 · The cosine rule

A 12.0-cm segment makes a 72.0-degree angle with a 16.0-cm segment. To the nearest tenth of a cm, find the third side of the triangle determined by this SAS information.

1. Identify which angle is known

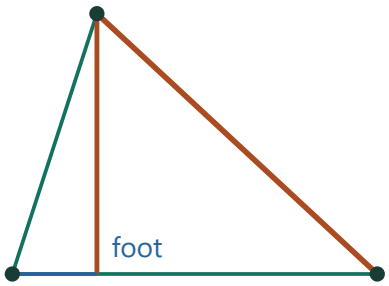
The 72° angle is formed by the two known sides, 12 and 16 cm. The unknown side c lies opposite that angle.



Known sides 16 and 12 meet at 72°.

2. Use the cosine rule

The included angle controls the projection of one known side onto the other. Accordingly $c^2=12^2+16^2-2\cdot12\cdot16\cos 72°=400-384\cos 72°$.



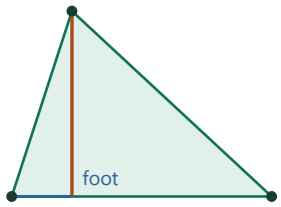
The opposite side faces 72°.

Its square is $16^2+12^2-2\cdot16\cdot12\cos 72° = 400-384\cos 72°$.

3. Choose the physical root

The side length is positive, giving $c \approx 16.773$ cm or 16.8 cm to the nearest tenth. Its value is below the 20-cm side produced by a right included angle.

Included angle · 72°



Base 16 cm, sloping side 12 cm.

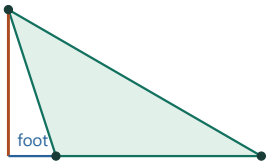
Altitude 11.413 cm; triangle area 91.301 cm^2; opposite triangle side 16.773 cm.

1. $c^2 = 144 + 256 - 384 \cos 72° = 400 - 384 \cos 72°$
Square each known side and double their product before applying the cosine.

2. $c = \sqrt{(400 - 384 \cos 72°)} \approx 16.773$ cm
The side length is the positive root; round only after taking it.

Compare settings · Included angle

108°

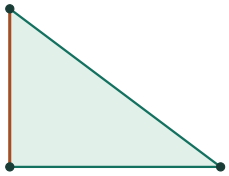


Base 16 cm, sloping side 12 cm.

Altitude 11.413 cm; triangle area 91.301 cm^2; opposite triangle side 22.774 cm.

Base 16 cm, sloping side 12 cm. Altitude 11.413 cm; triangle area 91.301 cm^2; opposite triangle side 22.774 cm.

90°



Base 16 cm, sloping side 12 cm.

Altitude 12 cm; triangle area 96 cm^2; opposite triangle side 20 cm.

Base 16 cm, sloping side 12 cm. Altitude 12 cm; triangle area 96 cm^2; opposite triangle side 20 cm.

Check. It is shorter than the 20-cm diagonal for a right included angle.

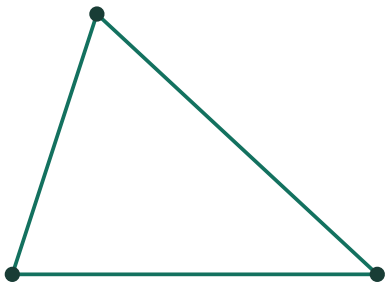
Exercise 79 · Area with a 72-degree angle

Printed page 7 · Triangle area using sine

(Continuation) Find the area of the triangle, to the nearest square centimeter.

1. Choose a base

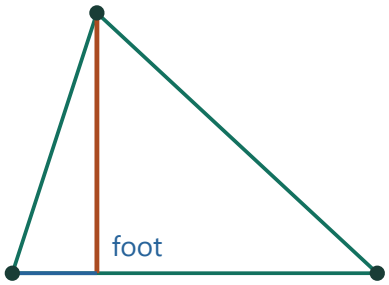
Place the 16-cm side horizontally. Drop a perpendicular from the end of the 12-cm side. This perpendicular distance is the height for that choice of base.



Known sides 16 and 12 meet at 72°.

2. Extract the height from the sloping side

The 12-cm side is the hypotenuse of the altitude triangle. Its perpendicular component is 12 sin 72°.



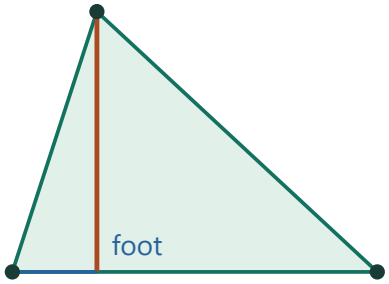
Drop the perpendicular from the 12-unit side.

Its height is 12 sin 72°; its signed projection is 12 cos 72°.

3. Use base times height divided by two

The triangle's area is ½·16·12 sin 72°=96 sin 72°≈91.30 cm², or 91 cm² to the nearest square centimeter.

Included angle · 72°

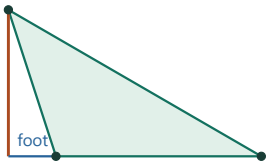


Base 16 cm, sloping side 12 cm.

Altitude 11.413 cm; triangle area 91.301 cm²; opposite triangle side 16.773 cm.

Compare settings · Included angle

108°

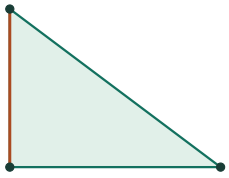


Base 16 cm, sloping side 12 cm.

Altitude 11.413 cm; triangle area 91.301 cm²; opposite triangle side 22.774 cm.

Base 16 cm, sloping side 12 cm. Altitude 11.413 cm; triangle area 91.301 cm²; opposite triangle side 22.774 cm.

90°



Base 16 cm, sloping side 12 cm.

Altitude 12 cm; triangle area 96 cm²; opposite triangle side 20 cm.

Base 16 cm, sloping side 12 cm. Altitude 12 cm; triangle area 96 cm²; opposite triangle side 20 cm.

Check. Using 12 as base and 16 sin72° as altitude gives the same result.

Exercise 80 · Altitude and projection

Printed page 7 · Vector projections / Right-triangle trigonometry

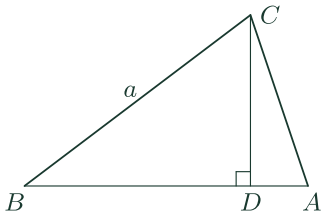
In the diagram at right, CD is the altitude from C.

(a) Express CD in terms of angle B and side a.

(b) Express BD in terms of angle B and side a.

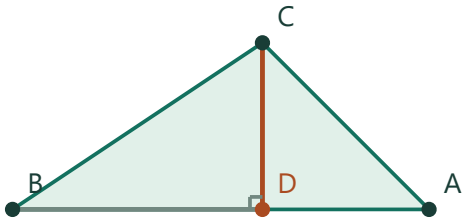
(c) Simplify the expression $(a\sin B)^2 + (a\cos B)^2$ and discuss its relevance to the diagram.

(d) Why was $a\sin B$ used instead of $\sin B \cdot a$?



1. Choose the right triangle in the drawing

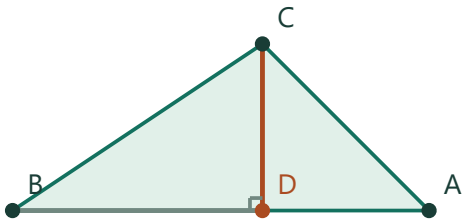
Trace BC from B to C and then CD straight down to D. The angle mark at D tells us CD is perpendicular to the base line BA. In the smaller triangle BCD the side opposite the right angle is the hypotenuse BC=a; the angle at B is the original angle B.



Trace BC and CD; BC=a is opposite the right angle at D.

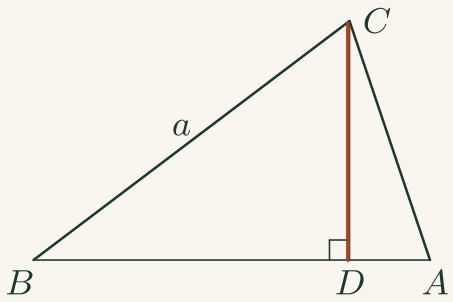
2. Find the altitude CD in (a)

Viewed from B, the far leg is CD. By the definition of sine, sin B=opposite/hypotenuse=CD/a. Multiply both sides by a to obtain CD=a sin B.



CD is the leg opposite B.

sin B = CD/a, hence CD = a sin B.

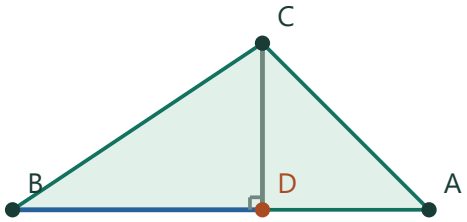


Vertical component CD

The small square at D establishes that this segment is perpendicular to the base line. It is opposite the angle at B.

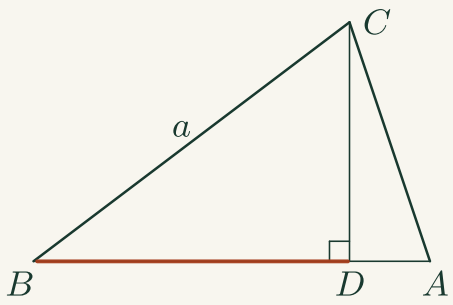
3. Find the base segment BD in (b)

The leg touching angle B along the base is BD, ending where the altitude meets the base. Cosine divides the adjacent leg by the hypotenuse, so cos B=BD/a. Hence BD=a cos B.



BD is the adjacent leg.

cos B = BD/a, hence BD = a cos B.

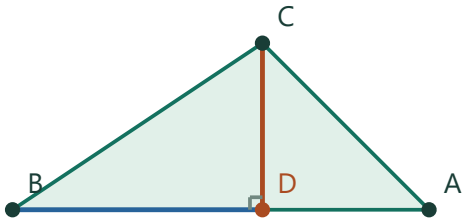


Horizontal projection BD

This is the segment of the base from B to the foot D, adjacent to angle B. BA extends farther along the same line.

4. Connect the squared sum to the picture in (c)

Substitute these lengths. The expression is CD^2+BD^2, the sum of the squared legs of right triangle BCD. Pythagoras makes it $BC^2=a^2$. Algebra gives the same result: factor a^2 from $a^2 \sin^2 B+a^2 \cos^2 B$ and use $\sin^2 B+\cos^2 B=1$.



$BC^2 = BD^2 + CD^2$.

The two squared projections add to a^2.

5. Explain the notation in (d)

First evaluate the sine of angle B, then multiply this number by a. Multiplication is commutative, so (sin B)·a gives the same length. Writing a sin B places the multiplier before the function and makes B, the sine's argument, easy to identify. The different expression sin(aB) would change the angle first.

B	sin B
then	× a

The angle B belongs inside the sine function.

a·sin B and (sin B)·a give the same length.

Check. Dividing both lengths by a recovers the unit-circle identity.

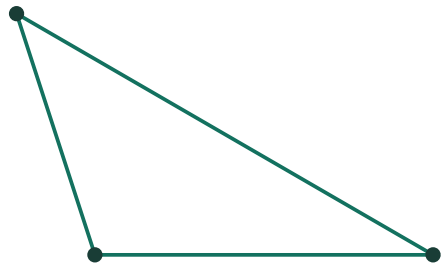
Exercise 81 · An obtuse SAS triangle

Printed page 7 · The cosine rule

A 12.0-cm segment makes a 108.0-degree angle with a 16.0-cm segment. To the nearest tenth of a cm, find the third side of the triangle determined by this SAS information.

1. Identify the changed angle

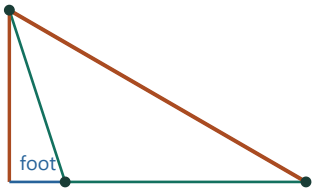
The lengths 12 and 16 cm are the same as before, but the included angle is now 108°, which is obtuse. Its cosine is negative.



Known sides 16 and 12 meet at 108°.

2. Apply the cosine rule with that sign

For the opposite side c, $c^2=12^2+16^2-2\cdot12\cdot16\cos 108°=400-384\cos 108°$. Subtracting a negative increases the squared length above 400.



With a 108° corner the altitude falls left of the included vertex.

The signed projection is negative and $400-384\cos 108°$ exceeds 400.

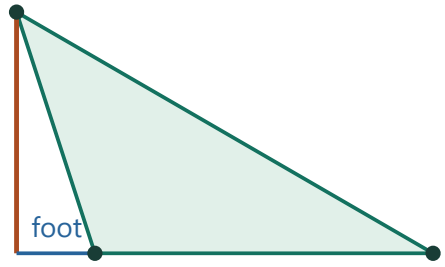
1. $\cos 108°=-\cos 72°$
The two included angles are supplementary.

2. $c^2=400-384(-\cos 72°)=400+384\cos 72°$
Subtracting a negative adds to the square, making this side longer.

3. Convert the square into a length

Take the positive root to get $c\approx22.774$ cm, or 22.8 cm. An obtuse included angle opens the triangle more widely than the right-angle comparison with side 20 cm.

Included angle · 108°

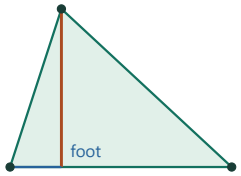


Base 16 cm, sloping side 12 cm.

Altitude 11.413 cm; triangle area 91.301 cm^2; opposite triangle side 22.774 cm.

Compare settings · Included angle

72°

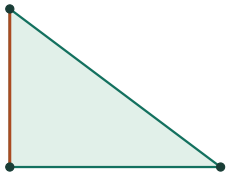


Base 16 cm, sloping side 12 cm.

Altitude 11.413 cm; triangle area 91.301 cm²; opposite triangle side 16.773 cm.

Base 16 cm, sloping side 12 cm. Altitude 11.413 cm; triangle area 91.301 cm²; opposite triangle side 16.773 cm.

90°



Base 16 cm, sloping side 12 cm.

Altitude 12 cm; triangle area 96 cm²; opposite triangle side 20 cm.

Base 16 cm, sloping side 12 cm. Altitude 12 cm; triangle area 96 cm²; opposite triangle side 20 cm.

Check. Since $\cos 108° < 0$ the result must exceed 20 cm.

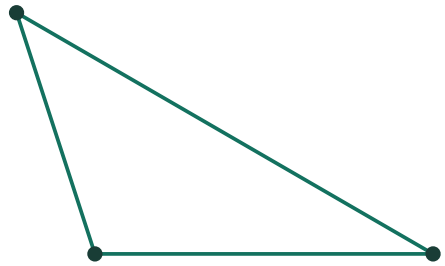
Exercise 82 · Area with an obtuse included angle

Printed page 7 · Triangle area using sine

(Continuation) Find the area of the triangle, to the nearest square centimeter.

1. Find an altitude even with an obtuse corner

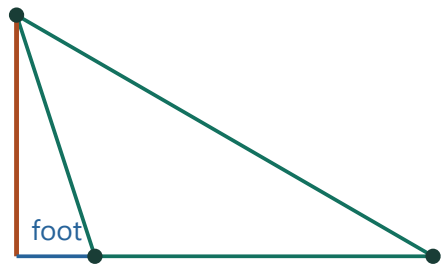
Use the 16-cm side as base. The perpendicular from the end of the 12-cm side lands on the extended base line; the height is still the perpendicular distance to that line.



Known sides 16 and 12 meet at 108°.

2. Relate the heights for the two triangles

Because 108° and 72° are supplementary, $\sin 108° = \sin 72°$. Both triangles have the same height, $12 \sin 108°$.



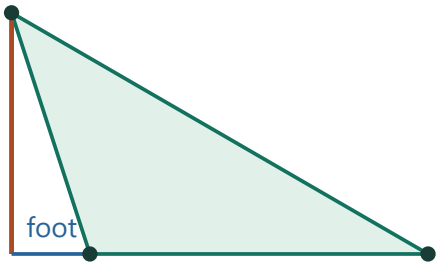
Drop the perpendicular from the 12-unit side.

Its height is $12 \sin 108°$; its signed projection is $12 \cos 108°$.

3. Compute the area

One half of base times height gives ½·16·12 sin 108°=96 sin 72°≈91 cm² to the nearest square centimeter.

Included angle · 108°

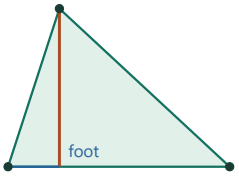


Base 16 cm, sloping side 12 cm.

Altitude 11.413 cm; triangle area 91.301 cm²; opposite triangle side 22.774 cm.

Compare settings · Included angle

72°

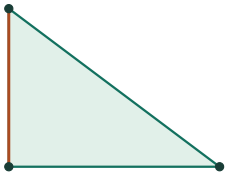


Base 16 cm, sloping side 12 cm.

Altitude 11.413 cm; triangle area 91.301 cm²; opposite triangle side 16.773 cm.

Base 16 cm, sloping side 12 cm. Altitude 11.413 cm; triangle area 91.301 cm²; opposite triangle side 16.773 cm.

90°



Base 16 cm, sloping side 12 cm.

Altitude 12 cm; triangle area 96 cm²; opposite triangle side 20 cm.

Base 16 cm, sloping side 12 cm. Altitude 12 cm; triangle area 96 cm²; opposite triangle side 20 cm.

Check. The altitude to the extended base has the same length in both triangles.

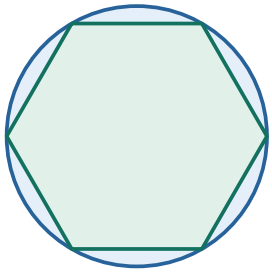
Exercise 83 · Packing a hexagonal prism

Printed page 7 · Packing / Prism volume / Volume

Kamāl has made some glass prisms to be sold as window decorations. Each prism is four inches tall, and has a regular hexagonal base with half-inch sides. They are to be shipped in cylindrical tubes that are 4 inches tall. What radius should Kamāl use for the tubes? Once a prism is inserted into its tube, what volume remains for packing material?

1. Fit the hexagonal end

A regular hexagon splits into six equilateral center triangles. Its center-to-vertex radius equals its side, 1/2 in, so an upright prism fits a radius-1/2 cylinder of height 4 in.



Every hexagon vertex touches the tube wall.

Radius ½ in fits the upright glass.

2. Find a lower bound valid for a tilted prism

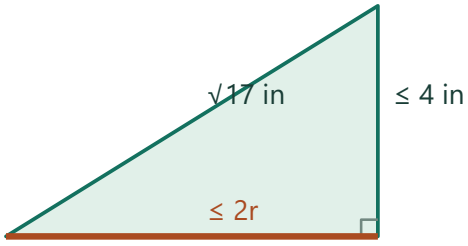
Two opposite hexagon vertices are 1 in apart horizontally. Choose them on opposite prism ends, 4 in apart vertically. Those two points are $\sqrt{(1^2+4^2)}=\sqrt{17}$ in apart in every orientation.



Opposite vertices on opposite ends have invariant separation √17 in.

3. Compare with distances possible in the tube

In a height-4 tube of radius r, two points differ horizontally by at most 2r and vertically by at most 4. Thus $17 \le (2r)^2+16$, requiring $r \ge 1/2$. The upright fit reaches that lower bound.



Schematic section

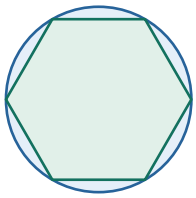
The tube bounds both components.

$17 \le 4r^2+16$ implies $r \ge$ ½ in.

1. $17 \le 4^2+(2r)^2=16+4r^2$	A diameter bounds the horizontal separation and 4 bounds the vertical separation.
2. $1 \le 4r^2 \Rightarrow 1/4 \le r^2$	Subtract 16 and divide by positive 4.
3. $r \ge 1/2$	Radius is nonnegative, so the lower bound survives taking square roots.

4. Subtract the occupied volume

Tube volume is $\pi(1/2)^2 \cdot 4 = \pi$. The glass volume is 4 times six equilateral triangle areas, or $4 \cdot 6 \cdot (\sqrt{3}/4)(1/2)^2 = 3\sqrt{3}/2$. The empty space is $\pi - 3\sqrt{3}/2 \approx 0.544$ in^3.



Packing material occupies the six circular segments.

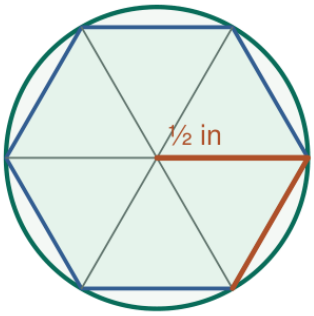
Multiply the leftover area by 4 in.

1. $B = 6(\sqrt{3}/4)(1/2)^2 = 3\sqrt{3}/8$
Tile the base with six equilateral triangles of side one half.

2. $V_{prism} = 4B = 3\sqrt{3}/2$, $V_{tube} = 4\pi(1/2)^2 = \pi$
Both upright solids have height 4.

3. $V_{empty} = \pi - 3\sqrt{3}/2$
Subtract occupied volume from the containing tube.

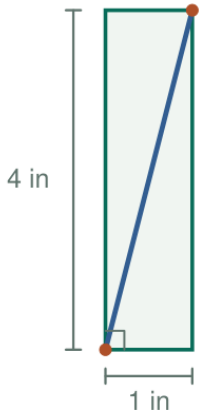
Looking straight down the tube



The six center triangles are equilateral.

A hexagon vertex reaches the tube wall.

Opposite vertices at opposite ends



Across the hexagon the span is 2 × ½ in.

Joining the two vertices uses both the full span and the 4-in height.

This separation stays fixed when the prism tilts.

Check. Opposite rim points, one on each end of the radius-1/2 tube, attain the diameter $\sqrt{17}$ used in the bound.

Exercise 84 · A second revolution

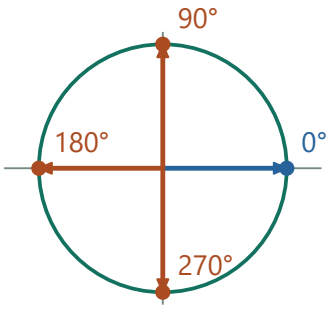
Printed page 8 · Trigonometric equations / Periodic functions

Use the unit circle and, if necessary, a calculator to find all solutions t between 360 and 720, inclusive:

(a) $\cos t = \sin t$ **(b)** $\tan t = -4.3315$ **(c)** $\sin t = -0.9397$

1. Move the requested interval back one turn

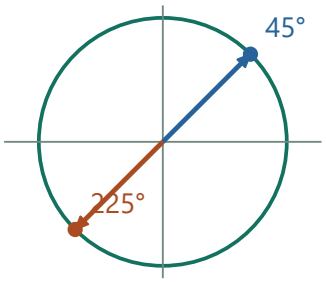
Subtracting 360° from 360° to 720° maps it to an ordinary full turn without changing sine, cosine or tangent. Solve there and add 360° at the end.



Subtract a full turn from the requested interval.

2. Solve (a) using equal coordinates

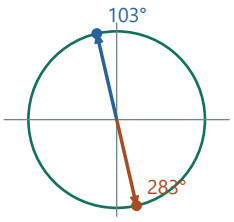
cos t=sin t means y=x on the unit circle. The diagonal meets it at 45° and 225° in the first turn. Return to the requested turn for 405° and 585°.



y=x at 45° and 225° in the first turn. Add 360°.

3. Solve (b) by tangent periodicity

arctan(−4.3315)≈−77°. Add multiples of 180°, the tangent period, to place angles inside the required interval. This gives about 463° and 643°.



The negative slope repeats every half-turn.

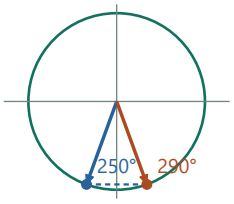
In the second revolution: about 463° and 643°.

1. $t \approx -77° + 180°k$
Tangent repeats every half-turn.

2. $k=3,4 \Rightarrow t \approx 463°, 643°$
These are the only integers giving angles in the second revolution.

4. Solve (c) using two lower quadrants

arcsin(−0.9397)≈−70.001°. The positive first-turn angles are about 250.001° and 289.999°. Add 360° to get approximately 610.001° and 649.999°.



A horizontal line of height −0.9397 hits twice.

Add 360° to both first-turn directions.

1. $t_0 \approx -70.001°$, $180° - t_0 \approx 250.001°$
Inverse sine supplies one reference value; reflection gives the other height match.

2. $t \approx 250.001° + 360°$, $-70.001° + 720°$
Add whole turns until each value lies in the required revolution.

Check. Subtract 360° and check quadrant signs; tangent is negative in II and IV, sine negative in III and IV.

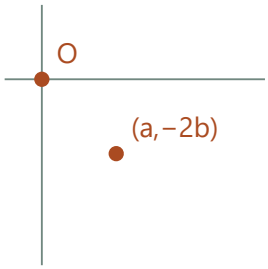
Exercise 85 · A parameterized circle equation

Printed page 8 · Standard form of a circle / Completing the square

Find the center and the radius of the circle $x^2 + y^2 - 2ax + 4by = 0$.

1. Recognize the square-completion pattern

In x^2-2ax the missing constant is a^2, since $(x-a)^2=x^2-2ax+a^2$. In y^2+4by the missing constant is $4b^2$.



Completing the squares displaces the center horizontally by a and vertically by −2b.

2. Add matching terms to both sides

The equation becomes $(x-a)^2+(y+2b)^2=a^2+4b^2$. The left is a squared distance.

x^2-2ax	$+a^2$
y^2+4by	$+4b^2$

Add a^2 and $4b^2$ to both sides.

The completed squares have center (a,−2b).

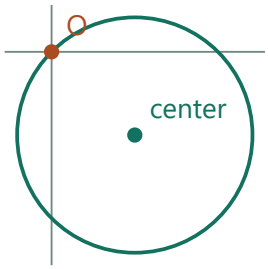
1. $x^2-2ax+a^2=(x-a)^2$
Half the x coefficient is −a.

2. $y^2+4by+4b^2=(y+2b)^2$
Half the y coefficient is 2b.

3. $(x-a)^2+(y+2b)^2=a^2+4b^2$
Add both square-completion constants to the original right side zero.

3. Read the geometric parameters

The center is $(a,-2b)$, and the radius is the nonnegative root $\sqrt{(a^2+4b^2)}$. If $a=b=0$ the only solution is the origin.



Example $a=2$, $b=1$.

The radius $\sqrt{(a^2+4b^2)}$ reaches the origin; zero parameters leave one point.

Check. The origin satisfies the original equation and lies one radius from the stated center.

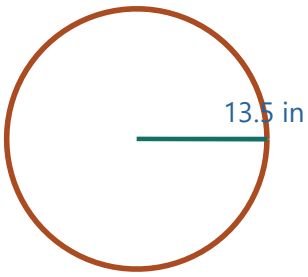
Exercise 86 · Bike speed

Printed page 8 · Angular and linear speed / Unit conversion

The wheels of a moving bicycle have 27-inch diameters, and they are spinning at 200 revolutions per minute. How fast is the bicycle traveling, in miles per hour? Through how many degrees does a wheel turn each second?

1. Connect one wheel turn to a distance

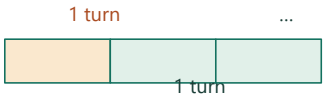
A 27-inch diameter gives circumference 27π inches. With rolling without slipping, each full revolution carries the wheel that distance along the road.



One wheel turn rolls out its circumference, 27π inches.

2. Convert revolutions per minute to road speed

At 200 rpm the distance per minute is $200\cdot 27\pi$ inches. Multiply by 60 min/h, then divide by 63,360 in/mi, to obtain about 16.065 mph.



200 turns each minute × 27π in per turn.

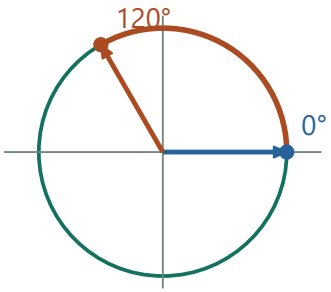
This is approximately 16.065 mph.

1. $v=5400\pi$ in/min·60 min/h$=324000\pi$ in/h
Multiply distance per revolution by the revolution count and change minutes to hours.

2. $v=324000\pi/(12\cdot 5280)$ mi/h$=(225\pi/44)$ mi/h
There are 12 inches per foot and 5280 feet per mile. Divide out both conversion factors.

3. Measure turning speed separately

Each revolution sweeps 360°. Thus 200·360 degrees per minute equals 200·360/60=1200° per second.



Angular speed = 200·360°/60 s = 1,200°/s.

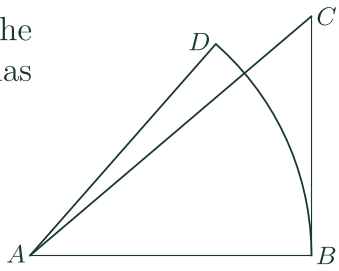
In 0.1 s the wheel turns 120°.

Check. 200 rpm is 10/3 revolutions per second.

Exercise 87 · Sector versus triangle

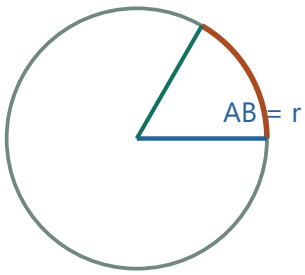
Printed page 8 · Circular sectors / Triangle area

In the figure at right, arc BD is centered at A, and it has the same length as tangent segment BC. Explain why sector ABD has the same area as triangle ABC.



1. Name the matched lengths

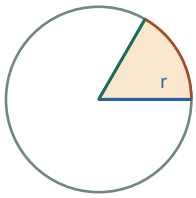
Let r=AB be the radius and s be both the arc length BD and the straight tangent length BC. The assumption matches lengths along two different paths.



Compare equal lengths: arc BD and tangent segment BC both have length s.

2. Measure the sector

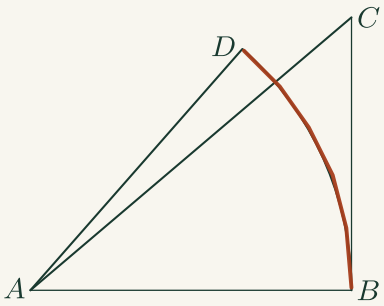
For radius r and arc length s, the area is rs/2. This follows by taking the arc's fraction of a complete circumference times the disk area.



Sector area = rs/2.

1. f=s/(2πr)
Compare the curved edge BD with the circumference of its own circle.

2. $A_{sector}=f\pi r^2=[s/(2\pi r)]\pi r^2=sr/2$
The disk area and circumference have a common positive factor πr.

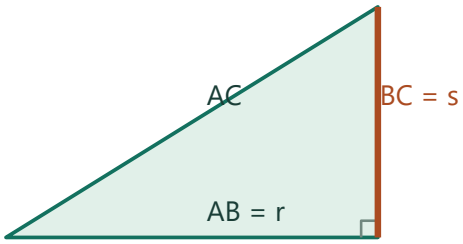


Curved sector boundary BD

Trace the curved path from B up to D. Its length is an arc length for the sector, whereas BC is a straight length.

3. Measure triangle ABC

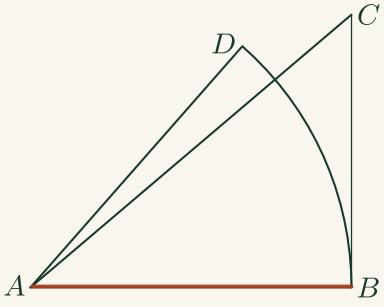
The tangent at B is perpendicular to the radius AB. Using BC=s as triangle base makes AB=r the altitude. The triangle's area is again sr/2, proving the equality.



Schematic section

The radius is perpendicular to the tangent at B.

Triangle ABC also has area rs/2.



Radius AB

The radius from A to B is shared as the sector radius and a perpendicular height for triangle ABC.

Check. Both figures use exactly the same two lengths in the area product.

Exercise 88 · Symmetric trigonometric values

Printed page 8 · Trigonometric equations / Reference angles and symmetry

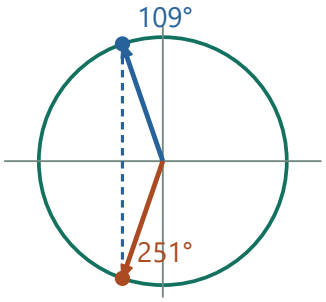
Find all solutions A between 0 and 360 without a calculator:

(a) $\cos A = \cos 251$ **(b)** $\cos A = 1.5$

(c) $\sin A = \sin 220$ **(d)** $\cos A = \cos(-110)$

1. Use the coordinate meaning of cosine in (a)

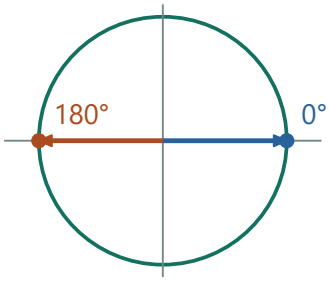
Equal cosines give equal horizontal coordinates, so the rays are reflections across the x-axis. The angles are 251° and 360°−251°=109°.



Reflection across the x-axis preserves cos 251°.

2. Test the proposed height or width in (b)

Every point on the unit circle has horizontal coordinate between −1 and 1. Consequently cos A=1.5 has no real solution.

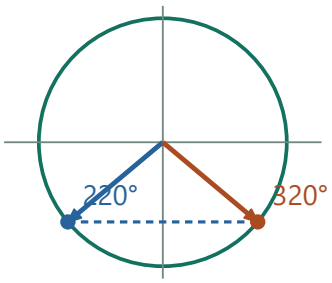


Cosine is a horizontal coordinate between −1 and 1.

There is no point with x=1.5.

3. Use equal heights in (c)

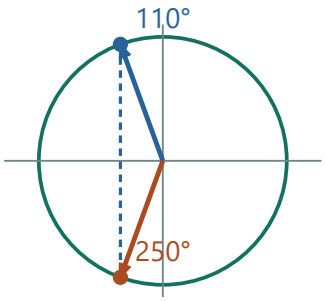
Equal sine means equal vertical coordinate. At 220° the 40° reference angle is below the x-axis in quadrant III. Its reflection into quadrant IV is 320°.



Equal negative heights give both sine solutions.

4. Reduce the negative angle in (d)

A clockwise turn of 110° is coterminal with 250°. Its x-axis reflection gives the other equal cosine at 110°. Thus the pair is 110° and 250°.



The −110° ray ends at 250°. Its x-axis reflection ends at 110°.

Check. Equal cosine points reflect across the horizontal axis; equal sine points reflect across the vertical axis.

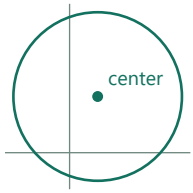
Exercise 89 · When a quadratic is a circle

Printed page 8 · Standard form of a circle / Completing the square

Assuming m, n, and p are constants, do all equations of the form $x^2 + mx + y^2 + ny = p$ represent circles? Explain.

1. Complete the x and y squares

Half of the coefficient m is m/2, so add $m^2/4$ to form $(x+m/2)^2$. Add $n^2/4$ similarly to form $(y+n/2)^2$.



Squared distance fixes the possible locations.

The center is the point where the two squares vanish.

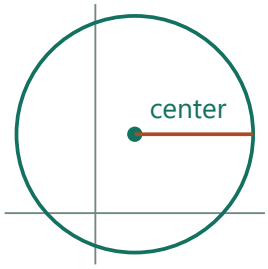
1. $x^2+mx=(x+m/2)^2-m^2/4$
Add and subtract the square of half the x coefficient.

2. $y^2+ny=(y+n/2)^2-n^2/4$
Use the same construction for y.

3. $(x+m/2)^2+(y+n/2)^2=p+(m^2+n^2)/4$
Shift both subtracted constants to the other side.

2. Read the minimum possible left side

The equation is now $(x+m/2)^2+(y+n/2)^2=p+(m^2+n^2)/4$. The left side is a squared distance from $(-m/2,-n/2)$ and can take any nonnegative value.

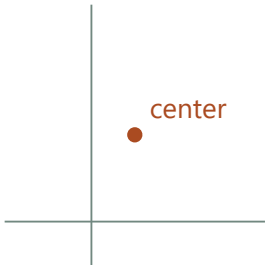


Positive squared radius

The points lie at one fixed distance from the center.

3. Classify by the right side

A positive right side gives a circle with radius its square root. Zero gives only the center. A negative right side gives no real points.



A zero radius leaves a single point. A negative squared radius has no real points.

Check. The left side has minimum zero exactly at the center $(-m/2,-n/2)$.

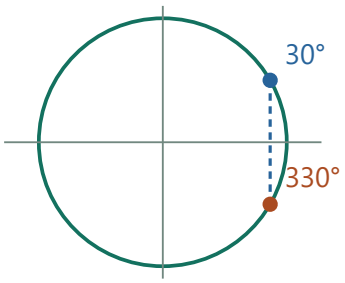
Exercise 90 · A cosine inequality

Printed page 8 · Trigonometric inequalities / The unit circle

Find all solutions between 0 and 360 of $\cos t < \frac{1}{2}\sqrt{3}$. Provide a diagram that supports your results.

1. Mark the comparison on the circle

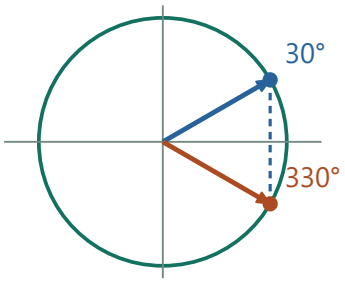
Cos t is the horizontal coordinate, so cos t<√3/2 means that the unit point lies strictly to the left of the vertical line x=√3/2.



A vertical boundary x = √3/2 cuts the circle.

2. Identify the boundary angles

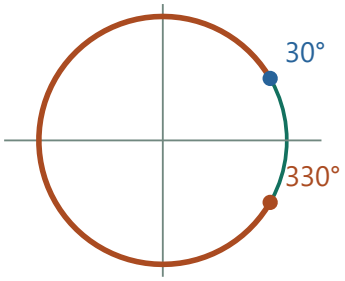
At 30° the x-coordinate is √3/2. Reflection across the x-axis gives the other boundary at 330°. These angles are excluded by the strict inequality.



The boundary rays are 30° and 330°.

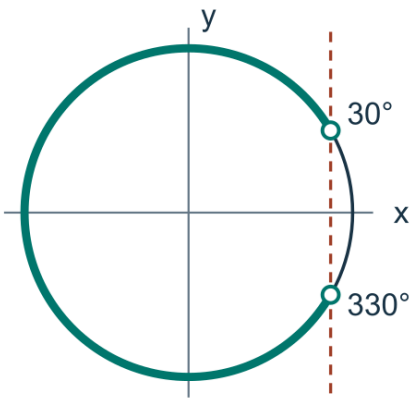
3. Select the correct arc

The point at 180° has x=−1 and satisfies the test; at 0° it has x=1 and fails. Moving along the selected arc gives 30°<t<330° within the requested turn.



Use the arc strictly to the left of the vertical cutoff.

Exclude its two endpoints.



Dashed line: x = √3/2

Green arc: 30° < t < 330°

Open ends exclude equality.

Check. 180° lies inside the arc and works; 0° and the two equality endpoints fail.

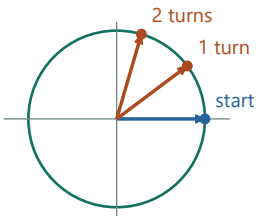
Exercise 91 · Iterated rotation

Printed page 8 · Rotations / Sequences and recursion

Consider the transformation $\mathcal{T}(x,y) = \left(\frac{4}{5}\,x - \frac{3}{5}\,y\,,\ \frac{3}{5}\,x + \frac{4}{5}\,y\right)$, which is a rotation centered at the origin. Describe the sequence of points that arise when $\mathcal{T}$ is applied repeatedly, starting with the point $A_0 = (5,0)$. In other words, A_1 is obtained by applying $\mathcal{T}$ to A_0, then A_2 is obtained by applying $\mathcal{T}$ to A_1, and so forth. Give a detailed description.

1. Trace the first two turns

The coefficients 4/5 and 3/5 are a cosine and sine because their squares sum to 1. Each application turns by the same α=arctan(3/4)≈36.87°. From (5,0) the first step gives (4,3), and the next gives (7/5,24/5).



The step angle is arctan(3/4).

The first points are (5,0),(4,3),(7/5,24/5).

1. x′=(4/5)4−(3/5)3=(16−9)/5
Rotate the first image (4,3) by the given coefficients.

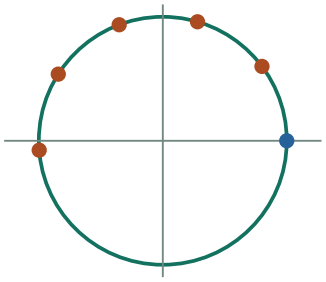
2. y′=(3/5)4+(4/5)3=(12+12)/5
Use the second row of the same rotation.

3. P_2=(7/5,24/5)
Keep the x and y results in that order.

2. Describe all the steps

Rotations preserve distance to the origin and their angles add. After n steps the point is (5 cos nα,5 sin nα). It stays on a circle of radius 5. Whether it ever returns exactly depends on whether some nα is a whole number of turns.

Number of repeated turns · 5 steps

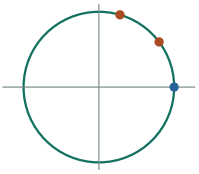


Each step adds arctan(3/4) to the direction.

After 5 steps: (−4.986, −0.379), still radius 5.

Compare settings · Number of repeated turns

2 steps

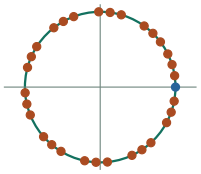


Each step adds arctan(3/4) to the direction.

After 2 steps: (1.4, 4.8), still radius 5.

Each step adds arctan(3/4) to the direction. After 2 steps: (1.4, 4.8), still radius 5.

30 steps



Each step adds arctan(3/4) to the direction.

After 30 steps: (4.49, 2.199), still radius 5.

Each step adds arctan(3/4) to the direction. After 30 steps: (4.49, 2.199), still radius 5.

3. Test for an exact return

Write a point (x,y) as x+iy, with $i^2=-1$. Multiplying by (4+3i)/5 implements the rotation. An exact return after $n>0$ steps would force $(4+3i)^n=5^n$. Reduce real and imaginary coefficients modulo 5. Since $(4+3i)^2=7+24i\equiv 3(4+3i)$, the nth power is congruent to $3^{n-1}(4+3i)$, whose coefficients cannot both vanish modulo 5. The right side does vanish, ruling out a return.

modulo 5	residue
4+3i	≠ 0
3^{n-1}	≠ 0

An exact return requires $(4+3i)^n=5^n$.

Modulo 5 its nonzero scalar multiple of 4+3i rules this out.

1. $z=4+3i$; $z^2=16+24i-9=7+24i$
Expand the complex square using $i^2=-1$.

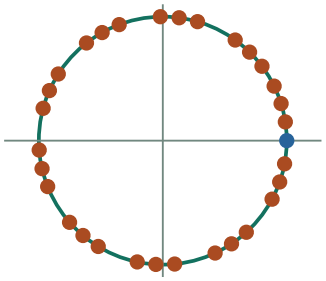
2. $z^2-3z=(7+24i)-(12+9i)=-5+15i$
Both coefficients of the difference are divisible by 5.

3. $z^2\equiv 3z \Rightarrow z^n\equiv 3^{n-1}z \pmod 5$, $n\geq 1$
Multiply repeatedly by z and replace each z^2 by 3z.

4. $3^{n-1}\cdot 4 \not\equiv 0 \pmod 5$
Neither 3 nor 4 is divisible by 5; the real coefficient alone rules out $z^n=5^n$.

4. Explain the dense-looking picture

Divide the circle into N equal arcs. Among the first N+1 points two must share an arc, so a positive number of rotations acts like a small turn forward or backward of less than 360°/N. It is nonzero by the preceding proof. Repeating that many rotations walks around the circle in small equal increments, getting within 360°/N of every target. Taking N larger makes that gap arbitrarily small.



The points approach every direction.

Many near returns occur without an exact repeat.

Check. Each step preserves length because $(4/5)^2+(3/5)^2=1$, and the cross terms cancel.

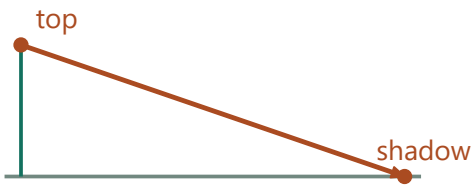
Exercise 92 · Shadow of an upright pole

Printed page 8 · Projection onto a plane / Parametric lines

How long is the shadow cast on the ground (represented by the xy-plane) by a pole that is eight meters tall, given that the sun's rays are parallel to the vector $[5, 3, -2]$?

1. Place the pole and the light ray

Put the foot of the pole at (0,0,0) and its tip at (0,0,8). From that tip a light ray changes position by t(5,3,−2).



Vertical section along the shadow.

The pole top is 8 m above ground.

2. Find when the ray reaches the ground

Its height is 8−2t. Setting this equal to zero gives t=4. The ray's horizontal changes during those four steps are 20 m and 12 m.



The ray drops 2 m per direction-vector step.

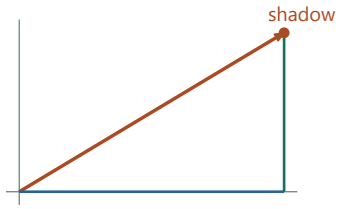
It reaches the ground in four steps.

1. z=8−2t=0 ⇒ 2t=8 ⇒ t=4
The tip ray reaches ground level when its height falls to zero.

2. (Δx,Δy)=4(5,3)=(20,12)
Use the same ray parameter in both horizontal coordinates.

3. Measure the shadow

The shadow runs from the foot to (20,12,0). Its length is $\sqrt{(20^2+12^2)}=4\sqrt{34}\approx 23.32$ m.



Ground displacement = (20,12) m.

Its length is $4\sqrt{34} \approx 23.32$ m.

1. $L^2=20^2+12^2=544=16\cdot 34$
The shadow's horizontal components are perpendicular.

2. $L=4\sqrt{34}$ m
Extract the positive square root of the square factor.

Check. Its horizontal-to-vertical ratio is $\sqrt{34}/2$, so multiplying the height by that ratio agrees.

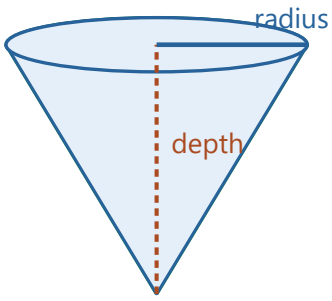
Exercise 93 · Conical cup capacity

Printed page 8 · Volume

A conical cup has a 10-cm diameter and is 12 cm deep. How much can this cup hold?

1. Read the cone's dimensions

The opening spans its diameter, 10 cm. Its radius is half that, 5 cm. The 12-cm depth measures perpendicular height from the tip to the opening.

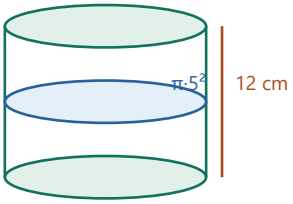


Opening diameter 10 cm gives radius 5 cm.

The perpendicular depth to the tip is 12 cm.

2. Find the comparison cylinder

A cylinder with that base and height would hold $\pi\cdot 5^2\cdot 12=300\pi$ cm³. A cone of the same height and base occupies one third of the cylinder.



The comparison cylinder holds 300π cm³.

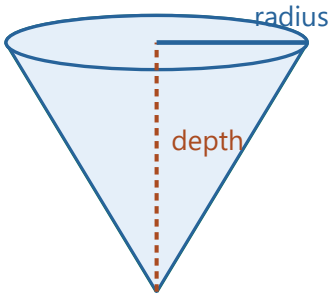
1. $B=\pi(10/2)^2=25\pi$ cm²
Square the radius to obtain the circular opening's area.

2. $V_{cyl}=B\cdot 12=25\pi\cdot 12=300\pi$ cm³
The cylinder with this opening keeps its base area throughout the full depth.

3. Express the capacity

The cone holds $100\pi\ \text{cm}^3 \approx 314\ \text{cm}^3$. Each cubic centimeter is one milliliter, so its capacity is about 314 mL.

Water depth as a fraction of the full cup · 100%

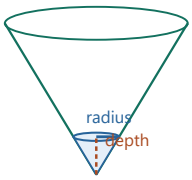


Full radius 5 cm, depth 12 cm.

At depth 12 cm, radius is 5 cm. Volume = 314.159 cm^3.

Compare settings · Water depth as a fraction of the full cup

25%

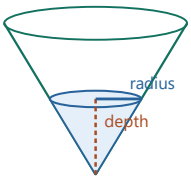


Full radius 5 cm, depth 12 cm.

At depth 3 cm, radius is 1.25 cm. Volume = 4.909 cm^3.

Full radius 5 cm, depth 12 cm. At depth 3 cm, radius is 1.25 cm. Volume = 4.909 cm^3.

50%



Full radius 5 cm, depth 12 cm.

At depth 6 cm, radius is 2.5 cm. Volume = 39.27 cm^3.

Full radius 5 cm, depth 12 cm. At depth 6 cm, radius is 2.5 cm. Volume = 39.27 cm^3.

Check. A cylinder of the same radius and height would hold exactly three times as much.

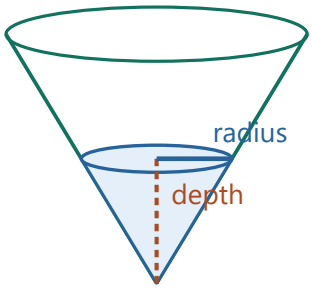
Exercise 94 · Half the depth

Printed page 8 · Volume / Area and volume scaling

(Continuation) Water in the cup is 6 cm deep. What percentage of the cup is filled?

1. Compare the two axial triangles

A vertical slice through the tip makes two similar triangles, one outlining the full cone and one outlining the water. The water surface is parallel to the opening.

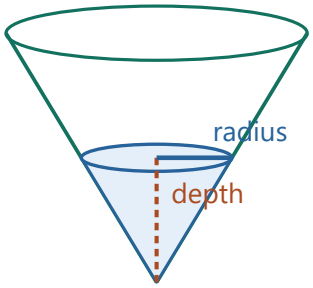


The water surface is parallel to the cup opening.

The small cone and the whole cone share the tip.

2. Scale each dimension

The water depth is 6 cm and the full depth is 12 cm. The scale is 1/2, so the water radius is half of 5 cm, or 2.5 cm.



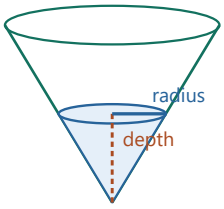
Depth 6 cm is half of 12 cm.

The water radius is 2.5 cm, half the opening radius.

3. Scale the volume

The base area contributes two factors of 1/2, and the height a third. The water volume is $(1/2)^3 = 1/8$ of the capacity, or 12.5%. The volume itself is 12.5π cm^3.

Water depth as a fraction of the full cup · 50%



Full radius 5 cm, depth 12 cm.

At depth 6 cm, radius is 2.5 cm. This is 12.5% of full capacity.

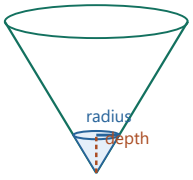
1. $V_{water}/V_{cup} = [\pi(2.5)^2 \cdot 6/3]/[\pi \cdot 5^2 \cdot 12/3]$
Write both cones' volumes in the same units.

2. $= (2.5/5)^2(6/12) = (1/2)^2 \cdot (1/2) = 1/8$
Cancel common cone constants and group the three scale factors.

3. $100 \cdot (1/8) = 12.5\%$
Measure the volume fraction in parts per hundred.

Compare settings · Water depth as a fraction of the full cup

25%

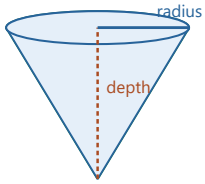


Full radius 5 cm, depth 12 cm.

At depth 3 cm, radius is 1.25 cm. This is 1.562% of full capacity.

Full radius 5 cm, depth 12 cm. At depth 3 cm, radius is 1.25 cm. This is 1.562% of full capacity.

100%



Full radius 5 cm, depth 12 cm.

At depth 12 cm, radius is 5 cm. This is 100% of full capacity.

Full radius 5 cm, depth 12 cm. At depth 12 cm, radius is 5 cm. This is 100% of full capacity.

Check. Substituting $r = 2.5, h = 6$ gives 12.5π cm^3, one eighth of 100π.

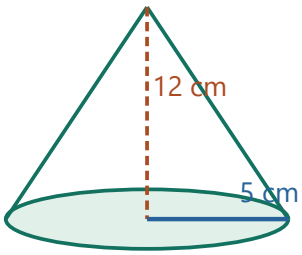
Exercise 95 · Unroll the cup

Printed page 8 · Nets and surface development / Arc length / Circular sectors

(Continuation) Dana takes a paper cone of the given dimensions, cuts it along a straight line from the rim to the vertex, then flattens the paper out on a table. Find the radius, the arc length, and the central angle of the resulting circular sector.

1. Identify what opens out

Cut the cone's lateral surface from its tip to the rim. The tip becomes the center of a circular sector. A straight generator of the cone becomes a radius of the sector.

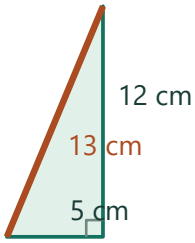


Cut from vertex to rim.

The curved surface can open into a plane.

2. Measure its radius and arc

The slant length is $\sqrt{(5^2+12^2)}=13$ cm. The rim becomes the outer curved arc without stretching, so the arc length is $2\pi\cdot5=10\pi$ cm.

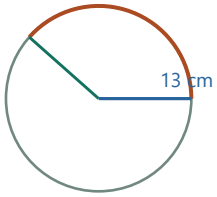


Slant height becomes the sector radius.

The rim becomes arc length 10π cm.

3. Find the opening angle

The sector's complete radius-13 circle would have circumference 26π. The arc takes the fraction $10\pi/(26\pi)=5/13$, so the angle is $(5/13)\cdot360°=1800/13°\approx138.46°$.



Only 5/13 of the full radius-13 circumference is needed.

Angle ≈ 138.46°.

1. $s/(2\pi\ell)=10\pi/(26\pi)=5/13$
Compare the arc with a full circle of slant radius 13.

2. $\theta=(5/13)360°=1800°/13$
The angle is the same fraction of a complete turn.

Check. Sector area is $(13)(10\pi)/2=65\pi$ cm², also π times cone radius times slant height.

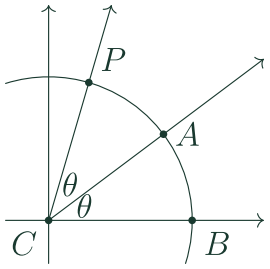
Exercise 96 · Double-angle coordinates

Printed page 8 · Trigonometric identities / Reflections / The unit circle

Point $A = (\cos\theta, \sin\theta)$ is at the intersection of $x^2 + y^2 = 1$ and a ray starting at the origin that makes an angle, θ, with the positive x-axis. The ray starting at the origin through point P makes an angle of 2θ with the positive x-axis.

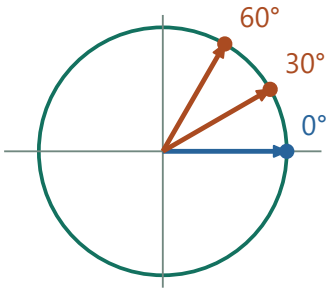
(a) Explain why $P = (\cos 2\theta, \sin 2\theta)$.

(b) Reflect $B = (1, 0)$ over the line CA to get an equivalent form of the coordinates of P written in terms of $\cos\theta$ and $\sin\theta$.



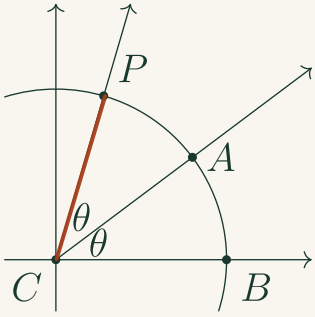
1. Measure the turn caused by reflection

The starting radius CB points along the positive x-axis from the origin C. Its angle to the mirror CA is θ. Reflecting it makes the same angle on the far side, so CP has directed angle 2θ and length 1. Hence P=(cos 2θ,sin 2θ).



B starts at direction 0°, the mirror is θ, and its image is at 2θ.

θ=30° shown.

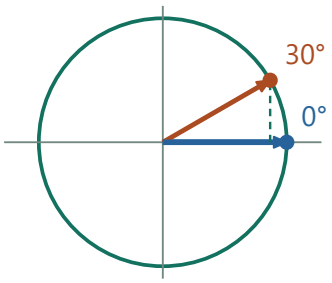


Reflected unit radius CP

The steeper ray ends on the unit circle at P. The reflection fixes C, so the radius length stays the same.

2. Find B′s projection on the mirror

The mirror has unit direction u=(cos θ,sin θ). The component of B=(1,0) along u is B·u=cos θ. The foot M of the perpendicular is therefore cos θ·u=(cos²θ,sin θ cos θ).

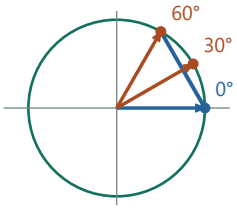


Project the unit vector CB onto the mirror CA.

The projected length equals cos θ.

3. Use the mirror as a perpendicular bisector

M is the midpoint of B and P, so P=2M−B=(2cos²θ−1,2 sin θ cos θ). Comparing matching coordinates yields the cosine and sine double-angle formulas.

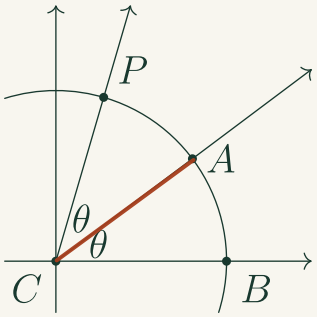


The mirror bisects the segment joining the two unit points.

Image = 2(projection) − B.

1. H=(cos θ)(cos θ,sin θ)
Scale the mirror's unit direction by the projection of B=(1,0).

2. P=2H−B=(2cos²θ−1,2cosθ sinθ)
H is the midpoint of the segment joining B to its reflected point.



Mirror ray CA

The middle sloping ray passes from C through A. It lies at the same angle from the initial and reflected radii.

Check. At θ=45° reflection in y=x sends (1,0) to (0,1); both expressions agree.

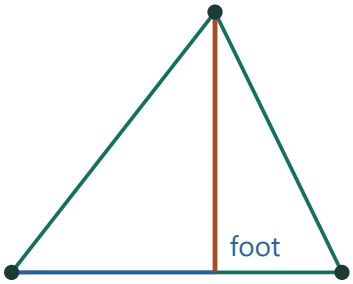
Exercise 97 · Projecting a javelin

Printed page 9 · Vector projections / Right-triangle trigonometry

A javelin lands with six feet of its length sticking out of the ground, making a 52-degree angle with the ground. The sun is directly overhead. The javelin's shadow on the ground is an example of a perpendicular projection. Find its length, to the nearest inch. Henceforth, whenever *projection* appears, "perpendicular" will be understood.

1. Choose the ground as the horizontal base

In overhead light, the javelin's shadow is its horizontal projection. The javelin, its shadow, and the vertical from its tip form a right triangle.

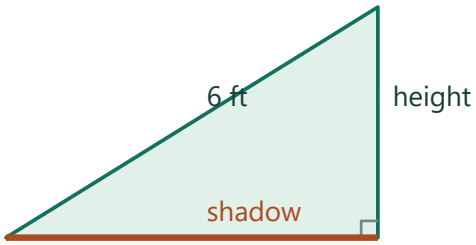


The javelin is the 6-ft sloping segment.

Sun rays drop perpendicular to the ground.

2. Use the angle with the ground

The 6-ft javelin is the hypotenuse and the shadow is adjacent to the 52° angle. Thus shadow/6=cos 52°.



Schematic section

Horizontal shadow length = 6 cos 52° ft.

3. Convert to the requested units

In inches the length is 6·12 cos 52°≈44.33 in. Rounded to a whole inch it is 44 in.



Six feet equals 72 inches.

Shadow ≈ 72 cos 52° ≈ 44.33 in.

1. Lft=6 cos52°
Multiply both sides of the shadow-to-javelin ratio by 6 ft.

2. Lin=(12 in/ft)(6 ft)cos52°=72 cos52° in
The foot units cancel before rounding in inches.

Check. The javelin, vertical drop, and ground shadow form a right triangle with hypotenuse 72 in.

Exercise 98 · Vector arithmetic

Printed page 9 · Vectors and components

The *dot product* of vectors $\mathbf{u} = [a, b]$ and $\mathbf{v} = [m, n]$ is the number $\mathbf{u}\bullet\mathbf{v} = am + bn$. The *dot product* of vectors $\mathbf{u} = [a, b, c]$ and $\mathbf{v} = [m, n, p]$ is the number $\mathbf{u}\bullet\mathbf{v} = am + bn + cp$. In general, the dot product of two vectors is the sum of all the products of corresponding components. Let $\mathbf{u} = [-2, 3, 1]$, $\mathbf{v} = [0, 1, 2]$, and $\mathbf{w} = [1, 2, -1]$. Calculate

(a) $4\mathbf{u}$ **(b)** $\mathbf{u} + \mathbf{v}$ **(c)** $4\mathbf{u} - \mathbf{v}$ **(d)** $\mathbf{u}\bullet(\mathbf{v} + \mathbf{w})$ **(e)** $\mathbf{u}\bullet\mathbf{v} + \mathbf{u}\bullet\mathbf{w}$

1. Read the three coordinate positions

The vectors are u=(−2,3,1), v=(0,1,2) and w=(1,2,−1). Keep x, y and z entries in the same order. Scaling, addition and subtraction return vectors. A dot product multiplies matching entries and adds those products, returning a single number.

x	y	z
-2	3	1
0	1	2
1	2	-1

Rows show u, v, w in that order.

Keep the three coordinate positions aligned.

2. Scale u in (a)

To form 4u, multiply each of u's entries by 4. The first component is 4(−2)=−8, the second is 4·3=12 and the third is 4·1=4. Collecting them in order gives (−8,12,4).

-2	3	1
-8	12	4

Multiply each entry of u by 4.

3. Add u and v in (b)

Add only entries in the same coordinate position: −2+0=−2, 3+1=4 and 1+2=3. Therefore u+v=(−2,4,3).

-2	3	1
0	1	2
-2	4	3

Add down each column to produce u+v.

4. Subtract v from the scaled vector in (c)

Start with the result (−8,12,4) from (a). Subtract v=(0,1,2) component by component. This gives (−8−0,12−1,4−2)=(−8,11,2). Adding v back recovers 4u.

-8	12	4
0	1	2
-8	11	2

Subtract the second row from the first.

This is 4u−v.

5. Form the inner sum in (d)

The parentheses in u·(v+w) tell us which vector to build first. Add v and w to obtain (0+1,1+2,2+(−1))=(1,3,1). The third component decreases because w points in the negative z direction there.

0	1	2
1	2	-1
1	3	1

Compute v+w before dotting with u.

6. Finish the dot product in (d)

Pair u=(−2,3,1) with (1,3,1). Their x entries contribute −2, their y entries contribute 9 and their z entries contribute 1. Adding gives u·(v+w)=−2+9+1=8.

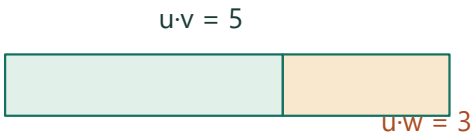
-2	3	1
1	3	1
-2	9	1

Multiply down each column.

Add the products: −2+9+1=8.

7. Evaluate the two separate dot products in (e)

For u·v the products are (−2)·0=0, 3·1=3 and 1·2=2, totaling 5. For u·w they are (−2)·1=−2, 3·2=6 and 1·(−1)=−1, totaling 3. Their sum is 5+3=8. This agrees with (d) because each component product distributes over the addition in v+w.



Their scalar sum is 8.

This matches u·(v+w).

Check. Agreement in (d),(e) illustrates distributivity.

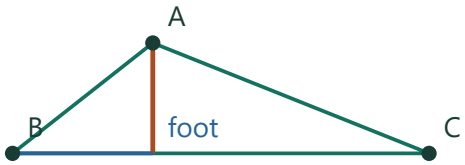
Exercise 99 · Two projections

Printed page 9 · Vector projections / Trigonometric ratios

In triangle ABC, it is given that $BC = 7$, $AB = 3$, and $\cos B = \frac{11}{14}$. Find the length of the projection of **(a)** segment AB onto line BC; **(b)** segment BC onto line AB.

1. Interpret a projected length

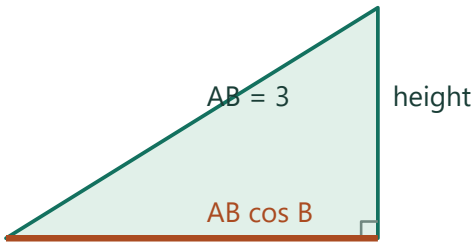
A segment's projection on a second line is the length of its perpendicular shadow along that line. In the resulting right triangle its length is the segment length times |cos B|.



Angle B is the angle between the two segments.

2. Project AB onto BC

The segment casting the shadow has length 3. Since cos B=11/14 is positive, its shadow has length 3·11/14=33/14.



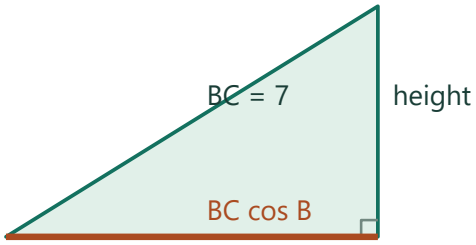
Schematic section

Project AB onto BC.

Projected length = 3·11/14.

3. Reverse the projection

Now the starting segment has length 7. The included angle is the same, so its projected length is 7·11/14=11/2. The two lengths need not agree because the original segments have different lengths.



Schematic section

Reverse the target direction.

Now use the length of BC: 7·11/14.

Check. The lengths are in the ratio 3:7, just like the original sides.

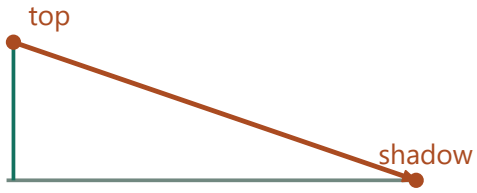
Exercise 100 · Oblique shadow of a point

Printed page 9 · Parametric lines / Projection onto a plane

Find the coordinates of the shadow cast on the xy-plane by a small object placed at the point $(10, 7, 20)$, assuming that the sun's rays are parallel to the vector $[5, 3, -2]$.

1. Parameterize the sunray

Start at (10,7,20). A multiple t of the light direction (5,3,−2) produces the coordinates (10+5t,7+3t,20−2t). Increasing t moves along the ray toward the ground.

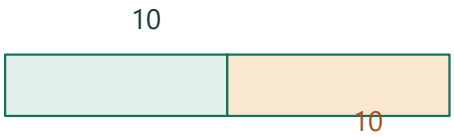


The ray begins at (10,7,20).

It travels +5,+3 horizontally and −2 vertically per unit t.

2. Use the ground's equation

The shadow is where this ray meets z=0. Solve 20−2t=0 to get t=10.

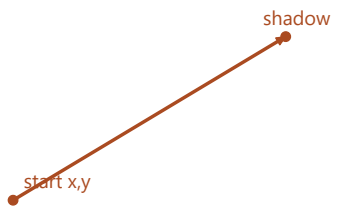


Set 20−2t=0.

Ten parameter units remove 20 height units.

3. Evaluate all three coordinates at the same t

Substitution gives x=60, y=37, z=0. The shadow is (60,37,0) and its displacement from the original point is ten copies of the light vector.



At t=10 the shadow is (60,37,0).

1. x=10+5(10)=60, y=7+3(10)=37
Use the same ground-intersection parameter for both horizontal coordinates.

2. z=20−2(10)=0
The computed point really is in the xy-plane.

Check. Subtracting the original point gives 10(5,3,−2).

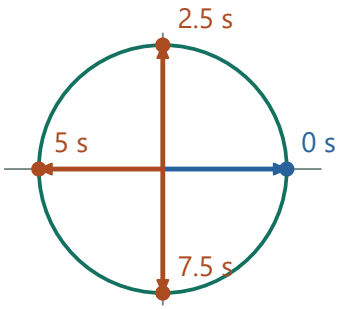
Exercise 101 · Carousel speed

Printed page 9 · Angular and linear speed / Unit conversion

Andy is riding a merry-go-round, whose radius is 25 feet and which is turning 36 degrees per second. Seeing a friend in the crowd, Andy steps off the outer edge of the merry-go-round and suddenly finds it necessary to run. At how many miles per hour?

1. Turn the angle rate into a lap time

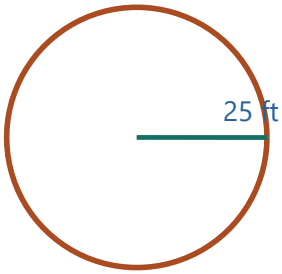
At 36° each second it takes 360/36=10 s to make one complete turn.



The angle rate makes one full revolution in 10 s.

2. Turn one lap into a distance

A radius of 25 ft gives circumference 2π·25=50π ft. Traveling that distance in 10 s gives tangential speed 5π ft/s.

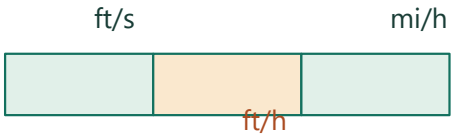


One turn covers 50π ft.

Tangential speed is 5π ft/s.

3. Convert to miles per hour

Multiply by 3600 seconds per hour and divide by 5280 feet per mile. The result is 5π·3600/5280≈10.71 mph.



Multiply by 3,600 s/h and divide by 5,280 ft/mi.

Speed ≈ 10.71 mph.

Check. The angular speed is π/5 rad/s, and multiplying by 25 gives 5π ft/s.

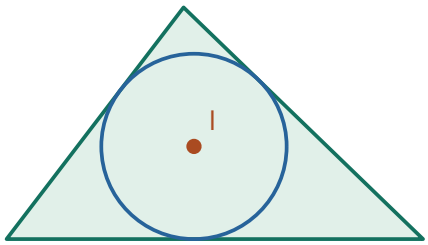
Exercise 102 · Incircle and triangle area

Printed page 9 · Incircles and circumcircles / Triangle area

The perimeter of a triangle, its area, and the radius of the circle inscribed in the triangle are related in an interesting way. Prove that the radius of the circle times the perimeter of the triangle equals twice the area of the triangle.

1. Locate the three perpendicular radii

Let I be the center of the inscribed circle and r its radius. The radius to each side ends at tangency and meets that side at right angles. The distance from I to each side is r.

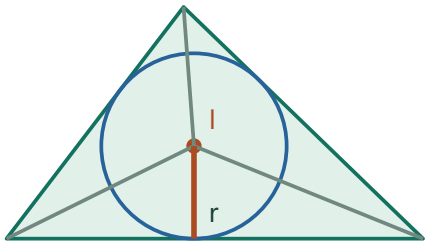


The incircle touches all three sides.

Its center is equally distant from their lines.

2. Partition the original triangle

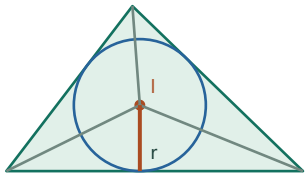
Join I to the three vertices. The resulting triangles have bases equal to the original side lengths a,b,c. Each has height r although the perpendicular feet can occupy different positions along the sides.



Joining the center to the three vertices partitions the triangle.

3. Add their areas

A=ar/2+br/2+cr/2=(a+b+c)r/2. Since P=a+b+c, this gives 2A=Pr. Equivalently, area is inradius times semiperimeter.



Three bases, one altitude r.

A = ½ar + ½br + ½cr.

1. A=ar/2+br/2+cr/2
Add the three triangles; each perpendicular height is the common radius.

2. 2A=r(a+b+c)=rP
Double both sides and factor out r.

Check. For a 3-4-5 right triangle, r=1, P=12 and A=6.

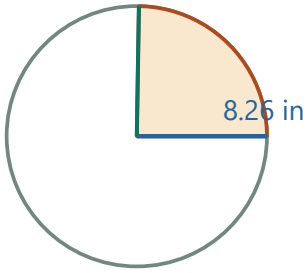
Exercise 103 · Another equal sector

Printed page 9 · Circular sectors / Quadratic equations

A circular sector has an 8.26-inch radius and a 12.84-inch arc length. There is another sector that has the same area and the same perimeter. What are its measurements?

1. Express the measurements as a sum and a product

The perimeter of a sector is 2r+s and its area is rs/2. Thus the two lengths 2r and s determine both quantities by their sum and product.

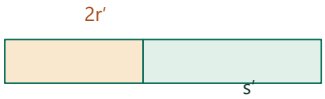


The original perimeter is 2r+s.

The area is rs/2.

2. Exchange the two lengths

Let 2r′=12.84 and s′=2·8.26. This gives r′=6.42 in and s′=16.52 in. The sum stays 29.36 in and the area stays 53.0292 in².



Swap the lengths s and 2r.

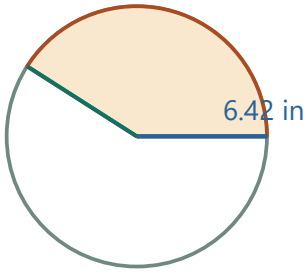
The sum and product stay fixed.

1. P′=2(s/2)+2r=s+2r=P
The new arc and the two new radii exchange their contributions.

2. A′=(s/2)(2r)/2=rs/2=A
The product is preserved by the same exchange.

3. Convert the new dimensions into an angle

Since s′=r′θ in radians, θ=16.52/6.42≈2.573 radians, about 147.4°. The dimensions differ from the first sector while meeting both constraints.



The partner has arc 16.52 in and angle about 147.4°.

Check. Both sectors have perimeter 29.36 in and area 53.0292 in².

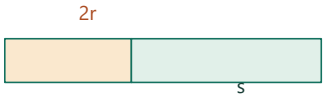
Exercise 104 · Does the second sector exist?

Printed page 9 · Circular sectors / Quadratic equations / Domain and range

(Continuation) Given a circular sector, is there always a different sector that has the same area and the same perimeter? Explain your answer.

1. Fix the two boundary measurements

For a sector let X=2r and Y=s. Knowing its perimeter P fixes X+Y=P, and knowing its area A=rs/2 fixes XY=4A.



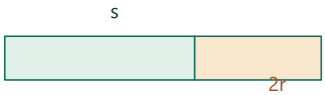
The two lengths have fixed sum P and fixed product 4A.

1. XY=(2r)s=2rs
Use twice the radius for one of the two boundary pieces.

2. A=rs/2 ⇒ rs=2A ⇒ XY=4A
Double the area equation and insert it into the product.

2. Count the possibilities

X and Y are the two roots of $z^2-Pz+4A=0$. Hence a second sector must exchange their roles. If X=Y, the exchange changes nothing. This happens at s/r=2 radians.



Swapping produces the only possible partner.

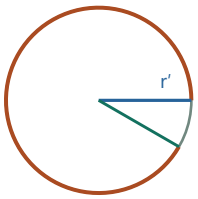
Equal lengths give the same sector.

1. $(z-X)(z-Y)=z^2-(X+Y)z+XY$
Expand the product whose roots are the two boundary measurements.

2. $=z^2-Pz+4A$
Substitute their fixed sum and product. Any new pair must use the same two roots.

3. Test whether the exchanged sector exists

Write θ=s/r. Under the exchange r′=s/2 and s′=2r, so θ′=s′/r′=4/θ. To represent less than a full disk it must satisfy θ′<2π, or θ>2/π radians.



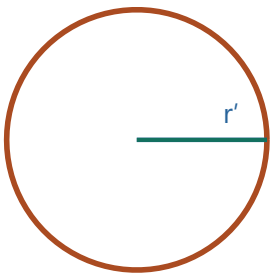
For an ordinary partner the angle 4/θ cannot exceed 2π.

1. θ′=(2r)/(s/2)=4r/s=4/θ
Insert the swapped dimensions into arc divided by radius.

2. $4/\theta<2\pi \Rightarrow 4<2\pi\theta \Rightarrow \theta>2/\pi$
Multiply by the positive original angle before dividing by 2π.

4. Handle the endpoint

A starting angle smaller than 2/π would require more than a full turn, so it has no ordinary partner. Exactly 2/π gives a full turn as partner, a limiting case whose coincident radii are counted only if that perimeter convention is adopted.



At θ=2/π the partner reaches one full turn.

Its two radial edges coincide.

Check. Solving the quadratic from the fixed sum and product leaves no other candidates.

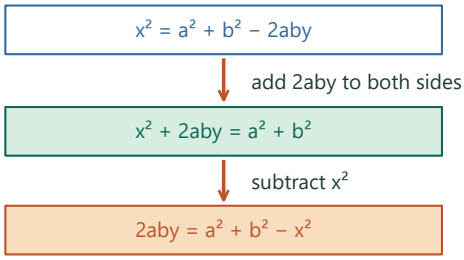
Exercise 105 · Isolate the cosine term

Printed page 9 · Algebraic manipulation

Solve for y: $x^2 = a^2 + b^2 - 2aby$

1. Collect the term containing y

Starting with $x^2=a^2+b^2-2aby$, add 2aby and subtract x^2 from both sides. This produces $2aby=a^2+b^2-x^2$.



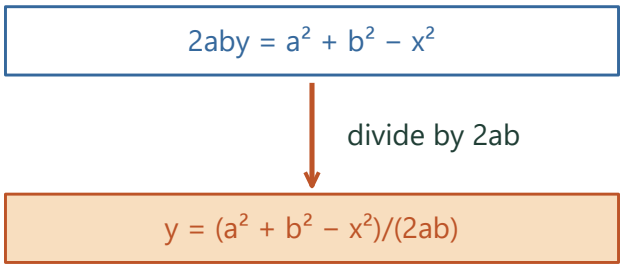
1. $x^2+2aby=a^2+b^2$
Add 2aby to both sides to leave a positive coefficient of y.

2. $2aby=a^2+b^2-x^2$
Subtract x^2 so the entire unknown term stands alone.

Keep the terms visible as the equation is rearranged. The coefficient of y is the product 2ab.

2. Divide when the coefficient is nonzero

If a and b are both nonzero, divide by 2ab to obtain $y=(a^2+b^2-x^2)/(2ab)$.

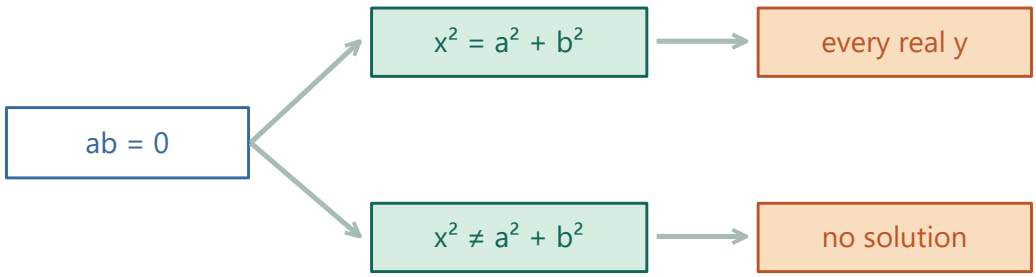


This step requires $ab \neq 0$.

The division arrow is valid only when ab is nonzero.

3. Inspect the possible zero coefficient

If ab=0, the original equation no longer involves y. Every y works when $x^2=a^2+b^2$. Otherwise no value of y can satisfy it.



If that coefficient vanishes, first test the remaining equality. Its truth decides whether every y works or none do.

Check. Substitution into the original equation returns x^2.

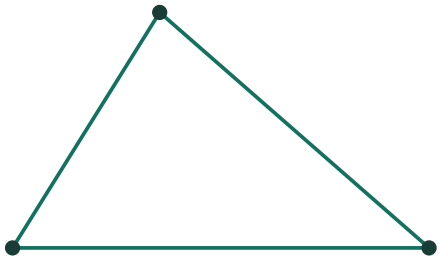
Exercise 106 · Law of cosines from height

Printed page 9 · The cosine rule / The Pythagorean theorem

A segment that is a units long makes a C-degree angle with a segment that is b units long. In terms of a, b, and C, find the third side of the triangle defined by this SAS description. You have done numerical versions of this question. Start by finding the length of the altitude drawn to side b, as well as the length of the perpendicular projection of side a onto side b. The resulting formula is known as the *Law of Cosines*.

1. Place the known sides on coordinate axes

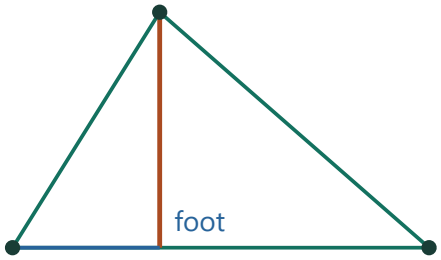
Use the included-angle vertex as origin and place the side of length b on the positive x-axis. Its endpoint is (b,0). The length-a side ends at (a cos C,a sin C).



Select the horizontal base and the included angle.

2. Read horizontal and vertical components

The perpendicular altitude is a sin C and the signed projection is a cos C. The length along the base line corresponding to that projection is a|cos C|. For obtuse C the signed projection is negative, putting the foot beyond the origin.

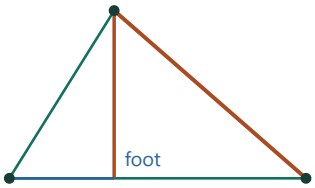


Drop the altitude from the end of the sloping side.

Only the vertical component enters the height.

3. Use the distance formula for the missing side

The squared distance between endpoints is $(b-a\cos C)^2+(a\sin C)^2$. Expanding leaves $b^2-2ab\cos C+a^2(\cos^2C+\sin^2C)$. The parentheses equal 1.



Squared distance uses the remaining horizontal run and the altitude.

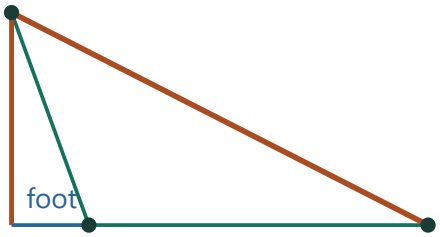
1. $c^2=(b-a\cos C)^2+(a\sin C)^2$
The difference of x coordinates and the vertical coordinate are perpendicular legs.

2. $=b^2-2ab\cos C+a^2\cos^2C+a^2\sin^2C$
Expand the binomial square and the altitude square.

3. $=a^2+b^2-2ab\cos C$
The last two squared trigonometric terms add to a^2.

4. Take the positive root

Thus $c^2=a^2+b^2-2ab\cos C$ and $c=\sqrt{(a^2+b^2-2ab\cos C)}$. With $C=90°$ the cosine term vanishes, recovering Pythagoras.



For an obtuse included angle $a\cos C$ is negative.

The same distance formula works; take the positive square root for the side length.

Check. At a right angle the formula reduces to the Pythagorean theorem.

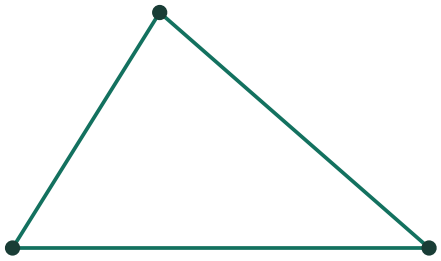
Exercise 107 · SAS area rule

Printed page 9 · Triangle area using sine

(Continuation) What is the area of the triangle defined by a, b, and C?

1. Select the base and corresponding height

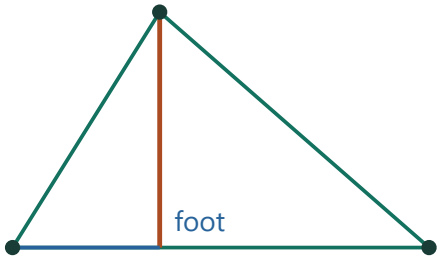
Place the length-b side horizontally. From the far end of the length-a side, drop a perpendicular to the line containing that base.



Select the horizontal base and the included angle.

2. Resolve the sloping side

The height is the perpendicular component of the length-a side, so h=a sin C. This height stays positive for an interior obtuse angle even if its foot lies on the extended base.

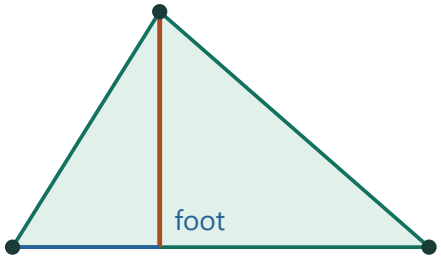


Drop the altitude from the end of the sloping side.

Only the vertical component enters the height.

3. Insert the height into the area formula

A=bh/2=b(a sin C)/2. Thus the area depends on the two adjacent side lengths and the sine of their included angle: A=ab sin C/2.



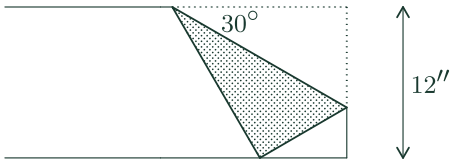
The triangle occupies half the parallelogram: A = ½ab sin C.

Check. Switching which side serves as base gives the same product.

Exercise 108 · A folded strip at 30 degrees

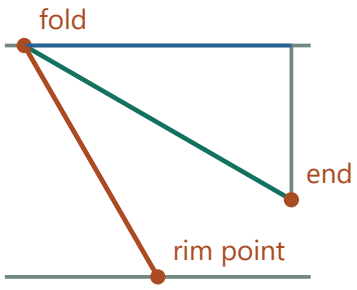
Printed page 10 · Paper-folding geometry / Right-triangle trigonometry

The figure at right shows a long rectangular strip of paper, one corner of which has been folded over to meet the opposite edge, thereby creating a 30-degree angle. Given that the width of the strip is 12 inches, find the length of the crease.



1. Identify what reflection preserves

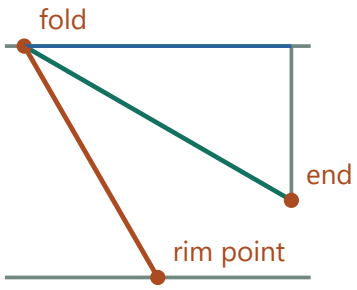
A fold is a reflection across its crease. At the upper endpoint the top edge makes 30° with the crease, so its reflected direction makes twice this angle, 60°, with the original horizontal.



The upper edge and its reflected image lie at equal angles to the crease.

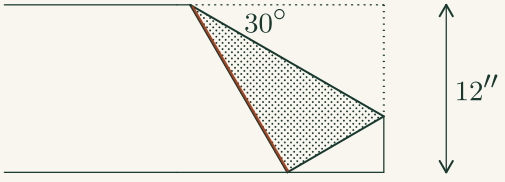
2. Find the reflected edge segment

It spans the 12-inch strip vertically. If its length is e, then e sin 60°=12. Reflection preserves distance, so the original top-edge segment also has length e=12/sin 60°.



Across the full width the folded edge drops 12 in.

The edge angle is twice the folding angle.



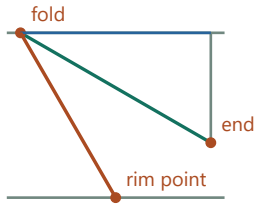
Folded edge segment

The other sloping side from the top endpoint reaches the bottom of the strip. Its vertical projection spans the strip width.

3. Recover the crease length

This original horizontal segment is the crease's horizontal projection, L cos 30°. Thus L=12/(sin 60° cos 30°)=12/(3/4)=16 in.

Crease angle · 30°

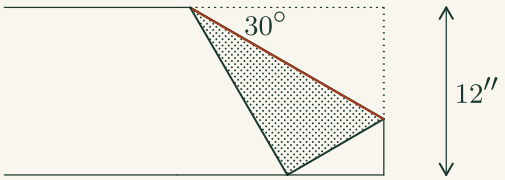


A 30° crease reflects the edge through 60°.

Across the 12-in strip the crease length is 16 in.

1. L cos30°=12/sin60°
The horizontal projection of the crease equals the measured reflected edge length.

2. L=12/[(√3/2)(√3/2)]=12/(3/4)=16 in
Divide by the cosine; both trigonometric factors are √3/2.

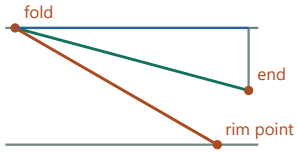


Crease at 30 degrees

Follow the slanted fold line from the top of the strip to the right edge. Its angle with the top edge is the marked 30 degrees.

Compare settings · Crease angle

15°

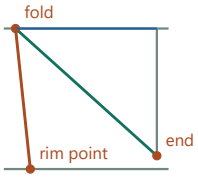


A 15° crease reflects the edge through 30°.

Across the 12-in strip the crease length is 24.847 in.

A 15° crease reflects the edge through 30°. Across the 12-in strip the crease length is 24.847 in.

42°



A 42° crease reflects the edge through 84°.

Across the 12-in strip the crease length is 16.237 in.

A 42° crease reflects the edge through 84°. Across the 12-in strip the crease length is 16.237 in.

Check. A 16-in crease has horizontal projection 8√3, and the reflected edge of length 8√3 drops exactly 12 in.

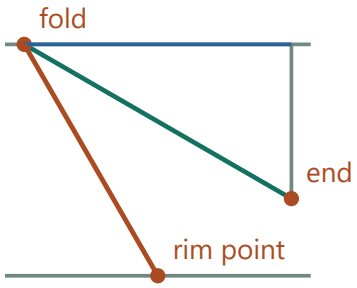
Exercise 109 · Crease as a function

Printed page 10 · Paper-folding geometry / Trigonometric ratios

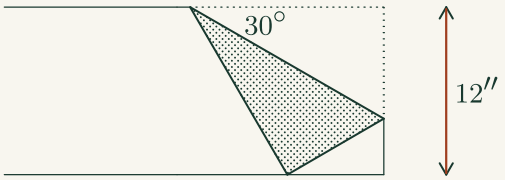
(Continuation) Suppose that the size of the folding angle is θ degrees. Use trigonometry to express the length of the crease as a function of θ. Check using the case $\theta = 30$.

1. Track the doubled angle at the fold

With crease angle θ in place of 30°, reflection puts the folded edge at angle 2θ to the horizontal. It still spans 12 inches vertically.



The upper edge and its reflected image lie at equal angles to the crease.

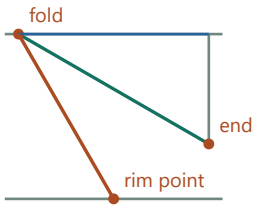


Twelve-inch strip width · Figure from exercise 108

The upright dimension on the right measures the separation of the two parallel edges. It is a vertical component of the folded edge.

2. Relate the edge to the crease

If e is the folded edge length then e sin(2θ)=12. The original length e along the top is also the horizontal projection L cos θ of the crease.

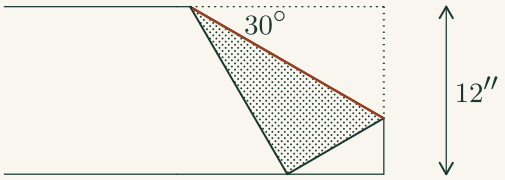


Across the full width the folded edge drops 12 in.

The edge angle is twice the folding angle.

1. e=12/sin(2θ)
Solve for the folded edge length from its vertical projection.

2. L cos θ=12/sin(2θ)
Replace e by that value in the crease's horizontal projection.



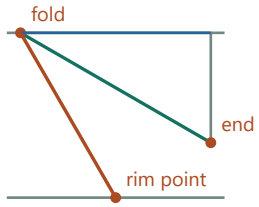
Crease at 30 degrees · Figure from exercise 108

Follow the slanted fold line from the top of the strip to the right edge. Its angle with the top edge is the marked 30 degrees.

3. Simplify the resulting expression

Combining gives $L=12/[\sin(2\theta)\cos\theta]$. Since $\sin(2\theta)=2\sin\theta\cos\theta$, the same formula is $L=6/[\sin\theta\cos^2\theta]$ inches. Substituting $\theta=30°$ gives 16.

Crease angle · 30°



A 30° crease reflects the edge through 60°.

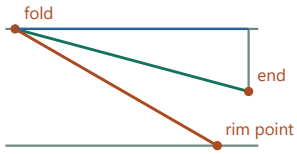
Across the 12-in strip the crease length is 16 in.

1. $L=12/[\sin(2\theta)\cos\theta]$
Divide the edge length by the cosine of the crease angle.

2. $=12/[2\sin\theta\cos\theta\cos\theta]=6/(\sin\theta\cos^2\theta)$
Expand the double-angle sine and cancel 2.

Compare settings · Crease angle

15°

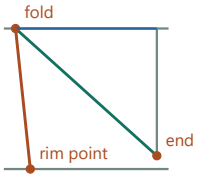


A 15° crease reflects the edge through 30°.

Across the 12-in strip the crease length is 24.847 in.

A 15° crease reflects the edge through 30°. Across the 12-in strip the crease length is 24.847 in.

42°



A 42° crease reflects the edge through 84°.

Across the 12-in strip the crease length is 16.237 in.

A 42° crease reflects the edge through 84°. Across the 12-in strip the crease length is 16.237 in.

Check. At 30° the denominator in the last form is (1/2)(3/4), giving 16 in.

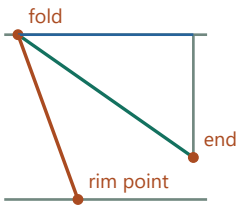
Exercise 110 · Minimum crease

Printed page 10 · Optimization / Paper-folding geometry

(Continuation) Find approximately that value of θ that makes the crease as short as it can be. Restrict your attention to angles that are smaller than 45 degrees. (Why is this necessary?)

1. Identify the geometric range

The crease's vertical drop is $L\sin\theta=6/\cos^2\theta$ using the preceding formula. For the end-edge intersection in this configuration it must be less than 12. This requires $0<\theta<45°$.



For the crease to hit the end first, $L\sin\theta$ must be below 12 in.

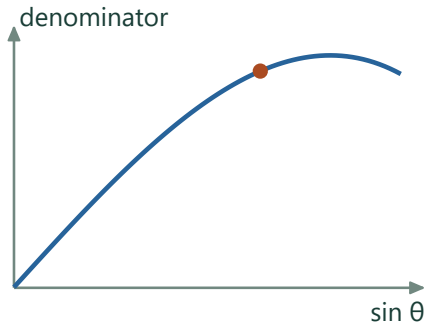
1. $6/\cos^2\theta<12 \Rightarrow 6<12\cos^2\theta$
Multiply by the positive squared cosine in the pictured acute fold.

2. $\cos^2\theta>1/2 \Rightarrow \cos\theta>1/\sqrt{2}$
The cosine is positive here.

3. $0<\theta<45°$
Acute-angle cosine decreases as the angle increases.

2. Convert minimization into maximization

The crease length is the positive fraction $L=6/(\sin\theta\cos^2\theta)$. A larger denominator gives a shorter crease. With $u=\sin\theta$, $\cos^2\theta=1-u^2$, so the denominator is $u-u^3$.

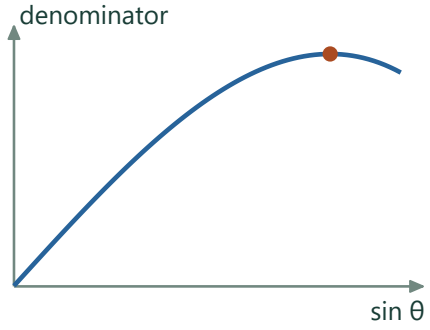


Substitute $u=\sin\theta$.

A larger $u-u^3$ makes the positive length $6/(u-u^3)$ smaller.

3. Locate the peak

The derivative is $1-3u^2$. It changes from positive to negative at $u=1/\sqrt{3}$, inside the allowed interval. The denominator is therefore largest at that u.



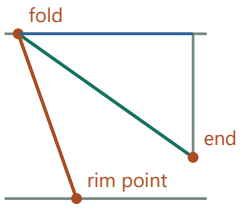
At $u=1/\sqrt{3}$ the slope changes sign.

This is the global peak in the permitted interval.

4. Translate back to an angle and length

$\theta=\arcsin(1/\sqrt{3})\approx35.264°$. At this angle $\cos^2\theta=2/3$, giving $L=6/[(1/\sqrt{3})(2/3)]=9\sqrt{3}\approx15.59$ in.

Crease angle · 35.264°



A 35.264° crease reflects the edge through 70.529°.

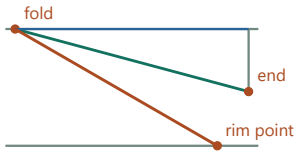
Across the 12-in strip the crease length is 15.588 in.

1. $u^2=1/3 \Rightarrow \cos^2\theta=1-u^2=2/3$
Use the complementary squares at the maximizing u.

2. $L=6/[(1/\sqrt{3})(2/3)]=6\cdot3\sqrt{3}/2=9\sqrt{3}$ in
Dividing by a positive fraction multiplies by its reciprocal.

Compare settings · Crease angle

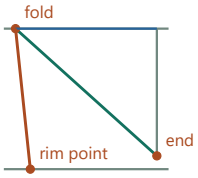
15°



A 15° crease reflects the edge through 30°.

Across the 12-in strip the crease length is 24.847 in.

42°



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A 42° crease reflects the edge through 84°. Across the 12-in strip the crease length is 16.237 in.

Check. Plugging 30° gives 16 in, longer than the minimum; θ→0 sends length to infinity.

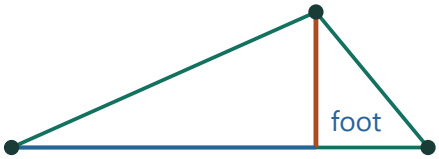
Exercise 111 · Two area-compatible angles

Printed page 10 · Triangle area using sine / The ambiguous SSA case

A triangle has an 8-inch side, a 10-inch side, and an area of 16 square inches. What can you deduce about the angle formed by these two sides?

1. Use the two known sides as a base and a sloping side

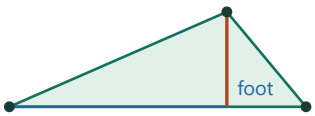
For included angle C the perpendicular height to the 10-inch side is 8 sin C. Thus the area is ½·10·8 sin C.



Fix the two side lengths, then vary their included angle.

2. Use the given area

Equating this expression to 16 gives 40 sin C=16 and sin C=0.4. Inverse sine returns the acute choice C≈23.58°.



Area 16 requires sin C = 16/40 = 0.4.

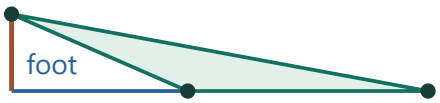
1. 16=(10·8/2)sin C=40 sin C
Insert the known area and multiply the fixed dimensions.

2. sin C=16/40=2/5 ⇒ C_1=arcsin(2/5)
Divide by the positive 40. Inverse sine selects the acute candidate.

3. Find the second triangle

An obtuse angle supplementary to the acute choice has the same sine and hence the same altitude. Thus $C \approx 180° - 23.58° = 156.42°$ is also possible.

Included angle · 156.422°

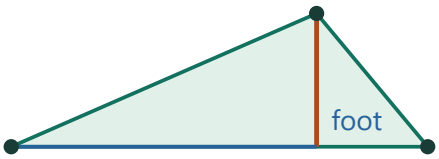


Base 10 in, sloping side 8 in.

Altitude 3.2 in; triangle area 16 in^2; opposite triangle side 17.625 in.

Compare settings · Included angle

23.578°



Base 10 in, sloping side 8 in.

Altitude 3.2 in; triangle area 16 in^2; opposite triangle side 4.166 in.

Base 10 in, sloping side 8 in. Altitude 3.2 in; triangle area 16 in^2; opposite triangle side 4.166 in.

Check. The corresponding altitudes to the 10-inch side are both 3.2 in.

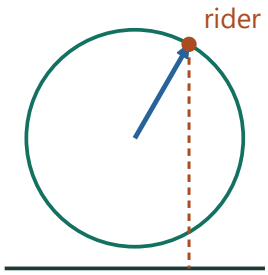
Exercise 112 · Ocean-view window

Printed page 10 · Sinusoidal functions / Trigonometric equations

Jamie rides a Ferris wheel for five minutes. The diameter of the wheel is 10 meters, and its center is 6 meters above the ground. Each revolution of the wheel takes 30 seconds. Being more than 9 meters above the ground enables Jamie to see the ocean. For how many seconds does Jamie see the ocean?

1. Set the angular reference at the top

The wheel's center is 6 m above ground and its radius is 5 m. At offset angle α from the top, the vertical component of the radius is 5 cos α, so height=6+5 cos α.



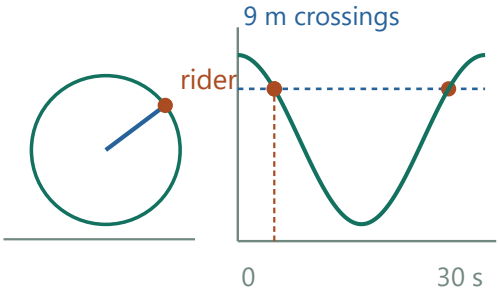
The center is 6 m above ground.

At angular offset α from the top, h=6+5 cos α.

2. Mark the two viewing limits

To exceed 9 m requires cos α>3/5. On either side of the top the boundary is α_0=arccos(3/5). Thus the allowed arc measures $2\alpha_0$ radians out of 2π.

Wheel position · 53.13°



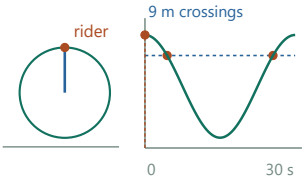
At t=4.428 s, the rider is 9 m high.

The 9-m threshold cuts the curve twice each turn.

1. 6+5 cos α>9 ⇒ 5 cos α>3 ⇒ cos α>3/5	Subtract the hub height and divide by the positive radius.
2. −arccos(3/5)<α<arccos(3/5)	Within a turn centered at the top, the two boundary angles enclose the viewing window.

Compare settings · Wheel position

0°

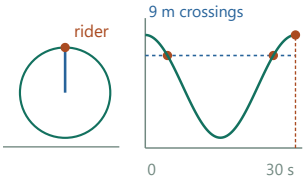


At t=0 s, the rider is 11 m high.

The 9-m threshold cuts the curve twice each turn.

At t=0 s, the rider is 11 m high. The 9-m threshold cuts the curve twice each turn.

360°



At t=30 s, the rider is 11 m high.

The 9-m threshold cuts the curve twice each turn.

At t=30 s, the rider is 11 m high. The 9-m threshold cuts the curve twice each turn.

3. Translate the fraction of a turn into time

Uniform angular speed makes the fraction of time α_0/π. Five minutes contain 300 s, exactly ten 30-s turns, so the start phase makes no difference. Total time above 9 m is 300 arccos(3/5)/π≈88.55 s.



Uniform speed converts arc fraction to time fraction.

Across 300 s the total is about 88.55 s.

1. time fraction=2 arccos(3/5)/(2π)
At constant speed the angular fraction equals the time fraction.

2. t=300 arccos(3/5)/π s
Multiply the fraction by the elapsed time over ten complete turns.

Check. It is above 9 for about 29.52% of each 30-s cycle.

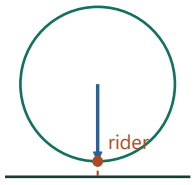
Exercise 113 · Ferris wheel parameters

Printed page 10 · Circular motion / Parametric equations

(Continuation) What graph is traced by the equation $(x, y) = (5\sin 12t\,,\ 6 - 5\cos 12t)$? How do the constants in this equation relate to Jamie's ride? What does t represent?

1. Locate the circle from the coordinates

The coordinate pair separates as (0,6)+(5 sin(12t),−5 cos(12t)). Squaring the displacement gives $x^2+(y-6)^2=25$. The center is (0,6) and the radius is 5 m.



The circle has radius 5 m and center 6 m above ground.

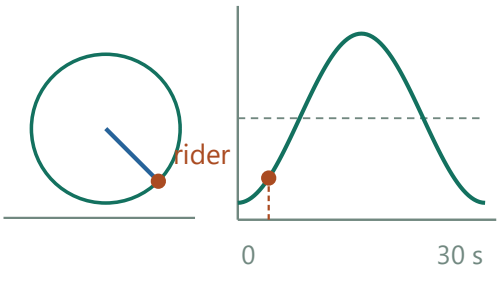
1. $x^2+(y-6)^2=25\ \sin^2(12t°)+25\ \cos^2(12t°)$
Square the displacements from the hub; the minus sign in the vertical one disappears.

2. $=25[\sin^2(12t°)+\cos^2(12t°)]=25$
The radius is therefore 5 at every time.

2. Find the start and initial direction

At t=0 the displacement is (0,−5) so the rider begins at (0,1), the bottom. For small positive times the sine becomes positive so the rider moves right.

Wheel position · 45°

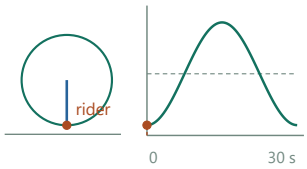


At t=3.75 s, the rider is 2.464 m high.

The graph dot and wheel rim mark the same instant. The 6-m midline is hub height.

Compare settings · Wheel position

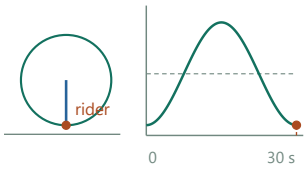
0°



At t=0 s, the rider is 1 m high.
The graph dot and wheel rim mark the same instant. The 6-m midline is hub height.

At t=0 s, the rider is 1 m high. The graph dot and wheel rim mark the same instant. The 6-m midline is hub height.

360°

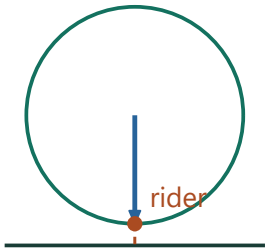


At t=30 s, the rider is 1 m high.
The graph dot and wheel rim mark the same instant. The 6-m midline is hub height.

At t=30 s, the rider is 1 m high. The graph dot and wheel rim mark the same instant. The 6-m midline is hub height.

3. Read the period

In degree mode the angular change is 12° per second. One complete revolution takes 360/12=30 s. At 15 s the rider is at (0,11) at the top.



Angular speed is 12 degrees per second.

After 30 s it returns to the bottom.

Check. At t=15 s it reaches (0,11), the top.

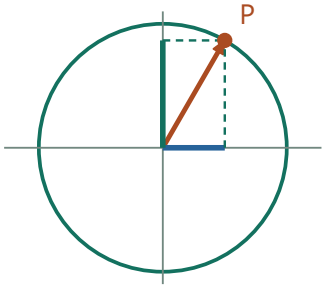
Exercise 114 · Two descriptions of one circle

Printed page 10 · Parametric equations / Circle geometry

Find two different parametric descriptions for the circle of radius 4 centered at $(-3, 2)$. Use your graphing device to check that your equations produce the same graph.

1. Choose a direction vector of fixed length

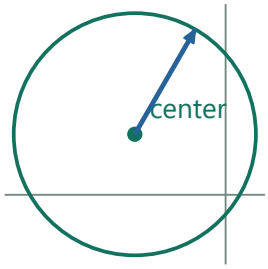
The vector (4 cos t,4 sin t) has length 4 for all t. Over 0≤t≤2π it points in every direction once.



Start with a direction of unit length.

2. Move the center to $(-3,2)$

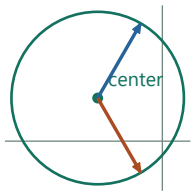
Adding the center gives the first parametrization $(-3+4\cos t, 2+4\sin t)$. At $t=0$ this starts at $(1,2)$, then rises, tracing the circle counterclockwise.



Scale the radius to 4 and place its tail at $(-3,2)$.

3. Reverse the direction for another description

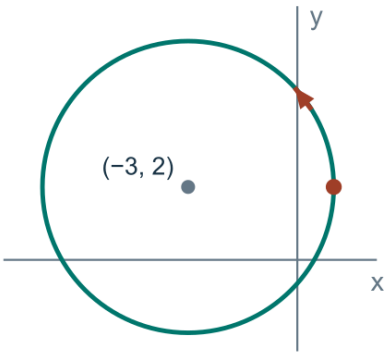
Replace t by $-t$. Cosine is even and sine odd, giving $(-3+4\cos t, 2-4\sin t)$. This starts at the same place and moves downward, tracing the same circle clockwise.



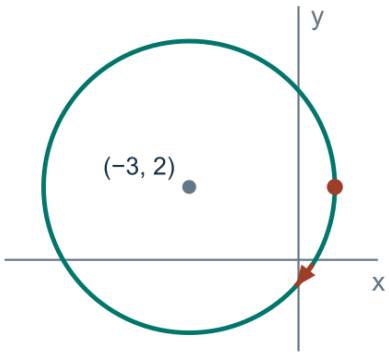
Opposite signs on the sine trace the same circle in opposite directions.

1. $x+3=4\cos t$, $y-2=-4\sin t$
Subtract the circle center from the second parametrization.

2. $(x+3)^2+(y-2)^2=16(\cos^2 t+\sin^2 t)=16$
Squaring removes its reversed vertical sign, giving the very same Cartesian circle.



$2 + 4\sin t$: counterclockwise



$2 - 4\sin t$: clockwise

Check. Both eliminate to $(x+3)^2+(y-2)^2=16$.

Exercise 115 · Dot-product properties

Printed page 10 · The dot product / Vectors and components

Let $\mathbf{u} = [a, b, c]$, $\mathbf{v} = [p, q, r]$, and $\mathbf{w} = [k, m, n]$ for the following questions:

(a) Verify that $\mathbf{u} \bullet \mathbf{v}$ is the same number as $\mathbf{v} \bullet \mathbf{u}$, for any vectors $\mathbf{u}$ and $\mathbf{v}$.

(b) What is the significance of the number $\mathbf{u} \bullet \mathbf{u}$?

(c) What does the equation $\mathbf{u} \bullet \mathbf{v} = 0$ tell us about the vectors $\mathbf{u}$ and $\mathbf{v}$?

(d) Is it true that $\mathbf{u} \bullet (\mathbf{v} + \mathbf{w}) = \mathbf{u} \bullet \mathbf{v} + \mathbf{u} \bullet \mathbf{w}$ holds for all vectors $\mathbf{u}$, $\mathbf{v}$ and $\mathbf{w}$?

(e) If $\mathbf{u}$ and $\mathbf{v}$ represent the sides of a parallelogram, $\mathbf{u}+\mathbf{v}$ and $\mathbf{u}-\mathbf{v}$ represent the diagonals. Justify this. Explain what the equation $(\mathbf{u} + \mathbf{v}) \bullet (\mathbf{u} - \mathbf{v}) = 0$ tells us about the parallelogram. Give an example of nonzero vectors $\mathbf{u}$ and $\mathbf{v}$ that fit this equation.

1. Check symmetry and self-products

A dot product sums products of coordinates in matching positions. Swapping the two factors in each scalar product leaves the sum unchanged, so u·v=v·u. Pairing u with itself gives a sum of squares, equal to ‖u‖².

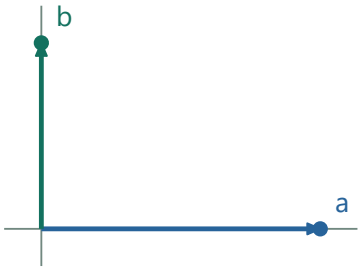
a·p	b·q	c·r
p·a	q·b	r·c

Switching vector order preserves every scalar product.

Pairing a vector with itself gives its squared length.

2. Interpret a zero dot product

For nonzero vectors u·v=‖u‖‖v‖cos θ. It vanishes exactly at a right angle. A zero vector also has zero dot product with every vector and has no direction of its own.



A perpendicular pair has dot product zero.

The zero vector also has zero dot product with every vector.

3. Distribute a dot product over a sum

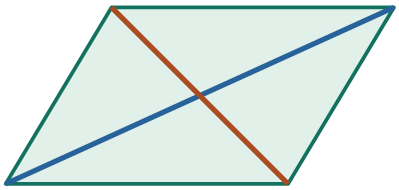
In each coordinate $u_i(v_i+w_i)=u_iv_i+u_iw_i$. Adding over all coordinates gives $u\cdot(v+w)=u\cdot v+u\cdot w$.

a(p+k)	b(q+m)	c(r+n)

Distribute within each component product, then group the terms belonging to v and to w.

4. Represent the parallelogram's diagonals

Place u and v at a common tail. The far vertex is u+v. The arrow from the head of v to the head of u is u−v. Their dot product expands to $\|u\|^2-\|v\|^2$ because the two cross terms cancel.

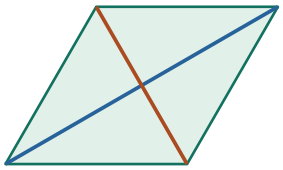


Opposite vertices differ by u+v or u−v.

The two colored segments are the diagonals.

5. Characterize perpendicular diagonals

For a nondegenerate parallelogram the diagonals are perpendicular exactly when this difference is zero, i.e. when adjacent side lengths agree. This characterizes a rhombus.



When adjacent lengths coincide, the two diagonal vectors are perpendicular.

1. $(u+v)\cdot(u-v)=u\cdot u-u\cdot v+v\cdot u-v\cdot v$
Distribute over the diagonal sum and difference.

2. $=\|u\|^2-\|v\|^2$
Symmetry cancels the two cross terms.

3. $=0 \Leftrightarrow \|u\|=\|v\|$
Both side lengths are nonnegative, so equality of their squares means equal lengths.

Check. Example: u=(1,0,0), v=(0,1,0) give diagonals (1,1,0) and (1,−1,0) with zero dot product.

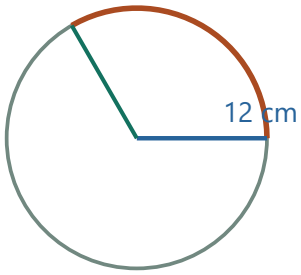
Exercise 116 · Cone from a 120-degree sector

Printed page 10 · Nets and surface development / Circular sectors / Volume

Dana takes a sheet of paper, cuts a 120-degree circular sector from it, then rolls it up and tapes the straight edges together to form a cone. Given that the sector radius is 12 cm, find the height and volume of this paper cone.

1. Match edges before and after rolling

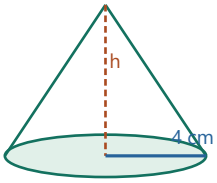
The sector's two straight radii join along the seam to form a generator of length 12 cm. The 120° arc, one third of a circumference, has length (1/3)·2π·12=8π cm.



The 120° arc becomes the cone's circular rim.

2. Recover the cone radius

That curved arc becomes the base rim. Set 2πr=8π, giving r=4 cm. The original 12 cm is the slant height, not this radius.



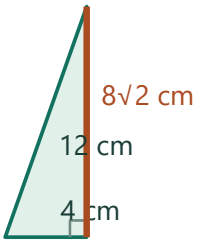
Arc = 8π cm, so the base radius is 4 cm.

1. $2\pi r=(120/360)(2\pi\cdot 12)=8\pi$
The sector arc becomes the complete cone rim.

2. $r=8\pi/(2\pi)=4$ cm
Divide by the full-turn factor to recover the cone radius.

3. Recover height and volume

In an axial right triangle $h^2+4^2=12^2$, so $h=8\sqrt{2}$ cm. The volume is $\pi r^2 h/3=128\pi\sqrt{2}/3$ cm³.



Slant radius becomes slant height.

Pythagoras gives the vertical height for the volume.

Check. The cone's lateral area π·4·12 equals the sector's area 48π.

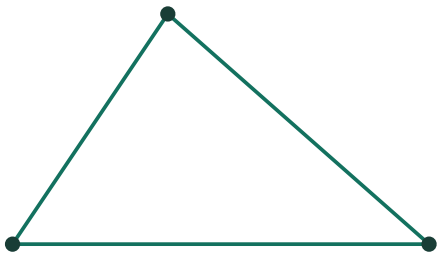
Exercise 117 · Third side to three figures

Printed page 10 · The cosine rule

Find the third side of a triangle in which a 4.00-inch side and a 6.00-inch side are known to make a 56.0-degree angle. Round your answer to three significant digits.

1. Identify the opposite side

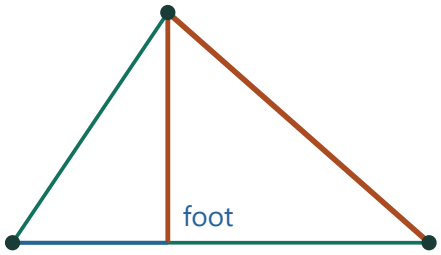
The sides 4.00 in and 6.00 in enclose the 56.0° angle. Their far endpoints determine the unknown opposite side c.



Known sides 6 and 4 meet at 56°.

2. Calculate the squared length

The cosine rule gives $c^2=4^2+6^2-2\cdot4\cdot6\cos 56°=52-48\cos 56°$. The cosine term accounts for the horizontal projection of one side onto the other.



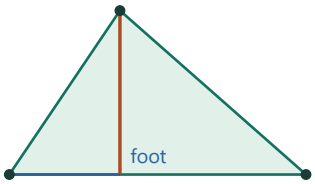
The highlighted side faces the included 56° angle.

Its square is $6^2+4^2-2\cdot6\cdot4\cos 56° = 52-48\cos 56°$.

3. Round after taking the root

The positive length is 5.01585… in. To three significant figures the result is 5.02 in.

Included angle · 56°



Base 6 in, sloping side 4 in.

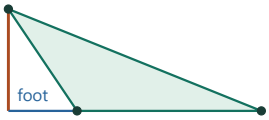
Altitude 3.316 in; triangle area 9.948 in²; opposite triangle side 5.016 in.

1. $c^2=52-48\cos56°\approx25.15874$
Evaluate the cosine term in degrees and keep the squared units at this stage.

2. $c=\sqrt{(52-48\cos56°)}\approx5.01585$ in
A length is positive; round to three significant digits only here.

Compare settings · Included angle

124°

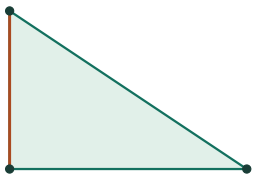


Base 6 in, sloping side 4 in.

Altitude 3.316 in; triangle area 9.948 in²; opposite triangle side 8.879 in.

Base 6 in, sloping side 4 in. Altitude 3.316 in; triangle area 9.948 in²; opposite triangle side 8.879 in.

90°



Base 6 in, sloping side 4 in.

Altitude 4 in; triangle area 12 in²; opposite triangle side 7.211 in.

Base 6 in, sloping side 4 in. Altitude 4 in; triangle area 12 in²; opposite triangle side 7.211 in.

Check. Its squared length also equals $(6-4\cos 56°)^2+(4\sin 56°)^2$.

Exercise 118 · Unit-circle solution sets

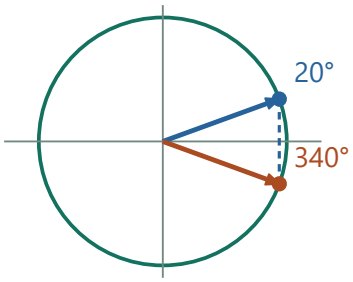
Printed page 11 · Trigonometric equations / Trigonometric inequalities / The unit circle

Without a calculator, find all solutions w between 0 and 360, inclusive, providing diagrams that support your results.

(a) $\cos w = \cos(-340)$ **(b)** $\cos w = \sin 20$ **(c)** $\sin w = \cos(-10)$
(d) $\sin w < -\frac{1}{2}$ **(e)** $1 < \tan w$

1. Sketch equal horizontal coordinates for (a)

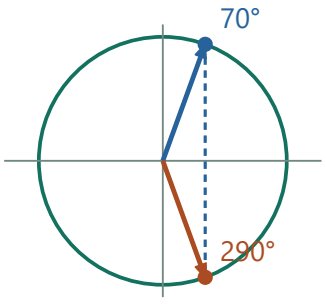
An angle of −340° ends at the same point as 20°. The line x=cos 20° meets the circle again at its x-axis reflection, so w=20° or 340°.



Both rays share cos 20°.

2. Change sine to cosine for (b)

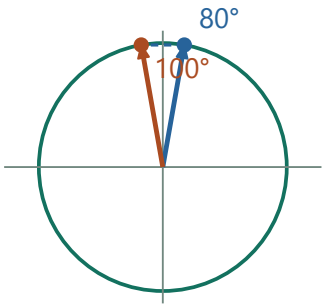
sin 20°=cos 70°. Draw x=cos 70° and read its two intersections at 70° and 290°.



sin 20° is cos 70°. Select both matching x coordinates.

3. Sketch equal heights for (c)

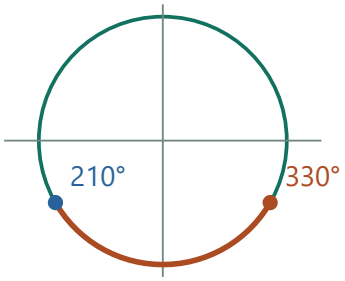
cos(−10°)=cos 10°=sin 80°. The horizontal line at this height intersects at 80° and 180°−80°=100°.



cos(−10°) = sin 80°. Both selected points share that y value.

4. Shade the correct arc for (d)

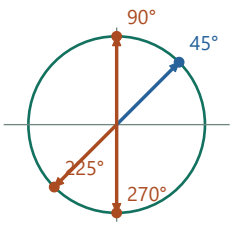
Draw y=−1/2. Equality occurs at 210° and 330°. The arc below this line passes through the bottom of the circle, giving 210°<w<330°.



Below y=−½ the open lower arc runs from 210° to 330°.

5. Compare slopes for (e)

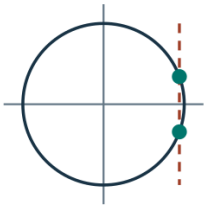
Draw y=x as the slope-one boundary. The positive slopes exceeding 1 lie between this diagonal and the vertical axis in quadrants I and III. Thus 45°<w<90° or 225°<w<270°. Boundary lines are excluded; vertical slope is undefined.



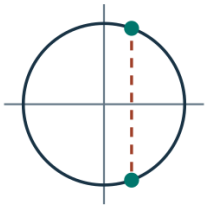
Positive slopes larger than 1 lie strictly between each diagonal and the next vertical ray.

1. tan w=1 at w=45° and 225°
The ray y=x has slope one in two opposite directions.

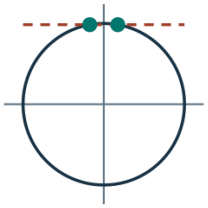
2. 45°<w<90° or 225°<w<270°
On either positive-slope branch the ratio rises past 1; the vertical endpoints have undefined slope.



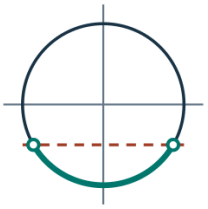
(a) 20°, 340°



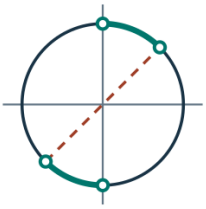
(b) 70°, 290°



(c) 80°, 100°



(d) 210° < w < 330°



(e) (45°, 90°) or (225°, 270°)

Check. The endpoints of these open arcs are equality points or vertical slopes, so are excluded.

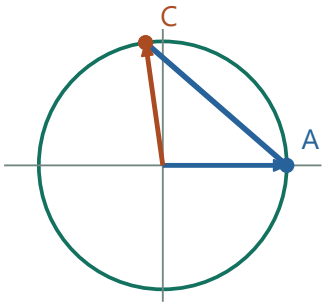
Exercise 119 · A chord seen at 49 degrees

Printed page 11 · Chords and inscribed angles / The sine rule

The radius of the circumscribed circle of the triangle ABC is 15 cm. Given that B is a 49-degree angle, find the length of side AC.

1. Identify the chord and its arc

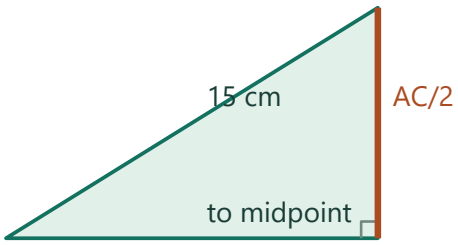
The 49° inscribed angle at B intercepts chord AC. The central angle subtending the same arc is twice as large, 98°.



The chord AC cuts off a central angle twice the inscribed angle.

2. Bisect the central triangle

Draw radii of length 15 cm to A and C. The perpendicular from the center to AC halves the chord and the central angle, making a right triangle with angle 49°.

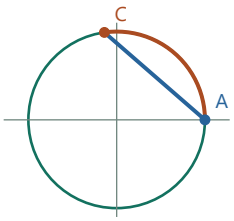


Schematic section

The radius through the chord midpoint divides it into two equal legs.

3. Restore the full chord

In that right triangle half of AC is 15 sin 49°. Thus AC=30 sin 49°≈22.64 cm.



AC = 30 sin 49° ≈ 22.64 cm.

1. sin49°=(AC/2)/15
The half-chord is opposite the 49-degree angle at the center.

2. AC/2=15 sin49° ⇒ AC=2·15 sin49°
First multiply by the radius, then double to restore the full chord.

Check. The extended sine rule AC/sin B=2R agrees.

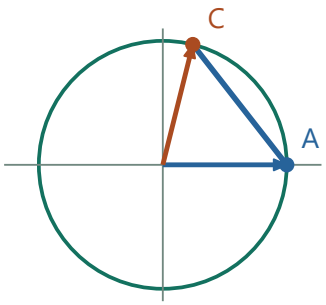
Exercise 120 · General chord formula

Printed page 11 · Chords and inscribed angles / The sine rule

(Continuation) The radius of the circumscribed circle of the triangle ABC is R cm. Given that B is a β-degree angle, find the length of side AC, in terms of R and β.

1. Represent the side as a chord

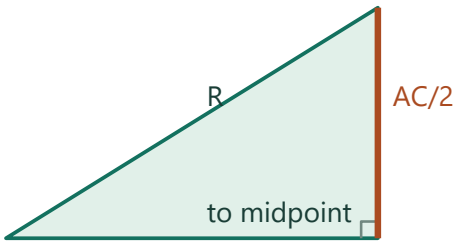
With the triangle on a circle of radius R, side AC is the chord cut off by inscribed angle β at B. For acute β the central angle over the same arc is 2β.



The chord AC cuts off a central angle twice the inscribed angle.

2. Use the chord midpoint

A perpendicular from the center to AC splits the central triangle into two right triangles. One half-chord measures R sin β. Thus AC=2R sin β.



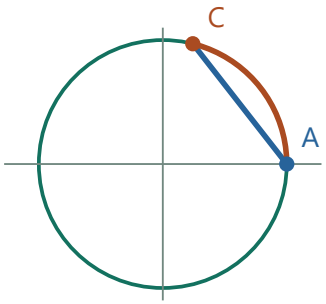
Schematic section

The radius through the chord midpoint divides it into two equal legs.

1. sin β=(AC/2)/R	In the acute case the radius is the hypotenuse of the bisected central triangle.
2. AC/2=R sin β ⇒ AC=2R sin β	Multiply out the radius denominator, then combine the two equal half-chords.

3. Extend to the other angle sizes

For obtuse β use the smaller central triangle with half-angle 180°−β, whose sine equals sin β. For β=90° the chord is a diameter. The same expression works throughout.



AC = 2R sin β. Supplementary angles have equal sine.

Check. A 90° inscribed angle subtends a diameter 2R.

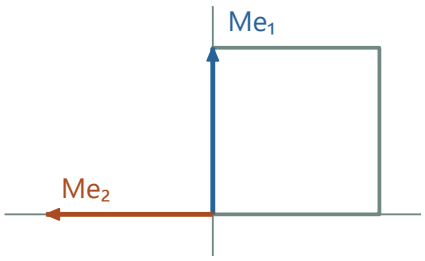
Exercise 121 · Quarter-turn operator

Printed page 11 · Rotations / Matrix transformations

A counterclockwise quarter-turn $\mathcal{Q}$ about the origin is applied to the point (x, y). What are the coordinates of the image point? Answer in the form $\mathcal{Q}(x, y) = (ax + by, cx + dy)$.

1. Rotate the coordinate directions

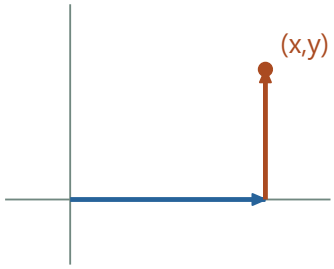
A counterclockwise 90° turn sends the unit horizontal vector (1,0) to (0,1) and the upward vector (0,1) to (−1,0).



Rotate e_1 to (0,1) and e_2 to (−1,0).

2. Decompose an arbitrary vector

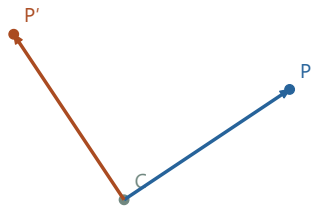
Write (x,y)=x(1,0)+y(0,1). A rotation about the origin acts linearly on these component vectors.



Represent an arbitrary arrow as x horizontal units plus y vertical units.

3. Recombine after the turn

The resulting vector is x(0,1)+y(−1,0)=(−y,x). Twice applying this rule gives (−x,−y), the expected half-turn.



Applying the quarter-turn to each part sends (x,y) to (−y,x).

1. x(0,1)=(0,x), y(−1,0)=(−y,0)
Multiply each rotated basis vector by its original component.

2. (0,x)+(−y,0)=(0x−1y,1x+0y)
Add corresponding coordinates to read off the four coefficients in the requested form.

Check. Two applications give (−x,−y), a half-turn.

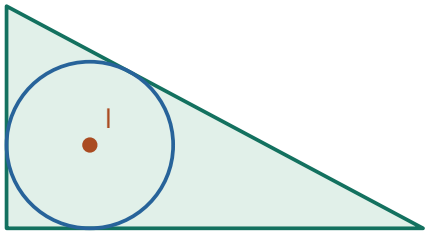
Exercise 122 · Coin inside a triangular tile

Printed page 11 · Geometric probability / Incircles and circumcircles

A coin of radius 1 cm is tossed onto a plane surface that has been tesselated by right triangles whose sides are 8 cm, 15 cm, and 17 cm long. What is the probability that the coin lands within one of the triangles?

1. Find the distance the center needs from a tile edge

The coin radius is 1 cm. Offset each of the tile's three sides inward by 1 cm. Centers inside the remaining triangle allow the entire coin to stay in the tile.



The center of the coin must stay at least 1 cm from every side.

Schematic illustration.

2. Measure the original inradius

The 8–15–17 tile is right-angled, giving area $8\cdot15/2=60$ cm² and semiperimeter $(8+15+17)/2=20$ cm. Since area=r times semiperimeter, r=3 cm.



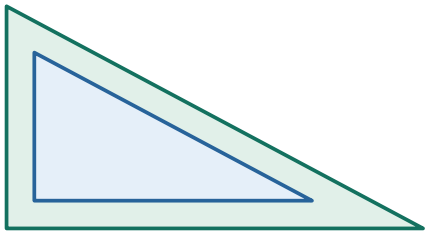
Original inradius = area/semiperimeter = 60/20 = 3 cm.

1. $60=r\cdot20 \Rightarrow r=60/20=3$ cm
The three triangles with the incenter as a vertex share height r, so their total area is r times the semiperimeter.

2. $r_{safe}=r-1=2$ cm
The inward offset uses one coin radius along each perpendicular.

3. Compare the safe triangle with the tile

Its sides are parallel to the original sides and its incenter is the same point, now $3-1=2$ cm from each shifted side. The triangles are similar with length scale 2/3.

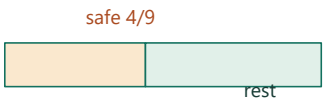


The inward parallels form a similar right triangle.

Its inradius is $3-1=2$ cm.

4. Turn area into probability

A uniform center gives probability equal to the safe area divided by the tile area. Area scales as the square of length, giving $(2/3)^2=4/9$.



Area ratio is the square of the length ratio, giving probability 4/9.

1. $A_{safe}=60\cdot(2/3)^2=60\cdot4/9=80/3$ cm²
Squaring the length factor gives the safe triangle's share of the tile.

2. $P=(80/3)/60=80/180=4/9$
Divide the safe center area by the tile area; a boundary line has zero area.

Check. In coordinates with legs on the axes the interior offset triangle has legs 16/3 and 10, giving area 80/3; dividing by 60 gives 4/9.

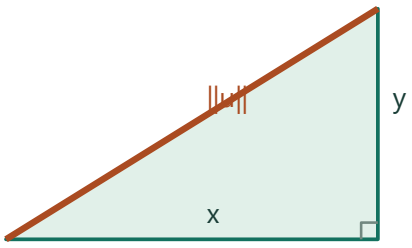
Exercise 123 · Vector magnitude notation

Printed page 11 · The dot product / Vectors and components

Given a vector $\mathbf{u}$, the familiar *absolute-value* notation $|\mathbf{u}|$ is often used for its *magnitude*. Thus the expressions $\mathbf{u}\bullet\mathbf{u}$ and $|\mathbf{u}|^2$ both mean the same thing. What exactly *do* they mean? Why do you think absolute value notation is used?

1. Read the coordinates geometrically

If u has components $u_1,u_2,\ldots$, these are displacements in mutually perpendicular directions. Applying Pythagoras gives squared distance from tail to head equal to Σu_i^2.

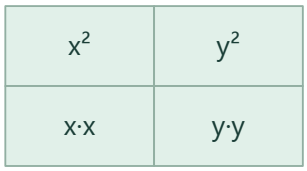


Schematic section

On the coordinate plane the vector supplies the hypotenuse.

2. Compare with the dot-product definition

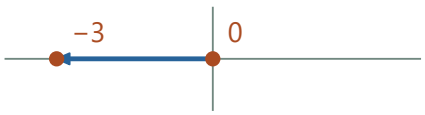
The self-product u·u is $\Sigma u_i u_i = \Sigma u_i^2$. Hence $u \cdot u = \|u\|^2$, and the magnitude is the nonnegative square root of that number.



u·u sums the same squares used by Pythagoras.

3. Connect with absolute value

On a line the vector consists of one coordinate a. Its magnitude is $\sqrt{(a^2)} = |a|$, the unsigned distance to zero. Vector magnitude extends this same distance idea to more dimensions.



On the number line the magnitude of −3 is its distance 3 from zero.

Check. For one coordinate a, its magnitude is the nonnegative square root of a^2, including when a is negative.

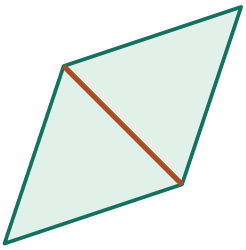
Exercise 124 · Square of a vector difference

Printed page 11 · The dot product / Polynomial identities

For any two numbers a and b, the product of $a - b$ times itself is equal to $a^2 - 2ab + b^2$. Does this familiar algebraic result hold for dot products of a vector $\mathbf{u} - \mathbf{v}$ with itself? In other words, is it true that $(\mathbf{u} - \mathbf{v}) \bullet (\mathbf{u} - \mathbf{v}) = \mathbf{u} \bullet \mathbf{u} - 2\mathbf{u} \bullet \mathbf{v} + \mathbf{v} \bullet \mathbf{v}$? Justify your conclusion, trying not to express vectors $\mathbf{u}$ and $\mathbf{v}$ in component form.

1. Interpret squaring in this setting

For real numbers, squaring a−b multiplies the difference by a second identical copy. Here the stated operation is a dot product: take (u−v)·(u−v). Its value is a scalar, the squared length of the vector difference.



The segment joining the vector heads represents u−v.

2. Distribute over the first difference

The law (a−b)·c=a·c−b·c applies to vectors. Use c=u−v as a complete second factor. This gives u·(u−v)−v·(u−v).

u·(u−v)
−v·(u−v)

Distribute once over the first difference.

3. Expand the second factors and track the signs

The first dot product becomes u·u−u·v. The second becomes v·u−v·v and has a minus sign in front of the whole expression. Subtracting it therefore gives u·u−u·v−v·u+v·v.

u·u	−u·v
−v·u	v·v

Two negative signs give the positive lower-right product.

4. Combine matching scalar products

The dot product is symmetric, so v·u=u·v. The two negative terms add to −2u·v. Thus (u−v)·(u−v)=u·u−2u·v+v·v, proving the proposed identity without choosing coordinate axes. When u=v, both sides give zero, as a check.

u·u	−2u·v	v·v

The two off-diagonal scalar products are equal.

Combine them to obtain −2u·v.

Check. If u=v, both sides vanish.

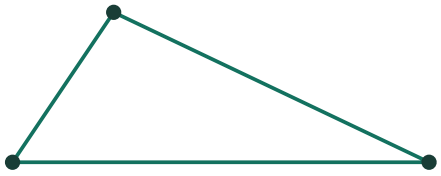
Exercise 125 · Unknown side from area

Printed page 11 · Triangle area using sine

A triangle has a 56-degree angle, formed by a 10-inch side and an x-inch side. Given that the area of the triangle is 18 square inches, find x.

1. Find the required altitude from the area

Using the 10-inch side as base, 18=½·10·h. Hence h=3.6 in.



Select the horizontal base and the included angle.

2. Relate that altitude to the missing side

If the unknown adjacent side is x, it makes 56° with the base. Its perpendicular component is x sin 56°, which must equal h.



Drop the altitude from the end of the sloping side.

Only the vertical component enters the height.

3. Recover the full side length

Solve x sin 56°=3.6 to obtain x=3.6/sin 56°≈4.342 in. This is longer than its perpendicular component, as the triangle requires.



A=18 gives altitude 3.6 in.

The sloping side is 3.6/sin 56° ≈ 4.342 in.

1. 18=(10x sin56°)/2 ⇒ 36=10x sin56°
Double the area equation to remove its denominator.

2. x=36/(10 sin56°)=3.6/sin56°
Divide by the positive sine factor and base.

Check. The altitude to the 10-in side equals x sin56°=3.6 in.

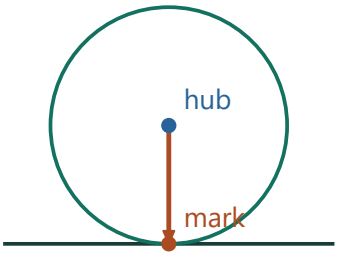
Exercise 126 · A tack on a rolling tire

Printed page 11 · Circular motion / Radian measure

Devon's bike has wheels that are 26 inches in diameter. After the front wheel picks up a tack, Devon rolls another 100 feet (1200 inches) and stops. How far above the ground in inches is the tack?

1. Measure rotation from the instant of pickup

The tire radius is 26/2=13 in. With no slipping, ground distance equals the arc length rolled off the wheel. Traveling 1200 inches turns it through θ=1200/13 radians.

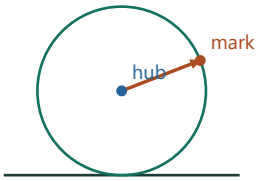


At pickup the tack is directly below the hub.

The initial displacement is (0,−13) inches.

2. Track the tack relative to the hub

Initially the tack is 13 in below the center, the lowest point. After rotation the vertical component of this initially downward radius is −13 cos θ.



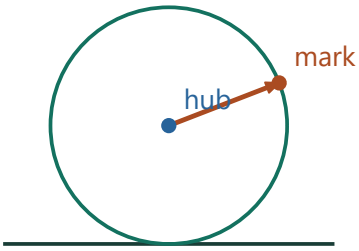
Rolling distance 1,200 in turns the wheel by 1,200/13 radians clockwise.

1. yrelative=0·sin(−θ)+(−13)cos(−θ)
Rotate the initial downward radius by the wheel's clockwise angle.

2. yrelative=−13 cos θ
Cosine has the same value at opposite angles, so the sign comes from the initial downward component.

3. Add the hub height

The center remains 13 in above the ground. Thus the final height is 13−13 cos(1200/13)≈17.69 in using radian mode. At zero travel this same expression gives zero.



The hub is 13 in above the road.

Tack height = 13−13 cos(1200/13) ≈ 17.69 in.

Check. At zero travel the formula gives zero, and all heights stay between 0 and 26 in.

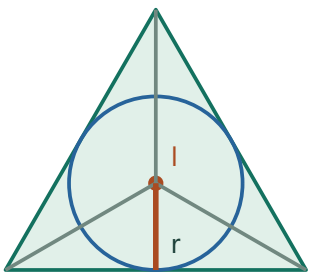
Exercise 127 · Three inscribed circles

Printed page 11 · Incircles and circumcircles / Triangle area

An isosceles triangle has two 10-inch sides and a $2w$-inch side. Find the radius of the inscribed circle of this triangle, in the cases $w = 5$, $w = 6$, and $w = 8$.

1. Use symmetry to find the altitude

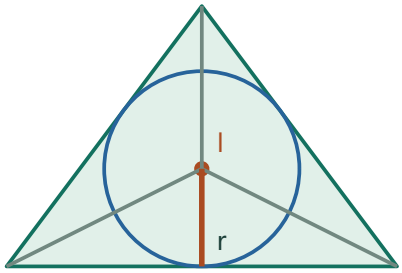
Bisect the base 2w. Either half-triangle has horizontal leg w and hypotenuse 10. Its vertical leg is $\sqrt{(100-w^2)}$.



For half-width w, the altitude is $\sqrt{(100-w^2)}$.

2. Relate triangle area to inradius

The area is $\frac{1}{2}\cdot 2w\cdot\sqrt{(100-w^2)}=w\sqrt{(100-w^2)}$. The semiperimeter is $(10+10+2w)/2=10+w$. Divide the area by this semiperimeter for r.



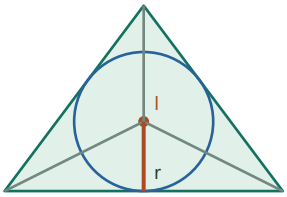
The same area is w times altitude and r times semiperimeter.

Thus $r=A/(10+w)$.

3. Evaluate the three widths

At w=5 this gives $25\sqrt{3}/15=5\sqrt{3}/3$ in. At w=6 the altitude is 8 and $r=48/16=3$ in. At w=8 the altitude is 6 and $r=48/18=8/3$ in.

Half-base w · 6 in



Equal sides stay 10 in. The altitude is 8 in.

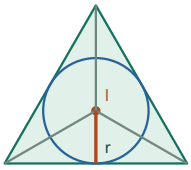
Inradius = w·height/(10+w) = 3 in.

1. w=5: $h=\sqrt{75}=5\sqrt{3}$, $A=25\sqrt{3}$
Insert the half-base in the right triangle, then multiply by w for area.

2. $r=A/(10+5)=25\sqrt{3}/15=5\sqrt{3}/3$
Divide by the semiperimeter and reduce by 5.

Compare settings · Half-base w

5 in

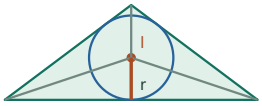


Equal sides stay 10 in. The altitude is 8.66 in.

Inradius = w·height/(10+w) = 2.887 in.

Equal sides stay 10 in. The altitude is 8.66 in. Inradius = w·height/(10+w) = 2.887 in.

8 in



Equal sides stay 10 in. The altitude is 6 in.

Inradius = w·height/(10+w) = 2.667 in.

Equal sides stay 10 in. The altitude is 6 in. Inradius = w·height/(10+w) = 2.667 in.

Check. Radius times semiperimeter reconstructs the corresponding areas $25\sqrt{3}$,48,48.

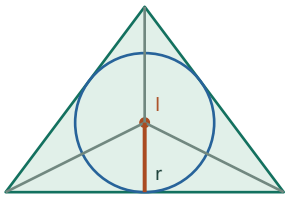
Exercise 128 · Largest incircle

Printed page 11 · Optimization / Incircles and circumcircles

(Continuation) Write an expression for the inscribed radius r in terms of the variable w, then find the value of w, to the nearest hundredth, that gives the maximum value of r.

1. Simplify the inradius function

The preceding formula $r=w\sqrt{(100-w^2)/(10+w)}$ simplifies to $w\sqrt{[(10-w)/(10+w)]}$. Real nondegenerate triangles require $0<w<10$.



As the base width grows the triangle becomes flatter.

The radius depends on both changes.

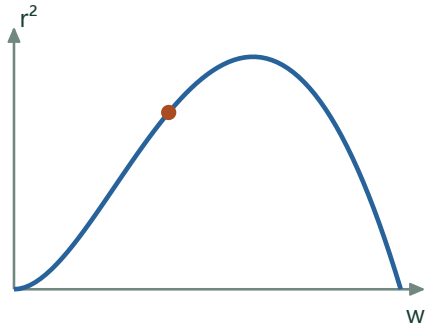
1. $r^2=w^2(100-w^2)/(10+w)^2$
Square the positive inradius formula before simplifying.

2. $r^2=w^2(10-w)(10+w)/(10+w)^2$
Factor the difference of squares.

3. $r=w\sqrt{[(10-w)/(10+w)]}$
Cancel the positive $10+w$ and take the nonnegative root.

2. Maximize a simpler quantity

Since r is positive, its square has the same maximizer. Differentiate $r^2=w^2(10-w)/(10+w)$. Its derivative is $2w(100-10w-w^2)/(10+w)^2$, whose sign on the allowed interval comes entirely from the quadratic.



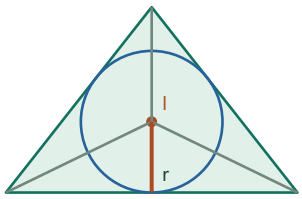
Maximizing r^2 gives the same location as maximizing the positive r.

1. $F=w^2(10-w)/(10+w)=(10w^2-w^3)/(10+w)$	Expand the numerator of the squared radius.
2. $F'=[(20w-3w^2)(10+w)-(10w^2-w^3)]/(10+w)^2$	Apply the quotient rule; the denominator has derivative one.
3. $F'=(200w-20w^2-2w^3)/(10+w)^2$	Distribute, subtract and collect powers of w.
4. $F'=2w(100-10w-w^2)/(10+w)^2$	The outside factors are positive for $0<w<10$, isolating the sign test.

3. Locate and classify the critical point

Solve $w^2+10w-100=0$. Only $w=5(\sqrt{5}-1)\approx 6.18$ in is positive. The derivative changes from positive to negative there, and the inradius tends to zero at both ends, proving this is the global maximum.

Half-base w · 6.18 in



Equal sides stay 10 in. The altitude is 7.862 in.

Inradius = w·height/(10+w) = 3.003 in.

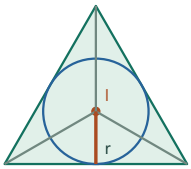
1. $w^2+10w-100=0 \Rightarrow (w+5)^2=125$
Add 100 and complete the square with 25.

2. $w=-5\pm5\sqrt{5}$
Take both square roots before subtracting 5.

3. $w=5(\sqrt{5}-1)$
The minus choice is negative and cannot be a half-base.

Compare settings · Half-base w

5 in

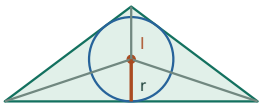


Equal sides stay 10 in. The altitude is 8.66 in.

Inradius = w·height/(10+w) = 2.887 in.

Equal sides stay 10 in. The altitude is 8.66 in. Inradius = w·height/(10+w) = 2.887 in.

8 in



Equal sides stay 10 in. The altitude is 6 in.

Inradius = w·height/(10+w) = 2.667 in.

Equal sides stay 10 in. The altitude is 6 in. Inradius = w·height/(10+w) = 2.667 in.

Check. The derivative is positive before this root and negative afterward; both degenerate endpoints have limiting radius zero.

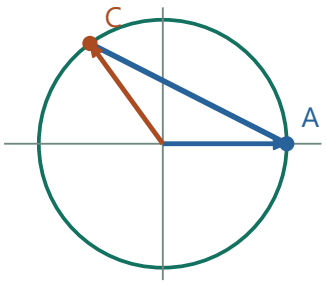
Exercise 129 · Circumcircle diameter

Printed page 11 · The sine rule / Circumcenters and circumcircles

Triangle ABC has a 63.0-degree angle at B, and side AC is 13.6 cm long. What is the diameter of the circle circumscribed about ABC?

1. Identify which angle faces the given chord

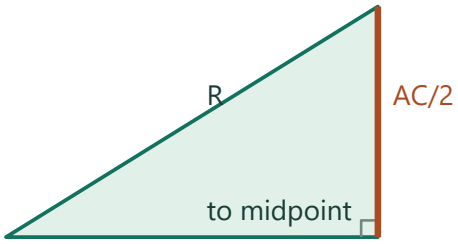
Side AC subtends the inscribed angle B=63.0°. For circumdiameter D, the half-chord construction gives AC=D sin B.



The chord AC cuts off a central angle twice the inscribed angle.

2. Solve for the circle size

The known chord is 13.6 cm, so D sin 63°=13.6. Divide by sin 63°, a positive number.

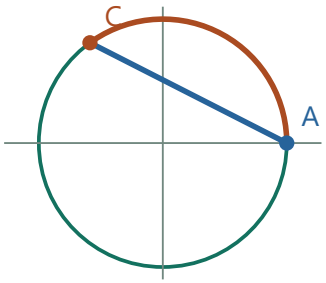


Schematic section

The radius through the chord midpoint divides it into two equal legs.

3. Calculate and check the scale

D=13.6/sin 63°≈15.3 cm. A chord must fit across its circle, so a diameter larger than 13.6 cm makes sense.



Diameter = 13.6/sin 63° ≈ 15.3 cm.

Check. A diameter must exceed the chord when the inscribed angle is acute.

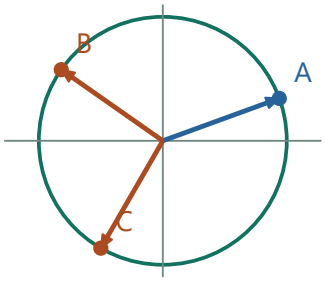
Exercise 130 · Extended sine rule

Printed page 11 · The sine rule / Incircles and circumcircles

(Continuation) Given any triangle ABC, with sides a, b, and c opposite angles A, B, and C, respectively, what can be said about the three ratios $\frac{a}{\sin A}$, $\frac{b}{\sin B}$, and $\frac{c}{\sin C}$? This result is known as the *Law of Sines*.

1. Use one circle for all three sides

Every vertex of the triangle lies on its circumcircle, so a,b,c are three chords of the same radius R. Side a lies opposite angle A, b opposite B and c opposite C.

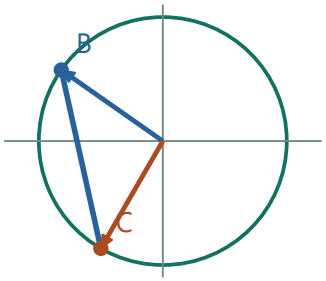


Take the circumcircle of triangle ABC.

Every side is a chord of this one circle.

2. Apply the chord formula three times

The half-chord argument gives a=2R sin A, b=2R sin B and c=2R sin C.



The chord opposite A has length 2R sin A.

3. Expose the common quantity

Divide each equality by its corresponding sine. Then a/sin A=b/sin B=c/sin C=2R. All three ratios equal the diameter of one circle, explaining why they agree.

a/sin A
b/sin B
c/sin C

Each row equals the same diameter 2R.

Check. For a right triangle the ratio at the right angle equals the hypotenuse, its circumdiameter.

Exercise 131 · Lengths and differences

Printed page 12 · Vectors and components / Inequalities

Let $\mathbf{u} = [3, -4]$ and $\mathbf{v} = [8, 15]$. Calculate the four numbers $|\mathbf{u}|$, $|\mathbf{v}|$, $|\mathbf{u} - \mathbf{v}|$, and $|\mathbf{u}| - |\mathbf{v}|$. Do the same with the vectors $\mathbf{u} = [2, 6, -3]$ and $\mathbf{v} = [2, 2, 1]$.

1. Distinguish the two operations

$\|u-v\|$ first subtracts coordinate vectors, then finds the length of the joining arrow. The scalar $\|u\|-\|v\|$ finds two lengths first, then subtracts them. It can even be negative.



Lengths of u and v are 5 and 17.

2. Compute the two-dimensional example

For $(3,-4)$ and $(8,15)$ the lengths are 5 and 17. The difference vector is $(-5,-19)$, of length $\sqrt{(25+361)}=\sqrt{386}$. The scalar difference is $5-17=-12$.



$u-v$ is the arrow from the head of v to the head of u.

Its length is $\sqrt{386}$, while $5-17$ is -12.

1. $\|u\|=\sqrt{(3^2+(-4)^2)}=\sqrt{25}=5$; $\|v\|=\sqrt{(8^2+15^2)}=\sqrt{289}=17$
Use perpendicular components to compute the two original lengths separately.

2. $u-v=(3-8,-4-15)=(-5,-19)$
Subtract components before finding the length of the vector difference.

3. Compute the three-dimensional example

For $(2,6,-3)$ and $(2,2,1)$ the lengths are $\sqrt{49}=7$ and $\sqrt{9}=3$. Their difference vector $(0,4,-4)$ has length $\sqrt{32}=4\sqrt{2}$, whereas the difference of lengths is 4.

2	6	-3
2	2	1
0	4	-4

For the 3D pair subtract column by column.

$\|u-v\|=4\sqrt{2}$, while $\|u\|-\|v\|=4$.

1. $\|u\|=\sqrt{(2^2+6^2+(-3)^2)}=7$; $\|v\|=\sqrt{(2^2+2^2+1^2)}=3$
Use all three components in the separate spatial distances.

2. $u-v=(2-2,6-2,-3-1)=(0,4,-4)$
The first components cancel and the last subtraction makes the vertical component more negative.

Check. In both cases the norm of the difference exceeds or equals the absolute difference of the norms.

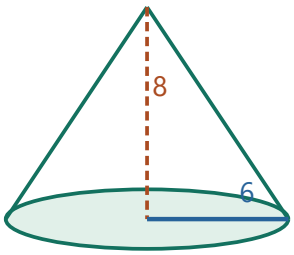
Exercise 132 · Side area of a cup

Printed page 12 · Surface area / Nets and surface development

An 8-inch tall conical cup has a 6-inch base radius. Explain how to find the surface area of the cup.

1. Read the cone’s axial section

The perpendicular height is 8 in and the base radius is 6 in. The slant height ℓ is their right triangle’s hypotenuse, $\ell=\sqrt{(8^2+6^2)}=10$ in.

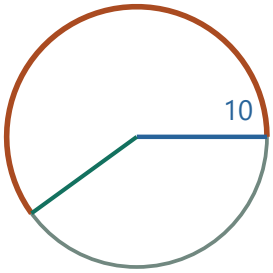


Cone radius 6; perpendicular height 8.

The slant edge runs from rim to vertex.

2. Unroll the cup wall

It forms a sector with radius 10 in and arc length equal to the base circumference 12π in. The sector area is half the radius times arc length, or $\tfrac{1}{2}\cdot 10\cdot 12\pi=60\pi$ in^2.

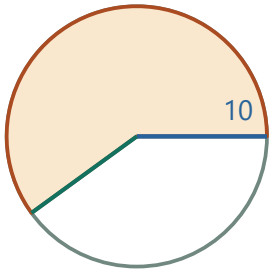


Unroll the lateral surface without stretching.

Its outer arc is exactly the old rim, length $2\pi r$.

3. Decide which surfaces count

An open conical cup has this curved wall only. If the intended object includes a circular face across the opening, add $\pi\cdot 6^2=36\pi$ to obtain 96π in^2.



Area of the sector = ½·slant height·rim circumference.

60π in^2 (open cup).

Check. The lateral-area rule $\pi r\ell$ gives 60π.

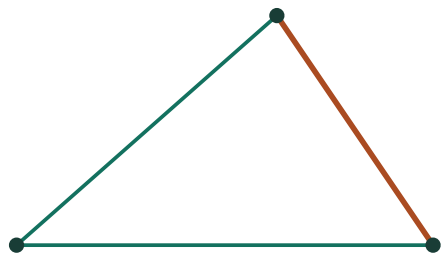
Exercise 133 · SSS form of cosine rule

Printed page 12 · The cosine rule / Inverse trigonometric functions

If a triangle has sides of lengths a and b, which make a C-degree angle, then the length of the side opposite C is c, where $c^2 = a^2 + b^2 - 2ab\cos C$. This is the SAS version of the Law of Cosines. Explain the terminology. Derive an equivalent SSS version of the Law of Cosines, which gives the cosine of the angle in terms of the lengths of the three sides. Now use it to find the angles of the triangle whose sides have lengths 4 cm, 5 cm, and 6 cm.

1. Use the cosine rule in the forward direction

With two sides a,b and their included angle C, $c^2=a^2+b^2-2ab\cos C$ determines the opposite side. This handles SAS data.

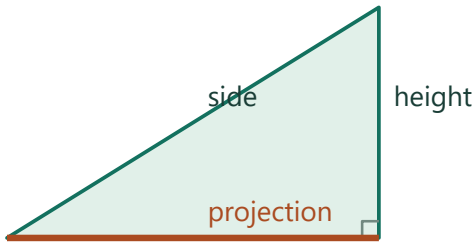


SAS gives the third side from the included angle.

SSS reverses this relationship.

2. Reverse the rule for three known sides

Rearrangement gives $\cos C=(a^2+b^2-c^2)/(2ab)$. Use inverse cosine to recover the unique interior angle opposite any chosen side.



Schematic section

The cosine term records how far one adjacent side projects along the other.

1. $2ab\cos C=a^2+b^2-c^2$	Move the cosine term to the left and c^2 to the right.
2. $\cos C=(a^2+b^2-c^2)/(2ab)$	The lengths a,b of a nondegenerate triangle are positive.
3. $C=\arccos[(a^2+b^2-c^2)/(2ab)]$	The interior angle lies between 0° and 180°, the principal cosine-inverse range.

3. Pair each angle with its opposite side

Opposite side 4 the cosine is $(5^2+6^2-4^2)/(2\cdot5\cdot6)=45/60$. Opposite side 5 it is $(4^2+6^2-5^2)/(2\cdot4\cdot6)=27/48$. Opposite side 6 it is $(4^2+5^2-6^2)/(2\cdot4\cdot5)=5/40$.

4	5	6
A	B	C

Name the opposite angle before applying the cosine rule.

A faces 4, B faces 5, and C faces 6.

4. Evaluate and compare

These give angles about 41.41°, 55.77° and 82.82°. Their total is 180° to rounding, and the longest side faces the largest angle.

4	5	6
41.41°	55.77°	82.82°

The largest side faces the largest angle.

The three angles add to approximately 180°.

Check. The angles sum to 180°, and their order matches the side lengths.

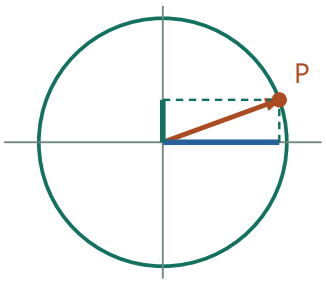
Exercise 134 · Constant-length rotated vector

Printed page 12 · Rotations / Vectors and components

What is the length of the vector $[5\cos\theta, 5\sin\theta]$? If the vector $[5,0]$ is rotated 36 degrees in the counterclockwise direction, what are the components of the resulting vector?

1. Find the squared length of the moving vector

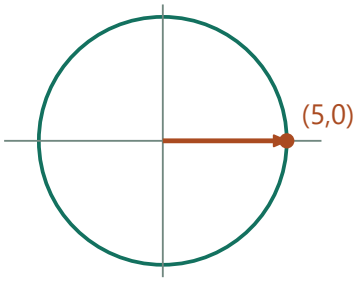
The squared horizontal and vertical components sum to $25\cos^2\theta+25\sin^2\theta=25$. Thus every vector of this form has length 5.



A radius 5 has projections 5 cos θ and 5 sin θ.

2. Read the initial direction

At $\theta=0$ the point is (5,0), directly to the right of the origin. Increasing θ to 36° changes the direction while preserving that radius.

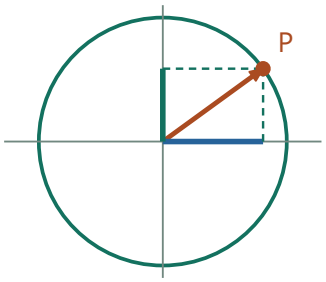


The length-5 initial vector points along positive x before its 36° turn.

3. Project at the new angle

The new components are $(5\cos 36°, 5\sin 36°)\approx(4.045, 2.939)$. Both are positive in the first quadrant.

Radius direction · 36°

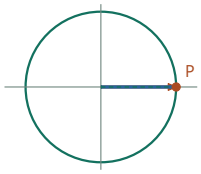


Radius 5; angle 36°.

Projection onto x = 4.045. Projection onto y = 2.939.

Compare settings · Radius direction

0°

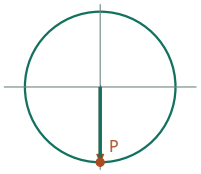


Radius 5; angle 0°.

Projection onto x = 5. Projection onto y = 0.

Radius 5; angle 0°. Projection onto x = 5. Projection onto y = 0.

270°



Radius 5; angle 270°.

Projection onto x = −0. Projection onto y = −5.

Radius 5; angle 270°. Projection onto x = −0. Projection onto y = −5.

Check. The resulting direction makes a 36° angle with the positive x-axis.

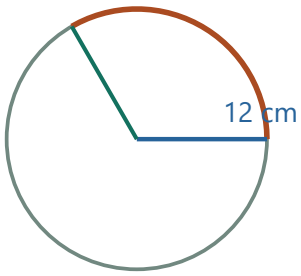
Exercise 135 · Optimizing a paper cone

Printed page 12 · Optimization / Volume / Circular sectors

Infinitely many different sectors can be cut from a circular piece of paper with a 12-cm radius, and any such sector can be fashioned into a paper cone with a 12-cm *slant height*.
(a) Show that the volume of the cone produced by the 180-degree sector is larger than the volume of the cone produced by the 120-degree sector.
(b) Find a sector of the same circle that will produce a cone whose volume is even larger.
(c) Express the volume of a cone formed from this circle as a function of the central angle of the sector used to form it, then find the sector that produces the cone of greatest volume.

1. Convert sector angle into cone dimensions

The arc of the radius-12 sector is $(\theta/360)\cdot 24\pi$ for θ degrees. Equate this to the new rim length $2\pi r$, obtaining $r=\theta/30$. The sector radius becomes slant height 12, so perpendicular height is $\sqrt{(144-r^2)}$.



The sector radius stays 12 cm.

Its arc becomes the new cone rim.

2. Compare the trial sectors

Volume is $V=(\pi/3)r^2\sqrt{(144-r^2)}$. For 120°, r=4 gives $128\pi\sqrt{2}/3$. For 180°, r=6 gives $72\pi\sqrt{3}$, which is larger. For 240°, r=8 gives $256\pi\sqrt{5}/3$, larger still.

120°	180°	240°
4 cm	6 cm	8 cm

Larger sector angle makes a larger base radius.

The slant height remains fixed.

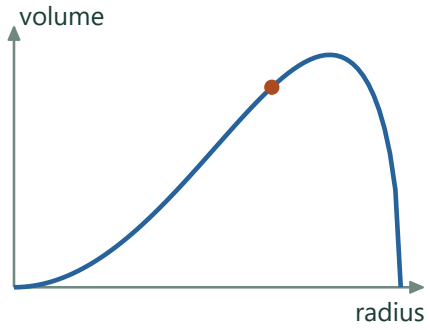
1. $V_{120}=(\pi/3)\cdot16\cdot\sqrt{128}=(128\pi/3)\sqrt{2}$
For r=4 the height is $\sqrt{(144-16)}=8\sqrt{2}$.

2. $V_{180}=(\pi/3)\cdot36\cdot\sqrt{108}=72\pi\sqrt{3}$
For r=6 the height becomes $6\sqrt{3}$.

3. $V_{240}=(\pi/3)\cdot64\cdot\sqrt{80}=(256\pi/3)\sqrt{5}$
For r=8 the height is $4\sqrt{5}$. Comparing these volumes gives about 190, 392 and 599 cm^3, respectively.

3. Track both competing effects

Increasing θ enlarges the base but reduces the perpendicular height. Put $u=r^2$, with $0\le u\le144$. Squaring the volume and dropping its positive constant gives the simpler objective $u^2(144-u)$.

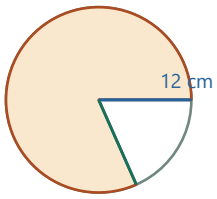


Broadening the base shortens the perpendicular height.

4. Prove which angle is best

Its derivative is $3u(96-u)$, positive then negative at u=96. Both endpoints give zero volume. Thus $r=\sqrt{96}=4\sqrt{6}$, $\theta=30r=120\sqrt{6}°\approx293.94°$, and $h=\sqrt{48}=4\sqrt{3}$. The maximum volume is $128\pi\sqrt{3}$ cm^3.

Paper sector used · 293.939°



Radius after rolling = 9.798 cm; perpendicular height = 6.928 cm.

Volume = 696.499 cm^3.

1. $F(u)=144u^2-u^3$
Expand the square-volume objective $u^2(144-u)$.

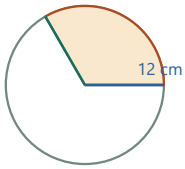
2. $F'(u)=288u-3u^2=3u(96-u)$
Differentiate both powers and factor out 3u.

3. $F'>0$ for $0<u<96$; $F'<0$ for $96<u<144$
Since u is positive, 96−u fixes the sign and confirms the maximum.

4. $V=(\pi/3)\cdot96\cdot\sqrt{48}=128\pi\sqrt{3}$ cm^3
Put the maximizing squared radius back into the original volume formula.

Compare settings · Paper sector used

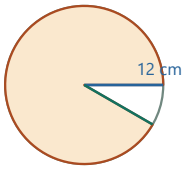
120°



Radius after rolling = 4 cm; perpendicular height = 11.314 cm.
Volume = 189.563 cm³.

Radius after rolling = 4 cm; perpendicular height = 11.314 cm. Volume = 189.563 cm³.

330°



Radius after rolling = 11 cm; perpendicular height = 4.796 cm.
Volume = 607.684 cm³.

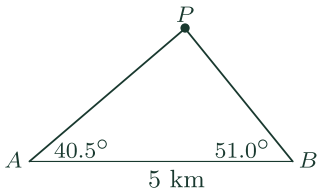
Radius after rolling = 11 cm; perpendicular height = 4.796 cm. Volume = 607.684 cm³.

Check. Both endpoint radii 0 and 12 yield zero volume, and the sole interior peak exceeds the sample volumes.

Exercise 136 · Triangulating an airplane

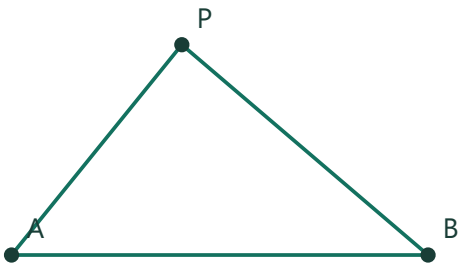
Printed page 12 · Triangulation / Right-triangle trigonometry

Two observers who are 5 km apart simultaneously sight a small airplane flying between them. One observer measures a 51.0-degree inclination angle, while the other observer measures a 40.5-degree inclination angle, as shown in the diagram. At what altitude is the airplane flying?



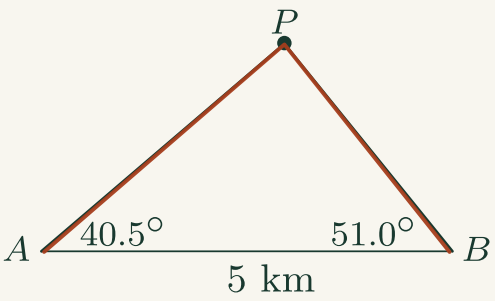
1. Drop a vertical from the airplane

Let h be its altitude. Because the airplane is between the observers, its perpendicular foot splits their 5-km separation into two horizontal segments x and 5−x.



Both sight lines meet at the aircraft P.

The observers are 5 km apart.

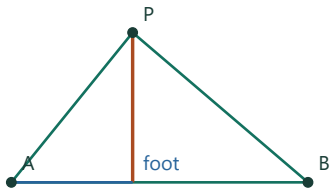


Elevation rays to P

The two sloping sight lines meet at the airplane. Both right triangles created by lowering its altitude share this same vertical height.

2. Use each angle of elevation

Each right triangle has opposite side h. From the 51° observer the horizontal distance is h/tan 51°=h cot 51°. From the 40.5° observer it is h cot 40.5°.

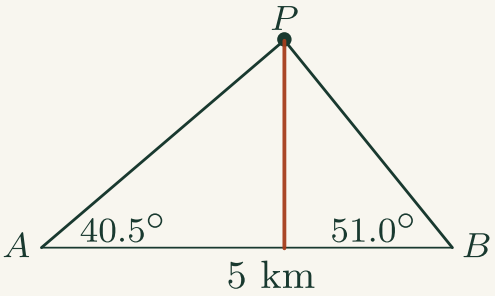


The two horizontal runs add to 5 km.

They are h cot 51° and h cot 40.5°.

1. $\tan 51° = h/d_1$, $\tan 40.5° = h/d_2$
Each sight line and the vertical altitude bound a right triangle.

2. $d_1 = h \cot 51°$, $d_2 = h \cot 40.5°$
Solve each tangent equation for its horizontal leg.

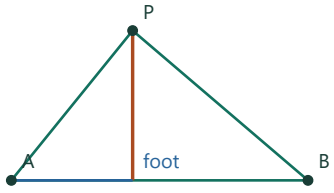


Vertical altitude from P

This added construction shows the height being sought. It lands between A at 40.5° and B at 51.0° on the horizontal baseline.

3. Use the full ground distance

Add those horizontal lengths: h(cot 51°+cot 40.5°)=5. Solving gives h≈2.524 km.



The shared vertical segment is h ≈ 2.524 km.

1. $5 = h \cot 51° + h \cot 40.5°$
The two ground segments partition the 5-km baseline.

2. $h = 5/(\cot 51° + \cot 40.5°)$
Factor h and divide by the positive sum.

Check. The two distances reconstructed from this altitude add to 5 km.

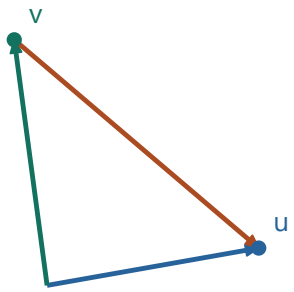
Exercise 137 · Difference vector in a triangle

Printed page 12 · Vectors and components / The dot product

Let $\mathbf{u} = [2, -3, 1]$ and $\mathbf{v} = [0, 1, 4]$. Calculate the vector $\mathbf{u} - \mathbf{v}$. Place $\mathbf{u}$ and $\mathbf{v}$ tail-to-tail to form two sides of a triangle. With regard to this triangle, what does $\mathbf{u} - \mathbf{v}$ represent? Calculate the number $\mathbf{u} \bullet \mathbf{u}$ and discuss its relevance to the diagram you have drawn. Do the same for the number $(\mathbf{u} - \mathbf{v}) \bullet (\mathbf{u} - \mathbf{v})$.

1. Identify the third side as an arrow

Start u=(2,−3,1) and v=(0,1,4) at the same tail. To travel from the head of v to the head of u, first reverse v and then follow u. The displacement is u−v=(2,−4,−3).



Start the blue u and green v at the same tail.

The orange arrow from v to u is u−v = (2,−4,−3).

2. Read a self-product as squared length

The u side has squared length $u\cdot u=2^2+(-3)^2+1^2=14$; its length is $\sqrt{14}$.

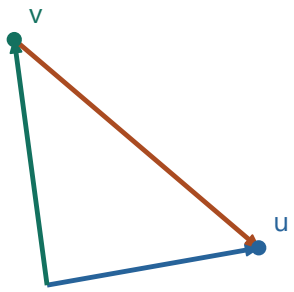
2	-3	1
4	9	1

Square the entries of u and sum.

u·u = 14.

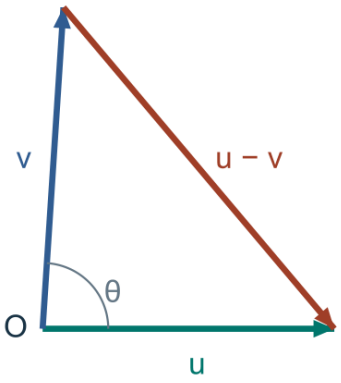
3. Measure the joining side

The self-product of u−v is $2^2+(-4)^2+(-3)^2=29$. Hence the two arrowheads are $\sqrt{29}$ units apart.



Measure the orange displacement using its three spatial components.

Its squared length is 4+16+9=29.



u = (2, −3, 1)

v = (0, 1, 4)

u − v = (2, −4, −3)

The triangle lies in the plane of u and v.

Check. The squared distance between the two heads by the coordinate distance formula is 29.

Exercise 138 · Designing a data matrix

Printed page 13 · Data matrices

A matrix can be used to display and process certain kinds of data. For example, during its final weekend of operation in 2008, the *IOKA* sold tickets to 186 adults on Friday, 109 adults on Saturday, 111 adults on Sunday, 103 children on Friday, 127 children on Saturday, 99 children on Sunday, 77 senior citizens on Friday, 67 senior citizens on Saturday, and 58 senior citizens on Sunday. This data is displayed in the 3×3 sales matrix $\mathbf{S}$ shown above. The descriptive labels given in the margin allow the reader to remember what all the numbers mean. Invent your own example of numerical data that can be displayed like this in a rectangular array.

$$\mathbf{S} = \begin{array}{c} \\ Fr \\ Sa \\ Su \end{array} \begin{array}{c} \begin{array}{ccc} Ch & Ad & Se \end{array} \\ \begin{bmatrix} 103 & 186 & 77 \\ 127 & 109 & 67 \\ 99 & 111 & 58 \end{bmatrix} \end{array}$$

1. Read the printed ticket matrix

The row label Fr means Friday and the column label Ad means adults. Their intersection, 186, is the number of adult tickets sold on Friday. The day and category together identify one number.

child	adult	senior
103	186	77

Friday in the source table

The adult-ticket entry is 186.

2. Choose two labels for each observation

For example, record library loans by both day and subject. Use rows for Monday and Tuesday, and columns for fiction, science and history, in that order.

fiction	science	history
12	8	5
9	10	6

Example library matrix

Rows are Monday and Tuesday; columns are book categories.

$$\begin{array}{c} \\ Fr \\ Sa \\ Su \end{array} \begin{array}{c} \begin{array}{ccc} Ch & Ad & Se \end{array} \\ \begin{bmatrix} 103 & 186 & 77 \\ 127 & 109 & 67 \\ 99 & 111 & 58 \end{bmatrix} \end{array} = \mathbf{S}$$

Day labels select a row

Fr, Sa and Su identify separate days in the printed sales array, just as day labels identify rows in the library example.

3. Enter the counts consistently

For Monday take (12,8,5) loans and for Tuesday (9,10,6). Writing the two rows one above the other forms a rectangular matrix with two rows and three columns.

fiction	science	history
12	8	5
9	10	6

Use the same column order on Monday and Tuesday so each category remains comparable.

4. Interpret an individual cell

Row 2, column 2 is 10. The row identifies Tuesday and the column identifies science, so this records ten science loans on Tuesday. The labels give meaning to each number.

fiction	science	history
12	8	5
9	10	6

Second row, second column

Ten science loans on Tuesday.

$$\begin{array}{c} \\ Fr \\ Sa \\ Su \end{array}\begin{array}{c} \begin{array}{ccc} Ch & Ad & Se \end{array} \\ \begin{bmatrix} 103 & 186 & 77 \\ 127 & 109 & 67 \\ 99 & 111 & 58 \end{bmatrix} \end{array} = \mathbf{S}$$

One cell needs both labels

The middle entry in the first row, 186, counts Friday adult tickets. Use its row and column together to interpret it.

Check. Each row counts one day; each column counts the same category on both days.

Exercise 139 · Ticket revenue as a dot product

Printed page 13 · The dot product / Data matrices

(Continuation) The IOKA's ticket prices can be read from the 3×1 matrix $\mathbf{P}$ shown at right. Such a matrix is often called a *column vector*. The first row of matrix $\mathbf{S}$ is a 3-component *row vector*. What is the dot product of these two vectors? What does it mean? What about the dot products of the other rows of $\mathbf{S}$ with $\mathbf{P}$?

$$\begin{array}{c} \\ Ch \\ Ad \\ Se \end{array}\begin{array}{c} \begin{array}{c} \$ \end{array} \\ \begin{bmatrix} 4.50 \\ 7.50 \\ 6.00 \end{bmatrix} \end{array} = \mathbf{P}$$

1. Match counts and prices by category

Each day’s row gives child, adult and senior counts, in that order. Multiply each count by its matching dollar price, then add the three category receipts. Combining counts first would lose the price differences.

child	adult	senior
103	186	77
4.5	7.5	6

Pair Friday counts with the matching three prices.

$$\begin{matrix} & \$ \\ Ch & 4.50 \\ Ad & 7.50 \\ Se & 6.00 \end{matrix} = \mathbf{P}$$

Price column in dollars

The three prices all share the same units and are listed in child, adult, senior order. Keep this order when multiplying by a day’s sales row.

2. Total Friday’s receipts

The three contributions are 103·$4.50=$463.50, 186·$7.50=$1395.00 and 77·$6.00=$462.00. Their sum is $2320.50.

463.50	1395	462

Category receipts add to $2,320.50.

1. 103·4.50=463.50; 186·7.50=1395; 77·6=462
Multiply each Friday count by the price for its own ticket category.

2. 463.50+1395+462=2320.50 dollars
Add category receipts to obtain one day's revenue.

$$\begin{matrix} & \$ \\ Ch & 4.50 \\ Ad & 7.50 \\ Se & 6.00 \end{matrix} = \mathbf{P}$$

Adult ticket price

The second entry belongs with the second count in the chosen daily row. Its price times that count is one contribution to receipts.

3. Repeat for Saturday and Sunday

Saturday gives 127·4.50+109·7.50+67·6=$1791.00. Sunday gives 99·4.50+111·7.50+58·6=$1626.00. Each result measures one day's total dollars.

Fri	2320.50
Sat	1791.00
Sun	1626.00

Repeat the same dot product for each day.

Column at right is revenue in dollars.

1. Saturday: 571.50+817.50+402=1791 dollars
Use 127 child, 109 adult and 67 senior ticket sales at the three stated prices.

2. Sunday: 445.50+832.50+348=1626 dollars
The same prices multiply Sunday's category counts 99, 111 and 58.

Check. Each term counts tickets multiplied by dollars per ticket, yielding dollars for that day.

Exercise 140 · Shape of a revenue matrix

Printed page 13 · Matrix multiplication / Data matrices

(Continuation) *Matrix multiplication* consists of calculating all possible dot products of row vectors from the first matrix and column vectors from the second matrix. How many dot products can be formed by multiplying matrix $\mathbf{S}$ times matrix $\mathbf{P}$? How would you organize them into a new matrix, called $\mathbf{SP}$? What do the entries of $\mathbf{SP}$ mean?

1. Read the two matrix shapes

S has three day rows and three category columns. P has three rows of prices and one column. The shared inner dimension 3 matches categories to prices.

103	186	77
127	109	67
99	111	58

S has three day rows and three categories.

P has one price column.

$$\begin{array}{c} \\ Fr \\ Sa \\ Su \end{array} \begin{array}{c} \begin{array}{ccc} Ch & Ad & Se \end{array} \\ \begin{bmatrix} 103 & 186 & 77 \\ 127 & 109 & 67 \\ 99 & 111 & 58 \end{bmatrix} \end{array} = \mathbf{S}$$

Day labels select a row · Figure from exercise 138

Fr, Sa and Su identify separate days in the printed sales array, just as day labels identify rows in the library example.

2. Form one output entry

For a fixed day, take its row in S and the only column in P. Multiply matching entries and add. This dot product is exactly that day’s ticket revenue.

103	186	77
127	109	67
99	111	58

Another row times the same price column produces Saturday’s revenue.

3. Assemble the revenue column

There are three day rows and one price column, hence three dot products. In the original day order SP=(2320.50,1791.00,1626.00)ᵀ dollars. Adding its entries gives weekend receipts of $5737.50.

2320.50
1791.00
1626.00

SP is a 3 × 1 column.

The days remain Friday, Saturday, Sunday in order.

1. Saturday=127(4.50)+109(7.50)+67(6)=571.50+817.50+402=1791
Use the second day's child, adult and senior counts in price-column order.

2. Sunday=99(4.50)+111(7.50)+58(6)=445.50+832.50+348=1626
Repeat with Sunday's row; add the daily totals only after pricing each category.

$$\begin{array}{c} \\ Fr \\ Sa \\ Su \end{array} \begin{array}{c} \begin{array}{ccc} \cancel{Ch} & \cancel{Ad} & \cancel{Se} \end{array} \\ \begin{bmatrix} 103 & 186 & 77 \\ 127 & 109 & 67 \\ 99 & 111 & 58 \end{bmatrix} \end{array} = \mathbf{S}$$

Ticket categories select a column · Figure from exercise 138

Ch, Ad and Se distinguish child, adult and senior sales. Each category keeps the same place across every day.

Check. The daily totals sum to $5737.50; summing counts by category first gives the same weekend receipts.

Exercise 141 · Seven features of one triangle

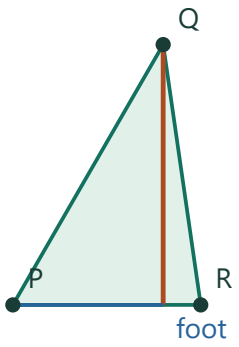
Printed page 13 · Triangle geometry / The cosine rule / Triangle area

The lengths of segments PQ and PR are 8 inches and 5 inches, respectively, and they make a 60-degree angle at P.

(a) Find the area of triangle PQR.
(b) Find the length of the projection of segment PQ onto segment PR.
(c) Find the length of segment QR.
(d) Find the sizes of the other two angles of triangle PQR.
(e) Find the length of the median drawn to side PQ.
(f) Find the length of the bisector of angle R.
(g) Find the third side of another triangle that has a 5-inch side, an 8-inch side, and the same area as triangle PQR.

1. Find the height, area and projection

In triangle PQR the 8-in side PQ and 5-in side PR meet at 60°. Relative to PR, PQ has perpendicular component 8 sin 60° and parallel component 8 cos 60°=4 in. Thus (a) area=½·5·8 sin 60°=10√3 in² and (b) the projection of PQ is 4 in.

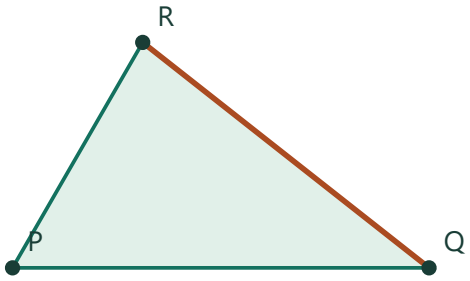


Project the 8-in side PQ onto the 5-in side PR.

Run = 4 in; altitude = 8 sin 60°.

2. Find the third side in (c)

QR is opposite the included angle. The cosine rule gives $QR^2=8^2+5^2-2\cdot8\cdot5\cos 60°=49$, so QR=7 in.

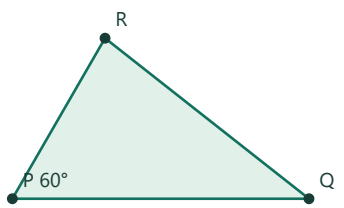


The side facing 60° is QR.

$QR^2 = 8^2+5^2-2\cdot8\cdot5\cos 60°$.

3. Find the other two angles in (d)

At Q the adjacent lengths are 8 and 7, with opposite length 5. Hence cos Q=(8²+7²−5²)/(2·8·7)=88/112. This gives Q≈38.21° and R≈180°−60°−38.21°=81.79°.



At Q the adjacent sides are 8 and 7, opposite side 5.

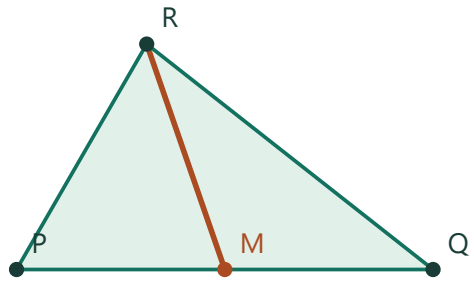
1. $5^2=8^2+7^2-2\cdot8\cdot7\cos Q$
The side of length 5 lies opposite Q.

2. $112\cos Q=64+49-25=88$
Isolate the cosine product by moving terms.

3. $Q=\arccos(11/14)$
Divide by 112 and take the interior angle.

4. Measure the median in (e)

Let M be the midpoint of PQ. In triangle RPM, PM=4, PR=5 and their included angle is 60°. Thus RM²=5²+4²−2·5·4 cos 60°=21, giving RM=√21 in.

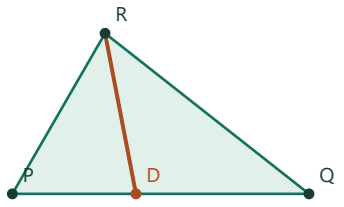


M is halfway along PQ; PM=4.

The median RM measures √21 in.

5. Locate and measure the bisector in (f)

Let D be where the R-angle bisector meets PQ. The angle-bisector theorem gives PD:DQ=PR:RQ=5:7, so PD=(5/12)·8=10/3. In RPD use the 60° angle again: RD²=5²+(10/3)²−2·5·(10/3)cos 60°=175/9. Thus RD=5√7/3 in.



The R-bisector divides PQ in ratio 5:7.

PD=10/3 in.

1. $PD/(8-PD)=5/7 \Rightarrow 7PD=40-5PD$
Use the two portions of PQ and cross-multiply the bisector proportion.

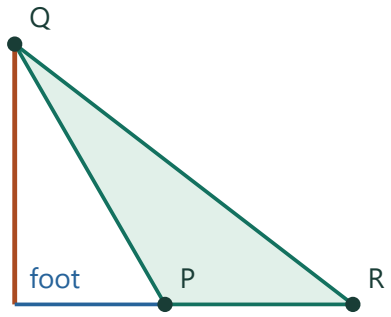
2. $12PD=40 \Rightarrow PD=10/3$
Collect the unknown lengths on one side.

3. $RD^2=25+100/9-50/3=(225+100-150)/9$
Use the cosine rule with cos60°=1/2 and a common denominator.

4. $RD=\sqrt{175/3}=5\sqrt{7/3}$ in
Extract the square factor 25 from the positive root.

6. Keep the area but change the triangle in (g)

The area depends on sin P, which is the same for 60° and 120°. Keep the two adjacent sides and use 120°. Their new opposite side is √(89−80 cos 120°)=√129 in, so this triangle has a different shape.



Use 120° at P instead.

Height is unchanged but the third side changes.

Check. The angles sum to 180°. Heron's formula with semiperimeter 10 gives √(10·5·3·2)=10√3.

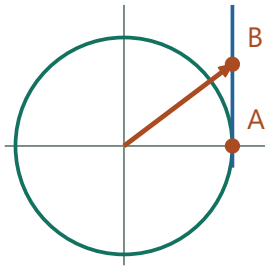
Exercise 142 · Tangent on the unit circle

Printed page 13 · The tangent function / The unit circle

Draw the unit circle and a first-quadrant ray from the origin that makes an angle θ with the positive x-axis. Let B be the point on this ray whose x-coordinate is 1, and let $A = (1, 0)$. Segment AB is tangent to the circle. In terms of θ, find its length. Hmm . . .

1. Read the reference triangle

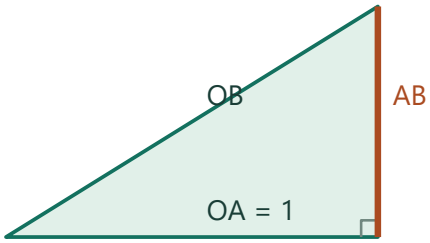
The ray from O meets the vertical line x=1 at B. The point A=(1,0) lies directly below B. Thus OA is a horizontal unit segment and AB is the vertical rise.



The ray meets the vertical tangent line x=1 at B.

2. Use the definition of tangent

At O, angle θ has opposite side AB and adjacent side OA=1. Hence tan θ=AB/1.

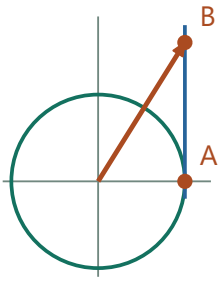


Schematic section

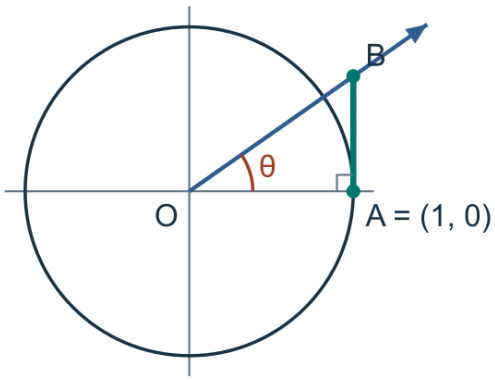
tan θ = opposite / adjacent = AB/1.

3. Interpret the segment

For a first-quadrant angle, B is above A and AB=tan θ is positive. Its growth as the ray steepens illustrates why tangent increases toward a vertical ray.



A steeper ray reaches a greater height at the same run of 1.



OA = 1

AB = tan θ

Check. In right triangle OAB adjacent length is 1, and opposite/adjacent equals tan θ.

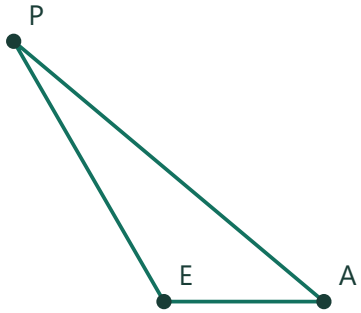
Exercise 143 · Two missing sides

Printed page 13 · The sine rule

Triangle PEA has a 20-degree angle at P and a 120-degree angle at E, and the length of side EA is 6 inches. Find the lengths of the other two sides of this triangle.

1. Complete the angle information

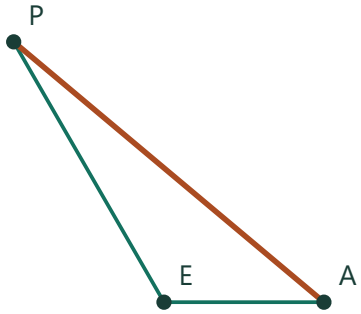
In triangle PEA the interior angles sum to 180°, so A=180°−20°−120°=40°. The given side EA=6 lies opposite P=20°.



Angles E=120° and P=20° leave A=40°.

2. Match sides to their opposite angles

The sine rule says EA/sin P=PA/sin E=PE/sin A. The common ratio is 6/sin 20°. This pairing avoids using the sine of an angle adjacent to a side.

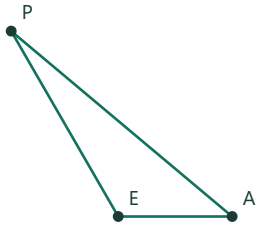


EA=6 lies opposite P=20°.

Use 6/sin 20° as the common side/sine ratio.

3. Calculate both unknowns

PA=6 sin 120°/sin 20°≈15.193 in and PE=6 sin 40°/sin 20°≈11.276 in. The 120° corner indeed faces the longest side.



The side opposite 120° is the longest.

PA≈15.193 in; PE≈11.276 in.

1. PA/sin120°=PE/sin40°=6/sin20°
Each ratio uses the angle opposite its side.

2. PA=(6/sin20°)sin120°, PE=(6/sin20°)sin40°
Multiply the shared ratio by the matching sine to recover each length.

Check. The longest side is opposite the 120° angle, and the shortest lies opposite 20°.

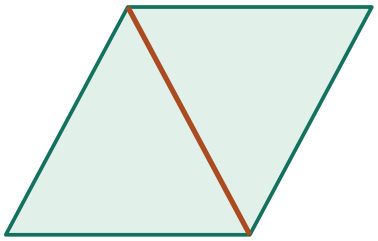
Exercise 144 · Pythagoras in dot products

Printed page 13 · The dot product / The Pythagorean theorem

If two vectors $\mathbf{u}$ and $\mathbf{v}$ fit the equation $(\mathbf{u}-\mathbf{v})\bullet(\mathbf{u}-\mathbf{v}) = \mathbf{u}\bullet\mathbf{u} + \mathbf{v}\bullet\mathbf{v}$, how must these vectors $\mathbf{u}$ and $\mathbf{v}$ be related? What familiar theorem does this equation represent?

1. Interpret the left side as a squared distance

The vector u−v joins the two heads when u and v share a tail. Its dot product with itself measures the square of the joining side's length.



The joining side is the vector difference.

2. Expand before comparing

Distributivity gives (u−v)·(u−v)=u·u+v·v−2u·v. For this to equal u·u+v·v, the cross term must vanish, so u·v=0.



The equality can hold only if the middle term vanishes.

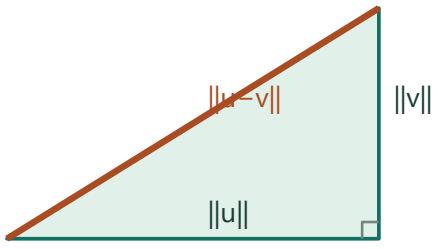
1. $\|u-v\|^2 = \|u\|^2 - 2u\cdot v + \|v\|^2$
Distribute both factors of the difference dot product.

2. $\|u\|^2 + \|v\|^2 = \|u\|^2 - 2u\cdot v + \|v\|^2$
Substitute the problem's equality.

3. $0 = -2u\cdot v \Rightarrow u\cdot v = 0$
Subtract equal squared-length terms and divide by −2.

3. Translate the conclusion to geometry

For nonzero vectors the angle must have cosine zero: the two sides meet at 90°. The original equality is then Pythagoras. If a vector is zero, the algebra still holds in a degenerate case.



Schematic section

Zero dot product for nonzero inputs gives perpendicular legs.

The given equality is Pythagoras.

Check. A zero vector satisfies the algebraic identity as a degenerate case.

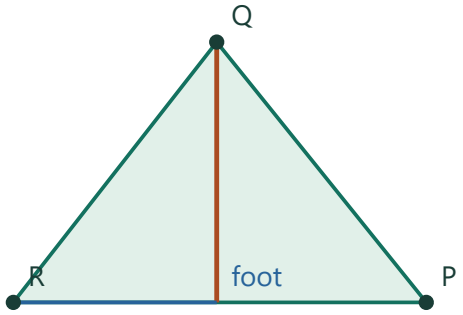
Exercise 145 · Area and the sine rule

Printed page 13 · Triangle area / The sine rule

The lengths QR, RP, and PQ in triangle PQR are often denoted p, q, and r, respectively. What do the formulas $\frac{1}{2}\,pq\sin R$ and $\frac{1}{2}\,qr\sin P$ mean? After you justify the equation $\frac{1}{2}\,pq\sin R = \frac{1}{2}\,qr\sin P$, simplify it to a familiar form.

1. Identify the sides enclosing each angle

At R the adjacent sides are p and q, so ½pq sin R measures the area. At P the adjacent sides are q and r, giving the same area as ½qr sin P.

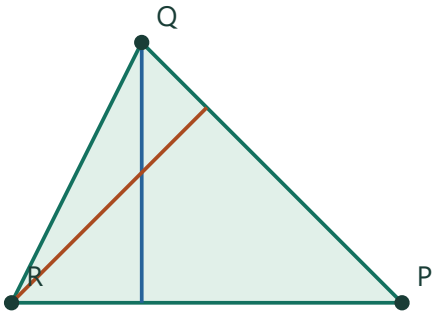


Use two sides adjoining R and the altitude from Q.

A = ½pq sin R.

2. Equate the two measurements

Both products describe one triangle. Equate them and cancel q/2, which is positive, obtaining p sin R=r sin P.



Same triangle, two perpendicular heights.

Equate its two area expressions and cancel their common q/2 factor.

3. Pair sides and opposite angles

Divide by sin P sin R to get p/sin P=r/sin R. These are two equal ratios from the sine rule.

p	sin P
r	sin R

Match each side with the sine of its opposite angle.

p/sin P = r/sin R.

1. p sin R=r sin P
Cancel the common positive half-side factor from the equal areas.

2. (p sin R)/(sin P sin R)=(r sin P)/(sin P sin R)
Both sines of interior triangle angles are positive.

3. p/sin P=r/sin R
Cancel one different sine on each side.

Check. Both ratios equal the circumcircle diameter.

Exercise 146 · Seven pizzas and five salads

Printed page 14 · Matrix multiplication / Data matrices

The price of a large pepperoni pizza is \$13.45 at New England Pizza, \$13.50 at Romeo's, and \$15 at Front Row. These shops charge \$6.25, \$10.50, and \$8, respectively, for a Greek salad. What would the bill be at each shop for seven pizzas and five salads?

1. Set up a single shop's calculation

The order contains seven pizzas and five salads. For each shop multiply its pizza price by 7, its salad price by 5, then sum in dollars.

pizza	salad
13.45	6.25
13.5	10.5
15	8

Price rows correspond to three different shops.

Prices within a row belong to that shop.

2. Compute the first two prices

Shop 1 gives $7\cdot13.45+5\cdot6.25=94.15+31.25=\$125.40$. Shop 2 gives $7\cdot13.50+5\cdot10.50=94.50+52.50=\$147.00$.

pizza	salad
13.45	6.25
13.5	10.5
15	8

For the first shop multiply its prices by 7 pizzas and 5 salads.

The first order costs \$125.40.

3. Finish the comparison

Shop 3 gives $7\cdot15+5\cdot8=105+40=\$145.00$. Among these offers the first is least expensive for this order.

125.40
147.00
145.00

Rows preserve shop order.

Compare \$125.40, \$147, and \$145.

Check. For the third store, $105 of pizza plus $40 of salad gives $145.

Exercise 147 · Two orders in one product

Printed page 14 · Matrix multiplication / Data matrices

(Continuation) What would each shop charge for two pizzas and a dozen salads? Show how this problem, as well as the previous one, can be solved by forming two suitable matrices and then multiplying them.

1. Choose the meaning of each matrix dimension

Put shops in rows of a price matrix, with pizza and salad prices in columns. Its rows are (13.45,6.25),(13.50,10.50),(15,8). Put each order in its own column of a second matrix: first (7,5), then (2,12).

pizza	salad
13.45	6.25
13.5	10.5
15	8

Price rows correspond to three different shops.

Prices within a row belong to that shop.

2. Interpret an entry of the product

A price row dotted with an order column gives one shop's charge for one order. For example the new order at the first shop is $2\cdot 13.45+12\cdot 6.25=\$101.90$.

pizza	salad
13.45	6.25
13.5	10.5
15	8

For the first shop multiply its prices by 7 pizzas and 5 salads.

The first order costs $125.40.

3. Complete and read the table

The product's rows are (125.40,101.90), (147.00,153.00), and (145.00,126.00), all in dollars. Column 1 reproduces the prior order. Column 2 prices the new order at all three shops.

125.40	101.90
147.00	153.00
145.00	126.00

Rows preserve shop order.

The second column changes the order to 2 pizzas and 12 salads.

1. 2(13.45)+12(6.25)=26.90+75.00=101.90
Dot the first shop's price row with the second order.

2. 2(13.50)+12(10.50)=27.00+126.00=153.00
Use both second-shop prices for the same order.

3. 2(15)+12(8)=30+96=126.00
The three receipts form column two in shop order.

Check. At the second shop, two pizzas cost $27 and twelve salads $126, totaling $153.

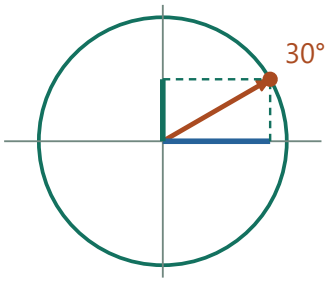
Exercise 148 · Supplementary angles

Printed page 14 · Reference angles and symmetry / The unit circle

If two angles are supplementary, then their sines are equal. Explain why. What about the cosines of supplementary angles? If you are not sure, calculate some examples.

1. Compare the two radii

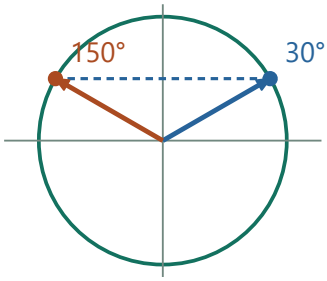
On the unit circle the rays at θ and 180°−θ are reflections of one another in the vertical axis. Follow their endpoints to see which coordinate changes.



Start with the radius at 30°.

2. Read their heights

Reflection in the vertical axis preserves the vertical coordinate. Because sine is height, sin(180°−θ)=sin θ.

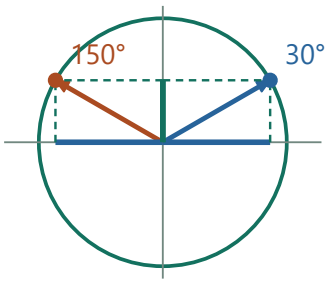


Reflect across the vertical axis.

Horizontal sign reverses; vertical height stays fixed.

3. Read their horizontal components

The same reflection reverses the horizontal coordinate, so $\cos(180° - \theta) = -\cos\theta$. For example, 60° and 120° share a height while lying on opposite sides of the axis.



Reflect across the vertical axis.

Horizontal sign reverses; vertical height stays fixed.

Check. For $\theta = 60°$, 120° has the same $\sqrt{3}/2$ sine and the opposite 1/2 cosine.

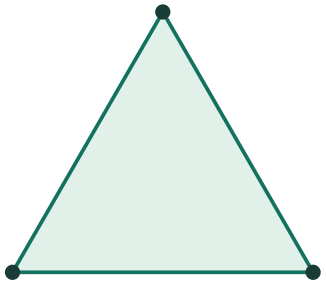
Exercise 149 · Double-angle sine

Printed page 14 · Trigonometric identities / Triangle area

An isosceles triangle has two sides of length w that make a 2α-degree angle. Write down two different formulas for the area of this triangle, in terms of w and α (*Greek* "alpha"). By equating the formulas, discover a relation involving $\sin 2\alpha$, $\sin\alpha$, and $\cos\alpha$.

1. Measure the area using the included angle

An isosceles triangle has two sides of length w enclosing angle 2α. The two-side area formula gives $A = \tfrac{1}{2}w^2 \sin(2\alpha)$.

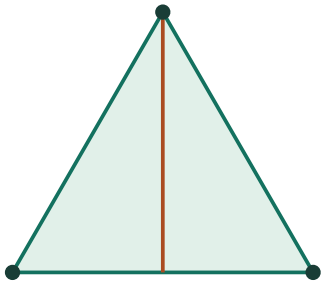


Two sides w meet at the full angle 2α.

Area = $\tfrac{1}{2}w^2 \sin 2\alpha$.

2. Drop the altitude from the apex

The altitude bisects the apex angle and the base. In either right half, half the base is w sin α and the altitude is w cos α. The whole base is twice that half-base.

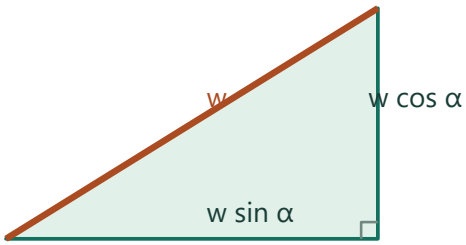


The angle bisector halves the base.

Half-base = w sin α; altitude = w cos α.

3. Equate the two area formulas

From base and height, $A=\frac{1}{2}(2w \sin\alpha)(w \cos\alpha)=w^2 \sin\alpha \cos\alpha$. Equate to the first formula and cancel $w^2/2$. Thus $\sin(2\alpha)=2 \sin\alpha \cos\alpha$.



Schematic section

Match the two area calculations.

sin(2α) = 2 sin α cos α.

1. $(w^2/2)\sin(2\alpha)=w^2 \sin\alpha \cos\alpha$	Set the included-angle and base-height areas equal.
2. $\sin(2\alpha)=2 \sin\alpha \cos\alpha$	Multiply by 2 and divide by w^2; w is a positive side length.

Check. At α=45° both expressions give area $w^2/2$.

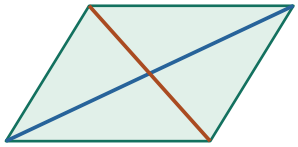
Exercise 150 · Other parallelogram diagonal

Printed page 14 · The cosine rule / Triangle geometry

A parallelogram has a 7-inch side and a 9-inch side, and the longer diagonal is 14 inches long. Find the length of the other diagonal. Do you need a calculator to do it?

1. Account for the two diagonals

With side vectors u,v, the parallelogram's diagonals are u+v and u−v. Squaring their lengths and adding cancels the opposite cross terms. The sum is twice the sum of the two side squares.



The opposite sides are parallel and have equal lengths.

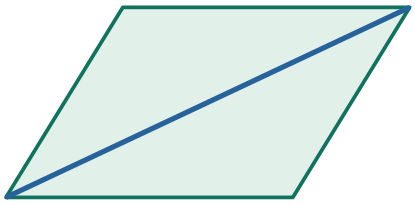
1. $d_1^2 = a^2 + b^2 + 2ab\cos C$
Expand the squared length of the sum of the two side vectors.

2. $d_2^2 = a^2 + b^2 - 2ab\cos C$
The other diagonal uses their difference.

3. $d_1^2 + d_2^2 = 2a^2 + 2b^2$
The opposite cross terms cancel in the sum.

2. Use the known lengths

With sides 7 and 9, the diagonal squares sum to $2(7^2+9^2)=260$. The longer diagonal contributes $14^2=196$.

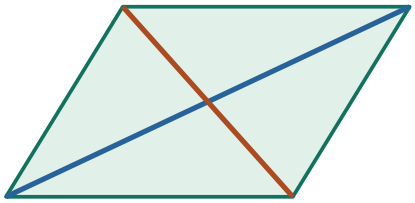


The first diagonal adds the side vectors.

Its square uses a positive 2ab cos C term.

3. Solve for the other length

Subtract 196, leaving 64 as the squared shorter diagonal. A length is positive, so the missing diagonal is $\sqrt{64}=8$ in.



The second diagonal subtracts the vectors.

The missing diagonal is 8 in.

Check. $8^2+14^2=260=2\cdot130$, entirely integer arithmetic.

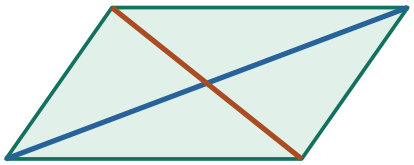
Exercise 151 · Parallelogram law

Printed page 14 · Vectors and components / Polynomial identities

(Continuation) Write an equation that expresses the relationship between the side lengths, a and b, and the diagonals, d_1 and d_2. One method is to use the Law of Cosines.

1. Name the included angle

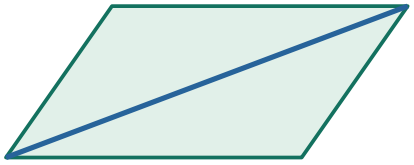
Let two neighboring side lengths be a and b and let C be the angle between them. One diagonal is opposite C, the other opposite the supplementary angle $180°-C$.



The opposite sides are parallel and have equal lengths.

2. Apply the cosine rule twice

The first diagonal has square $a^2+b^2-2ab\cos C$. Since the cosine of the supplement is $-\cos C$, the other diagonal has square $a^2+b^2+2ab\cos C$.

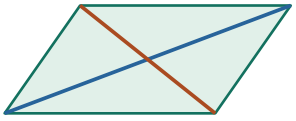


The first diagonal adds the side vectors.

Its square uses a positive $2ab\cos C$ term.

3. Remove the angle by addition

The cosine terms cancel. Thus $d_1^2+d_2^2=2a^2+2b^2$ for any parallelogram with these side lengths.



The second diagonal subtracts the vectors.

Adding the squares cancels the cosine terms.

1. $d_1^2+d_2^2=(a^2+b^2-2ab\cos C)+(a^2+b^2+2ab\cos C)$
Add corresponding squared diagonal formulas before simplifying.

2. $=a^2+a^2+b^2+b^2=2(a^2+b^2)$
The two included-angle terms have equal magnitude and opposite signs.

Check. For a rectangle the two diagonals each have square a^2+b^2.

Exercise 152 · Row-by-column products

Printed page 14 · Matrix multiplication / The dot product

Multiplying two matrices consists of calculating several dot products and then arranging them to form a new matrix. There is a natural way to arrange these dot products, each of which combines a row vector from the first matrix and a column vector from the second matrix. Use the example as a model to calculate the matrix products that follow.

$$\text{Example:}\quad \begin{bmatrix}1 & 2\\ 3 & 4\end{bmatrix}\begin{bmatrix}5 & 6 & 7\\ 8 & 9 & 0\end{bmatrix}=\begin{bmatrix}21 & 24 & 7\\ 47 & 54 & 21\end{bmatrix}$$

$$\textbf{(a)}\ \begin{bmatrix}2 & -3\\ 3 & 5\end{bmatrix}\begin{bmatrix}4\\ -1\end{bmatrix}\qquad \textbf{(b)}\ \begin{bmatrix}4 & 3\\ -2 & 5\end{bmatrix}\begin{bmatrix}6 & 0\\ -3 & 4\end{bmatrix}\qquad \textbf{(c)}\ \begin{bmatrix}2 & -3 & 6\end{bmatrix}\begin{bmatrix}4\\ 4\\ -7\end{bmatrix}$$

1. Read (a) row by column

Dot the first row (2,−3) with (4,−1) to get 8+3=11. Dot the second row (3,5) with the same column to get 12−5=7. The product is the column (11,7)ᵀ.

2	-3
3	5

Dot row (2,−3) with (4,−1) to obtain 11.

The next row yields 7.

2. Build the first product row in (b)

The two columns on the right are (6,−3) and (0,4). Dot (4,3) with them to obtain 24−9=15 and 0+12=12.

15	12
?	?

Use the first row of the left matrix twice, once against each column.

The first output row is (15,12).

3. Build the second row in (b)

Dot $(-2,5)$ with those same columns: $-12-15=-27$ and $0+20=20$. The resulting matrix has rows $(15,12),(-27,20)$.

15	12
-27	20

Now move to the second row of the left matrix.

Its two dot products give $(-27,20)$.

4. Compute the one entry in (c)

A 1-by-3 row times a 3-by-1 column gives a 1-by-1 result: $2\cdot4+(-3)\cdot4+6\cdot(-7)=8-12-42=-46$.

2	-3	6
4	4	-7
8	-12	-42

For (c), sum the three component products.

$8-12-42=-46$.

Check. The respective output shapes are 2×1, 2×2 and 1×1, matching the outer dimensions.

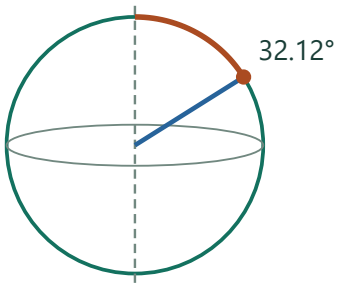
Exercise 153 · Distance from the pole

Printed page 14 · Spherical coordinates / Arc length

Describe all the points on the earth's surface that are exactly 4000 miles from the North Pole. Remember that the radius of the earth is 3960 miles.

1. Choose the surface route from the pole

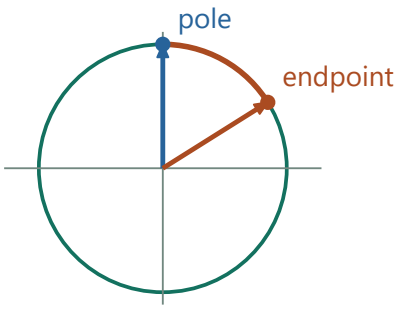
Each meridian is a great circle, giving a shortest route from the North Pole south along that longitude. Every point at the same angular distance down a meridian has the same latitude.



Start at the North Pole and travel south along any meridian for 4,000 surface miles.

2. Convert 4000 miles to a central angle

For surface distance $s=R\theta$, so $\theta=4000/3960$ radians$\approx 57.875°$. This is the angle down from the pole, called colatitude.



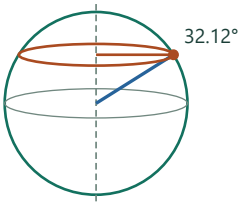
The central angle equals 4000/3960 radians, about 57.875°.

This is measured down from the north polar axis.

3. Identify the parallel of latitude

Subtract from 90° to obtain about 32.125° north. Allowing every longitude gives an entire circle of points. If the distance were a straight chord through Earth, its angular measure would instead satisfy $4000=2\cdot 3960\sin(\theta/2)$.

Latitude · 32.125°



At latitude 32.125° the radius of the parallel is 3353.684 mi.

Its circumference is 0.847 times the equatorial circumference.

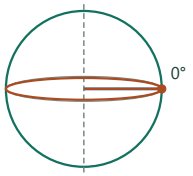
1. $c=2R\sin(\varphi/2)$
For a straight chord, bisect the central isosceles triangle; φ is its central angle.

2. $\varphi=2\arcsin[4000/(2\cdot 3960)]$
Divide half the chord by R and double the resulting half-angle.

3. latitude$=90°-\varphi$
Subtract the colatitude in degrees from the latitude of the North Pole.

Compare settings · Latitude

0°

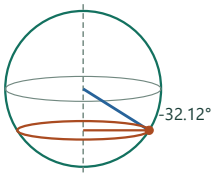


At latitude 0° the radius of the parallel is 3960 mi.

Its circumference is 1 times the equatorial circumference.

At latitude 0° the radius of the parallel is 3960 mi. Its circumference is 1 times the equatorial circumference.

−32.125°



At latitude −32.125° the radius of the parallel is 3353.684 mi.

Its circumference is 0.847 times the equatorial circumference.

At latitude −32.125° the radius of the parallel is 3353.684 mi. Its circumference is 0.847 times the equatorial circumference.

Check. Rotations around the polar axis preserve both distance and latitude.

Exercise 154 · Angles from dot products

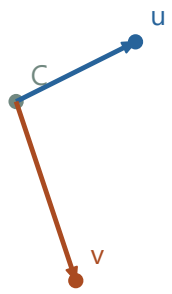
Printed page 14 · The dot product / Inverse trigonometric functions

In each of the following, find the angle formed by $\mathbf{u}$ and $\mathbf{v}$:

(a) $\mathbf{u}=[2,1]$ and $\mathbf{v}=[1,-3]$ **(b)** $\mathbf{u}=[-1,0,1]$ and $\mathbf{v}=[0,2,-2]$

1. Prepare the angle formula

For nonzero vectors u,v, cos θ=(u·v)/(‖u‖‖v‖). Use the component dot product for the numerator and the lengths of the individual vectors for the denominator.

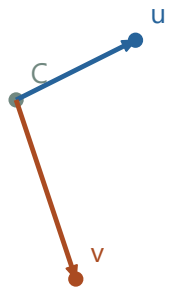


Measure the common-tail angle from the directions.

Normalize the dot product by the two lengths.

2. Apply it to the planar pair

For (2,1) and (1,−3), u·v=2−3=−1 and the lengths are √5 and √10. Thus θ=arccos(−1/√50)≈98.13°.

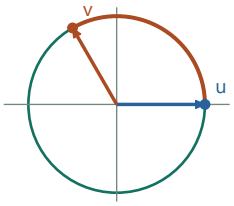


u·v=−1; ‖u‖‖v‖=√50.

The obtuse angle is about 98.13°.

3. Apply it in three dimensions

For (−1,0,1) and (0,2,−2), the dot product is −2 and the lengths are √2 and √8. The cosine is −2/4=−1/2, giving θ=120°. Both negative dot products correctly produce obtuse angles.



Their normalized dot product is −1/2.

The actual angle between the spatial directions is 120°.

1. u·v=(−1)0+0·2+1(−2)=−2
Multiply matching spatial components including their signs.

2. ‖u‖‖v‖=√(1+0+1)√(0+4+4)=4
Multiply the two positive lengths.

3. cos θ=−2/4=−1/2 ⇒ θ=120°
Take the angle in the interval from 0° to 180°.

Check. The negative dot product in each case requires an obtuse angle.

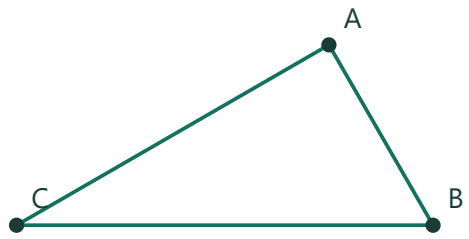
Exercise 155 · Double an angle, double a side?

Printed page 14 · The sine rule / Trigonometric identities

Given a triangle ABC in which angle B is exactly twice the size of angle C, must it be true that side AC is exactly twice the size of side AB? Could it be true?

1. Match the two sides with their opposite angles

By the sine rule, b/c=sin B/sin C. Since B=2C this becomes sin(2C)/sin C.

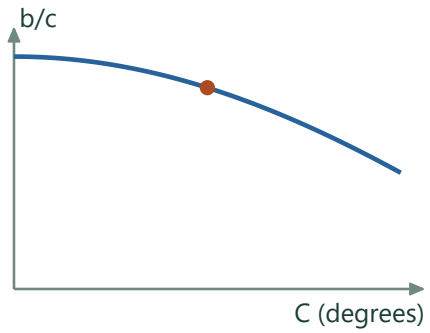


A triangle can have B = 2C.

Opposite side lengths depend on the sines of the angles.

2. Use the double-angle identity

sin(2C)=2 sin C cos C, so b/c=2 cos C. The cancellation is legal because C is a positive interior angle.



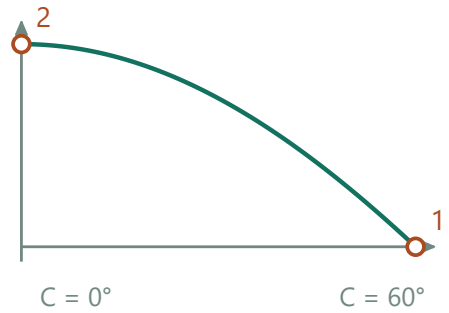
sin(2C)/sin C = 2 cos C.

The factor varies as C varies.

1. b/c=sin(2C)/sin C	Pair each side with its opposite angle in the sine rule.
2. =2 sin C cos C / sin C=2 cos C	The interior angle satisfies sin C>0, allowing cancellation.

3. Bound the actual ratio

B+C=3C must be less than 180°, so 0<C<60°. Then 1/2<cos C<1, giving 1<b/c<2. Doubling an angle does not exactly double its opposite side in any nondegenerate example.



C runs strictly between 0° and 60°.

b/c = 2 cos C drops from 2 toward 1. Hollow endpoints exclude both limiting triangles.

1. A=180°−(C+2C)=180°−3C>0	A nondegenerate triangle must leave a positive third angle.
2. 0<C<60° ⇒ 1/2<cos C<1	Cosine decreases through this acute interval.
3. 1<b/c<2	Double both cosine bounds for the side ratio.

Check. As C tends to zero the ratio approaches 2 but never attains it.

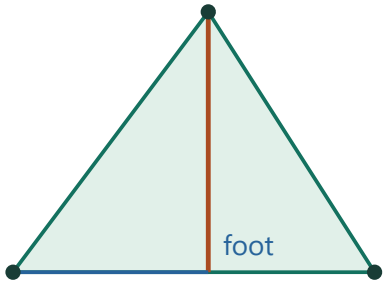
Exercise 156 · Two triangles with area 36

Printed page 14 · The ambiguous SSA case / Triangle area

There are two noncongruent triangles that have a 9-inch side, a 10-inch side, and that enclose 36 square inches of area. Find the length of the third side in each of these triangles.

1. Find the included angle's sine

With sides 9 and 10 and included angle C, area=½·9·10 sin C=45 sin C. Set this to 36, giving sin C=4/5.

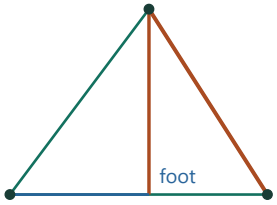


Area = ½·10·9 sin C.

The required 36 in^2 fixes sin C at 4/5.

2. Keep both angle possibilities

The same positive sine can occur for an acute angle or its obtuse supplement. From $\cos^2 C=1-(4/5)^2=9/25$, cos C can be 3/5 or −3/5.



With acute C the signed projection is positive.

cos C=3/5 gives $c^2=73$.

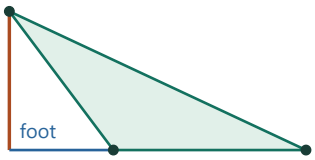
1. $\cos^2 C=1-(4/5)^2=9/25$
The unit-circle identity fixes the square of the cosine.

2. cos C=3/5 or −3/5
The same positive sine occurs at supplementary interior angles.

3. Compute both third sides

The cosine rule gives $c^2=9^2+10^2-2\cdot9\cdot10\cos C=181-180\cos C$. The acute angle gives $c^2=73$; the obtuse gives $c^2=289$. The possible lengths are √73 and 17 in.

Included angle · 126.87°



Base 10 in, sloping side 9 in.

Altitude 7.2 in; triangle area 36 in^2; opposite triangle side 17 in.

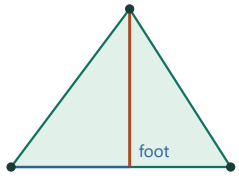
1. $c^2=181-180(3/5)=181-108=73$
The positive cosine gives the smaller squared side.

2. $c^2=181-180(-3/5)=181+108=289$
The negative choice changes subtraction into addition.

3. c=√73 or 17
Keep only positive physical lengths for each case.

Compare settings · Included angle

53.13°

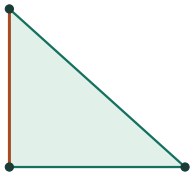


Base 10 in, sloping side 9 in.

Altitude 7.2 in; triangle area 36 in^2; opposite triangle side 8.544 in.

Base 10 in, sloping side 9 in. Altitude 7.2 in; triangle area 36 in^2; opposite triangle side 8.544 in.

90°



Base 10 in, sloping side 9 in.

Altitude 9 in; triangle area 45 in^2; opposite triangle side 13.454 in.

Base 10 in, sloping side 9 in. Altitude 9 in; triangle area 45 in^2; opposite triangle side 13.454 in.

Check. Both satisfy the triangle inequalities, and supplementary included angles give the same area.

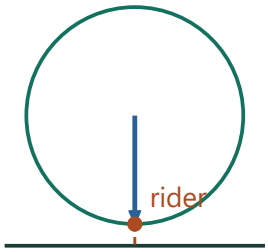
Exercise 157 · Slow Ferris wheel

Printed page 15 · Circular motion / Sinusoidal functions

Centered 6 meters above the ground, a Ferris wheel of radius 5 meters rotates at 1 degree per second. Assuming that Jamie's ride begins at the lowest point on the wheel, find how far Jamie is above the ground after 29 seconds; after 331 seconds; after t seconds.

1. Measure the rider's height relative to the hub

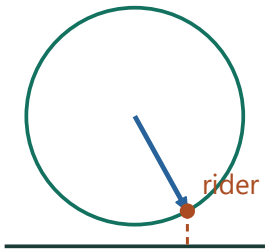
The hub is always 6 m high and the radius is 5 m. At t=0 the rider is 5 m below the hub, so a downward cosine component matches the start.



Start at the bottom, height 6−5=1 m.

2. Turn time into angle

At one degree per second the elapsed angle is t degrees. The vertical component is −5 cos(t°) and total height is h(t)=6−5 cos(t°).



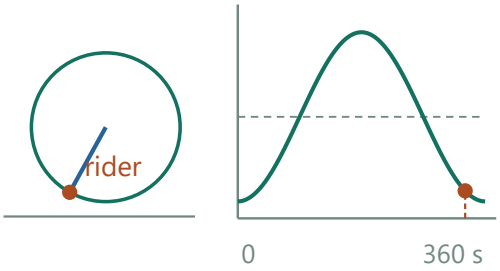
1 second corresponds to one degree.

At 29 s the radius is 29° from downward.

3. Evaluate the two times

The angles 29° and 331°=360°−29° have identical cosines. Hence at both times h=6−5 cos 29°≈1.627 m.

Wheel position · 331°

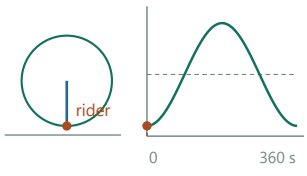


At t=331 s, the rider is 1.627 m high.

The graph dot and wheel rim mark the same instant. The 6-m midline is hub height.

Compare settings · Wheel position

0°

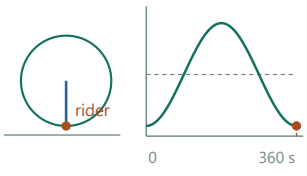


At t=0 s, the rider is 1 m high.

The graph dot and wheel rim mark the same instant. The 6-m midline is hub height.

At t=0 s, the rider is 1 m high. The graph dot and wheel rim mark the same instant. The 6-m midline is hub height.

360°



At t=360 s, the rider is 1 m high.

The graph dot and wheel rim mark the same instant. The 6-m midline is hub height.

At t=360 s, the rider is 1 m high. The graph dot and wheel rim mark the same instant. The 6-m midline is hub height.

Check. The rider starts at 1 m and reaches 11 m halfway through a 360-s revolution.

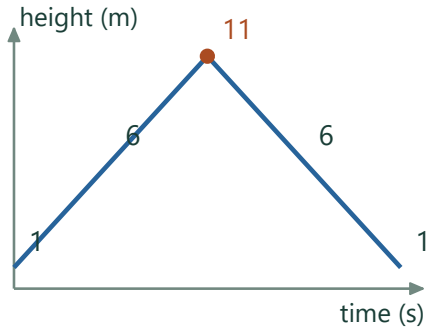
Exercise 158 · Graphing ride height

Printed page 15 · Sinusoidal functions / Graph interpretation

(Continuation) Graph the equation $y = 6 - 5\cos x$. What does this picture tell you about Jamie's ride? Would a graph of $y = 6 + 5\cos x$ mean anything?

1. Use quarters of a revolution as landmarks

The slow wheel turns 1° per second, so one turn lasts 360 s. For h=6−5 cos x, the heights at x=0,90,180,270,360 are 1,6,11,6,1 m.

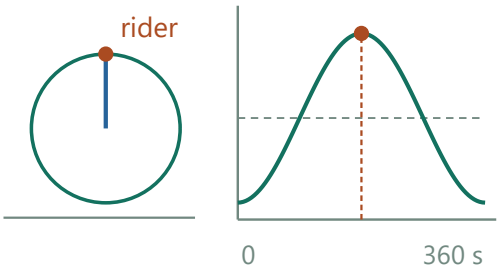


Mark quarter-turn heights before sketching the smooth curve.

2. Sketch the continuous motion

Place elapsed seconds on the horizontal axis and height on the vertical axis. Join the landmarks smoothly. The midline h=6 represents the hub and the 5-m amplitude is the radius.

Wheel position · 180°

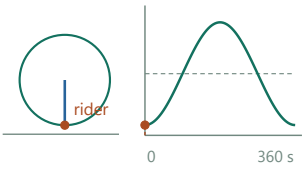


At t=180 s, the rider is 11 m high.

The graph dot and wheel rim mark the same instant. The 6-m midline is hub height.

Compare settings · Wheel position

0°

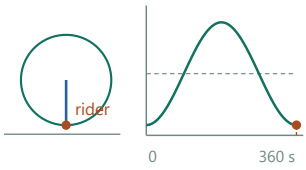


At t=0 s, the rider is 1 m high.

The graph dot and wheel rim mark the same instant. The 6-m midline is hub height.

At t=0 s, the rider is 1 m high. The graph dot and wheel rim mark the same instant. The 6-m midline is hub height.

360°



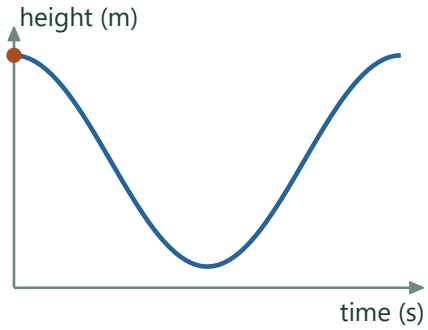
At t=360 s, the rider is 1 m high.

The graph dot and wheel rim mark the same instant. The 6-m midline is hub height.

At t=360 s, the rider is 1 m high. The graph dot and wheel rim mark the same instant. The 6-m midline is hub height.

3. Interpret the change of sign

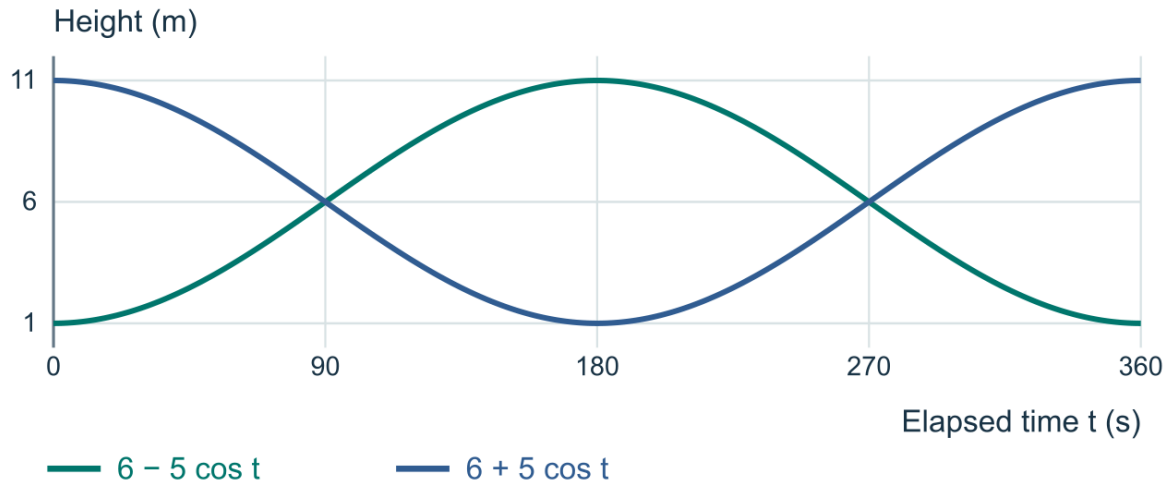
The graph of 6+5 cos x starts at 11 m and reaches 1 m at 180 s. It represents a rider half a turn ahead, at the top when the first begins at the bottom. Heights at the same time sum to 12 m.



Changing the sign starts the rider at the top.

This is a half-turn phase shift.

1. $h(x+180°)=6-5\cos(x+180°)=6+5\cos x$	After half a turn the radius points in the opposite direction and the cosine changes sign.
2. $h(x)+h(x+180°)=(6-5\cos x)+(6+5\cos x)=12$	Opposite riders' vertical displacements from the hub cancel.



Check. The two heights at equal times always sum to 12 m, twice the hub height.

Exercise 159 · Law of cosines for vectors

Printed page 15 · The cosine rule / The dot product

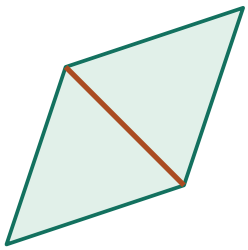
Draw nonzero vectors $\mathbf{u}$ and $\mathbf{v}$ tail-to-tail so that they make a θ-degree angle. Draw the vector $\mathbf{u} - \mathbf{v}$, the third side of the triangle, and check to see that it points in the right direction.

(a) Solve for $\cos\theta$ using the SSS version of the Law of Cosines, expressing all lengths in terms of $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{u} - \mathbf{v}$.

(b) If you use vector algebra to simplify the numerator as much as possible, you will discover an interesting new result connecting $\mathbf{u}\bullet\mathbf{v}$ to $\cos\theta$.

1. Build the triangle from the two vectors

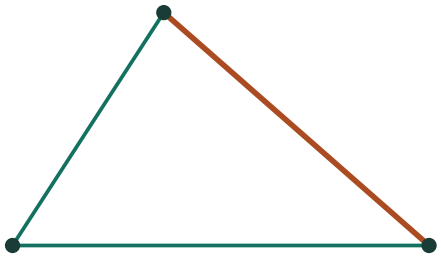
Place u and v tail to tail. The arrow from v's head to u's head is u−v, opposite the angle θ between them. Denote the three lengths by U=‖u‖, V=‖v‖, D=‖u−v‖.



Joining the heads makes a triangle with sides U, V and D=‖u−v‖.

2. Find the angle from the three side lengths

For $U,V>0$ the cosine rule yields $D^2=U^2+V^2-2UV\cos\theta$. Solving gives $\cos\theta=(U^2+V^2-D^2)/(2UV)$.



The included angle and three sides obey the ordinary cosine rule.

3. Express the same D^2 by a dot product

Distribute $(u-v)\cdot(u-v)$ to get $U^2+V^2-2u\cdot v$. Substitution in the angle formula cancels the self-products and gives $\cos\theta=(u\cdot v)/(UV)$, or $u\cdot v=UV\cos\theta$.

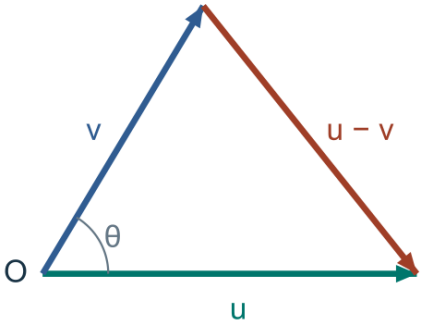
U^2	$-2u\cdot v$	V^2

Compare D^2 computed by dot product and by the cosine rule.

$u\cdot v = UV\cos\theta$.

1. $D^2=U^2+V^2-2u\cdot v=U^2+V^2-2UV\cos\theta$
The dot-product expansion and the cosine rule measure the same side.

2. $-2u\cdot v=-2UV\cos\theta \Rightarrow u\cdot v=UV\cos\theta$
Remove the equal squares from both sides, then divide by -2.



The arrows u and v share their tail.

The third arrow runs from v's head to u's head, so v + (u − v) = u.

Check. Parallel same-direction vectors give $\theta=0$ and dot product UV.

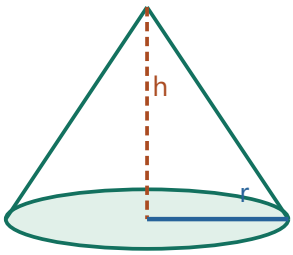
Exercise 160 · Lateral area of a cone

Printed page 15 · Surface area / Nets and surface development

A paper cone has an ℓ-inch slant height and an r-inch base radius. In terms of the quantities ℓ and r, write a formula for the *lateral area* of the cone. In other words, find the area of the circular sector obtained by cutting the cone from base to vertex and flattening it out.

1. Unroll the lateral surface

Cut the cone along one straight generator and lay it flat. The tip is now the center of a sector. Distance along the original generator becomes sector radius ℓ.

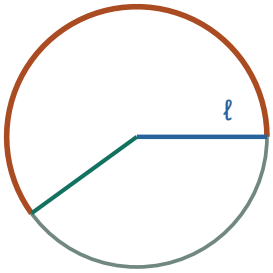


Cone radius r; perpendicular height h.

The slant edge runs from rim to vertex.

2. Trace the base rim

The curved sector edge was the circular base rim, so its arc length is $2\pi r$. Unrolling preserves the length of that edge and the area of the side.

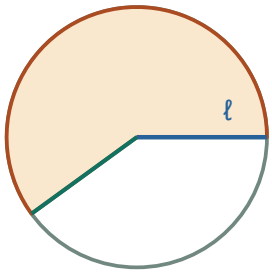


Unroll the lateral surface without stretching.

Its outer arc is exactly the old rim, length $2\pi r$.

3. Use the sector area formula

The area is $\tfrac{1}{2}(\text{radius})(\text{arc}) = \tfrac{1}{2}\ell \cdot 2\pi r = \pi r\ell$. This counts the side only; the base disk has its own area πr^2.



Area of the sector = ½·slant height·rim circumference.

This gives $\pi r\ell$.

Check. Its fraction of the full disk is r/ℓ, giving the same area.

Exercise 161 · Compatible two-matrix products

Printed page 15 · Matrix multiplication / Matrix dimensions

Let $\mathbf{M} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, $\mathbf{N} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$, and $\mathbf{P} = \begin{bmatrix} 0 & 1 \end{bmatrix}$. Evaluate all possible two-matrix products.

1. Check the shapes before multiplying

M is 2-by-2, N is a 2-by-1 column, P a 1-by-2 row. Adjacent inner sizes must match. Thus NM and MP are undefined, even though MN and PM are allowed.

M	2×2
N	2×1
P	1×2

Only products whose adjacent dimensions match can be formed.

2. Use zero and one as selectors

MN selects the second column (b,d)ᵀ. PM selects the second row (c,d). Multiplication order therefore matters both for values and for output shape.

a	b
c	d

N selects the second column of M.

P on the left selects the second row.

3. Multiply N and P in both orders

NP is an outer product with rows (0,0),(0,1). PN is the one-entry row-column product [0·0+1·1]=[1].

0	0
0	1

NP is a two-by-two matrix; PN is the one-entry matrix [1].

4. Check the repeated selections

If repeated factors are included, M^2 has rows $(a^2+bc, ab+bd), (ac+cd, bc+d^2)$ by row-column products. NN and PP have incompatible inner dimensions.

a	b
c	d

For M^2, form four ordinary row-by-column dot products.

1. row 1 of M^2=(a·a+b·c, a·b+b·d)
Dot the first row of M with each column of M in order.

2. row 2 of M^2=(c·a+d·c, c·b+d·d)
Repeat using the second row; each row-column dot product supplies one entry.

Check. Multiplying by a standard-basis column selects a column; multiplying on the left by its transpose selects a row.

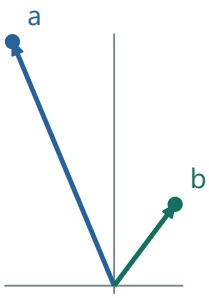
Exercise 162 · Projection onto (−5,12)

Printed page 15 · Vector projections / The dot product

Find the angle formed when $[3, 4]$ and $[-5, 12]$ are placed tail-to-tail, then find components for the *vector projection* that results when $[3, 4]$ is projected onto $[-5, 12]$.

1. Calculate the shared dot product

For u=(3,4) and v=(−5,12), u·v=−15+48=33. Their lengths are √25=5 and √169=13.

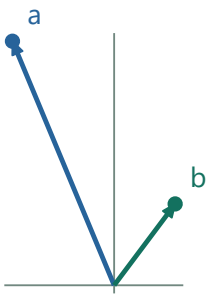


Blue target a=(−5,12); green input b=(3,4).

The dot product is 33.

2. Determine their angle

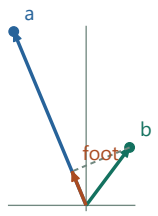
cos θ=33/(5·13)=33/65. Taking inverse cosine gives θ≈59.49°, an acute angle as expected from a positive dot product.



The included angle satisfies cos θ=33/(13·5), giving 59.49°.

3. Construct the projection vector

The projection of u onto v is some multiple kv, and the residual u−kv must be perpendicular to v. Thus 0=(u−kv)·v=33−169k, so k=33/169. The projection is (−165/169,396/169).



Drop a perpendicular to the target line.

The projected vector is (33/169)(−5,12).

1. $k(u\cdot u)=v\cdot u$
The residual v−ku must have zero dot product with projection direction u.

2. $k=33/[(-5)^2+12^2]=33/169$
Divide by the squared length of the nonzero direction.

3. ku=(−165/169,396/169)
Multiply both coordinates of (−5,12) by the scalar k.

Check. Subtracting the projection from (3,4) leaves a vector perpendicular to (−5,12).

Exercise 163 · Dimension rule

Printed page 15 · Matrix dimensions / Matrix multiplication

Two matrices can be multiplied only if their sizes are compatible. Suppose that $\mathbf{U}$ is an $m \times n$ matrix, and that $\mathbf{V}$ is a $p \times q$ matrix. In order for the matrix product $\mathbf{UV}$ to make sense, what must be true about the dimensions of these matrices?

1. Ask what one output entry requires

Entry (i,j) of UV is the dot product of row i of U with column j of V. The row has n entries and the column p entries.

u_1	u_2	u_3
v_1	v_2	v_3

One row of U must pair entry for entry with one column of V (shown horizontally).

2. Match the middle dimensions

A dot product pairs entries one by one, so it exists exactly when n=p. There is no need for the outside dimensions to agree.

m × n
p × q

The pairings exist if the inner counts n and p agree.

3. Read the output shape

One entry is produced for each pairing of an original row of U and column of V. There are m rows and q columns, giving an m-by-q product.

1,1	1,2	…
2,1	2,2	…

Use one result per row-column pair.

The output has m rows and q columns.

Check. For example, a 2×3 times a 3×4 matrix makes eight length-3 dot products.

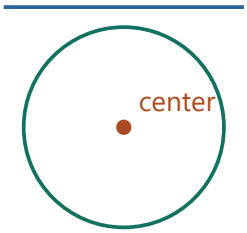
Exercise 164 · Sphere and the ground plane

Printed page 15 · Sphere geometry / Planes in space

A sphere consists of all the points that are 5 units from its center $(2, 3, -6)$. Write an equation that describes this sphere. Does the sphere intersect the xy-plane? Explain.

1. Write the three displacements from the center

From (2,3,−6) to a general point (x,y,z), the differences are x−2, y−3, and z+6. Their squared sum gives the squared radius for points on the sphere.

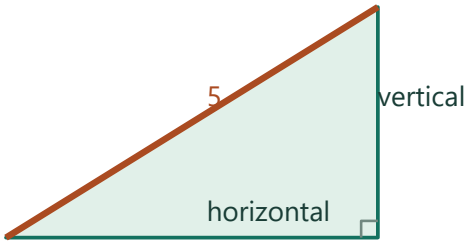


Vertical section through (2,3,−6).

The ground plane appears as the blue horizontal line.

2. Impose radius 5

The equation is $(x-2)^2+(y-3)^2+(z+6)^2=25$.

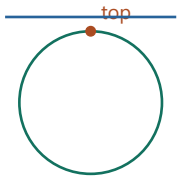


Schematic section

Squared distance to the center equals 25.

3. Test the horizontal plane

The center lies 6 units below z=0 and the radius reaches only 5 units. Equivalently, setting z=0 would force $(x-2)^2+(y-3)^2=25-36=-11$. The sphere misses that plane.



The top reaches z=−1, one unit below the ground.

The plane z=0 misses the sphere.

1. $(x-2)^2+(y-3)^2+6^2=25$
Set z=0 in the spatial sphere equation.

2. $(x-2)^2+(y-3)^2=-11$
Subtract 36. A sum of two real squares cannot be negative.

Check. Substituting z=0 would require the first two squares to sum to −11.

Exercise 165 · Four parameterizations

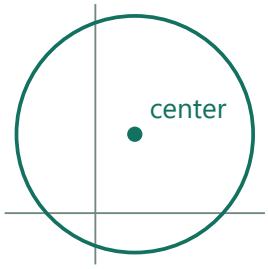
Printed page 15 · Parametric equations / Circular motion

The curve $(x-1)^2+(y-2)^2=9$ can be parameterized in many different ways. Investigate the differences in the following.

(a) $(x,y)=(1+3\cos t, 2+3\sin t)$
(b) $(x,y)=(1+3\sin t, 2+3\cos t)$
(c) $(x,y)=(1-3\sin t, 2+3\cos t)$
(d) $(x,y)=(1+3\cos 2t, 2+3\sin 2t)$

1. Identify the common path

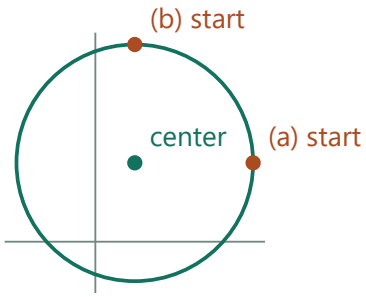
Each pair is (1,2) plus a displacement of squared length 9, because $\sin^2+\cos^2=1$. All four motions trace the radius-3 circle about (1,2). Compare where each starts and where it heads next.



All four parametrizations share center (1,2) and radius 3.

2. Read (a) and (b)

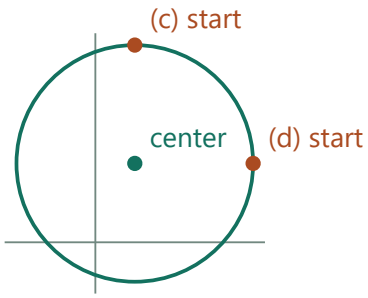
In (a) the displacement at t=0 is (3,0) and then its vertical component rises: the point starts at (4,2) and moves counterclockwise. In (b) it starts at (1,5) and moves right from the top, hence clockwise.



(a) begins at the right moving up; (b) at the top moving right.

3. Read (c) and (d)

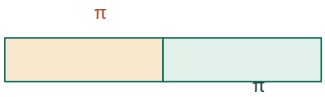
Changing the horizontal sine to negative in (c) makes the point start at (1,5) and move left, counterclockwise. Doubling the angle in (d) leaves the start at (4,2) and upward direction unchanged, but makes it turn twice as fast.



(c) starts leftward from the top; (d) starts upward from the right at double angular speed.

4. Compare periods

The first three need t to increase by 2π radians or 360 degree units to repeat. In (d) 2t must increase that amount, so t increases by π or 180 units.



In radian mode the first three periods are 2π.

The double-speed fourth takes π.

1. Δt=2π for t; 2Δt=2π for 2t
A full revolution requires the argument to change by 2π radians.

2. Δt=π for the doubled argument
Divide by the angular multiplier; in degrees divide 360° by 2 instead.

Check. After subtracting the center, each squared coordinate sum is 9.

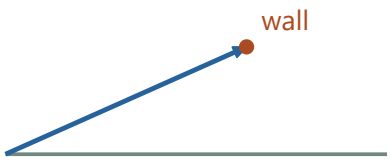
Exercise 166 · Two possible property areas

Printed page 15 · The ambiguous SSA case / Triangle area using sine

Sam owns a triangular piece of land on which the tax collector wishes to determine the correct property tax. Sam tells the collector that "the first side lies on a straight section of road and the second side is a stone wall. The wall meets the road at a 24-degree angle. The third side of the property is formed by a 180-foot-long fence, which meets the wall at a point that is 340 feet from the corner where the wall meets the road." After a little thought, the tax collector realizes that Sam's description of his property is ambiguous, because there are still two possible lengths for the first side. By means of a clear diagram, explain this situation, and calculate the two possible areas, to the nearest square foot.

1. Mark the road and the wall end

Use the road as x-axis and begin the wall at (0,0). Its endpoint 340 ft away at 24° is (m,h)=(340 cos 24°,340 sin 24°)≈(310.61,138.29). This fixed height is the altitude of either triangle.

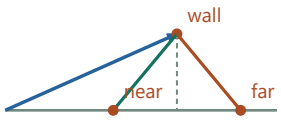


The fixed 340-ft wall meets the road at 24°.

2. Locate both possible fence ends

The far end of the 180-ft fence lies on the road and 180 ft from (m,h). At road coordinate x its squared distance is $(x-m)^2+h^2=180^2$. Because $h<180$, there are two horizontal offsets, $\pm\sqrt{(180^2-h^2)}$.

Fence length · 180 ft



The wall end is 138.29 ft above the road.

Road positions x ≈ 195.385 and 425.826 ft.

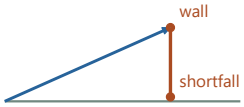
1. $(x-m)^2=180^2-h^2$
Remove the constant squared height from the distance equation.

2. $x-m=\pm\sqrt{(180^2-h^2)}$
Both signs are possible because the fence reaches the road on either side of its perpendicular foot.

3. $x=m\pm\sqrt{(180^2-h^2)}$
Add m to both sides to obtain the two base endpoints.

Compare settings · Fence length

130 ft



The wall end is 138.29 ft above the road.

A fence this short cannot reach it.

The wall end is 138.29 ft above the road. A fence this short cannot reach it.

220 ft



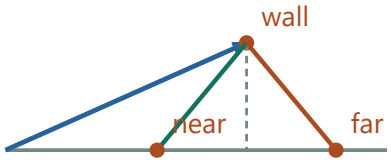
The wall end is 138.29 ft above the road.

Road positions x ≈ 139.504 and 481.707 ft.

The wall end is 138.29 ft above the road. Road positions x ≈ 139.504 and 481.707 ft.

3. Draw two distinct bases

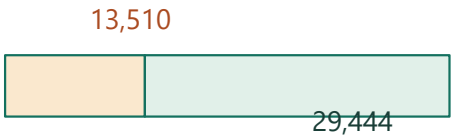
The road intersections occur near (195.39,0) and (425.83,0). Both x values are positive. Join each point to the wall end to see a shorter and a longer possible triangular base.



The two road coordinates are about 195.39 ft and 425.83 ft.

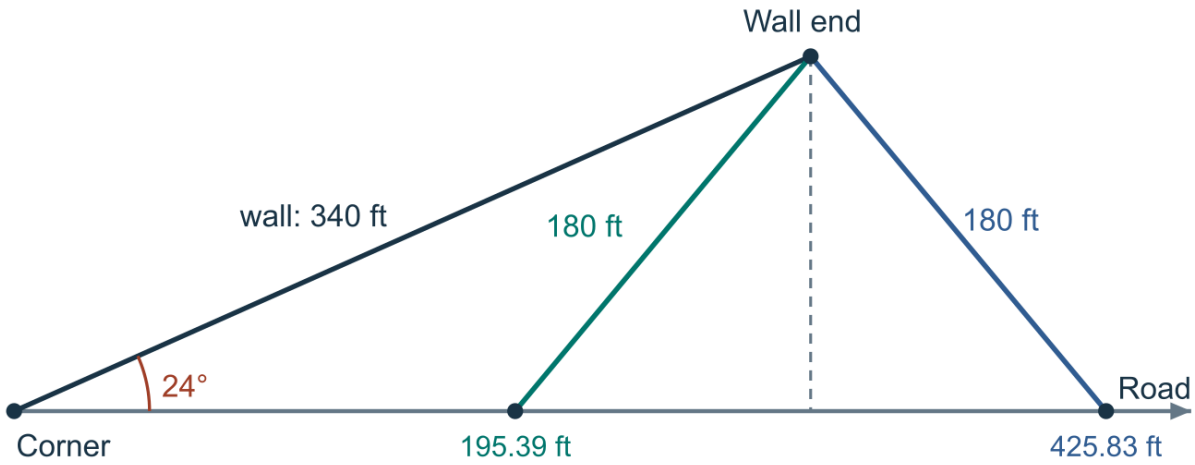
4. Compute the two areas

Each base begins at (0,0), so its length is x, and its altitude is h. Using A=xh/2 before rounding gives approximately 13,510 ft² and 29,444 ft².



Both triangles have height about 138.29 ft.

Areas differ because the road bases differ (ft²).



Check. Using either road point gives wall-to-road-point distance exactly 180 ft.

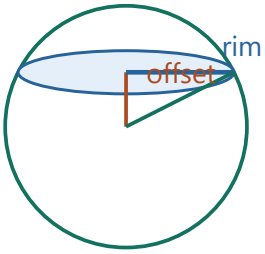
Exercise 167 · Sphere in standard form

Printed page 16 · Sphere geometry / Distance and loci

What does the graph of $(x-a)^2+(y-b)^2+(z-c)^2=r^2$ look like?

1. Recognize a distance formula

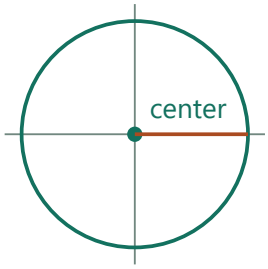
For P=(x,y,z), the differences x−a, y−b and z−c are the perpendicular coordinate components of its displacement from C=(a,b,c). Their squared sum equals CP².



Every radial segment measures the same distance abs(r) from the center.

2. Translate the equation into a locus

$CP^2=r^2$ means $CP=|r|$. Every direction from C at that distance gives a point on the sphere's surface, and every solution must be on that surface.



A plane through the center cuts the sphere in a circle of radius abs(r).

3. Check the zero-radius case

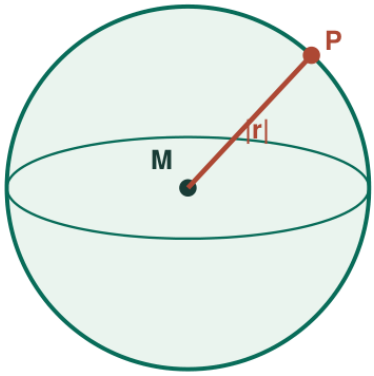
If r=0, all three coordinate differences must vanish. The locus then consists only of the center.



At r=0 all displacement components must vanish.

Only (a,b,c) remains.

A fixed distance in three dimensions



Choose a center and a point

M = (a, b, c) P = (x, y, z)

$MP^2 = (x-a)^2 + (y-b)^2 + (z-c)^2$.

Hold MP fixed

Turning the radius in every direction sweeps the surface of a sphere with radius |r|.

r = 0

Only the center then has the required distance.

Check. A coordinate plane through its center cuts a radius-abs(r) circle.

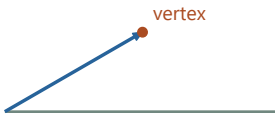
Exercise 168 · SSA cases

Printed page 16 · The ambiguous SSA case / The sine rule

Assuming $q = 6$ and angle $P = 30$ degrees, for each of the following, tell how many noncongruent triangles PQR fit the given description, and find the size of angle Q.
(a) $p = 4$ **(b)** $p = 8$ **(c)** $p = 2$ **(d)** $p = 3$

1. Turn the SSA data into a sine test

The sine rule gives sin Q=q sin P/p=6 sin 30°/p=3/p. A possible Q also needs 0<Q<150° so that the remaining angle is positive.



The fixed side of 6 makes angle P=30° with the base.

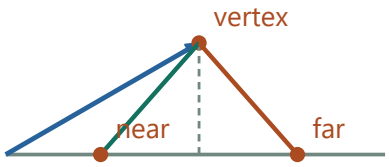
Its tip stands 3 units above that base.

1. sin Q/q=sin P/p
Match the given side-angle pair in the sine rule.

2. sin Q=(q sin P)/p=3/p
The known opposite-side and sine product is 3; p must be positive.

2. Test p=4

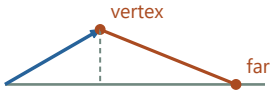
Sin Q=3/4 gives Q≈48.59° or its supplement 131.41°. Each is below 150°, giving two triangles.



With p=4 the side reaches two points on the positive base ray.

3. Test p=8

Sin Q=3/8 gives Q≈22.02° or 157.98°. The supplement added to P=30° already exceeds 180°, leaving only the acute Q.



With p=8 the other intersection lies behind the vertex.

Only one valid triangle remains.

1. Q_1=arcsin(3/8)≈22.02°
Use the acute solution from inverse sine.

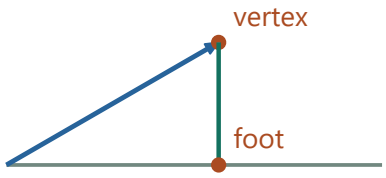
2. Q_2=180°−Q_1≈157.98°
The supplement has the same sine.

3. P+Q_2≈30°+157.98°>180°
The supplementary solution cannot leave a third angle.

4. Test p=2 and p=3

For p=2, 3/p=1.5 is outside the possible sine range. There is no triangle. For p=3, sin Q=1 forces Q=90°, giving one right triangle.

Opposite side p · 3 units

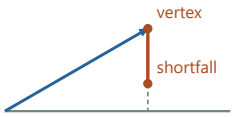


Fixed perpendicular distance = 3 units.

The side reaches one perpendicular foot: one right triangle.

Compare settings · Opposite side p

2 units

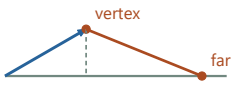


Fixed perpendicular distance = 3 units.

The side cannot reach the base ray: no triangles.

Fixed perpendicular distance = 3 units. The side cannot reach the base ray: no triangles.

8 units



Fixed perpendicular distance = 3 units.

One intersection is behind the ray origin: one triangle.

Fixed perpendicular distance = 3 units. One intersection is behind the ray origin: one triangle.

Check. The altitude threshold is q sin P=3. The two-solution case requires 3<p<6.

Exercise 169 · Identify four transformations

Printed page 16 · Matrix transformations / Coordinate transformations

Describe the effect of each of the following geometric transformations. To generate and test your hypotheses, transform some simple points.

(a) $\mathcal{T}(x, y) = (-3x, -3y)$

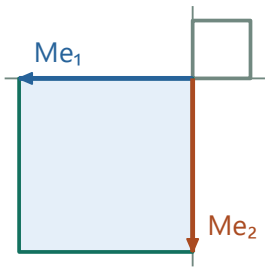
(b) $\mathcal{T}(x, y) = (-y, x)$

(c) $\mathcal{T}(x, y) = (-y, -x)$

(d) $\mathcal{T}(x, y) = (0.6x - 0.8y, 0.8x + 0.6y)$

1. Track both coordinate axes in (a) and (b)

Multiplying every coordinate by -3 reverses each displacement and triples its length: a half-turn and scale factor 3 at the origin. The map $(-y,x)$ sends right to up and up to left, a counterclockwise quarter-turn.

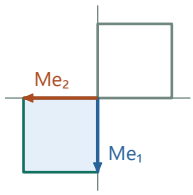


Map (a) reverses both axes and triples length.

(x,y) goes to $(-3x,-3y)$.

2. Find the mirror in (c)

Under $(x,y)\mapsto(-y,-x)$ the points with $y=-x$ stay fixed. Right moves to down and up moves to left, exactly the reflection across that diagonal.



Reflection in the descending diagonal $y=-x$ sends (x,y) to $(-y,-x)$.

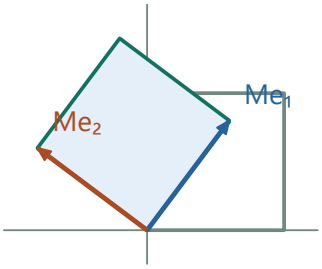
1. $P=(x,y)$, $P'=(-y,-x)$
Apply the stated coordinate rule.

2. midpoint$=((x-y)/2,(y-x)/2)$
The midpoint coordinates are negatives and thus lie on $y=-x$.

3. $P'-P=(-x-y,-x-y)$
Equal coordinate differences point along a normal to $y=-x$, verifying reflection.

3. Read the rotation in (d)

The images of the unit axes are $(0.6,0.8)$ and $(-0.8,0.6)$. They are perpendicular unit vectors with the second a quarter-turn ahead. The first turns counterclockwise by $\arctan(0.8/0.6)=\arctan(4/3)\approx53.13°$.



For map (d), the two unit columns are perpendicular.

The image square turns through $\arctan(4/3)$.

Check. Test $e_1=(1,0)$ and $e_2=(0,1)$. Their images in that order are (a) $(-3,0),(0,-3)$, (b) $(0,1),(-1,0)$, (c) $(0,-1),(-1,0)$, and (d) $(0.6,0.8),(-0.8,0.6)$. A linear map is determined by these two images.

Exercise 170 · Read off linear coefficients

(Continuation) Each transformation takes the general form $\mathcal{T}(x, y) = (ax+by, cx+dy)$, using suitable constants a, b, c, and d. For example, $a = -3$, $b = 0 = c$, and $d = -3$ in part (a). What are the values of a, b, c, and d for the remaining examples?

1. Fix the reading order

In (ax+by,cx+dy), take first the x- then y-coefficient in the first output; then do the same for the second output. Missing variables have coefficient zero.

a	b
c	d

Read each output in input order x,y.

The first row builds x′, the second y′.

2. Read the two coefficients for each output

For (−3x,−3y) this gives (−3,0,0,−3). For (−y,x) it gives (0,−1,1,0). The signs distinguish the scaling and turn.

-3	0
0	-3

Case (a) has x′=−3x and y′=−3y.

Missing coefficients are zero.

3. Repeat for the remaining maps

The reflection (−y,−x) gives (0,−1,−1,0). The rotation in decimal coefficients gives (0.6,−0.8,0.8,0.6). Applying the coefficients to (1,0) and (0,1) checks the two axis images.

0.6	-0.8
0.8	0.6

For case (d), read across rows: 0.6, −0.8, 0.8, 0.6.

Check. Substituting x=1,y=0 exposes the first and third coefficients; x=0,y=1 gives the other two.

Exercise 171 · Transformation matrices

Printed page 16 · Matrix transformations / Matrix multiplication

(Continuation) Since the expression $ax+by$ is equivalent to the dot product $[a,b]\bullet[x,y]$, matrices can be used to represent these transformations. Explain the connection between the matrix product $\begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}$ and the transformation $\mathcal{T}(x,y)=(ax+by,cx+dy)$. Write the *transformation matrix* $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ for each of the transformations above.

1. Turn coefficients into rows

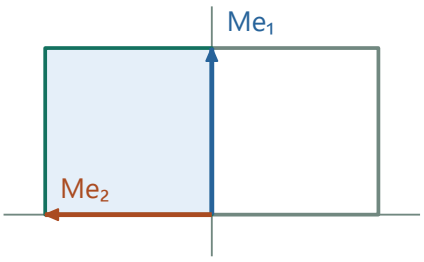
To act on the column (x,y)ᵀ, put the coefficients producing the first output into the top row and the coefficients producing the second into the bottom row. Rows (a,b),(c,d) give (ax+by,cx+dy)ᵀ.

a	b
c	d

Write x′=ax+by as row one and y′=cx+dy as row two.

2. Represent maps (a) and (b)

For the half-turn and expansion the rows are (−3,0),(0,−3). For the quarter-turn they are (0,−1),(1,0). Dotting each row with (x,y) yields the original formulas.

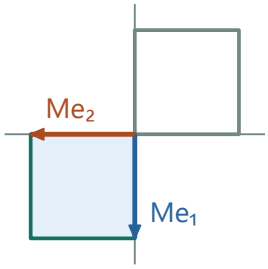


The matrices for (a) and (b) act by −3I and by a counterclockwise quarter-turn respectively.

The image of the square for (b) is shown.

3. Represent maps (c) and (d)

The reflection uses rows $(0,-1),(-1,0)$, and the rotation uses $(0.6,-0.8),(0.8,0.6)$. Each matrix's two columns are its images of the unit coordinate vectors.



Use rows $(0,-1),(-1,0)$ for the reflection illustrated.

For the rotation in (d), the rows are $(0.6,-0.8),(0.8,0.6)$.

Check. Each matrix's columns are the images of the two basis vectors.

Exercise 172 · A spatial projection

Printed page 16 · Vector projections / The dot product

Find the angle formed when $[4,4,2]$ and $[4,3,12]$ are placed tail-to-tail; then find the components of the vector that results when $[4,3,12]$ is projected onto $[4,4,2]$.

1. Name the two arrows and the angle

Let a=(4,4,2) and b=(4,3,12). Move their tails together. The usual angle θ between their directions lies between 0° and 180°. Its cosine is (a·b)/(||a||||b||).



The blue target and green input share a tail.

Axonometric view of the spatial vectors.

2. Compute the numerator and both lengths

Multiply matching components to obtain $a\cdot b=4\cdot4+4\cdot3+2\cdot12=52$. The lengths are $\|a\|=\sqrt{(4^2+4^2+2^2)}=6$ and $\|b\|=\sqrt{(4^2+3^2+12^2)}=13$.

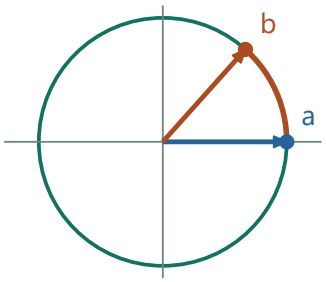
4	4	2
4	3	12
16	12	24

Sum the component products to get 52.

The vector lengths are 6 and 13.

3. Recover the angle

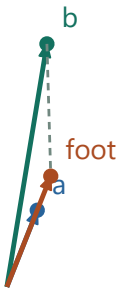
Dividing gives $\cos\theta=52/(6\cdot13)=2/3$. This is positive, so the angle must be acute. Using inverse cosine in degree mode gives $\theta=\arccos(2/3)\approx48.19°$.



In the plane of the arrows, the actual angle is 48.19°.

4. Choose the correct projection direction

The question projects b onto a. The desired vector must lie along a, so write it as ka. The leftover part b−ka must be perpendicular to a. Taking its dot product with a gives $0=b\cdot a-k(a\cdot a)$.



The orange projection lies on the target line.

The remaining spatial segment is perpendicular to it.

5. Compute the components and check the perpendicular part

Since $a\cdot a=6^2=36$, $k=52/36=13/9$. Scaling a gives $(52/9,52/9,26/9)$. Subtract this from b to obtain $(-16/9,-25/9,82/9)$. Dotting with a gives $(-64-100+164)/9=0$, confirming the required right angle.

52/9	52/9	26/9

Multiply all components of a by 13/9.

The residual has zero dot product with a.

Check. The residual $b-$projection has zero dot product with a.

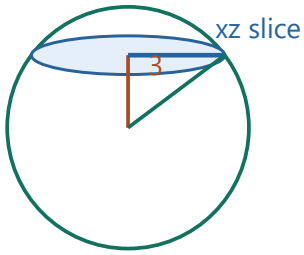
Exercise 173 · Circle cut from a sphere

Printed page 16 · Sphere geometry / Cross sections

Consider again the sphere of radius 5 centered at $(2, 3, -6)$. Describe the intersection of the sphere with the xz-plane. Write an equation (or equations) for this curve.

1. Read the plane's coordinate condition

The xz-plane consists of points $(x,0,z)$; hence set $y=0$ in the sphere equation $(x-2)^2+(y-3)^2+(z+6)^2=25$.

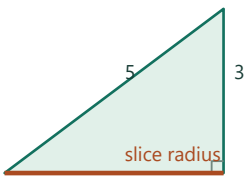


The xz plane sets $y=0$.

The center is 3 units off that plane.

2. Simplify the section

The middle term is 9, leaving $(x-2)^2+(z+6)^2=16$. This is a two-dimensional distance equation in the plane $y=0$.



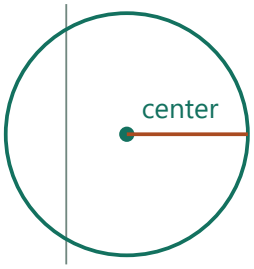
Subtract 3^2 from 5^2: the squared slice radius is 16.

1. $(x-2)^2+(0-3)^2+(z+6)^2=25$
The intersecting plane requires $y=0$.

2. $(x-2)^2+(z+6)^2=25-9=16$
Remove the squared offset from the sphere's squared radius.

3. Describe the intersection in space

Its center is (2,0,−6) and its radius is 4. The original center is 3 units away from this plane; section radius 4, center offset 3 and sphere radius 5 form a right triangle.



In the xz plane the center is (x,z)=(2,−6).

Every point also has y=0.

Check. The perpendicular distance to the center is 3, and $3^2+4^2=5^2$.

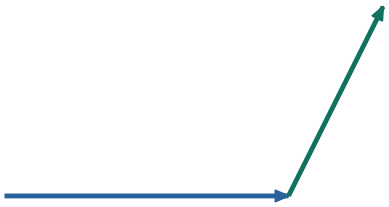
Exercise 174 · Vector triangle inequality

Printed page 16 · Vectors and components / Inequalities

The triangle inequality. Explain why $|\mathbf{u}+\mathbf{v}| \le |\mathbf{u}| + |\mathbf{v}|$ holds for any vectors $\mathbf{u}$ and $\mathbf{v}$.

1. Draw the vectors head to tail

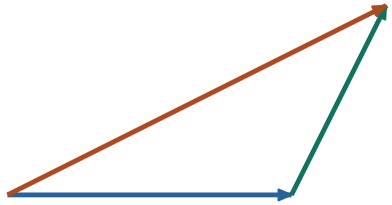
Follow u from a start point to an intermediate point, then v from there to the finish. The straight arrow from start to finish is u+v.



Travel along u, then along a translated copy of v.

2. Compare the path lengths

By the triangle inequality, the straight-line distance between two points is at most the sum of the lengths of a two-leg path between them. Thus $\|u+v\| \le \|u\| + \|v\|$.



The direct arrow u+v cannot be longer than the two-step route.

3. Identify equality

The two legs can match the direct length when they point in the same direction, or one has zero length. Algebraically the difference between the two squared sides is $2\|u\|\|v\|(1-\cos\theta)$, nonnegative whenever the angle is defined.

When the two input arrows point in the same direction both routes cover the same segment.

1. $(\|u\|+\|v\|)^2 = \|u\|^2 + 2\|u\|\|v\| + \|v\|^2$
Expand the square of the sum of the two lengths.

2. $\|u+v\|^2 = \|u\|^2 + 2\|u\|\|v\|\cos\theta + \|v\|^2$
With both vectors nonzero, expand the dot product of their sum with itself.

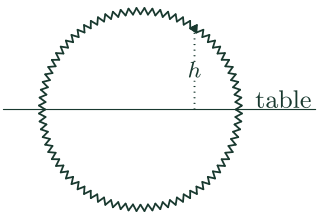
3. difference $= 2\|u\|\|v\| - 2\|u\|\|v\|\cos\theta \ge 0$
Cancel the two squared-length terms and use $\cos\theta \le 1$.

Check. Algebraically $(u+v)\cdot(u+v) = U^2 + 2UV\cos\theta + V^2 \le (U+V)^2$.

Exercise 175 · A saw tooth's height

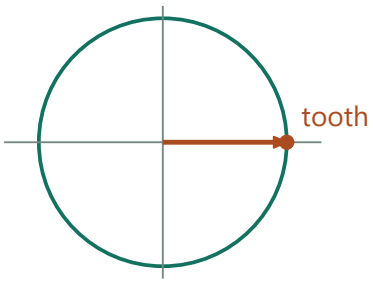
Printed page 17 · Sinusoidal functions / Circular motion

A large circular saw blade with a 1-foot radius is mounted so that exactly half of it shows above the table. It is spinning slowly, at one degree per second counterclockwise. One tooth of the blade has been painted red. This tooth is initially 0 feet above the table, and rising. What is the height after 37 seconds? After 237 seconds? After t seconds? Draw by hand a graph that shows how the height h of the red tooth is determined by the elapsed time t. It is customary to say that h is a function of t.



1. Set the tooth’s starting height to zero

The center is level with the table and the radius is 1 ft. The tooth starts on the right and rises. Relative to the horizontal the radius rotates t degrees in t seconds.



The tooth begins on the horizontal reference ray from the axle.

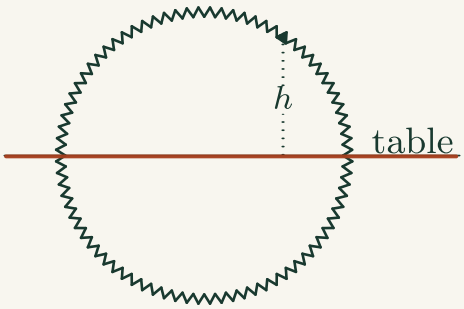
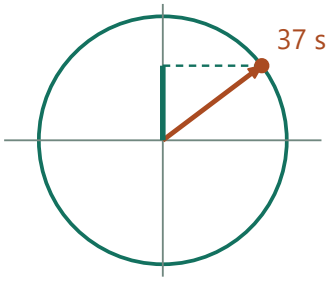


Table level through the axle

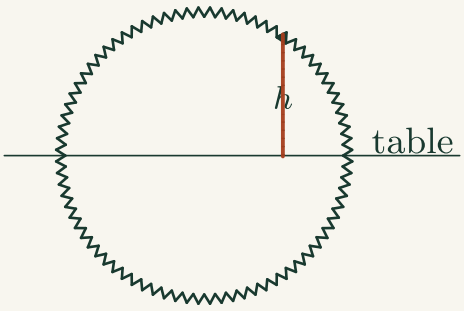
The straight horizontal line establishes height zero. A point below it has negative signed height.

2. Project vertically

The vertical component of a unit radius is h(t)=sin(t°) ft. A negative value means the tooth is below the table.



Use the vertical component of the one-foot radius.

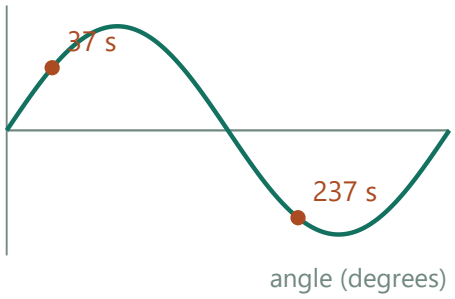


Vertical height h

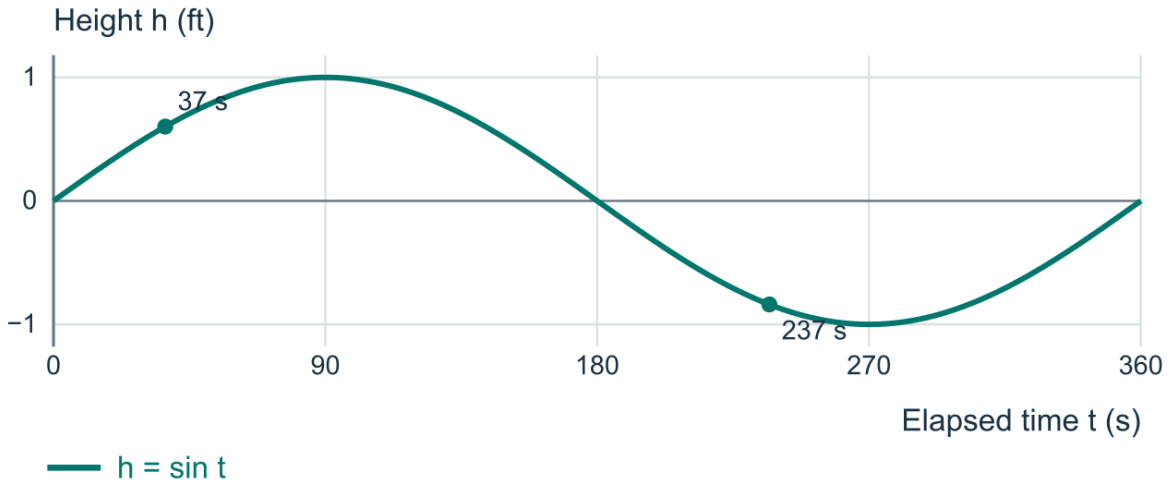
This dotted perpendicular measures the tooth’s vertical displacement from the table, even when the radius points diagonally.

3. Calculate and sketch

At 37 s, h≈0.602 ft. At 237 s, h≈−0.839 ft. Plot (0,0),(90,1),(180,0),(270,−1),(360,0), connect them smoothly and repeat after each 360 s.



At 37 s and 237 s the signed readings are 0.602 ft and −0.839 ft.

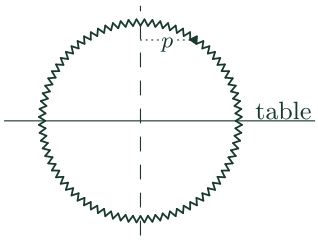


Check. After one turn height returns to zero and initially increases.

Exercise 176 · A saw tooth's horizontal position

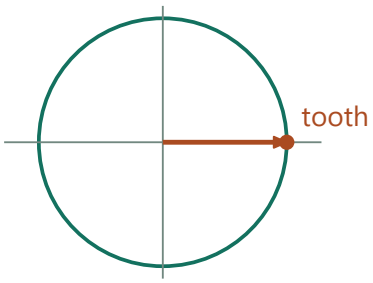
Printed page 17 · Sinusoidal functions / Circular motion

(Continuation) Now explore the position of the red saw tooth in reference to an imaginary vertical axis of symmetry of the circular blade. The red tooth is initially one foot to the right of the dotted line. How far to the right of this axis is the tooth after 37 seconds? After 237 seconds? After t seconds? Draw by hand a graph that shows how the displacement p of the red tooth with respect to the vertical axis is a function of the elapsed time t.

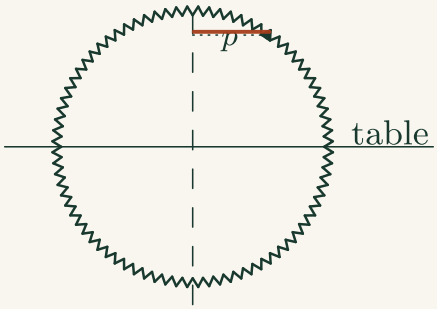


1. Read displacement horizontally

The marked tooth starts one radius to the right of the center. A projected radius along the horizontal axis has signed length r cos θ.



The tooth begins on the horizontal reference ray from the axle.

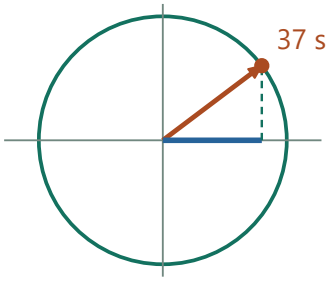


Horizontal distance p

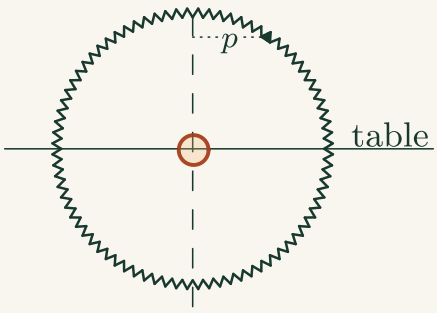
At the indicated tooth level, the dotted horizontal marks the distance from the vertical centerline to the tooth. Its sign records left or right.

2. Use the tooth’s radius and angular speed

Here r=1 ft and θ=t degrees after t seconds. Hence p(t)=cos(t°) ft. Rightward is positive and leftward negative.



Use the horizontal component of the one-foot radius.

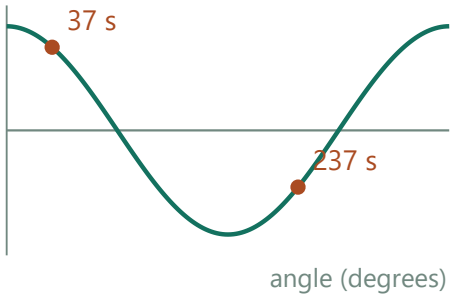


Blade axle

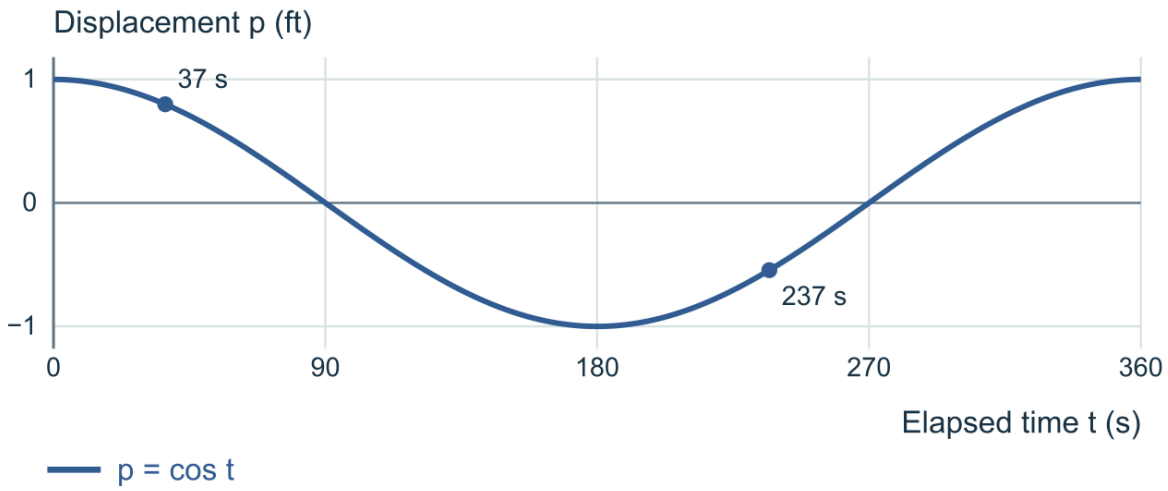
All radius directions begin here; rotating the tooth changes its projections while its distance to the axle stays constant.

3. Evaluate and build the graph

At 37 s, p≈0.799 ft; at 237 s, p≈−0.545 ft. Plot the quarter-turn values (0,1),(90,0),(180,−1),(270,0),(360,1) and repeat the smooth curve.



At 37 s and 237 s the signed readings are 0.799 ft and −0.545 ft.



Check. At all times $h^2+p^2=1$, the squared blade radius.

Exercise 177 · Times on the sine and cosine curves

Printed page 17 · Sinusoidal functions / Trigonometric equations

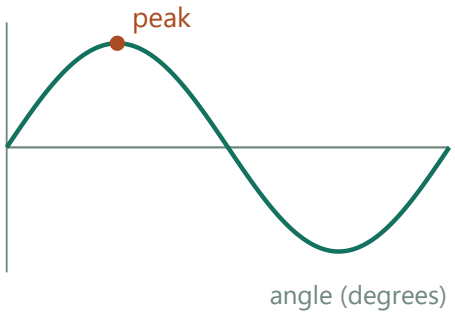
(Continuation) The graphs of the height h and the horizontal displacement p of the red saw tooth are examples of *sine* and *cosine* curves, respectively. Draw $y = \sin x$ and $y = \cos x$ on a graphing tool and compare them with the graphs that you drew in the preceding exercises. Use these graphs to answer the following questions:

(a) For what values of t is the red tooth 0.8 feet above the table? 0.8 feet below the table?

(b) When is the tooth 6 inches to the right of the vertical axis? When is it farthest left?

1. Connect the graph to the blade

In degrees the graph input equals time in seconds, since the blade turns 1°/s. Height is sin t and rightward displacement cos t in feet. The cosine peaks occur a quarter-turn, or 90 s, before the sine peaks.

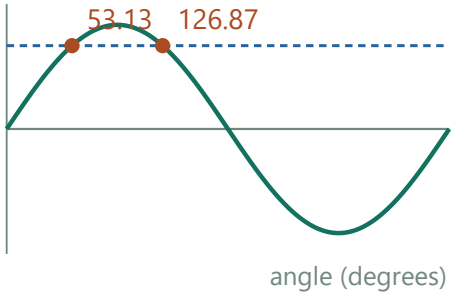


Time runs at one second per degree.

The sine peak occurs at 90 s.

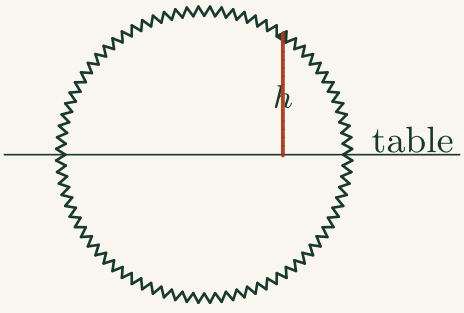
2. Read the two height crossings

For 0.8 ft above, sin t=0.8 has first solution arcsin(0.8)≈53.13° and its supplement 126.87°. A half-turn later the heights reverse, giving −0.8 ft at 233.13 s and 306.87 s.



A horizontal line at +0.8 ft meets the curve at two times.

1. α=arcsin(0.8)≈53.13°	Locate the first-quadrant angle that reaches height 0.8.
2. t≈α or 180°−α, thus 53.13 or 126.87 s	One degree corresponds to one second and the upper-quadrant heights agree.
3. t≈180°+α or 360°−α, thus 233.13 or 306.87 s	Reflect the two upper crossings to reach height −0.8.

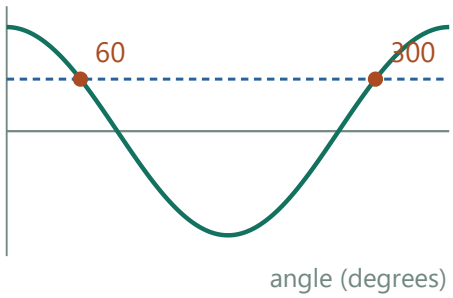


Vertical height h · Figure from exercise 175

This dotted perpendicular measures the tooth's vertical displacement from the table, even when the radius points diagonally.

3. Read horizontal positions from cosine

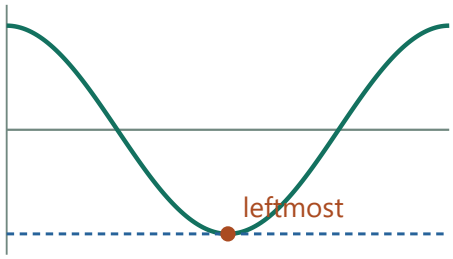
Six inches right is +0.5 ft, so cos t=0.5 gives times 60 s and 300 s. Farthest left means p=−1 ft, reached at 180 s.



Horizontal tooth position equals +0.5 ft twice each turn.

4. Account for later turns

These are all times in a single revolution. Repeat each listed time after every additional 360 s. Strictly negative heights or displacements indicate below-table or left-of-center positions.



The minimum x projection occurs at 180 s, then recurs every 360 s.

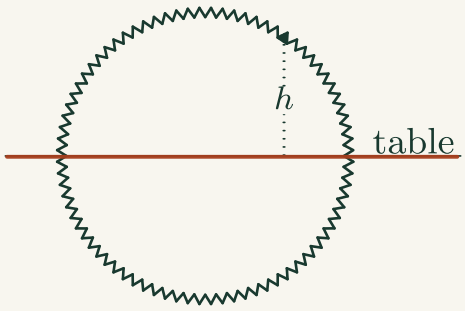
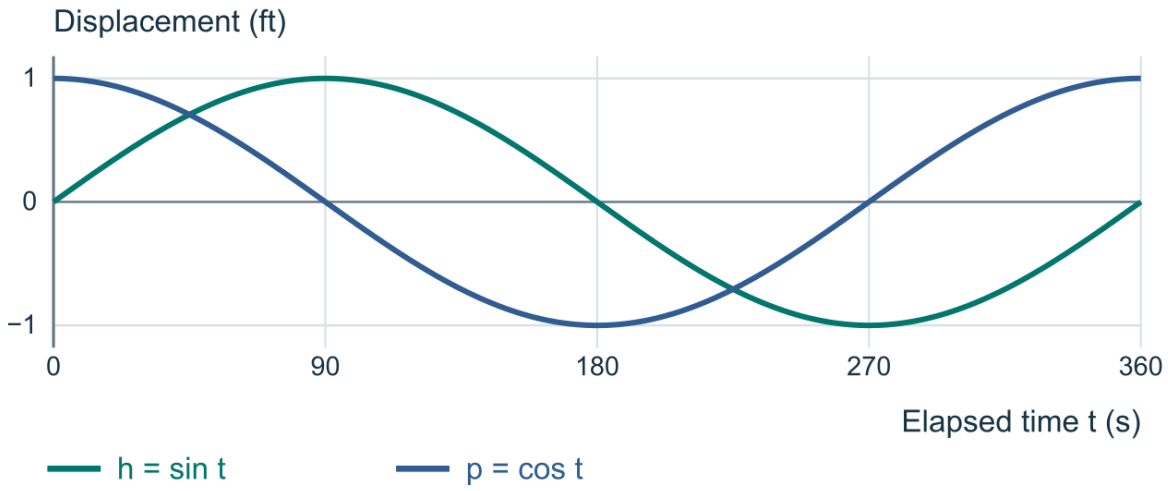


Table level through the axle · Figure from exercise 175

The straight horizontal line establishes height zero. A point below it has negative signed height.



Check. Intersecting horizontal lines at these heights with the sine and cosine graphs gives the same times.

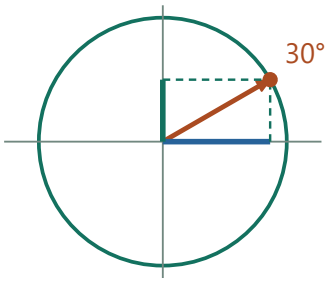
Exercise 178 · Sine of a supplement

Printed page 17 · Reference angles and symmetry / The unit circle

Asked to simplify the expression $\sin(180 - \theta)$, Rory volunteered the following solution: $\sin(180 - \theta) = \sin 180 - \sin\theta$, and, because $\sin 180$ is zero, it follows that $\sin(180 - \theta)$ is the same as $-\sin\theta$. Is this answer correct? If not, what is a correct way to express $\sin(180 - \theta)$ in simpler form? Answer the same question for $\cos(180 - \theta)$.

1. Test the suggested subtraction rule

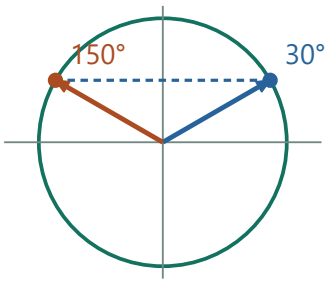
If sin(180°−θ) were sin 180°−sin θ, then at θ=30° it would equal −1/2. But the ray at 150° lies above the x-axis and has height +1/2.



Start with the radius at 30°.

2. Track the endpoints geometrically

The unit-circle points at θ and its supplement 180°−θ are reflections in the y-axis. Reflection leaves the vertical coordinate unchanged.

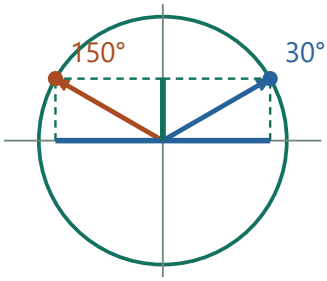


Reflect across the vertical axis.

Horizontal sign reverses; vertical height stays fixed.

3. Read sine and cosine separately

Therefore sin(180°−θ)=sin θ. Reflection reverses the horizontal coordinate, so cos(180°−θ)=−cos θ.



Reflect across the vertical axis.

Horizontal sign reverses; vertical height stays fixed.

Check. At θ=30°, sin150°=1/2, contradicting the proposed negative result.

Exercise 179 · Trigonometry after a half-turn

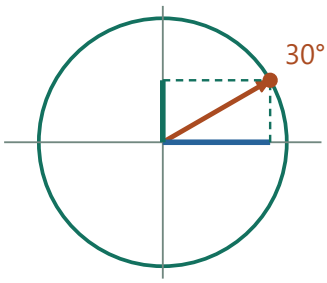
Printed page 17 · Reference angles and symmetry / Trigonometric identities

Find simpler, equivalent expressions for the following. Justify your answers.

(a) $\sin(180+\theta)$ **(b)** $\cos(180+\theta)$ **(c)** $\tan(180+\theta)$

1. Rotate the unit-circle radius by a half-turn

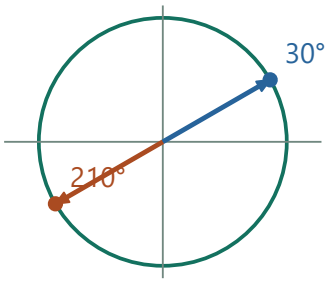
Starting at the point (cos θ,sin θ), a further 180° sends it to the diametrically opposite point. Both coordinates change sign.



Start with the radius at 30°.

2. Read the new trigonometric values

The new height is −sin θ and the new horizontal coordinate is −cos θ. Thus sin(180°+θ)=−sin θ and cos(180°+θ)=−cos θ.

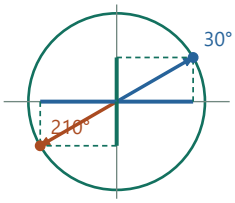


A half-turn reverses both components.

The signed ratio y/x stays unchanged.

3. Form the tangent quotient

Where cos θ≠0, the two minus signs cancel in (−sin θ)/(−cos θ). Hence tan(180°+θ)=tan θ; the opposite rays have the same slope.



A half-turn reverses both components.

The signed ratio y/x stays unchanged.

1. tan(180°+θ)=(−sin θ)/(−cos θ)
Divide the reflected vertical coordinate by the reflected horizontal coordinate.

2. =sin θ/cos θ=tan θ
The two minus signs cancel only where the denominator is nonzero.

Check. At θ=45°, angles 45° and 225° both have tangent 1.

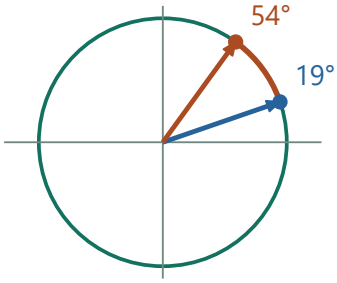
Exercise 180 · Angle between two unit directions

Printed page 17 · The dot product / The unit circle

Show that there are at least two ways to calculate the angle formed by the vectors $[\cos 19, \sin 19]$ and $[\cos 54, \sin 54]$.

1. Compare the two directions as angles

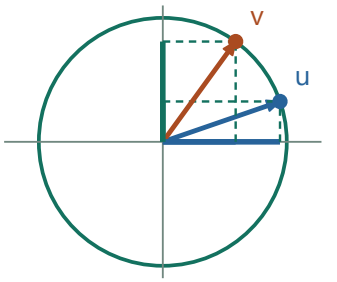
Both given directions are measured from the same positive axis. From 19° to 54° the change is 54°−19°=35°, the smaller angle between them.



Subtract the two direction angles to get the gap of 35°.

2. Represent the directions by unit vectors

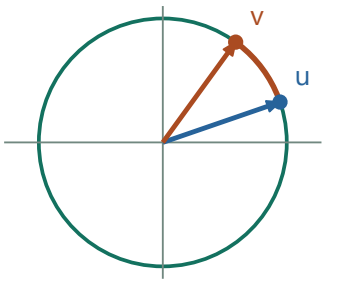
Use (cos 19°,sin 19°) and (cos 54°,sin 54°). Each has length 1, so their dot product equals the cosine of their included angle.



Each horizontal component is a cosine and each vertical component a sine.

3. Recover the angle a second way

Taking arccos of cos 19° cos 54°+sin 19° sin 54° gives 35°. These approaches agree because the sum of products is the cosine of the difference in directions.



Both vectors have length 1.

Their dot product therefore equals cos 35°.

Check. The cosine-difference identity guarantees the dot product equals cos(54°−19°).

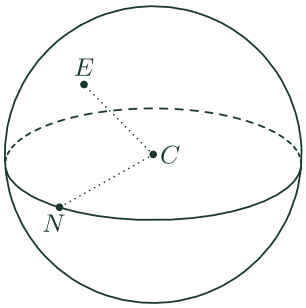
Exercise 181 · Exeter on the globe

Printed page 17 · Spherical coordinates / Coordinate geometry

Let N be the point on the equator closest to E = Exeter, and let C be the center of the earth. The central angle ECN is called the *latitude* of E; it is approximately 43 degrees. Take the radius of the earth to be 3960 miles as you answer the following distance questions:

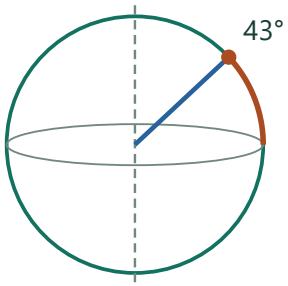
(a) How far from the equator is Exeter? Travel *on* the earth, not through it.

(b) How far does the earth's rotation on its axis carry the citizens of Exeter during a single day?

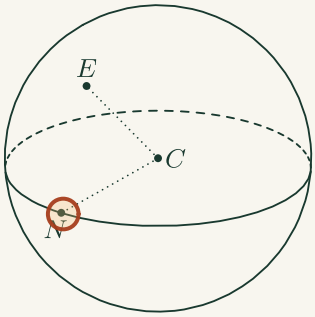


1. Measure the north-south path

In the picture E is Exeter, C is Earth's center, and N is the nearest point on the equator. The surface arc EN and the central angle ECN correspond. Traveling from E to N along a meridian changes latitude by 43°. In radians that is 43π/180. The meridian has radius 3960 mi, so its arc length is 3960·43π/180≈2972 mi.



Exeter's meridian arc to the equator subtends 43° at Earth's center.



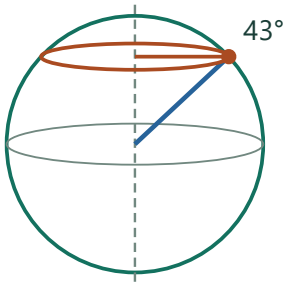
Equator through N

The ellipse in this perspective view represents the equator, a great circle. N marks a point on it along the relevant meridian.

2. Find the radius of Exeter's daily path

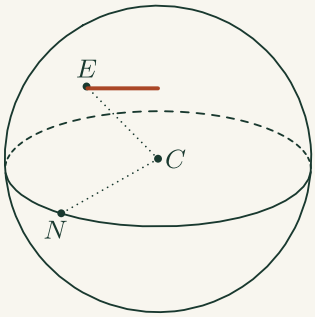
Earth's rotation carries Exeter along a parallel of latitude. In the meridian cross section, its distance to the polar axis is the equatorial projection of the Earth radius: 3960 cos 43°.

Latitude · 43°



At latitude 43° the radius of the parallel is 2896.161 mi.

Its circumference is 0.731 times the equatorial circumference.

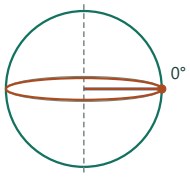


Axial distance of Exeter

In a front-facing meridian section, the radius of Exeter's daily parallel is its distance from the polar axis. A horizontal guide shows this projection.

Compare settings · Latitude

0°

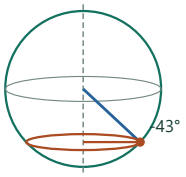


At latitude 0° the radius of the parallel is 3960 mi.

Its circumference is 1 times the equatorial circumference.

At latitude 0° the radius of the parallel is 3960 mi. Its circumference is 1 times the equatorial circumference.

−43°



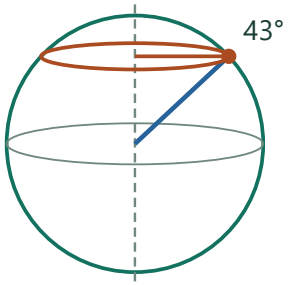
At latitude −43° the radius of the parallel is 2896.161 mi.

Its circumference is 0.731 times the equatorial circumference.

At latitude −43° the radius of the parallel is 2896.161 mi. Its circumference is 0.731 times the equatorial circumference.

3. Follow one full rotation

A point on this parallel travels its circumference in one turn. The distance is 2π·3960 cos 43°≈18,197 mi. This parallel is shorter than the equator.



Earth carries Exeter along this smaller circle once each day.

Check. At the equator the parallel radius would be 3960 and at a pole zero, consistent with the cosine factor.

Exercise 182 · Transform a unit square

Printed page 18 · Matrix transformations / Coordinate geometry

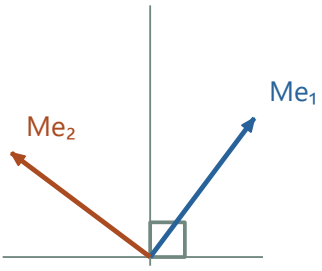
The matrix $\begin{bmatrix} 3 & -4 \\ 4 & 3 \end{bmatrix}$ represents the transformation $\mathcal{T}(x, y) = (3x - 4y, 4x + 3y)$.

(a) Apply transformation $\mathcal{T}$ to the *unit square* whose vertices are $(0, 0)$, $(1, 0)$, $(1, 1)$, and $(0, 1)$. In particular, notice what the images of the points $(1, 0)$ and $(0, 1)$ are, and compare them with the entries in the columns of the coefficient matrix.

(b) Confirm that the same results can be obtained by doing some matrix arithmetic. Calculate products $\begin{bmatrix} 3 & -4 \\ 4 & 3 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix}$, $\begin{bmatrix} 3 & -4 \\ 4 & 3 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix}$, and $\begin{bmatrix} 3 & -4 \\ 4 & 3 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix}$, and interpret.

1. Track the origin and the two basis corners

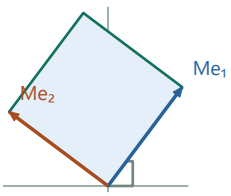
A linear map sends (0,0) to (0,0). Multiplying the matrix by (1,0) selects its first column (3,4); multiplying by (0,1) selects its second column (−4,3).



The basis corners move to (3,4) and (−4,3).

2. Recover the fourth corner

The original opposite corner is (1,1)=(1,0)+(0,1). Its image is the sum (3,4)+(−4,3)=(−1,7). In boundary order the four images are (0,0),(3,4),(−1,7),(−4,3).



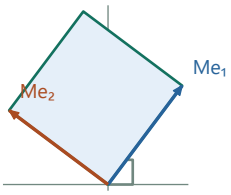
Adding the two basis images places the far vertex at (−1,7).

1. M(1,1)=M(1,0)+M(0,1)
The opposite corner of the square sums its basis directions.

2. =(3,4)+(−4,3)=(−1,7)
Add the two column images by coordinate.

3. Identify the image shape

The two image edges from the origin have equal length 5 and dot product 3(−4)+4·3=0. They are perpendicular, so the image parallelogram is indeed a square.



The columns are perpendicular and length 5.

The image is a larger rotated square.

1. (3,4)·(−4,3)=−12+12=0
The image edges meet at right angles.

2. ||(3,4)||²=||(−4,3)||²=25
Each image edge has length 5, so the output parallelogram is a square.

Check. The two image edges have equal length 5 and dot product zero, so the image really is a square.

Exercise 183 · Columns are images

Printed page 18 · Matrix transformations / Matrix columns and basis vectors

Calculate the products $\begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix}$, and interpret the results.

1. Apply the matrix to the first basis vector

For rows (a,b),(c,d), multiplication by (1,0) gives (a·1+b·0,c·1+d·0)=(a,c), the first column.

a	b
c	d

Multiplying by (1,0) selects the first column (a,c).

2. Apply it to the second

Multiplication by (0,1) gives (b,d), the second column. Each column has a geometric meaning as an image of a unit axis arrow.

a	b
c	d

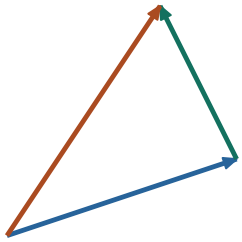
Multiplying by (0,1) selects the second column (b,d).

1. row 1·(0,1)=a·0+b·1=b
Only the second entry of the first row contributes.

2. row 2·(0,1)=c·0+d·1=d
The same choice in row two completes the second column.

3. Recover every other image

Write (x,y)=x(1,0)+y(0,1). By linearity its image is x(a,c)+y(b,d). Thus knowing the two basis-vector images determines the entire transformation.



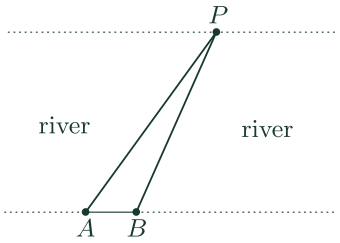
For a general input, scale each column by its input coordinate and add head to tail.

Check. Acting on (x,y) combines the columns as x(a,c)+y(b,d).

Exercise 184 · River triangulation

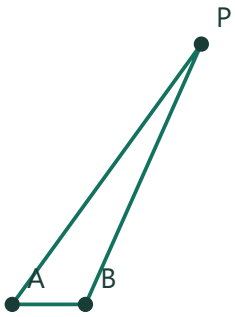
Printed page 18 · The sine rule / Right-triangle trigonometry

Two industrious PEA students are trying to find the distance across the Squamscott River. They mark points A and B on one bank, 60 meters apart. Then they sight the Powderhouse P on the opposite bank, and measure angles PAB and PBA to be 54 and 114 degrees, respectively. This enables them to calculate the altitude from P to the baseline AB. To the nearest meter, what is their result?



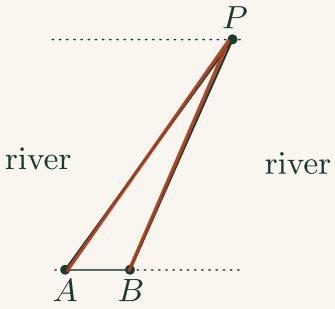
1. Complete the triangle of sight lines

Let A and B be the baseline endpoints and P the sighted point. The third interior angle is 180°−54°−114°=12°, opposite the known 60-m side AB.



The third angle is 12°.

The 60-m baseline is opposite this small angle.

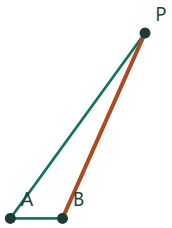


Sight lines AP and BP

Both observers sight the same point P across the river. The two rays enclose the small remaining angle at P.

2. Find a usable sight-line length

The sine rule compares AP opposite 114° with AB opposite 12°: AP=60 sin 114°/sin 12°. This gives the sloping distance from A to P.



AP is opposite 114°.

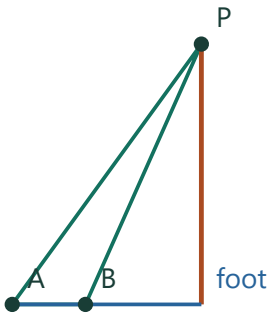
Use AP/sin114° = 60/sin12°.

1. AP/sin114°=60/sin12°
Pair the baseline and AP with their opposite angles.

2. AP=60 sin114°/sin12°
Multiply by sin114°; the remaining divisor is positive.

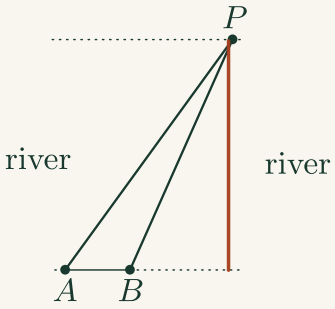
3. Project onto the perpendicular

At A the sight line makes angle 54° with the baseline, so the perpendicular distance is AP sin 54°≈213.285 m, about 213 m. The foot lies on the extension past the obtuse-angle endpoint.



The altitude foot lies beyond B.

Width = AP sin54° ≈ 213 m.



Perpendicular from P to the baseline line

This constructed vertical segment meets the extension beyond B. Its length is the perpendicular river-crossing distance used in the solution.

Check. Using BP=60 sin54°/sin12° and multiplying by sin114° gives the same height.

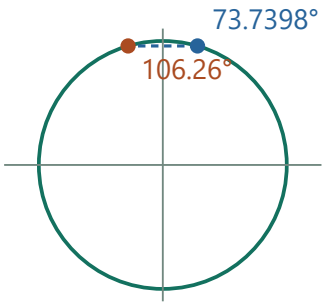
Exercise 185 · Recover cosine

Printed page 18 · Trigonometric ratios / Trigonometric identities

If $\sin A$ is known to be 0.96, then what can be said about $\cos A$? What if it is also known that angle A is obtuse?

1. Use the point's unit distance

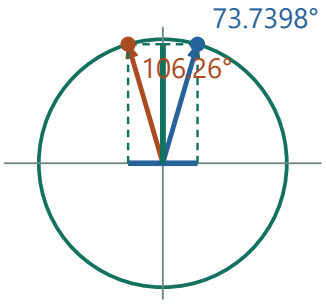
If sin A=0.96, then $\cos^2 A = 1 - 0.96^2 = 0.0784$ by the Pythagorean identity.



A horizontal line y = 0.96 meets the unit circle twice.

2. Consider the two square roots

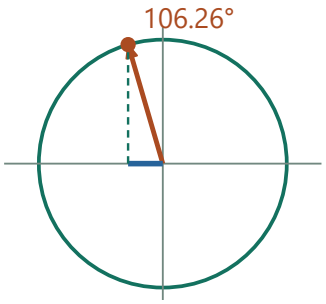
Taking a square root of $\cos^2 A$ gives $|\cos A| = 0.28$. A positive sine can occur in either upper quadrant, so cos A can be +0.28 or −0.28.



Both points have $x^2 = 1 - 0.96^2 = 0.0784$, but opposite x signs.

3. Use the added obtuse-angle condition

For $90° < A < 180°$ the point lies left of the y-axis. Its horizontal coordinate is negative, selecting $\cos A = -0.28$.



Select only the obtuse direction.

Its horizontal coordinate is −0.28.

Check. The associated right triangle has sides 7,24,25 after rescaling.

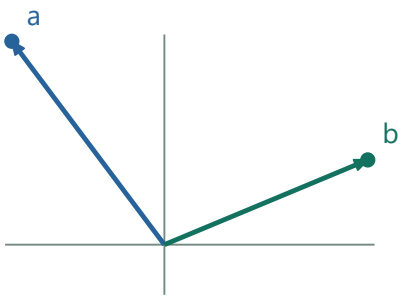
Exercise 186 · Project (12,5) onto (−9,12)

Printed page 18 · Vector projections / The dot product

Find components for the vector projection of $[12, 5]$ onto $[-9, 12]$.

1. Choose the projection direction

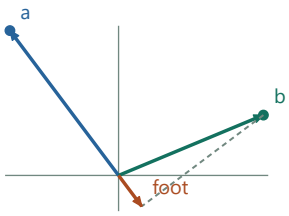
The requested shadow of a=(12,5) lies on the line of b=(−9,12), so it equals kb for some real k. A negative k would mean the shadow points opposite b.



Blue is the target (−9,12); green is the vector (12,5) being projected.

2. Make the residual perpendicular

The difference a−kb from the shadow to the original arrow is perpendicular to b. Thus 0=(a−kb)·b=a·b−k(b·b). Here a·b=12(−9)+5(12)=−48 and b·b=81+144=225.



The negative dot product puts the foot on the opposite extension of the target.

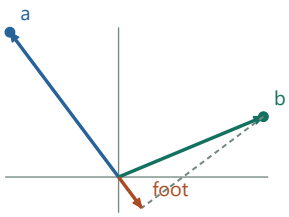
1. (v−ku)·u=0 ⇒ v·u−k(u·u)=0
The projection residual is perpendicular to the target direction.

2. 225k=12(−9)+5(12)=−48
Insert the squared direction length and dot product.

3. k=−48/225=−16/75
A negative coefficient means the projection points opposite u.

3. Return a vector

Multiply b by k=−48/225. The result is (48/25,−64/25). Its direction points down and right, as expected from the negative coefficient multiplying b.



Scale (−9,12) by −48/225.

The projection is (48/25,−64/25).

1. ku=(−16/75)(−9,12)=(144/75,−192/75)
Scale the target direction by the signed projection coefficient.

2. =(48/25,−64/25)
Reduce both fractions by 3.

Check. Its dot product with the target equals −48, so the discarded residual is perpendicular to the target.

Exercise 187 · Weighted test average

Printed page 18 · Weighted averages / The dot product

During one term in Math 310, Min Lee took seven tests, the last of which carried twice the weight of each of the others when averages were computed. Min's test-score vector for the term was $[84, 78, 91, 80, 72, 88, 83]$. Show that Min's final average, a *weighted average*, can be calculated as a dot product of this vector with another seven-component vector. How can the teacher obtain a class list of test averages by multiplying two matrices?

1. Translate “twice” into counts

Each of the first six scores has weight one and the last has weight two. That makes a total weight of eight, even though there are seven listed numbers.



The first six grades get one share each; the final gets two.

2. Form the weight column

Divide the column $(1,1,1,1,1,1,2)^{\mathsf{T}}$ by 8. Its entries sum to one. Multiplying the student’s score row on the right by this column computes a weighted average.

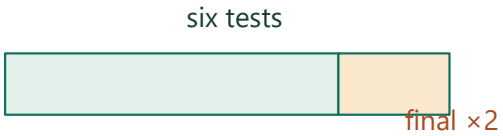
84	78	91	80
72	88	83	83

Eight equally weighted shares.

The final score appears twice.

3. Evaluate and generalize

The numerator is $84+78+91+80+72+88+2\cdot83=659$, so the mean is 82.375. For a class, place students in separate rows with assessments in the same column order and multiply the whole matrix by the weight column.



Total weighted score 659 divided by 8 gives 82.375.

Check. The normalized weights sum to 1, and duplicating the final score creates the equivalent eight-score mean.

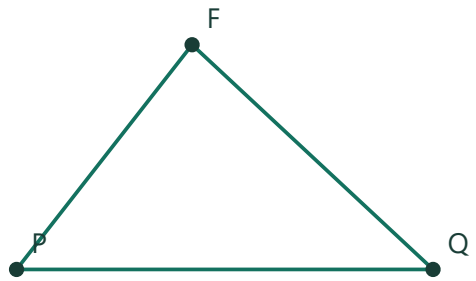
Exercise 188 · Fire wardens

Printed page 18 · Triangulation / The sine rule

Two fire wardens are stationed at locations P and Q, which are 45.0 km apart. Each warden sights the forest fire at F. Given that angle FPQ is 52.0 degrees and angle FQP is 43.0 degrees, find the distance from F to the nearer warden, to the nearest 0.1 km.

1. Find the missing angle

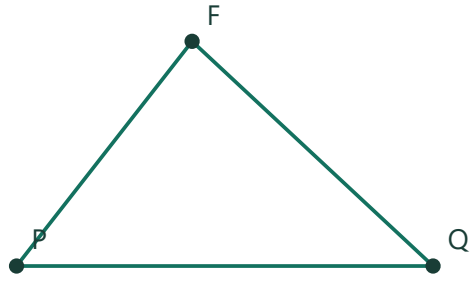
The wardens P and Q form the known 45-km baseline. At F the triangle angle is 180°−52°−43°=85°. The baseline is opposite this angle.



Fire sight lines meet at 85°.

2. Identify the nearer warden

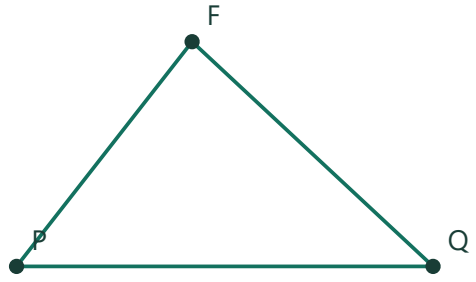
A smaller opposite angle faces a shorter side. PF lies opposite the 43° angle at Q, whereas QF lies opposite 52° at P, so PF is shorter.



PF faces the smaller angle, 43°, so it is the shorter sight line.

3. Apply the sine rule

PF/sin 43°=45/sin 85°. This gives PF=45 sin 43°/sin 85°≈30.8 km to the nearest tenth.



PF = 45 sin43°/sin85° ≈ 30.8 km.

Check. The other sight line uses sin52°, which is larger than sin43°.

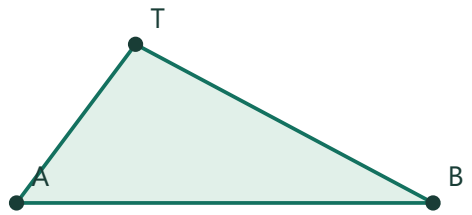
Exercise 189 · Area of a 10-17-21 triangle

Printed page 18 · Triangle area / The cosine rule

Find the area of a triangle that has a 10-inch side, a 17-inch side, and a 21-inch side.

1. Read the three side lengths

When all three sides are available, Heron's formula finds area directly from them. Here they are 10,17,21 inches, giving semiperimeter s=(10+17+21)/2=24 in.



The three sides are 10,17,21 inches.

The semiperimeter is 24.

2. Calculate each factor

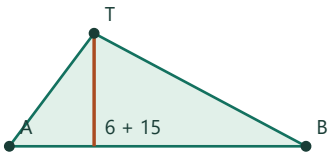
Heron gives $A^2=s(s-10)(s-17)(s-21)=24\cdot14\cdot7\cdot3=7056$. The product has fourth-power length units.

24	14	7	3

The Heron factors are s, s−10, s−17 and s−21.

3. Take the area root

$A=\sqrt{7056}=84$ in². As a geometric check, the 21-inch base can be divided into lengths 6 and 15 by an 8-inch altitude, making right triangles with hypotenuses 10 and 17 and area ½·21·8=84.



An 8-in altitude gives 6–8–10 and 8–15–17 right triangles.

Area = ½·21·8 = 84 in².

1. $x^2+h^2=100$, $(21-x)^2+h^2=289$
Measure x along the base from the 10-inch side's endpoint.

2. $441-42x=289-100=189$
Subtract the first equation from the second and cancel both squared unknowns.

3. $42x=252 \Rightarrow x=6$, $h=\sqrt{(100-36)}=8$
Isolate x and return to its right triangle to find the positive altitude.

Check. With 21 as base, an 8-inch altitude splits it into 6 and 15; these produce sides 10 and 17.

Exercise 190 · Four matrix transformations

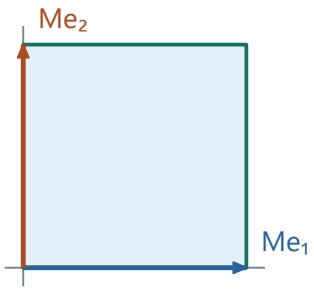
Printed page 18 · Matrix transformations / Coordinate transformations

For each of the following transformation matrices, describe the effect of its transformation:

(a) $\begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix}$ (b) $\begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$ (c) $\begin{bmatrix} 5/13 & -12/13 \\ 12/13 & 5/13 \end{bmatrix}$ (d) $\begin{bmatrix} 3 & -4 \\ 4 & 3 \end{bmatrix}$

1. Read scalar diagonal matrices first

Matrix (a) multiplies both components by 3, a dilation of scale 3 about the origin. Matrix (b) multiplies both by −1, which reverses each radius through a 180° turn.

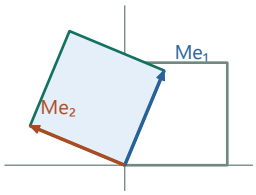


Scaling both axes by +3 enlarges all lengths threefold.

Scaling both by −1 is instead a half-turn.

2. Read the columns of (c) as axis images

The images are (5,12)/13 and (−12,5)/13. Both have length 1 and are perpendicular. The first makes angle arctan(12/5)≈67.38° above the positive x-axis. The second is a quarter-turn ahead, so this is a counterclockwise rotation.



The first image column points along (5,12).

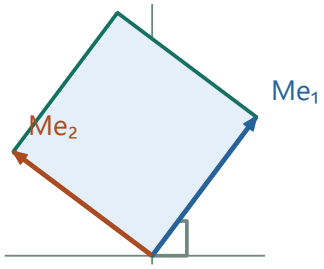
Normalizing by 13 leaves a unit direction.

1. $\|(5,12)/13\|^2=(25+144)/169=1$
The second column has the same squared entries in reverse order.

2. $(5,12)\cdot(-12,5)/169=(-60+60)/169=0$
The columns form a right angle as required for the rotation.

3. Separate length and angle in (d)

The column lengths are 5. Divide by 5 to get perpendicular unit columns (3/5,4/5) and (−4/5,3/5), describing a counterclockwise turn of arctan(4/3)≈53.13°. The original matrix combines this turn with a fivefold expansion.



Here both transformed basis vectors have length 5.

The map rotates and scales.

Check. For (c) and (d), the columns are perpendicular; their lengths determine whether a scale factor is present.

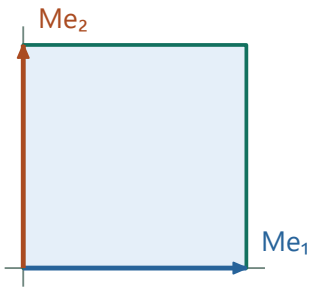
Exercise 191 · When is a matrix an isometry?

Printed page 18 · Matrix transformations / Isometries

(Continuation) Which of the preceding matrices represent isometries? In order for a matrix to represent an isometry, what must be true of its column vectors?

1. Identify what an isometry preserves

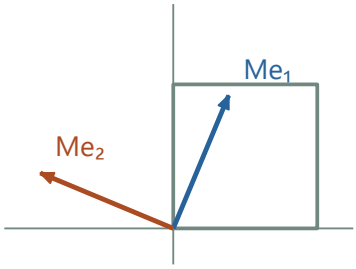
An isometry preserves distances between any two points. For a linear map it is enough to test all displacement lengths. The expansions in 190(a,d) change lengths, while its half-turn (b) and rotation (c) preserve them.



A tripled basis side witnesses failure to preserve lengths.

2. Track the two basis arrows

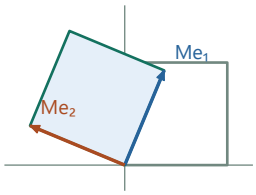
Originally the basis vectors are unit length and perpendicular. Their images, the matrix columns p and q, must still be unit length. Also the distance between their heads remains $\sqrt{2}$, so $\|p-q\|^2=2$ forces $p\cdot q=0$.



An isometry must carry the two basis arrows to perpendicular unit arrows.

3. Show that those conditions suffice

For unit perpendicular columns, the image $xp+yq$ has squared length $x^2\|p\|^2+2xy\ p\cdot q+y^2\|q\|^2=x^2+y^2$. Thus perpendicular unit columns are both necessary and sufficient.



With orthonormal columns, the squared length of any combination stays x^2+y^2.

1. $\|xu+yv\|^2=x^2(u\cdot u)+2xy(u\cdot v)+y^2(v\cdot v)$
Expand the dot product of the transformed vector with itself.

2. $=x^2\cdot1+2xy\cdot0+y^2\cdot1=x^2+y^2$
The two columns are perpendicular unit vectors, so all input lengths are preserved.

Check. Equivalently $M^TM=I$, ensuring every displacement retains length.

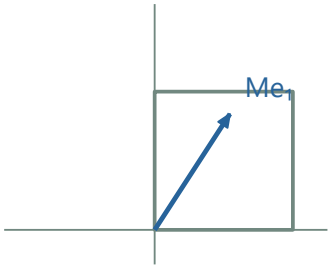
Exercise 192 · Rotation through 57 degrees

Printed page 19 · Rotations / Matrix transformations

Apply the 57-degree counterclockwise rotation about the origin to the vectors $[1,0]$ and $[0,1]$, then use the image vectors (written as columns) to form the coefficient matrix $\mathbf{M}$ for the rotation. Test $\mathbf{M}$ by calculating the products $\mathbf{M}\begin{bmatrix}1\\0\end{bmatrix}$ and $\mathbf{M}\begin{bmatrix}0\\1\end{bmatrix}$. Where does this transformation send the vector $[3,1]$? Does $\mathbf{M}$, when applied to $\begin{bmatrix}3\\1\end{bmatrix}$, do its job correctly?

1. Turn the horizontal and vertical axes

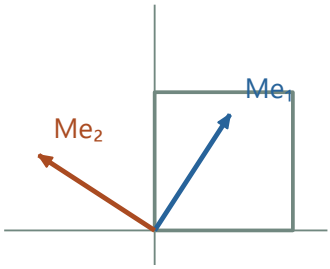
Write c=cos 57° and s=sin 57°. The horizontal unit vector rotates to (c,s); the upward one rotates to (−s,c), since it stays a quarter-turn ahead.



The first column gives the new horizontal unit arrow for 57°.

2. Assemble the matrix

Place the two image vectors in columns, producing rows (c,−s),(s,c). Multiplying by either original unit vector selects its intended column.

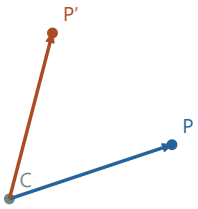


The other column stays a quarter turn ahead of the first; the columns have equal unit length.

3. Act on (3,1)

Take three copies of the first image plus one of the second: (3c−s,3s+c)≈(0.795,3.061). The squared length stays $3^2+1^2=10$.

Counterclockwise turn · 57°



Rotate by 57° about (0, 0).

New point (0.795, 3.061); distance to center remains 3.162.

1. $(3c-s)^2=9c^2-6cs+s^2$, $(3s+c)^2=9s^2+6sc+c^2$
Expand the image coordinates before the radius check.

2. sum=$10c^2+10s^2=10$
The mixed products cancel and $c^2+s^2=1$, reproducing the input's squared length.

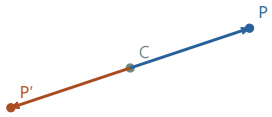
Compare settings · Counterclockwise turn

0°



Rotate by 0° about (0, 0).
New point (3, 1); distance to center remains 3.162.

180°



Rotate by 180° about (0, 0).
New point (−3, −1); distance to center remains 3.162.

Rotate by 0° about (0, 0). New point (3, 1); distance to center remains 3.162.

Rotate by 180° about (0, 0). New point (−3, −1); distance to center remains 3.162.

Check. Its squared length is 10 and the angle it makes with the original point's direction is 57°.

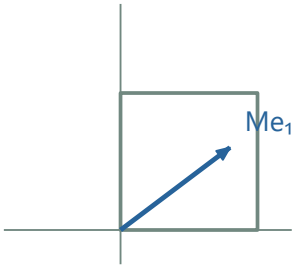
Exercise 193 · General rotation matrix

Printed page 19 · Rotations / Matrix transformations

Write the coefficient matrix for a θ-degree counterclockwise rotation about the origin.

1. Find where the horizontal basis vector goes

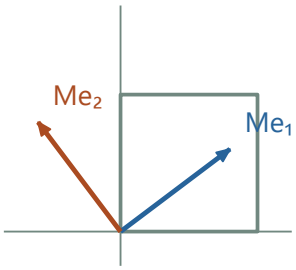
A counterclockwise turn of θ sends (1,0) to the unit-circle point (cos θ,sin θ). This must be the first column.



The first column gives the new horizontal unit arrow for a general angle θ.

2. Turn the vertical basis vector as well

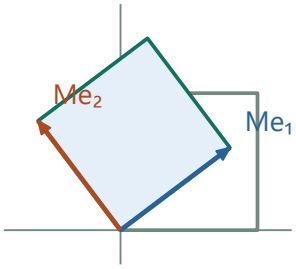
Its direction was 90° ahead and remains so after the rotation. A quarter-turn applied to (cos θ,sin θ) gives (−sin θ,cos θ), the second column.



The other column stays a quarter turn ahead of the first; the columns have equal unit length.

3. Write the row form

The rows are (cos θ,−sin θ),(sin θ,cos θ). Their dot products with (x,y) give the rotated horizontal and vertical components respectively.



The two columns together specify the rotation. In general the columns are (cos θ,sin θ) and (−sin θ,cos θ).

Check. At θ=0 it is I; at θ=90° it sends (x,y) to (−y,x).

Exercise 194 · Closest point on a spatial line

Printed page 19 · Vector projections / Parametric lines / Distance and loci

Let $A = (0, 0, 0)$, $B = (9, 8, 12)$, and $C = (6, 2, 3)$. Find coordinates for the point on line AB that is closest to C.

1. Constrain the answer to the given line

The line passes through the origin and B=(9,8,12). Every candidate foot is H=tB. From C=(6,2,3) to this line the shortest segment must meet it perpendicularly.

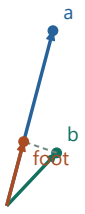


All candidates have the form t(9,8,12).

The green arrow locates C.

2. Set up orthogonality

The vector C−tB from the foot back to C satisfies (C−tB)·B=0. Compute C·B=54+16+36=106 and B·B=81+64+144=289.



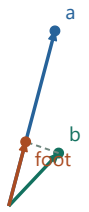
The perpendicular foot minimizes distance to the line.

1. 0=(C−tB)·B=C·B−t(B·B)
Express perpendicularity through the component dot product.

2. t(B·B)=C·B ⇒ t=106/289
Divide by the nonzero squared line-direction length.

3. Locate and justify the foot

Thus t=106/289 and H=(106/289)(9,8,12). Moving the foot by any nonzero amount along B adds a perpendicular leg to CH and increases the squared distance.



The nearest point is (106/289)(9,8,12).

1. H=(954/289,848/289,1272/289)
Multiply every component of (9,8,12) by the scale factor 106/289.

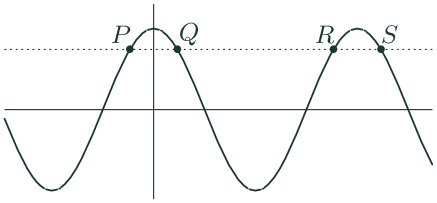
2. $\|C-(H+sB)\|^2=\|C-H\|^2+s^2\|B\|^2$
The cross term vanishes because CH is perpendicular to B, proving that any s≠0 gives a larger distance.

Check. For any displacement sB along the line, squared distance becomes $CH^2+s^2(B\cdot B)$, minimized at s=0.

Exercise 195 · Cosine intersections

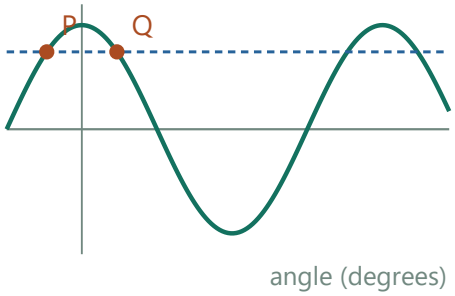
Printed page 19 · Trigonometric equations / Periodic functions

At right you see the graphs of $y = \cos x$ and $y = 0.7431$ (dotted). Given that $Q = (42, 0.7431)$, find coordinates for the intersection points P, R, and S without using a calculator. Use a calculator to check your answers.

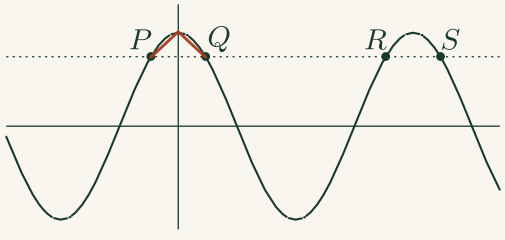


1. Find P by reflecting the cosine graph

Cosine is even, so y=cos x has the same height at x and −x. Given Q near (42°,0.7431), the intersection on the left of the peak is P near (−42°,0.7431).



Reflection across x=0 preserves cosine: P≈−42°, Q≈42°.

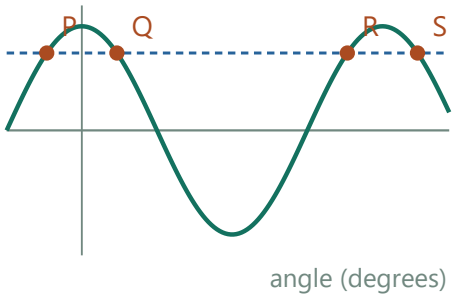


Reflected pair P and Q

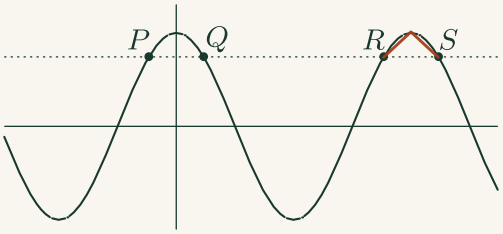
These are on opposite sides of the first shown peak, which sits on the vertical axis. Their input angles are opposites.

2. Follow the next complete cycle

The cosine curve repeats after 360°. Translate both points 360° to the right: R is near (318°,0.7431) and S near (402°,0.7431).



Add one full 360° period to each point.

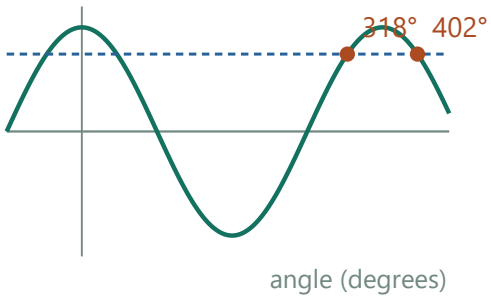


Next cycle pair R and S

These two crossings repeat the shape of the first pair one entire period to the right. Their common height is unchanged.

3. Interpret the calculator comparison

cos 42°≈0.743145, so the printed coordinates are rounded. If 0.7431 is specified exactly, set a=arccos(0.7431) in degrees. The four exact abscissas are −a, a, 360°−a and 360°+a.



The four crossings share the same rounded height 0.7431.

Check. The printed height is rounded: cos42°≈0.743145. If 0.7431 is taken exactly, use a=arccos(0.7431) in degrees and the exact abscissas −a, a, 360°−a and 360°+a.

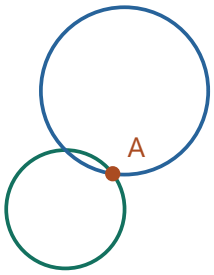
Exercise 196 · Intersection angle of circles

Printed page 19 · Circle tangency / Vectors and components

Verify that the circles $x^2+y^2=25$ and $(x-5)^2+(y-10)^2=50$ intersect at $A=(4,3)$. Find the size of the acute angle formed at A by the intersecting circles. You will first have to decide what is meant by the phrase *the angle formed by the intersecting circles.* To do so, graph the circles on a graphing device and zoom in on point A. What do you notice?

1. Check the marked intersection

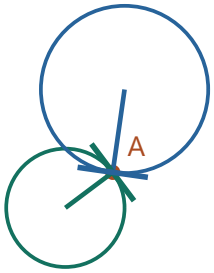
At (4,3) the first circle gives $4^2+3^2=25$. Relative to center (5,10) the point has displacement $(-1,-7)$, whose squared length is 50, so it lies on both circles.



The marked point (4,3) lies on both circles.

2. Replace each curve locally by its tangent

Zooming in makes each arc at the intersection look increasingly straight. Each tangent is perpendicular to its circle's radius there. Rotating both radius directions by 90° preserves the angle between the lines.

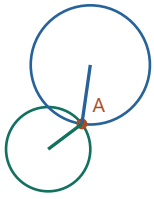


Each tangent is perpendicular to its own radius.

Read the crossing angle between the two tangents.

3. Compute the smaller crossing angle

The radii (4,3) and $(-1,-7)$ have dot product -25 and lengths $5,\sqrt{50}$. For the acute angle between unoriented lines take the absolute cosine, $25/(5\sqrt{50})=1/\sqrt{2}$. The acute crossing angle is 45°.

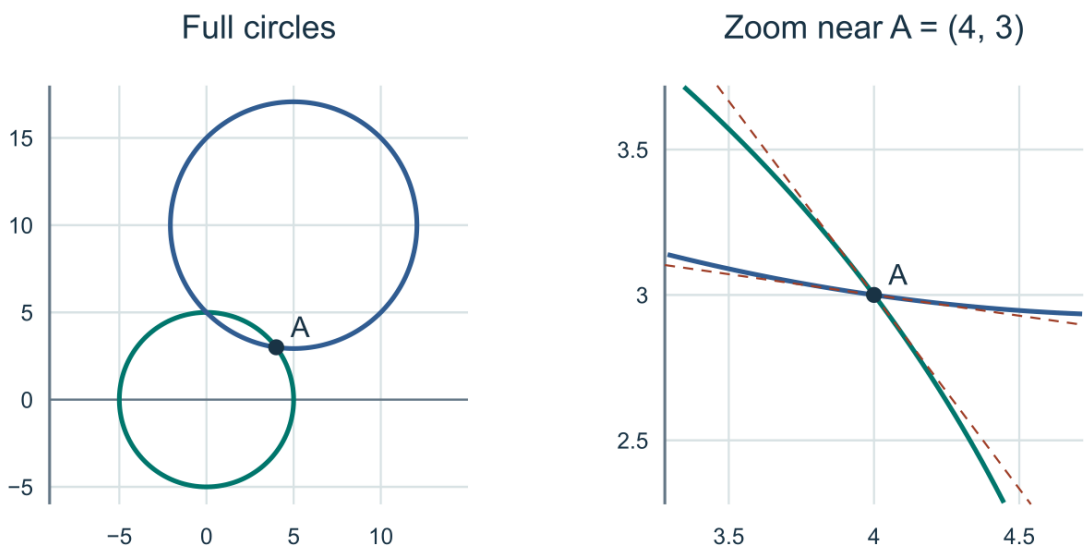


The two radius directions determine the same acute line angle, 45°.

1. $u\cdot v=4(-1)+3(-7)=-25$
Use the two radius vectors normal to the tangents.

2. $\cos\alpha=25/(5\sqrt{50})=5/(5\sqrt{2})=1/\sqrt{2}$
Take the absolute dot product to select the smaller line angle.

3. $\alpha=45°$
The normal lines and tangent lines have the same acute crossing angle.



Check. The tangent slopes −4/3 and −1/7 give an acute angle whose tangent is 1.

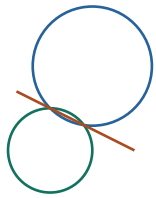
Exercise 197 · The second crossing

Printed page 19 · Systems of equations / Circle geometry

(Continuation) The circles intersect at a second point B. Find coordinates for B. What can be said about the angle of intersection formed by the circles at B?

1. Eliminate the squared terms

The two circle equations are $x^2+y^2=25$ and $(x-5)^2+(y-10)^2=50$. Expand the second and subtract the first. The difference reduces to $x+2y=10$, a line containing both intersections.



Subtracting circle equations gives the straight line $x+2y=10$ through both crossings.

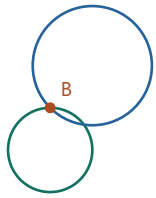
1. $x^2-10x+25+y^2-20y+100=50$
Expand the circle centered at (5,10).

2. $25-10x-20y+125=50$
Use $x^2+y^2=25$ from the first circle.

3. $-10x-20y=-100 \Rightarrow x+2y=10$
Subtract 150, then divide by −10.

2. Solve on the first circle

Substitute $x=10-2y$ into $x^2+y^2=25$. Simplifying gives $y^2-8y+15=(y-3)(y-5)=0$. The first point (4,3) is known. The second is B=(0,5).



Solving on that line gives the second crossing B=(0,5).

1. $(10-2y)^2+y^2=25$
Insert the line's expression for x in the first circle.

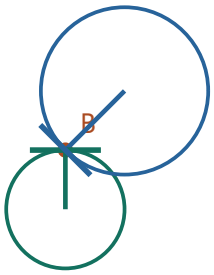
2. $100-40y+5y^2=25 \Rightarrow 5y^2-40y+75=0$
Expand, collect squared terms, and subtract 25.

3. $y^2-8y+15=(y-3)(y-5)=0$
Divide by 5 and factor.

4. $y=5 \Rightarrow x=10-2\cdot5=0$
The other root 3 gives the known point.

3. Check the angle at B

The two radii there are (0,5) and (−5,−5). Their absolute dot product is 25, and their lengths are 5 and √50. The acute crossing angle again has cosine 1/√2 and is 45°.



The tangents at B also make an acute angle of 45°.

Check. Reflection across the line of centers swaps the two crossing points, preserving the crossing angle.

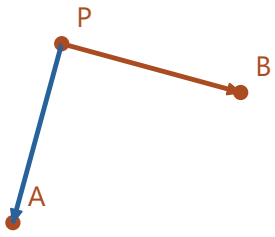
Exercise 198 · A dot-product locus

Printed page 19 · The dot product / Distance and loci

Let $A = (-7, -4)$ and $B = (7, 4)$, and consider the equation $\overrightarrow{PA} \cdot \overrightarrow{PB} = 0$. Describe the configuration of all points $P = (x, y)$ that solve this equation.

1. Express both displacements from P

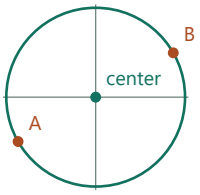
With P=(x,y), the arrows from P to A and B are (−7−x,−4−y) and (7−x,4−y). Being perpendicular corresponds to a zero dot product when both are nonzero.



At P, the two displacement vectors PA and PB form a right angle.

2. Expand the condition

The products in matching coordinate positions give $(x^2-49)+(y^2-16)=0$. Therefore $x^2+y^2=65$.



Expanding PA·PB=0 gives $x^2+y^2=65$.

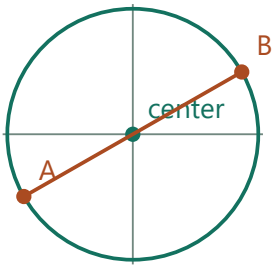
1. $(-7-x)(7-x)=x^2-49$
The cross terms +7x and −7x cancel.

2. $(-4-y)(4-y)=y^2-16$
The y cross terms cancel for the same reason.

3. $x^2-49+y^2-16=0 \Rightarrow x^2+y^2=65$
Collect constants in the perpendicularity equation.

3. Interpret and include the endpoints

This is the circle centered at the origin with radius √65 and diameter AB. Every point other than the endpoints gives a right angle APB. The endpoints also satisfy the dot-product equation because one arrow is then zero.



AB is a diameter.

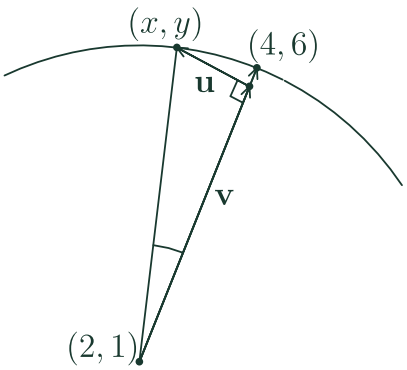
At either endpoint one vector is zero, still satisfying the equation.

Check. For every other point on that circle the inscribed angle subtending the diameter is a right angle.

Exercise 199 · Resolve a rotated displacement

Printed page 19 · Vectors and components / Rotations

Previously you found that a 15-degree counterclockwise rotation centered at $(2, 1)$ sends the point $(4, 6)$ to another point $(x, y) \approx (2.638, 6.347)$. The diagram on the right shows the vector $\mathbf{v}$ in the same direction as vector $[2, 5]$, only with a different length. Similarly, the vector $\mathbf{u}$ is perpendicular to $\mathbf{v}$ and a different length. Show how you can use $(2, 1) + \mathbf{v} + \mathbf{u}$ to find the rotated point (x, y).



1. Read the two directions in the diagram

From the center (2,1) to (4,6) the radius is d=(2,5). A counterclockwise quarter-turn gives the equal-length perpendicular e=(−5,2). These two directions support the illustrated components v and u.



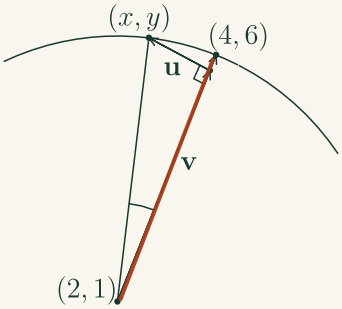
The original radius from C=(2,1) is d=(2,5).

2. Resolve the rotated radius

Relative to d, the new radius makes 15°. Its part along d is v=cos 15°·d. Its perpendicular part toward e is u=sin 15°·e. The cosine and sine multiply equal-length direction vectors here.



The first component along d is cos 15° times d.



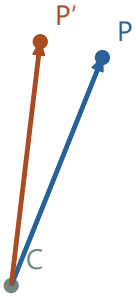
Original radius from the center

The starting arrow leads from (2,1) to (4,6). Its direction defines the parallel component v.

3. Add the two pieces at the center

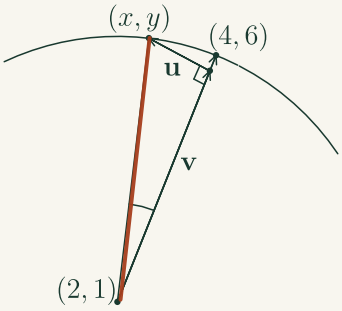
The rotated point is (2,1)+v+u=(2+2cos 15°−5sin 15°,1+5cos 15°+2sin 15°)≈(2.638,6.347).

Counterclockwise turn · 15°



Rotate by 15° about (2, 1).

New point (2.638, 6.347); distance to center remains 5.385.



Rotated radius

The line from the original center to the new endpoint (x,y) combines both components into one radius.

Compare settings · Counterclockwise turn

0°



Rotate by 0° about (2, 1).
New point (4, 6); distance to center remains 5.385.

Rotate by 0° about (2, 1). New point (4, 6); distance to center remains 5.385.

180°



Rotate by 180° about (2, 1).
New point (−0, −4); distance to center remains 5.385.

Rotate by 180° about (2, 1). New point (−0, −4); distance to center remains 5.385.

Check. v and u are perpendicular, and their squared lengths add to 29, so the radius is preserved.

Exercise 200 · The same rotation by matrix

Printed page 19 · Matrix transformations / Rotations

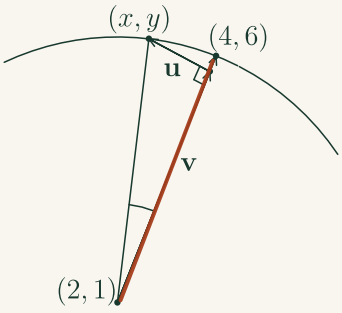
(Continuation) Show how a coefficient matrix with a 15-degree counterclockwise rotation can be used to find the coordinates for the point (x, y). Explain how this is (nearly) equivalent to the vector expression $(2, 1) + \mathbf{v} + \mathbf{u}$.

1. Measure the radius from the center shown in the drawing

The matrix for a counterclockwise turn acts on a displacement from the rotation center. Subtract C=(2,1) from the starting point (4,6) to get the radius d=(2,5). The desired point is C plus the rotated radius.



The original radius from C=(2,1) is d=(2,5).

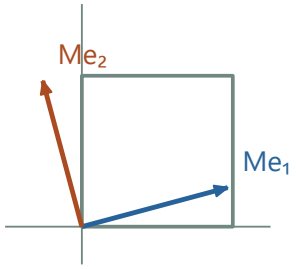


Original radius from the center · Figure from exercise 199

The starting arrow leads from (2,1) to (4,6). Its direction defines the parallel component v.

2. Build the coefficient matrix

Write c=cos 15° and s=sin 15°. During the turn the horizontal unit vector becomes (c,s) and the vertical unit vector becomes (−s,c). These are the first and second columns, so the matrix has rows (c,−s) and (s,c).

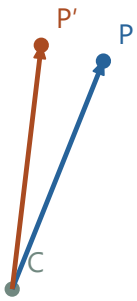


The rotated unit basis directions supply the matrix columns.

3. Multiply and restore the center

The first row times $(2,5)^{\mathsf{T}}$ gives 2c−5s. The second gives 2s+5c. After adding back C, the new coordinates are (2+2c−5s,1+2s+5c)≈(2.638,6.347).

Counterclockwise turn · 15°



Rotate by 15° about (2, 1).

New point (2.638, 6.347); distance to center remains 5.385.

Compare settings · Counterclockwise turn

0°



Rotate by 0° about (2, 1).

New point (4, 6); distance to center remains 5.385.

Rotate by 0° about (2, 1). New point (4, 6); distance to center remains 5.385.

180°



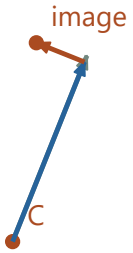
Rotate by 180° about (2, 1).

New point (−0, −4); distance to center remains 5.385.

Rotate by 180° about (2, 1). New point (−0, −4); distance to center remains 5.385.

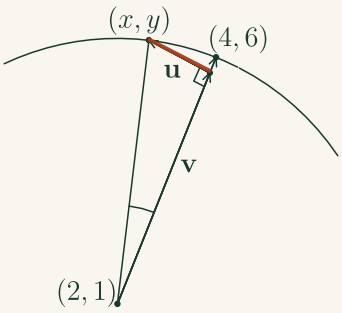
4. Read the very same calculation along the arrows v and u

Regroup the rotated radius as c(2,5)+s(−5,2). The first arrow, v=c(2,5), points along the original radius d. The perpendicular direction (−5,2) is d after a 90° counterclockwise turn, so the second arrow, u=s(−5,2), goes from the foot of the perpendicular to the rotated endpoint in the picture. The sum C+v+u gives exactly the coordinates found by the matrix.



Blue = cos 15°·(2,5).

Orange = sin 15°·(−5,2). Add both arrows after the center.



Perpendicular component u · Figure from exercise 199

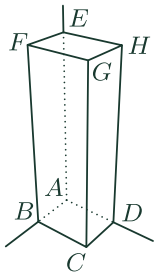
From the perpendicular foot, the short segment reaches the rotated endpoint. The little square marks the change to the perpendicular direction.

Check. Regrouping cosine and sine terms is exactly the parallel-plus-perpendicular decomposition of 199.

Exercise 201 · A box diagonal and its projection

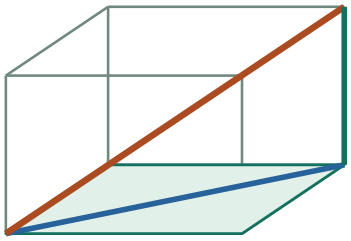
Printed page 20 · Vector projections / Vectors and components

The diagram at right shows a rectangular solid, two of whose vertices are $A = (0, 0, 0)$ and $G = (3, 4, 12)$. Find angle FBH and the vector projection of $\overrightarrow{BF}$ onto $\overrightarrow{BH}$.



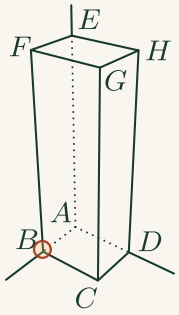
1. Read the two arrows from the common vertex B

Using the labeled box axes take B=(3,0,0), F=(3,0,12) and H=(0,4,12). Subtraction gives BF=(0,0,12) and BH=(−3,4,12).



Use B as the common tail.

BF is vertical; BH is the space diagonal.

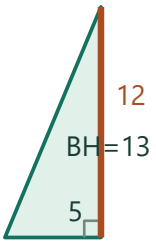


Common tail B

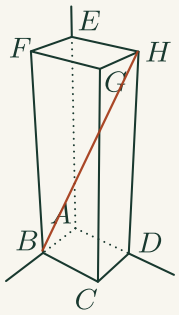
The named angle FBH has its vertex at the middle letter, B. Both directions must be measured from this same point.

2. Find the angle between them

BF has length 12. BH has length √(9+16+144)=13. Their dot product is 144, so cos∠FBH=144/(12·13)=12/13. The angle is about 22.62°.



The angle from the vertical satisfies cos θ=12/13.

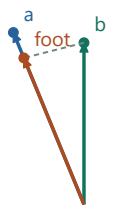


Space diagonal BH

The comparison direction joins B to the opposite top vertex H. Its length involves horizontal and vertical box dimensions together.

3. Project onto the box diagonal

A projection onto BH has the form k BH with k=(BF·BH)/(BH·BH)=144/169. Hence the vector projection is (144/169)(−3,4,12); it points in the diagonal's direction.



Project the vertical arrow onto BH.

The projected vector is 144/169 times (−3,4,12).

1. BF·BH=0(−3)+0·4+12·12=144
Use components with common tail B.

2. BH·BH=9+16+144=169
Square the length of the diagonal receiving the projection.

3. projection=(144/169)(−3,4,12)
Multiply the direction vector by dot product divided by its own squared length.

Check. The horizontal offset from B to H has magnitude 5, so the right triangle also gives tanθ=5/12.

Exercise 202 · Three roots in each family

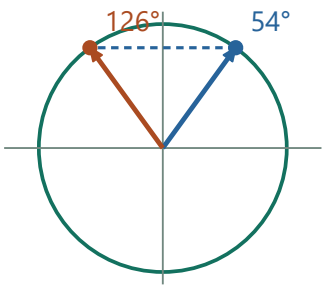
Printed page 20 · Trigonometric equations / Periodic functions

To the nearest degree, find at least three solutions to each of the following:

(a) $\sin A = 0.80902$ **(b)** $\cos B = -0.80902$ **(c)** $\tan C = 1.96261$.

1. Use inverse sine and the upper quadrants

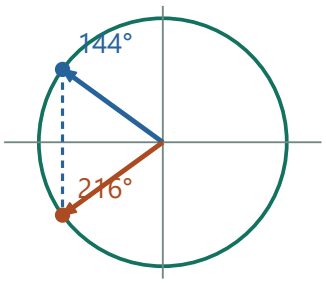
arcsin(0.80902)≈54°. The other upper intersection has angle 180°−54°≈126°. Adding a full turn gives another example, 414°. All reported angles are to the nearest degree.



The same positive sine height gives 54°,126°; one more turn gives 414°.

2. Use the negative cosine

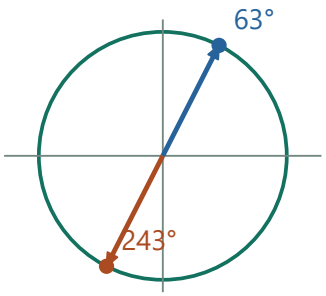
For cos B = −0.80902, the reference angle is about 36°. Negative x occurs in quadrants II and III, giving 144° and 216°. A further full turn from 144° gives 504°.



The two negative-cosine rays are 144° and 216°; 504° is another full-turn copy.

3. Use the slope's half-turn period

arctan(1.96261)≈63°. The same positive slope repeats every 180°, yielding 63°,243°,423°. Further sine or cosine roots repeat after 360°.



Positive slope repeats after 180°.

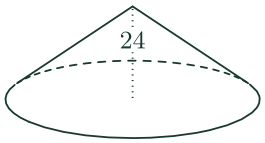
63°,243°,423° each fit to the nearest degree.

Check. The reference angles are approximately 54°,36°,63°, with signs selecting the relevant quadrants.

Exercise 203 · Painting a sand shed

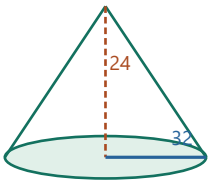
Printed page 20 · Surface area / Prism volume

The highway department keeps its sand in a conical storage building that is 24 feet high and 64 feet in diameter. To estimate the cost of painting the building, the lateral area of the cone is needed. To the nearest square foot, what is the area?



1. Distinguish the three lengths

The cone's 64-ft diameter gives a base radius of 32 ft. Its perpendicular height is 24 ft, while the slant height along the surface is $\ell=\sqrt{(32^2+24^2)}=40$ ft.

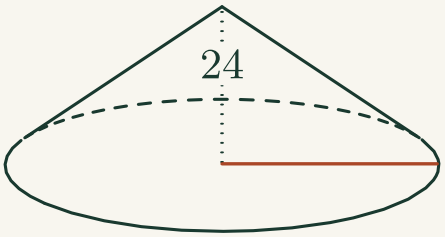


Cone radius 32; perpendicular height 24.

The slant edge runs from rim to vertex.

1. $r=64/2=32$, $\ell^2=32^2+24^2$
The axial slice gives a right triangle whose hypotenuse is the sloping wall.

2. $\ell^2=1024+576=1600 \Rightarrow \ell=40$ ft
Only the positive square root measures a length.

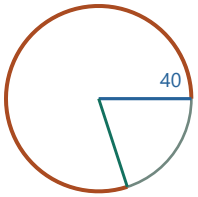


Base radius

In the axial section this radius reaches from the base center to the side of the rim. Its length is half the diameter.

2. Imagine opening the cone wall

The wall becomes a sector of radius 40 and arc length equal to the rim circumference, 64π. A sector's area is half its radius times its arc length.

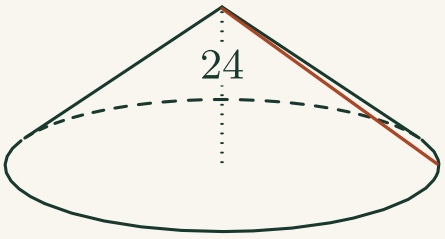


Unroll the lateral surface without stretching.

Its outer arc is exactly the old rim, length $2\pi r$.

1. $s=2\pi\cdot 32=64\pi$ ft
The sector's curved boundary is the building's base rim.

2. $A_{sector}=(s/(2\pi\ell))\pi\ell^2=s\ell/2$
Multiply the full slant disk's area by the rim's fraction of its circumference and cancel $\pi\ell$.

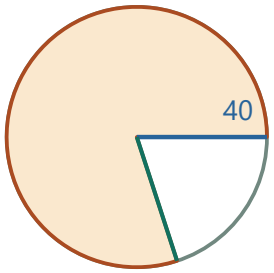


Slant height

The straight generator from the apex to the rim becomes the sector radius when the wall is unrolled.

3. Compute lateral area

A=½·40·64π=1280π ft²≈4021 ft². This measures the wall, as required by lateral area.



Area of the sector = ½·slant height·rim circumference.

1280π ft².

Check. Unrolling yields a sector of radius 40 and arc 64π, with area half their product.

Exercise 204 · Projection formula in any direction

Printed page 20 · Vector projections / The dot product

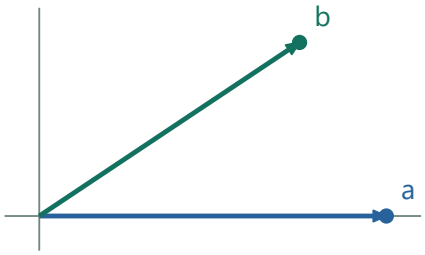
Let vectors $\mathbf{u}$ and $\mathbf{v}$ form an angle θ when placed tail-to-tail, and let $\mathbf{w}$ be the vector projection of $\mathbf{v}$ onto $\mathbf{u}$.

(a) Assume that θ is acute. Notice that $\mathbf{w}$ points in the same direction as $\mathbf{u}$. Find $|\mathbf{w}|$, then show that $\mathbf{w} = |\mathbf{v}|\,\dfrac{\mathbf{u}\bullet\mathbf{v}}{|\mathbf{u}|\,|\mathbf{v}|}\,\dfrac{1}{|\mathbf{u}|}\,\mathbf{u}$, which simplifies to just $\mathbf{w} = \dfrac{\mathbf{u}\bullet\mathbf{v}}{\mathbf{u}\bullet\mathbf{u}}\,\mathbf{u}$.

(b) If θ is obtuse, do $\mathbf{w}$ and $\mathbf{u}$ point in the same direction? Does formula (a) still work? By the way, the notation $\mathbf{proj_u v}$ is sometimes used for the vector projection of $\mathbf{v}$ onto $\mathbf{u}$.

1. Define a signed component along u

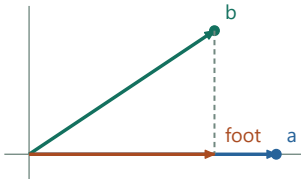
Take U=‖u‖>0 and V=‖v‖. The unit vector u/U points along the target line. For acute θ the length of the shadow is V cos θ.



In an acute angle the shadow points along the target vector.

2. Construct the projected vector

Multiplying that length by its unit direction gives w=(V cos θ)(u/U). Substitute cos θ=(u·v)/(UV) and U^2=u·u. Then w=[(u·v)/(u·u)]u.



The foot is V cos θ along the unit target.

The residual to the original tip is perpendicular.

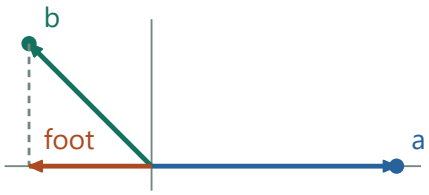
1. w=(V cos θ)(u/U)
Give the signed component the unit direction of u.

2. w=V[(u·v)/(UV)](u/U)
Substitute the dot-product formula for the cosine of the included angle.

3. w=[(u·v)/U^2]u
Cancel V for nonzero v and combine both U factors. If v=0 the projection is zero directly.

3. Interpret obtuse or right angles

At an obtuse angle the scalar coefficient is negative, so the shadow points opposite u and has ordinary length V|cos θ|. At 90° the shadow is zero. The formula also projects the zero vector to zero without needing an angle. In every case (v−w)·u=0 checks perpendicularity.



At an obtuse angle the foot lies beyond the origin on the opposite ray.

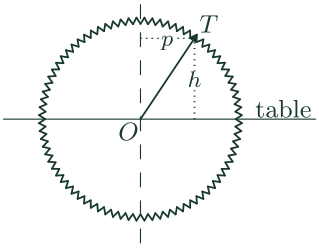
The projection coefficient is negative.

Check. The residual (v−w)·u vanishes exactly.

Exercise 205 · Slope of a rotating radius

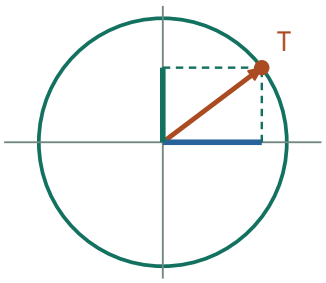
Printed page 20 · The tangent function / Circular motion

Revisit the 1-foot radius circular saw blade with the one red tooth. Look at the ratio m of the height h to the horizontal displacement p, that is $m = h/p$. The red tooth starts at the rightmost point of the saw and rotates at one degree per second counterclockwise. What is m after 37 seconds? After 137 seconds? After 237 seconds? After t seconds? Draw by hand a graph that shows how m is a function of the elapsed time t. What does the ratio $m = h/p$ tell you about the line OT from the saw center to the tooth?

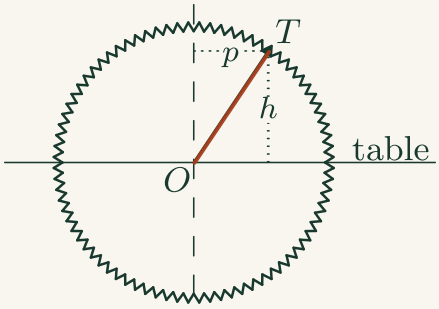


1. Read slope from the radius OT

The tooth T has rightward displacement p and signed height h from the blade center O. Its radial line has slope rise/run=h/p when p≠0.



Slope of OT is signed vertical displacement divided by horizontal displacement.

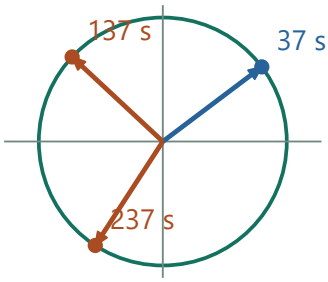


Radius OT

O is the axle and T is the marked tooth. The straight radius sets the line whose slope is being measured.

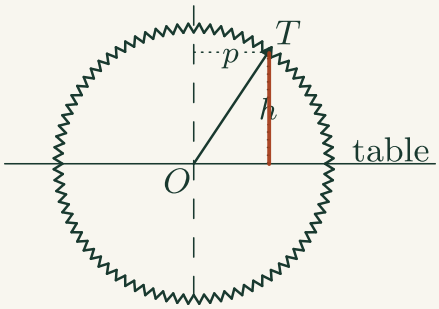
2. Substitute the rotating components

At t seconds the projections are h=sin(t°) and p=cos(t°). Thus m=tan(t°). At 37,137,237 s the slopes are approximately 0.754, −0.933, 1.540.



Substitute sin(t°)/cos(t°).

The signs of the components determine the signs of the slopes.

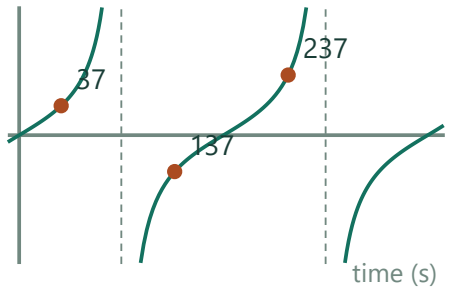


Vertical rise h

The perpendicular from the tooth to table level measures signed height relative to O, the numerator of the slope ratio.

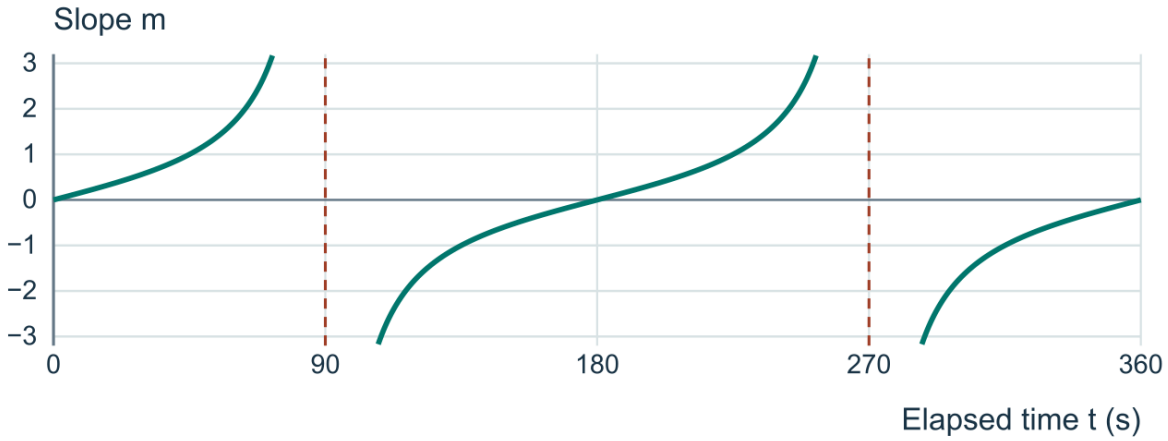
3. Build the tangent graph

A horizontal radius gives zero slope at t=180k. Approaching a vertical radius makes p tend to zero and the slope arbitrarily large in magnitude; the graph has vertical asymptotes at 90+180k. Each continuous branch increases and repeats every 180 s.



Tangent repeats in 180 seconds.

Vertical asymptotes occur at 90+180k seconds.



Dashed verticals: undefined at 90 and 270 seconds.

Check. Slope is undefined at a vertical radius where p=0.

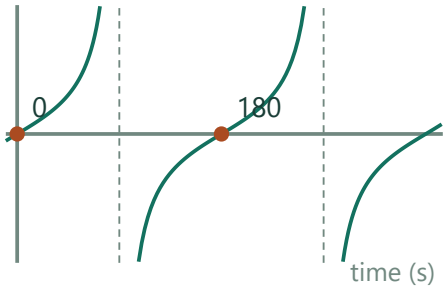
Exercise 206 · Reading the tangent curve

Printed page 20 · The tangent function / Asymptotes

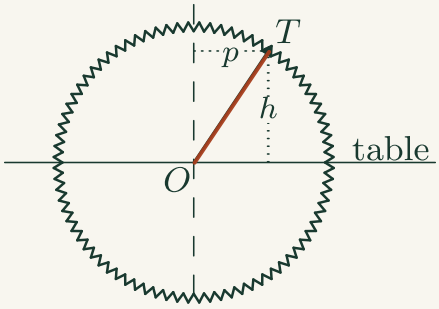
(Continuation) The graph of m versus t is an example of a *tangent* curve. Draw $y = \tan x$ on a graphing tool and compare it to the graph you drew in the previous exercise. Use this graph to determine the values of t for which m takes on the following values: 0, 0.5, and -2. How large can m be? Is m defined for *all* values of t?

1. Compare the tool’s graph with the tooth model

Both graphs evaluate sine divided by cosine in degrees. Thus each increasing tangent branch crosses zero at t=180k and is undefined when t=90+180k, with integer k and nonnegative times for the physical model.



Zeros occur when the tooth is level with the axle.

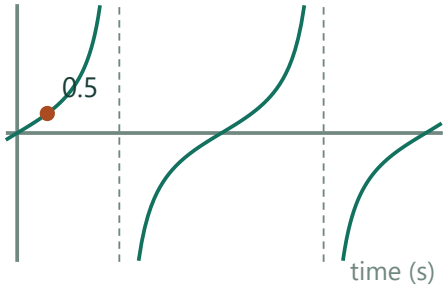


Radius OT · Figure from exercise 205

O is the axle and T is the marked tooth. The straight radius sets the line whose slope is being measured.

2. Read the positive requested slope

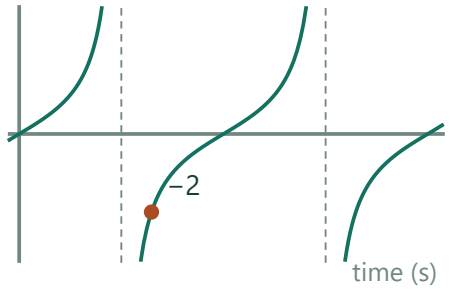
A line at m=0.5 intersects one branch at arctan 0.5≈26.565°. Add 180k to obtain all such times in seconds.



For slope 0.5 the first time is about 26.565 s.

3. Read the negative requested slope

arctan(−2)≈−63.435°. The positive representative one half-turn later is 116.565°, so t≈116.565+180k seconds gives all solutions allowed by the time range.

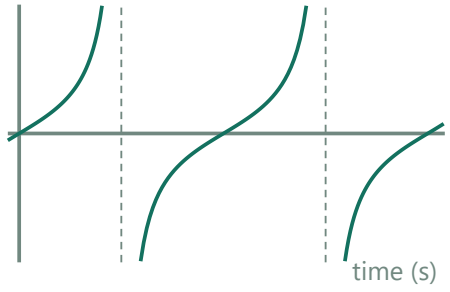


For slope −2 the first positive time is about 116.565 s.

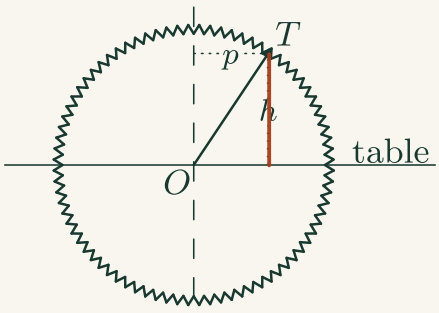
1. arctan(−2)≈−63.435°	The inverse tangent returns the principal negative angle.
2. −63.435°+180°≈116.565°	Add a half-turn for the first nonnegative time with that slope.

4. Interpret how large the slope can be

On any full branch the tangent sweeps all real values, so it has no finite upper bound. At a vertical radius it has no defined value at all.

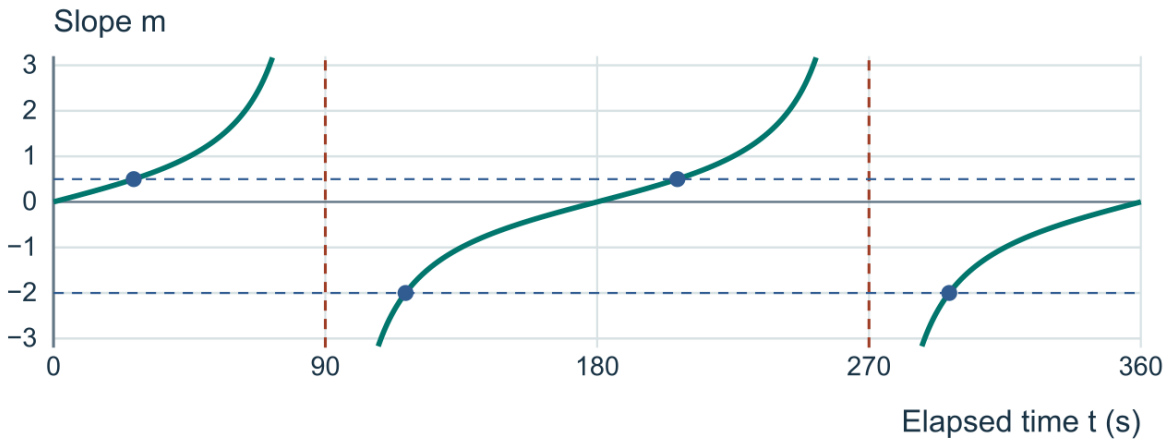


Just before a vertical direction, positive slope becomes arbitrarily large.



Vertical rise h · Figure from exercise 205

The perpendicular from the tooth to table level measures signed height relative to O, the numerator of the slope ratio.



Dashed verticals: undefined at 90 and 270 seconds.

Check. Using inverse tangent gives −63.435° for −2, coterminal modulo 180° with 116.565°.

Exercise 207 · Compose reflection and rotation

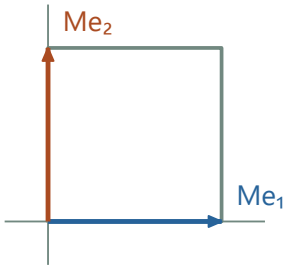
Printed page 20 · Matrix transformations / Reflections / Rotations

Find the entries of the following matrices:

(a) the 2×2 matrix $\mathbf{M}$ for the reflection across the line $y = x$.
(b) the 2×2 matrix $\mathbf{N}$ for the 90-degree counterclockwise rotation about the origin.
(c) the product $\mathbf{MN}$; what transformation does this represent?
(d) the product $\mathbf{NM}$; what transformation does this represent?
(e) the product $\mathbf{MM}$; what transformation does this represent?

1. Read a transformation from its two columns

Write $e_1=(1,0)$ and $e_2=(0,1)$, the two unit coordinate arrows. A two-by-two matrix has the image of e_1 as its first column and the image of e_2 as its second column. Acting on a column $(x,y)^T$ takes x copies of the first column plus y copies of the second.

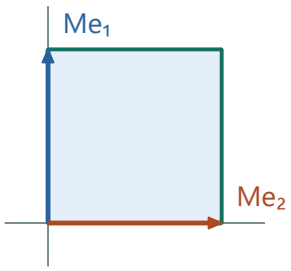


Follow the two basis arrows separately.

Their images become the columns.

2. Construct the reflection M in (a)

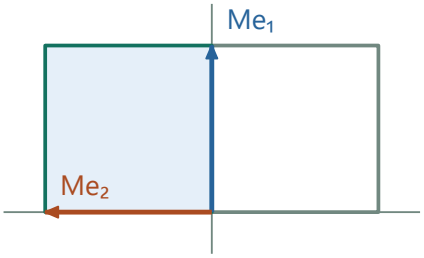
The line y=x bisects the angle between the positive axes. Reflection swaps e_1 and e_2, so the columns of M are (0,1) and (1,0). Thus M has rows (0,1) and (1,0). It sends (x,y) to (y,x), leaving points (t,t) on the mirror fixed.



The mirror y=x swaps the rightward and upward arrows.

3. Construct the quarter-turn N in (b)

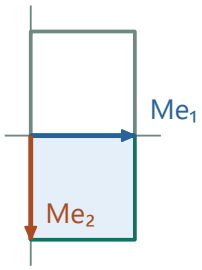
A 90° counterclockwise turn sends the rightward unit arrow to the upward arrow (0,1), and sends the upward unit arrow to the leftward arrow (−1,0). Putting these in the columns gives rows (0,−1) and (1,0), so N(x,y)=(−y,x).



A counterclockwise quarter-turn sends up to left.

4. Compose MN in (c)

For a column p, (MN)p=M(Np). Apply N first to take (x,y) to (−y,x), then M swaps those entries to give (x,−y). Its coordinate rules are x′=1x+0y and y′=0x−1y, giving rows (1,0) and (0,−1). It reflects across the x-axis because it preserves x and reverses height.

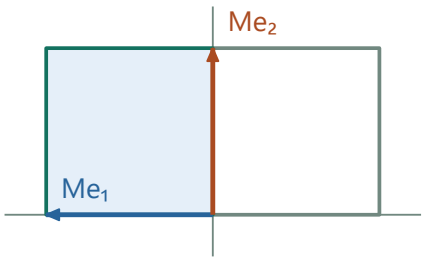


MN applies the turn first and then the mirror.

The outcome reverses height.

5. Compose NM in (d)

This time M acts first, taking (x,y) to (y,x). N then takes (y,x) to (−x,y). The rows are (−1,0) and (0,1). The y coordinate stays fixed while the horizontal displacement reverses, so this is reflection across the y-axis. For example (2,3) goes to (−2,3) here and to (2,−3) in (c).

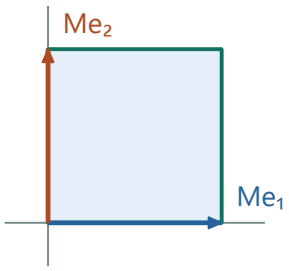


NM swaps first and turns second.

Now the horizontal coordinate reverses.

6. Apply the mirror twice in (e)

Following the two swaps gives (x,y)→(y,x)→(x,y). Each unit arrow returns to itself, so the columns of MM are e_1 and e_2. Its rows are (1,0) and (0,1), the identity matrix. Every point remains at its starting position.



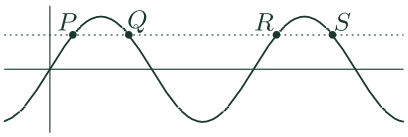
Swapping twice restores every point: MM=I.

Check. Following (x,y) through the two maps in order verifies each sign.

Exercise 208 · Successive sine intersections

Printed page 20 · Trigonometric equations / Periodic functions

The graphs of $y = \sin x$ and $y = k$ (dotted) are shown at right. Given that the coordinates of P are (θ, k), find the coordinates of Q, R, and S, in terms of θ and k.



1. Locate Q at the same sine height

A first crossing P=(θ,k) lies on the rising branch. Reflection in x=90° moves it the same distance past 90° to the falling branch. Thus Q=(180°−θ,k).

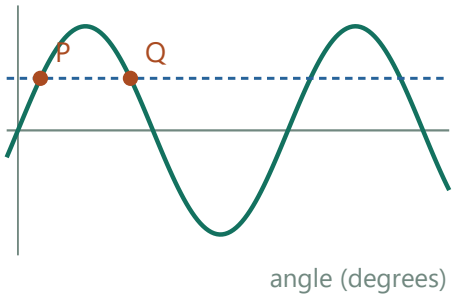
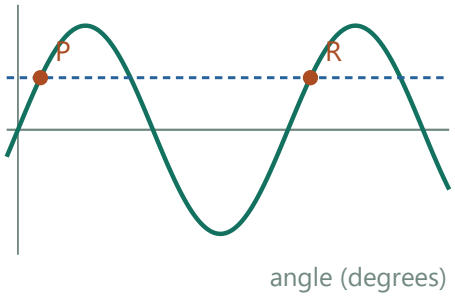


Illustration with k=½.

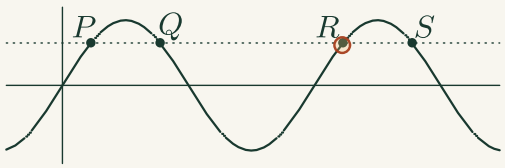
Reflection about 90° maps θ to 180°−θ.

2. Move P forward one full cycle

Sine repeats every 360° in degree mode. Its next rising crossing is R=(360°+θ,k).



Add 360° to P to reach the next point in its family.

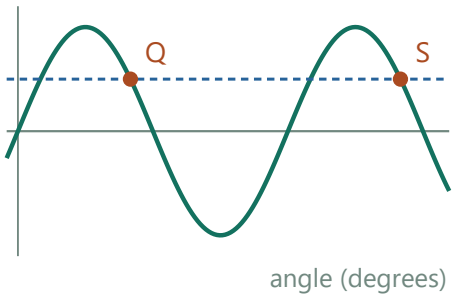


Next rising intersection R

R is one full cycle beyond P, at the same height and on a rising branch again.

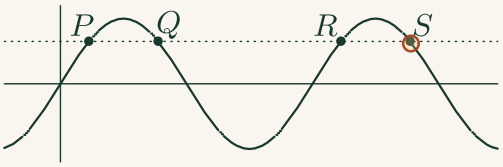
3. Move Q by the same amount

The next falling crossing is S=(360°+180°−θ,k)=(540°−θ,k). All four lie on the same horizontal line by construction.



Add 360° to Q too.

S has abscissa 540°−θ.



Corresponding falling intersection S

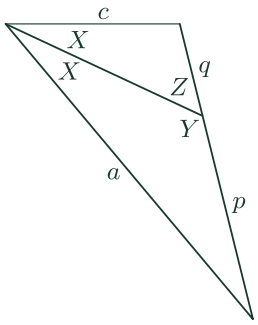
S plays the same role in its cycle as Q did in the first, and meets the same dotted line.

Check. The spacing pattern alternates 180°−2θ and 180°+2θ, totaling 360° per pair.

Exercise 209 · Angle-bisector theorem

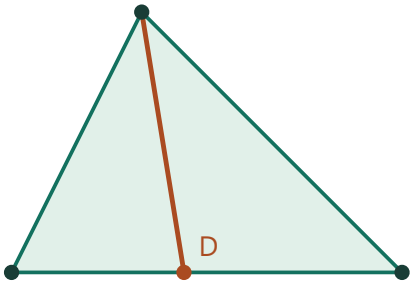
Printed page 21 · Angle bisectors / Triangle area using sine

Refer to the diagram at right for the following questions: Express the ratio $p : a$ in terms of $\sin X$ and $\sin Y$. Express the ratio $q : c$ in terms of $\sin X$ and $\sin Z$. Because angles Y and Z are supplementary, you can now combine the preceding answers to obtain a familiar result about angle bisectors.

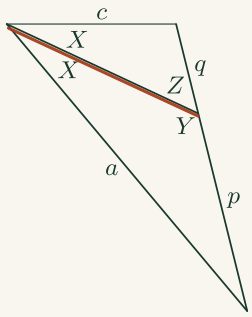


1. Use the two equal angles marked X

The bisector splits the vertex angle into two equal angles. In one small triangle the sine rule gives p/a=sin X/sin Y. In the other it gives q/c=sin X/sin Z.



The bisector creates two equal angles at the common apex.

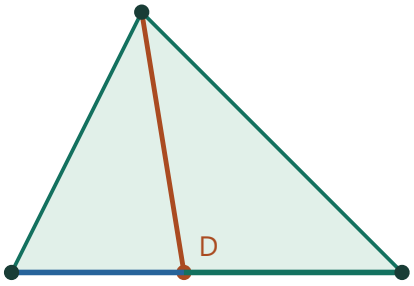


Angle bisector from the shared vertex

The segment from the left vertex to the opposite side passes between the two angles X. Each small triangle receives one of these equal angles.

2. Relate the angles on the opposite side

Y and Z sit adjacent on a straight line, so they sum to 180°. Supplementary angles have the same sine: sin Y=sin Z.

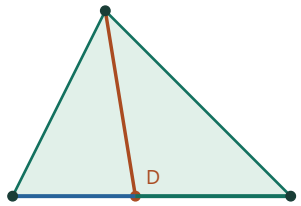


The neighboring angles at D are supplementary.

Their sine values therefore agree.

3. Compare the side portions

Both ratios therefore equal the same number, giving p/a=q/c, or p/q=a/c. The cuts p and q of the opposite side are proportional to the adjoining sides.

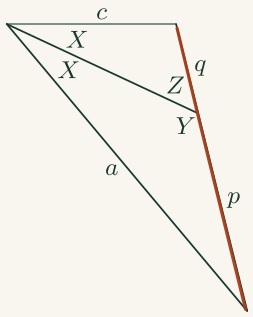


The two base portions are in the ratio of the adjacent sides.

In the source notation p/q = a/c.

1. p/a=q/c ⇒ pc=aq
Cross-multiply using positive triangle sides.

2. p/q=a/c
Divide by cq to compare the portions with the adjacent sides.



Divided side q and p

From the top-right corner down to the cut is q; from there to the bottom-right corner is p. They are opposite the two equal angles.

Check. The two small triangles have area ratio p/q from their common altitude and also a/c from equal included angles.

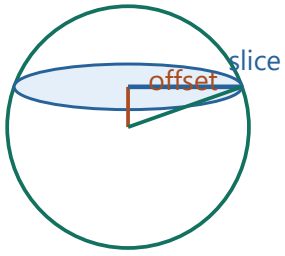
Exercise 210 · Off-center grapefruit slice

Printed page 21 · Sphere geometry / Cross sections

The diameter of a spherical grapefruit is 6.0 inches. An amateur fruit slicer misses its center by one inch. What is the radius of the circular slice?

1. Draw a radius to the slice rim

The sphere diameter is 6 in, so its radius is 3 in. Drop a perpendicular from the sphere center to the cutting plane. Its foot is the center of the circular slice.

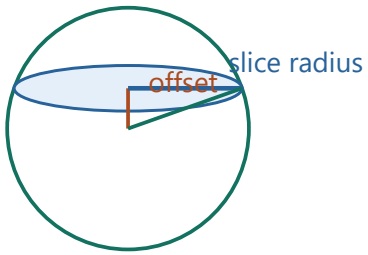


Cut the sphere with a plane. The circular cross section shrinks away from the center.

2. Identify the right triangle

The normal segment is 1 in, the segment from its foot to a rim point has unknown length a, and the sphere radius to that point is 3 in. The first two lie perpendicular to each other.

Offset from the sphere center · 1 in

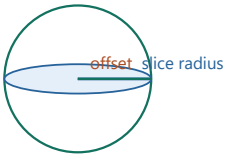


Sphere radius 3; offset 1.

Pythagoras gives slice radius $\sqrt{(3^2-1^2)} \approx 2.828$.

Compare settings · Offset from the sphere center

0 in



Sphere radius 3; offset 0.

Pythagoras gives slice radius $\sqrt{(3^2-0^2)} \approx 3$.

Sphere radius 3; offset 0. Pythagoras gives slice radius $\sqrt{(3^2-0^2)} \approx 3$.

3 in



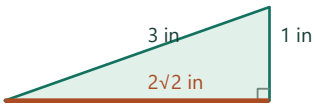
Sphere radius 3; offset 3.

Pythagoras gives slice radius $\sqrt{(3^2-3^2)} \approx 0$.

Sphere radius 3; offset 3. Pythagoras gives slice radius $\sqrt{(3^2-3^2)} \approx 0$.

3. Solve for the in-plane radius

Then $1^2+a^2=3^2$. Thus $a=\sqrt{8}=2\sqrt{2}\approx 2.83$ in, slightly less than the sphere radius as expected off center.



The circular section radius is $2\sqrt{2}$ in.

1. $a^2=3^2-1^2=9-1=8$
Subtract the squared offset from the squared sphere radius.

2. $a=\sqrt{(4\cdot 2)}=2\sqrt{2}$ in
A radius is nonnegative and the square factor 4 comes out as 2.

Check. The central slice would have radius 3; this displaced slice is smaller.

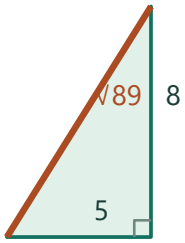
Exercise 211 · Polar forms for (−5,8)

Printed page 21 · Polar coordinates / Inverse trigonometric functions

The point $P = (-5, 8)$ is in the second quadrant. You are used to describing it by using the *rectangular coordinates* -5 and 8. It is also possible to accurately describe the location of P by using a different pair of coordinates, $P = (r; \theta)$, called *polar coordinates*. The distance from the origin is $|r|$ and θ is an angle in *standard position*. Calculate polar coordinates for P, and notice that there is more than one correct answer.

1. Read the radius before the angle

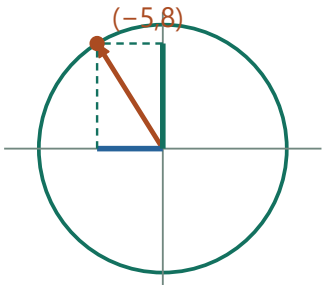
The point $(-5,8)$ is $\sqrt{((-5)^2+8^2)}=\sqrt{89}$ units from the origin. Its negative x and positive y place it in quadrant II.



The absolute coordinate changes have magnitudes 5 and 8.

2. Locate the terminal ray

The reference angle with the negative x-axis is $\arctan(8/5)$. Measured counterclockwise from positive x, $\theta=180°-\arctan(8/5)\approx122.005°$.

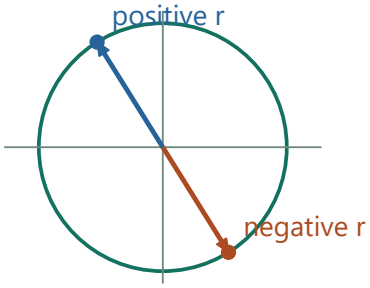


Quadrant II gives $\theta\approx122.005°$.

Radius $\sqrt{89}$.

3. Describe equivalent polar pairs

A positive radius $\sqrt{89}$ works with $\theta+360°k$ for every integer k. A negative radius reverses a ray, so $(-\sqrt{89},\theta-180°)$, approximately $(-\sqrt{89},-57.995°)$, locates the same point, again with full turns allowed.



A negative radius measures backward along the opposite direction to the same point.

Check. Multiplying each radial value by its cosine and sine gives $x=-5, y=8$.

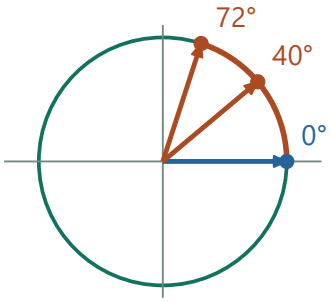
Exercise 212 · Compose 32° and 40° rotations

Printed page 21 · Matrix multiplication / Rotations

Verify that $\begin{bmatrix}\cos 32 & -\sin 32\\ \sin 32 & \cos 32\end{bmatrix}\begin{bmatrix}\cos 40 & -\sin 40\\ \sin 40 & \cos 40\end{bmatrix}=\begin{bmatrix}\cos 72 & -\sin 72\\ \sin 72 & \cos 72\end{bmatrix}$, and explain why this result could have been expected.

1. Read composition from right to left

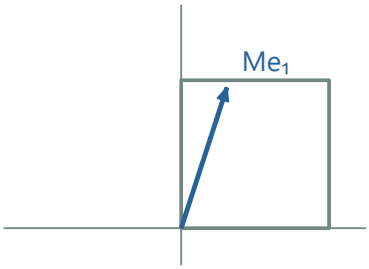
The right matrix rotates a column 40° and the left one rotates the result 32° about the same center. Geometrically the net rotation is 72°.



First rotate 40°; then add 32° about the same center.

2. Multiply the first column

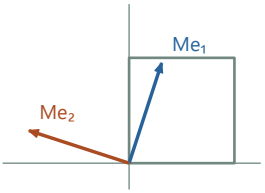
The new upper entry is cos 32° cos 40°−sin 32° sin 40°=cos 72°. The lower entry is sin 32° cos 40°+cos 32° sin 40°=sin 72°, by the angle-addition formulas.



The first column is the direction (cos72°,sin72°).

3. Multiply the second column

The resulting entries are −sin 72° and cos 72°. Thus the two columns are precisely the images of the coordinate unit vectors under a 72° rotation, verifying the entire matrix equality.



The perpendicular unit second column completes the 72° rotation.

1. upper right=−cos32° sin40°−sin32° cos40°=−sin72°
Dot the first rotation row with the second 40-degree column; both terms carry minus signs.

2. lower right=−sin32° sin40°+cos32° cos40°=cos72°
Use the cosine addition identity to complete the second column.

Check. Apply the product to both unit basis vectors; each has turned through 72°.

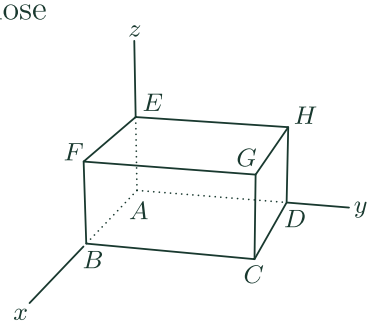
Exercise 213 · Projections in a second box

Printed page 21 · Vectors and components / Vector projections

The diagram at right shows a rectangular solid, two of whose vertices are $A = (0, 0, 0)$ and $G = (4, 6, 3)$.

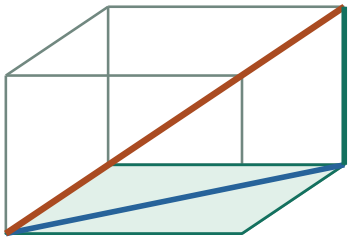
(a) Find vector projections of $\overrightarrow{AG}$ onto $\overrightarrow{AB}$, $\overrightarrow{AD}$, and $\overrightarrow{AE}$.

(b) Find the point on segment AC that is closest to the midpoint of segment GH.

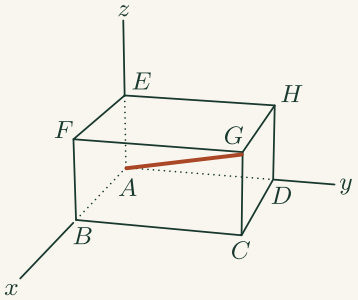


1. Project onto the three box edges

With A=(0,0,0) and G=(4,6,3), vector AG has three coordinate components. Its projections onto AB, AD and AE, the three coordinate-edge directions, are (4,0,0),(0,6,0),(0,0,3).



Resolve the box diagonal into the three mutually perpendicular edge directions.

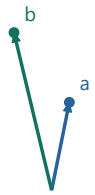


Space diagonal AG

The displacement from A to the top opposite vertex G has contributions along all three coordinate directions.

2. Locate the target point and line

The midpoint M of GH is (2,6,3). Diagonal AC has direction c=(4,6,0) from A. The point on its line closest to M must be tc, with M−tc perpendicular to c.



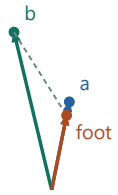
The target AC has displacement (4,6,0); the marked midpoint is M=(2,6,3).

1. G=(4,6,3), H=(0,6,3)
The edge GH runs parallel to the x-axis. Its far endpoint has the same y and z values.

2. M=((4+0)/2,(6+6)/2,(3+3)/2)=(2,6,3)
The three coordinate averages locate the midpoint used for the projection.

3. Solve for the nearest point

Compute t=(M·c)/(c·c)=(8+36)/(16+36)=11/13. Thus tc=(44/13,66/13,0). Because 0<t<1, the perpendicular foot is also on the actual segment AC.



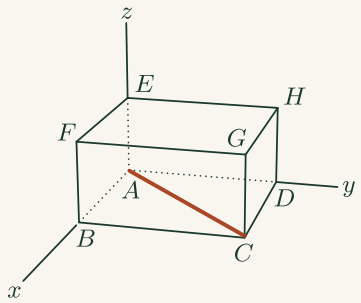
The nearest point is 11/13 of the way along AC.

The perpendicular residual checks the foot.

1. 0=(M−tc)·c=M·c−t(c·c)
The perpendicular residual identifies the foot on AC.

2. t(4²+6²)=2·4+6·6+3·0 ⇒ 52t=44
Compute the matching coordinate products.

3. t=11/13 ⇒ tc=(44/13,66/13,0)
The factor lies between 0 and 1, placing the foot on the segment.



Base diagonal AC

The line on which the nearest point must lie starts at A and travels across the base to C.

Check. Subtracting that point from M gives a vector perpendicular to AC.

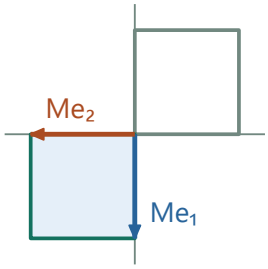
Exercise 214 · Rotate after reflecting

Printed page 21 · Matrix multiplication / Reflections

The result of reflecting across the line $y = -x$ and then rotating 330 degrees counterclockwise around the origin is an isometry $\mathcal{T}$. Represent $\mathcal{T}$ by a 2×2 matrix. There is more than one way to do it. Use the point $(1, 1)$ to check your answer.

1. Perform the reflection first

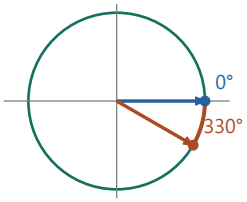
Across $y=-x$, the point (x,y) maps to $(-y,-x)$. Keep this order: these are the coordinates to which the rotation is applied.



Reflection in $y=-x$ sends (x,y) to $(-y,-x)$.

2. Rotate the reflected pair

Use rotation rows $(\cos 330°,-\sin 330°),(\sin 330°,\cos 330°)$, where $c=\sqrt{3}/2$ and $s=-1/2$. Acting on $(-y,-x)$ gives $(-x/2-\sqrt{3}y/2,-\sqrt{3}x/2+y/2)$.



Rotate the reflected output clockwise through 30°.

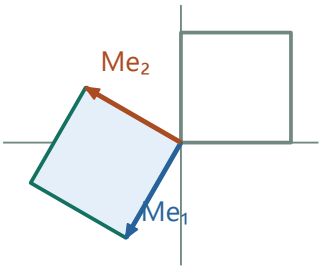
1. $x'=(\sqrt{3}/2)(-y)-(-1/2)(-x)$
Feed the reflected pair $(-y,-x)$ to the first rotation row.

2. $x'=-x/2-\sqrt{3}y/2$
The subtracted product of two negative factors contributes $-x/2$.

3. $y'=(-1/2)(-y)+(\sqrt{3}/2)(-x)=-\sqrt{3}x/2+y/2$
The second rotation row produces the other output.

3. Read the composite matrix

Collect the coefficients of x and y from each output. The first row is $(-1/2,-\sqrt{3}/2)$ and the second is $(-\sqrt{3}/2,1/2)$.



The result combines both transformations.

Track both unit directions to read the two columns.

Check. At $(1,1)$ this gives $(-(1+\sqrt{3})/2,(1-\sqrt{3})/2)$, also obtained by rotating $(-1,-1)$.

Exercise 215 · The diameter sphere

Printed page 21 · Sphere geometry / Midpoints / The dot product

Given points A and B in 3-dimensional space, describe the solutions to $\overrightarrow{PA}\bullet\overrightarrow{PB}=0$.

1. Choose the midpoint to simplify the geometry

Let M=(A+B)/2 and d=(B−A)/2. Measured from M, the endpoints sit at −d and d. Write the displacement to P as q=P−M.

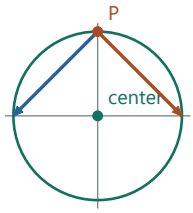


M is the midpoint of AB.

The two endpoints have opposite displacement vectors about M.

2. Compute the two arrows starting at P

They are PA=−d−q and PB=d−q. Expanding their dot product cancels the cross terms, leaving q·q−d·d.



Plane through A, B and P.

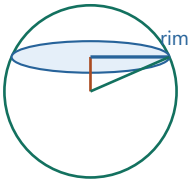
At the illustrated P, PA and PB are perpendicular.

1. $(-d-q)\cdot(d-q)=-d\cdot d+d\cdot q-q\cdot d+q\cdot q$
Distribute both differences using the displacements from P.

2. $=q\cdot q-d\cdot d$
The mixed dot products are equal, with opposite signs.

3. Describe every qualifying point

The zero condition becomes $\|q\|^2=\|d\|^2$, so P lies at distance AB/2 from M. This is the sphere with diameter AB, including both endpoints by the zero-vector convention. If A=B the locus reduces to that single point.



Rotating the diameter-circle plane about AB sweeps out the sphere.

Its radius is AB/2.

1. $q\cdot q-d\cdot d=0 \Rightarrow \|P-M\|^2=\|d\|^2$
Impose perpendicularity and replace self-products by squared lengths.

2. $\|P-M\|=\|B-A\|/2$
Take the nonnegative root and use d=(B−A)/2.

Check. Every plane through AB cuts this sphere in the plane's circle with diameter AB. If A=B, only that one point works.

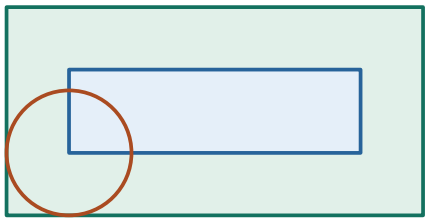
Exercise 216 · Carnival ring size

Printed page 21 · Geometric probability / Quadratic equations

To win the carnival game *Ring Ding*, you must toss a wooden ring onto a grid of rectangles so that it lands without touching any of the grid lines. The ring has a 3-inch diameter, the rectangles are twice as long as they are wide, and the game has been designed so that you have a 28% chance of winning. What are the dimensions of each rectangle?

1. Find the ring-center positions that win

The ring diameter is 3 in. In a cell of dimensions w and 2w, its center must stay 1.5 in from each boundary. Removing both border strips leaves safe dimensions w−3 and 2w−3.



Only center positions at least 1.5 in from the edges can win.

Shown for the admissible 2-to-1 shape.

2. Set the fraction of area to 28%

Under uniform landing, $(w-3)(2w-3)=0.28\cdot 2w^2$. Expanding gives $2w^2-9w+9=0.56w^2$, hence $1.44w^2-9w+9=0$.



Winning area / whole area = 0.28.

Use $(w-3)(2w-3)/(2w^2)$.

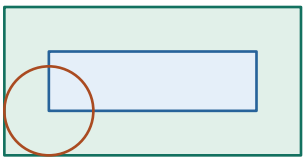
1. $(w-3)(2w-3)=0.28(2w^2)$
Safe-center area must be 28% of the whole tile area.

2. $2w^2-9w+9=0.56w^2$
Distribute the left factors.

3. $1.44w^2-9w+9=0$
Subtract the right side to collect squared terms.

3. Filter the algebraic roots by the geometry

The roots are 5 and 1.25. A cell narrower than the ring cannot have a positive safe area, so w=5. The cell is 5 by 10 in, with safe area 2·7=14 out of 50 in².



At w=5, inner dimensions are 2 × 7 and outer dimensions are 5 × 10.

14/50 = 28%.

1. $1.44w^2-9w+9=1.44(w-5)(w-1.25)$
These roots have sum 6.25 and product 6.25.

2. w=5 or w=1.25
Set the factors to zero.

3. $w-3>0 \Rightarrow$ w=5 in
A positive safe width rules out 1.25.

Check. The safe area is 2×7=14 in² out of 50, exactly 28%.

Exercise 217 · Symmetry through a full turn

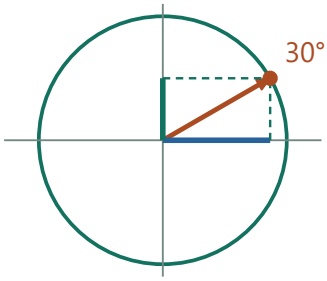
Printed page 21 · Reference angles and symmetry / The unit circle

Simplify the following:

(a) $\cos(360-\theta)$ **(b)** $\sin(360-\theta)$ **(c)** $\cos(360+\theta)$
(d) $\sin(360+\theta)$ **(e)** $\tan(360+\theta)$

1. Use a point on the unit circle

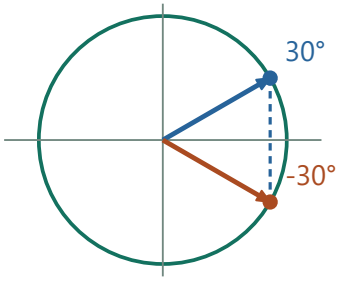
Take θ in degrees and let its terminal point be P=(x,y). By definition x=cos θ and y=sin θ. Reading the relevant coordinate after moving or reflecting this point will simplify each expression.



Start with the radius at 30°.

2. Read cosine in (a)

The direction 360°−θ is the same as −θ. From the positive horizontal axis, the turn −θ goes as far clockwise as θ goes counterclockwise. The new point is P reflected across the horizontal axis, with coordinates (x,−y). Thus cos(360°−θ)=x=cos θ.

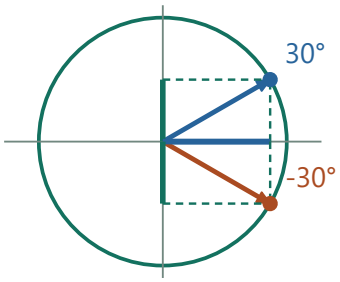


Reflect across the horizontal axis.

The horizontal coordinate stays fixed; the vertical sign reverses.

3. Read sine in (b)

Use the same reflected point (x,−y), now reading its vertical coordinate. Its height is the negative of the original height, so sin(360°−θ)=−sin θ. For instance θ=30° puts the reflection at 330°, below the axis.

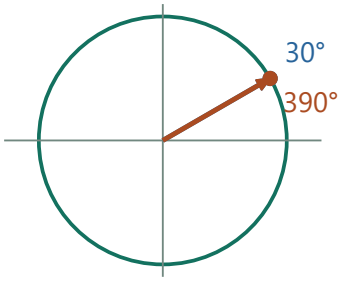


Reflect across the horizontal axis.

The horizontal coordinate stays fixed; the vertical sign reverses.

4. Read cosine and sine after a full turn in (c) and (d)

Moving from θ to 360°+θ takes one full revolution and lands back on P itself. The horizontal coordinate gives (c) cos(360°+θ)=cos θ. The vertical coordinate gives (d) sin(360°+θ)=sin θ. For θ=30°, the 390° point therefore has the same height as the 30° point.

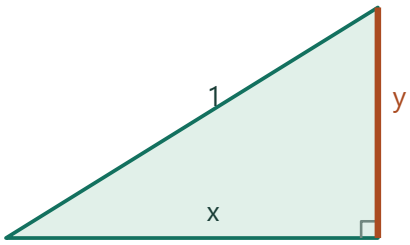


A whole extra turn returns to the original point.

Cosine and sine both stay unchanged.

5. Read the slope for tangent in (e)

On the unit circle tan θ=y/x whenever x≠0. A full full turn leaves both numerator and denominator unchanged, so tan(360°+θ)=tan θ. At θ=90°+180°k for an integer k, x=0 and both tangents are undefined.



Schematic section

The same y/x gives the same tangent when x≠0.

Check. At θ=30°, 330° has negative sine, whereas 390° has the same sine as 30°.

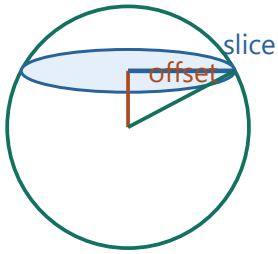
Exercise 218 · General radius of a sphere slice

Printed page 21 · Sphere geometry / Cross sections

A sphere of radius r inches is sliced by a plane that is k inches from the center. In terms of r and k, what is the radius of the circle of intersection?

1. Drop the center-to-plane normal

Let O be the sphere center and H the perpendicular foot in the cutting plane, distance OH=k. For any slice point P, HP lies in the plane and so is perpendicular to OH.

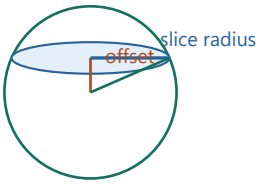


Cut the sphere with a plane. The circular cross section shrinks away from the center.

2. Find the common distance to the foot

With sphere radius OP=r, Pythagoras gives $HP^2=r^2-k^2$. For $0\le k<r$ this describes a circle centered at H with radius $\sqrt{(r^2-k^2)}$.

Offset from the sphere center · 2 units



Sphere radius 5; offset 2.

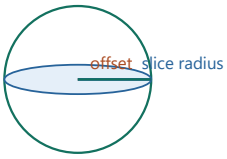
Pythagoras gives slice radius $\sqrt{(5^2-2^2)} \approx 4.583$.

1. $HP^2+k^2=r^2 \Rightarrow HP^2=r^2-k^2$
The perpendicular foot fixes a right triangle with hypotenuse r.

2. $HP=\sqrt{[(r-k)(r+k)]}$
Factor the difference of squares. For $0\le k\le r$ the factors are nonnegative, making the root real.

Compare settings · Offset from the sphere center

0 units



Sphere radius 5; offset 0.

Pythagoras gives slice radius $\sqrt{(5^2-0^2)} \approx 5$.

Sphere radius 5; offset 0. Pythagoras gives slice radius $\sqrt{(5^2-0^2)} \approx 5$.

5 units



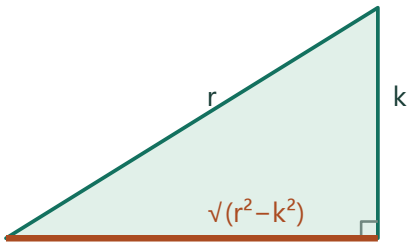
Sphere radius 5; offset 5.

Pythagoras gives slice radius $\sqrt{(5^2-5^2)} \approx 0$.

Sphere radius 5; offset 5. Pythagoras gives slice radius $\sqrt{(5^2-5^2)} \approx 0$.

3. Check the limiting cases

At k=0 the plane goes through the center and its circular radius is r. At k=r the slice is a point. At k>r a negative square would be required and the plane misses the sphere.



Schematic section

The circular section radius is $\sqrt{(r^2-k^2)}$.

Check. At k=0 this gives a great circle of radius r.

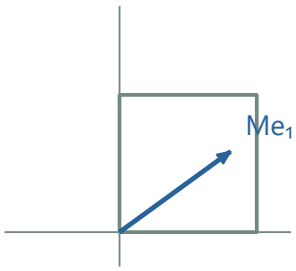
Exercise 219 · Compute a rotated point with a matrix

Printed page 21 · Rotations / Matrix transformations

A 36-degree counterclockwise rotation centered at the origin sends the point $A = (6, 3)$ to the image point A'. Use a rotation matrix to find coordinates for A'.

1. Write down the rotation directions

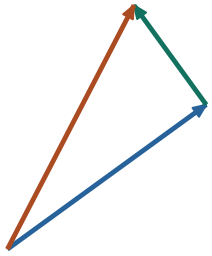
For 36° let c=cos 36°, s=sin 36°. A horizontal unit arrow rotates to (c,s) and a vertical unit arrow to (−s,c).



The first column gives the new horizontal unit arrow for 36°.

2. Act on the actual vector

The input point (6,3) combines six horizontal and three vertical units. The rotation matrix therefore gives $6(c,s)+3(-s,c)=(6c-3s,6s+3c)$.



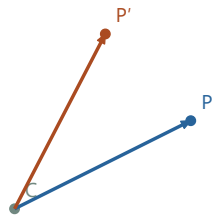
Six turned horizontal units plus three turned vertical ones.

Add head-to-tail to get (6c−3s, 6s+3c), with c=cos 36° and s=sin 36°.

3. Evaluate and check

The image is approximately (3.091,5.954). Its squared distance from the origin remains $6^2+3^2=45$, and its direction has increased by 36°.

Counterclockwise turn · 36°



Rotate by 36° about (0, 0).

New point (3.091, 5.954); distance to center remains 6.708.

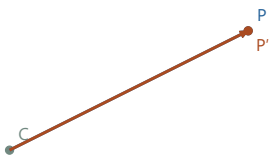
1. $(6c-3s)^2=36c^2-36cs+9s^2$
Square the first image component using c=cos36° and s=sin36°.

2. $(6s+3c)^2=36s^2+36sc+9c^2$
The second component contributes the opposite cross term.

3. $\text{sum}=45(c^2+s^2)=45$
The length check now follows from the unit-circle identity.

Compare settings · Counterclockwise turn

0°

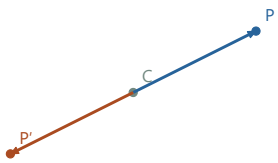


Rotate by 0° about (0, 0).

New point (6, 3); distance to center remains 6.708.

Rotate by 0° about (0, 0). New point (6, 3); distance to center remains 6.708.

180°



Rotate by 180° about (0, 0).

New point (−6, −3); distance to center remains 6.708.

Rotate by 180° about (0, 0). New point (−6, −3); distance to center remains 6.708.

Check. Adding 36° to the original polar angle arctan(1/2) at fixed radius √45 gives the same point.

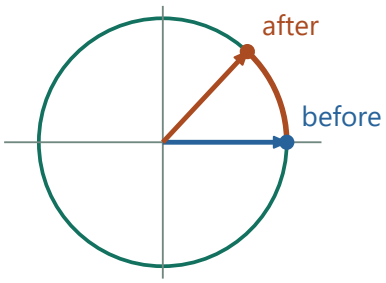
Exercise 220 · A rotation and its inverse

Printed page 22 · Matrix inverses / Rotations

$$\text{Calculate the product } \begin{bmatrix} \cos 47 & -\sin 47 \\ \sin 47 & \cos 47 \end{bmatrix} \begin{bmatrix} \cos 47 & \sin 47 \\ -\sin 47 & \cos 47 \end{bmatrix}. \text{ Interpret the result.}$$

1. Identify the reversed angle

Write c=cos 47°, s=sin 47°. A counterclockwise turn has rows (c,−s),(s,c). The reverse turn keeps c and changes the signs of s, giving rows (c,s),(−s,c).



One matrix turns through +47°; the other turns through −47°.

2. Compute the four row-column products

The two diagonal entries are $c^2+s^2=1$. The two off-diagonal entries are cs−sc and sc−cs, each zero.

c^2+s^2	$-cs+sc$
$sc-cs$	s^2+c^2

The two diagonal entries are 1; the off-diagonal ones vanish.

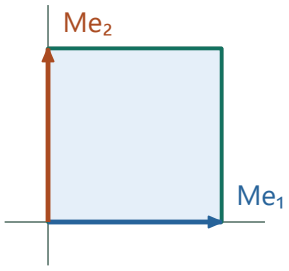
1. c=cos47°, s=sin47°
The inverse rotation has the same cosine and opposite sine.

2. diagonal entries: c^2+s^2, s^2+c^2
In each matching row and column, the two negative signs multiply to a plus.

3. off-diagonal entries: cs−sc, sc−cs
The two mixed products cancel. The diagonals both equal one.

3. Interpret the identity

The resulting rows (1,0),(0,1) form I. This fits the geometry: a 47° turn and its reverse about the same center restore any starting point.



Every point returns to its initial location.

Check. A 47° clockwise turn followed by a 47° counterclockwise turn restores every point.

Exercise 221 · A parallelogram from two sides

Printed page 22 · Triangle area / The cosine rule

A parallelogram has a 5-inch side and an 8-inch side that make a 50-degree angle. Find the area of the parallelogram and the lengths of its diagonals.

1. Calculate the perpendicular height

For a parallelogram with base 8 in, the 5-in side makes angle 50° with the base. Its perpendicular component is 5 sin 50°. Hence its area is 8·5 sin 50°≈30.642 in².

Included angle · 50°



Perpendicular height = 5 sin 50° ≈ 3.83 in.

Area = 8 × height ≈ 30.642 in².

Compare settings · Included angle

130°



Perpendicular height = 5 sin 130° ≈ 3.83 in.

Area = 8 × height ≈ 30.642 in².

Perpendicular height = 5 sin 130° ≈ 3.83 in. Area = 8 × height ≈ 30.642 in².

90°



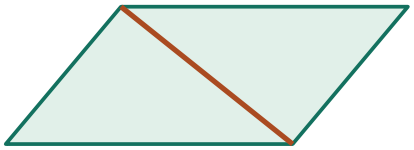
Perpendicular height = 5 sin 90° ≈ 5 in.

Area = 8 × height ≈ 40 in².

Perpendicular height = 5 sin 90° ≈ 5 in. Area = 8 × height ≈ 40 in².

2. Use the triangle across one diagonal

One diagonal is opposite the 50° angle in a triangle with sides 5 and 8. The cosine rule gives its squared length $5^2+8^2-2\cdot5\cdot8\cos 50°=89-80\cos 50°$.

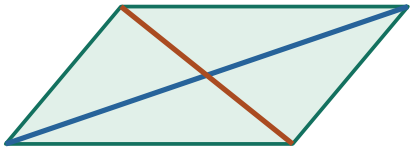


Opposite 50°, the shorter diagonal is the difference of the two side vectors.

Its square is 89 − 80 cos 50°.

3. Use the supplementary corner for the other

The other diagonal faces 130°, whose cosine is $-\cos 50°$. Its squared length is $89+80\cos 50°$. Taking roots gives about 6.130 in and 11.850 in, respectively.



The second diagonal subtracts the vectors.

Lengths ≈ 11.850 in and 6.130 in.

Check. Their squared lengths sum to 178, twice 5^2+8^2.

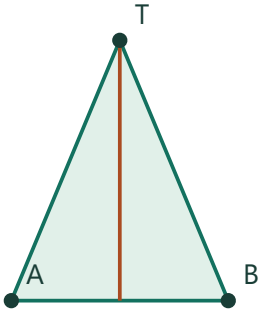
Exercise 222 · A 13-13-10 circumcircle

Printed page 22 · Incircles and circumcircles / The sine rule

Find the diameter of the circle that can be circumscribed around a triangle that has two 13-inch sides and one 10-inch side.

1. Split the isosceles triangle

Drop the altitude from the top to the 10-inch base. It halves the base into two 5-inch pieces, and its length is $\sqrt{(13^2-5^2)}=12$ in.

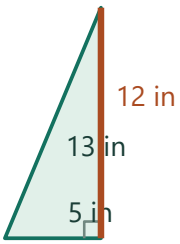


The altitude bisects the 10-in base.

Each half-triangle has lengths 5,12,13.

2. Read the sine of a base angle

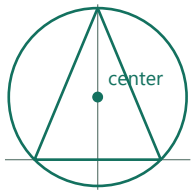
In the resulting right triangle the opposite leg is 12 and the hypotenuse is 13, so sin B=12/13. In the full triangle the side opposite this base angle has length 13.



A base angle has sine 12/13.

3. Use the circumdiameter formula

The sine rule in diameter form gives D=(opposite side)/sin B=13/(12/13)=169/12 in. This exceeds the longest chord as it should.



The opposite length 13 is a chord.

Circumdiameter = 13/(12/13) = 169/12 in.

1. 13=D sin B=D(12/13)
The opposite side is the diameter times the sine of its inscribed angle.

2. 169=12D ⇒ D=169/12 in
Multiply by 13 to clear the denominator and divide by 12.

Check. Triangle area is 60 in². The formula D=abc/(2A) also gives 169/12.

Exercise 223 · A straight included angle

Printed page 22 · The cosine rule / Triangle geometry

What does the SAS version of the Law of Cosines have to say about the "triangle" whose sides p and q form a 180-degree angle?

1. Draw the limiting angle

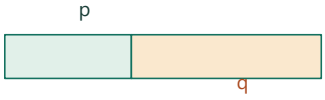
Place lengths p and q on opposite rays from the same vertex, making angle 180°. Their far endpoints lie on the same straight line on opposite sides of the vertex.



Start the two sides at O. Their rays point in opposite directions.

2. Use the cosine-rule expression

The third distance has square $p^2+q^2-2pq\cos 180°$. Since $\cos 180°=-1$, this is $p^2+2pq+q^2=(p+q)^2$.



With cos 180°=−1 the squared distance is $(p+q)^2$.

1. $c^2=p^2+q^2-2pq(-1)=p^2+2pq+q^2$
Insert the cosine of a straight angle.

2. $c^2=(p+q)^2 \Rightarrow c=p+q$
Factor the perfect square and use nonnegative lengths.

3. Interpret the positive root

The distance is p+q. Equality in the triangle inequality occurs here because reaching the far endpoint through the common vertex follows a straight line; the resulting figure has zero area.



The distance between endpoints is p+q.

Check. The triangle inequality becomes equality and the area is zero.

Exercise 224 · Inverse pairs

Printed page 22 · Matrix inverses / Matrix multiplication

For each of the following, calculate $\mathbf{MN}$ and $\mathbf{NM}$:

$$\textbf{(a)}\ \mathbf{M}=\begin{bmatrix}5 & -2\\ 3 & 1\end{bmatrix},\ \mathbf{N}=\begin{bmatrix}\frac{1}{11} & \frac{2}{11}\\ -\frac{3}{11} & \frac{5}{11}\end{bmatrix}\qquad \textbf{(b)}\ \mathbf{M}=\begin{bmatrix}1 & 2 & 0\\ -2 & 0 & 1\\ 1 & 0 & -1\end{bmatrix},\ \mathbf{N}=\begin{bmatrix}0 & -1 & -1\\ \frac{1}{2} & \frac{1}{2} & \frac{1}{2}\\ 0 & -1 & -2\end{bmatrix}$$

$$\textbf{(c)}\ \mathbf{M}=\begin{bmatrix}a & b\\ c & d\end{bmatrix},\ \mathbf{N}=\begin{bmatrix}\frac{d}{ad-bc} & \frac{-b}{ad-bc}\\ \frac{-c}{ad-bc} & \frac{a}{ad-bc}\end{bmatrix}$$

1. Check the scaled 2-by-2 pair in (a)

Keep 1/11 outside while multiplying. For M(11N) the entries are 5+6=11, 10−10=0, 3−3=0, 6+5=11. In the reverse product the rows similarly give (5+6,−2+2) and (−15+15,6+5). Dividing both products by 11 gives I_2.

11	0
0	11

For (a) the unscaled product is 11I.

Divide all entries by 11.

2. Compute the three output rows of MN in (b)

Each row of M forms a linear combination of rows of N. Row 1 uses $N_1+2N_2=(1,0,0)$. Row 2 uses $-2N_1+N_3=(0,1,0)$. Row 3 uses $N_1-N_3=(0,0,1)$. Thus $MN=I_3$.

1	0	0
0	1	0
0	0	1

Each row of MN in (b) selects one coordinate of the identity.

1. $N_1+2N_2=(0,-1,-1)+(1,1,1)=(1,0,0)$
The first row of M selects the first row of N plus twice its second.

2. $-2N_1+N_3=(0,2,2)+(0,-1,-2)=(0,1,0)$
Use the second row of M as combination coefficients.

3. $N_1-N_3=(0,-1,-1)-(0,-1,-2)=(0,0,1)$
The third output row completes the identity matrix.

3. Check the reverse product in (b)

Row 1 of NM uses $-M_2-M_3=(1,0,0)$. Row 2 uses $(M_1+M_2+M_3)/2=(0,1,0)$. Row 3 uses $-M_2-2M_3=(0,0,1)$, proving $NM=I_3$ as well.

1	0	0
0	1	0
0	0	1

Check NM separately.

The second row $(M_1+M_2+M_3)/2$ gives (0,1,0).

1. $-M_2-M_3=(2,0,-1)-(1,0,-1)=(1,0,0)$
Use the first row of N as coefficients for rows of M.

2. $(M_1+M_2+M_3)/2=(0,2,0)/2=(0,1,0)$
Each coefficient in the second row of N is one half.

3. $-M_2-2M_3=(2,0,-1)-(2,0,-2)=(0,0,1)$
The last row of N supplies the third unit row.

4. Explain the cancellation in (c)

Ignoring the common denominator, multiply rows (a,b),(c,d) by rows (d,−b),(−c,a). The diagonal entries are ad−bc; the off-diagonal entries are −ab+ab and cd−dc. The reverse multiplication has the same diagonal and vanishing off-diagonal entries. Dividing by ad−bc gives I_2 in each order when ad−bc≠0.

ad−bc	0
0	ad−bc

Multiply by the adjugate before dividing.

The determinant must be nonzero.

1. adj(M)M row 1=(da−bc, db−bd)
Dot the adjugate row (d,−b) with the two columns of M.

2. adj(M)M row 2=(−ca+ac, −cb+ad)
Use the other adjugate row (−c,a).

3. adj(M)M=(ad−bc)I
The mixed terms cancel; division yields I only for nonzero determinant.

Check. In (b) the first row (1,2,0) times the displayed second matrix gives (1,0,0), and its first row (0,−1,−1) times the first matrix gives (1,0,0).

Exercise 225 · Two matrix inverses

Printed page 22 · Matrix inverses / Determinants and area

(Continuation) The matrix that results each time that $\mathbf{M}$ is multiplied by $\mathbf{N}$ is called the *identity matrix* $\mathbf{I}$, and $\mathbf{N}$ is referred to as the *inverse matrix* of $\mathbf{M}$ and denoted $\mathbf{M}^{-1}$. By hand, compute $\mathbf{M}^{-1}$ for each of the matrices $\mathbf{M} = \begin{bmatrix} 4 & 0 \\ 0 & 3 \end{bmatrix}$ and $\mathbf{M} = \begin{bmatrix} 2 & -2 \\ 1 & 3 \end{bmatrix}$.

1. Undo the diagonal scaling

The matrix diag(4,3) multiplies the first coordinate by 4 and the second by 3. Undo both scalings independently with diag(1/4,1/3).

¼	0
0	⅓

Undo horizontal ×4 and vertical ×3 independently.

2. Find the second determinant

For rows (2,−2),(1,3), the determinant is 2·3−(−2)·1=8. It is nonzero, so the inverse exists.

2	-2
1	3

The determinant is 2·3−(−2)·1 = 8.

1. det=2·3−(−2)·1=8
Subtract the off-diagonal product from the diagonal product.

2. 8≠0
Therefore dividing the adjugate entries by 8 is permitted.

3. Form the inverse

Swap the diagonal entries and negate the off-diagonal entries, obtaining rows (3,2),(−1,2). Divide every entry by 8. Multiplying the unscaled pair gives diagonal entries 8 and off-diagonal entries zero, verifying the normalization.

3	2
-1	2

Divide this entire adjugate by 8 to obtain the second inverse.

1. row 1 products: 2·3+(−2)(−1)=8, 2·2+(−2)2=0
Use the two columns of the proposed unscaled inverse in order.

2. row 2 products: 1·3+3(−1)=0, 1·2+3·2=8
Together the products equal 8I, so multiplying every proposed inverse entry by 1/8 restores I.

Check. The unscaled product of the second matrix and its adjugate is diag(8,8).

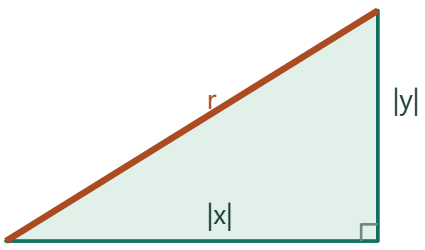
Exercise 226 · Five polar conversions

Printed page 22 · Polar coordinates / Trigonometric ratios

Polar coordinates for a point P in the xy-plane consist of two numbers, r and θ, where r is the distance from P to the origin O, and θ is the size of an angle in standard position that has OP as its terminal ray. Find polar coordinates for each of the following points:
(a) $(0, 1)$ **(b)** $(-1, 1)$ **(c)** $(4, -3)$ **(d)** $(1, 7)$ **(e)** $(-1, -7)$

1. Choose a positive radius and the correct quadrant

For (x,y) the radius is √(x²+y²). The reference angle comes from the absolute rise/run, and the signs of x,y locate the terminal ray.

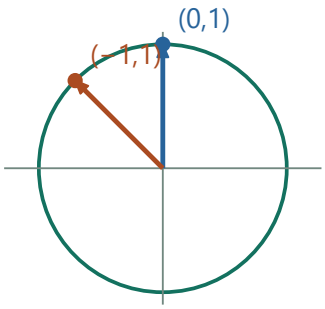


Schematic section

Radius is the distance from the origin, √(x²+y²).

2. Locate (a) and (b)

The point (0,1) is one unit up the vertical axis, giving (1,90°). At (−1,1) the reference angle is 45° in quadrant II, giving (√2,135°).

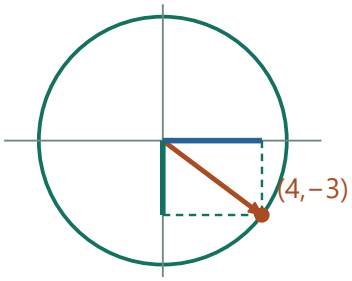


Direction first: up is 90°; up-left on the diagonal is 135°.

Each arrow is normalized here.

3. Locate (c)

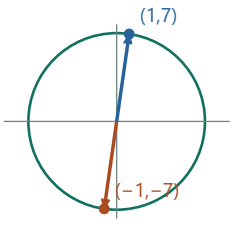
For (4,−3), r=5. The point is below the positive x-axis by arctan(3/4), giving (5,360°−arctan(3/4))≈(5,323.13°).



Radius 5 and a quadrant-IV ray at about 323.13°.

4. Locate (d) and (e)

For (1,7), r=√50 and θ=arctan 7≈81.87°. The point (−1,−7) has the same radius and the opposite direction, giving θ=180°+arctan 7≈261.87°.



The opposite points share radius √50 and their angles differ by 180°.

1. $r=\sqrt{(1^2+7^2)}=\sqrt{50}$, $r'=\sqrt{[(-1)^2+(-7)^2]}=\sqrt{50}$
Reversing both signs does not change the distance to the origin.

2. $\theta'=180°+\arctan(7/1)$
The opposite vector has the same slope but lies in quadrant III, requiring an extra half-turn.

Check. For example, multiplying 5 by the quadrant-IV direction (4/5,−3/5) reconstructs (4,−3).

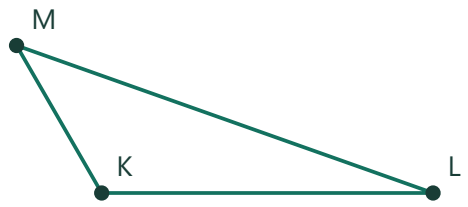
Exercise 227 · An obtuse triangle from a side ratio

Printed page 22 · The sine rule / Triangle geometry

Triangle KLM has a 120-degree angle at K and side KL is three fourths as long as side LM. To the nearest tenth of a degree, find the sizes of the other two angles of KLM.

1. Pair lengths and opposite angles

Length KL lies opposite angle M, whereas LM is opposite K. The given ratio KL/LM=3/4 therefore equals sin M/sin K by the sine rule.

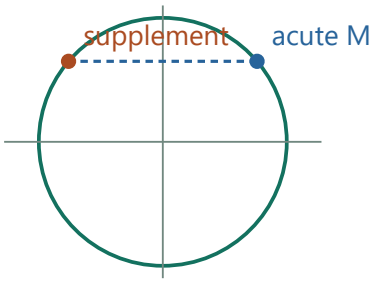


Side LM lies opposite 120°.

The side opposite M is three quarters as long.

2. Use the known obtuse angle

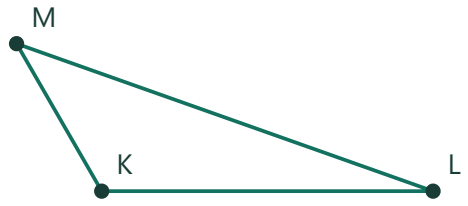
sin M=(3/4)sin 120°=3√3/8. The acute inverse-sine answer is M≈40.5°. Its supplement cannot be added to a 120° angle inside a triangle.



The sine equation allows two directions, but the obtuse M plus 120° exceeds 180°.

3. Find the last angle

Only 60° remains after K. Thus L=60°−M≈19.5°. Both unknown angles are positive and together complete the triangle.



Angle sum then gives L≈19.5° and M≈40.5°.

Check. Both unknown angles sum to the 60° remaining.

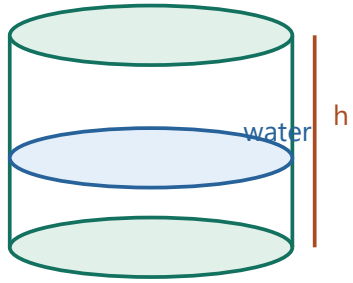
Exercise 228 · Liquid depths in two cups

Printed page 22 · Volume / Area and volume scaling

A drinking cup is 27/64 full of liquid. What is the ratio of the depth of the liquid to the depth of the cup, assuming that **(a)** the cup is cylindrical? **(b)** the cup is conical?

1. Relate height to filled volume in the cylinder

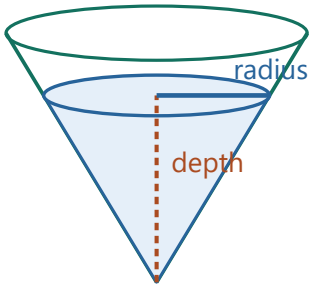
The horizontal base area never changes. If water reaches fraction t of the full height, its volume is fraction t of the total. A 27/64-full cylinder is filled to 27/64 of its height.



Equal horizontal areas make depth fraction = volume fraction, 27/64.

2. Identify the shape of the water in the cone

For a tip-down cone, the surface of the water cuts a smaller similar cone. If its depth is fraction t of the full height, its radius is also t times the full radius.



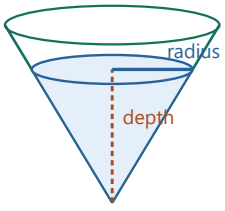
The water makes its own smaller cone at the bottom.

Its radius factor equals its depth factor t.

3. Scale all three dimensions

The water volume fraction is t^3 because base area scales by t^2 and depth by t. Solve $t^3=27/64$ to get $t=3/4$. The narrow lower part requires greater depth for the same volume fraction.

Water depth as a fraction of the full cup · 75%



Full radius 4 units, depth 8 units (example).

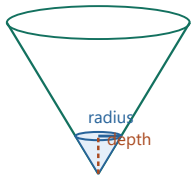
At depth 6 units, radius is 3 units. This is 42.188% of full capacity.

1. $V_{water}/V_{full}=(h/H)^3=27/64$
Cone radius scales with height, so volume scales in three dimensions.

2. $h/H=\sqrt[3]{27}/\sqrt[3]{64}=3/4$
Take the positive cube root of both sides.

Compare settings · Water depth as a fraction of the full cup

25%

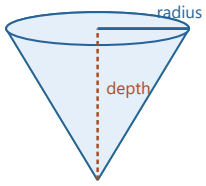


Full radius 4 units, depth 8 units (example).

At depth 2 units, radius is 1 units. This is 1.562% of full capacity.

Full radius 4 units, depth 8 units (example). At depth 2 units, radius is 1 units. This is 1.562% of full capacity.

100%



Full radius 4 units, depth 8 units (example).

At depth 8 units, radius is 4 units. This is 100% of full capacity.

Full radius 4 units, depth 8 units (example). At depth 8 units, radius is 4 units. This is 100% of full capacity.

Check. Cubing 3/4 gives 27/64; the cone needs the larger depth fraction since its lower part narrows.

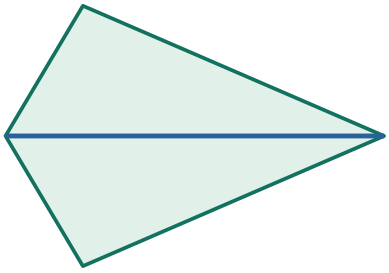
Exercise 229 · Diagonals of a kite

Printed page 22 · Triangle geometry / Triangle area

A kite has a 6.00-inch side and a 13.00-inch side, and one of the diagonals is 15.00 inches long. Find the length of the other diagonal, to the nearest hundredth of an inch.

1. Determine which diagonal can measure 15

The endpoints of the diagonal crossing the symmetry axis connect to an apex by two 6-in sides. Their separation cannot exceed 12 in by the triangle inequality. Therefore the 15-in diagonal must be the symmetry axis.

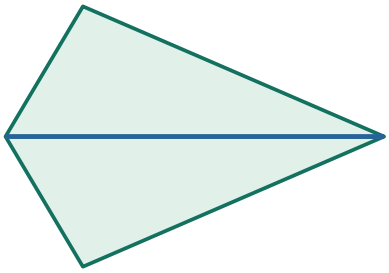


The 15-in diagonal must be the symmetry axis.

A chord joining two endpoints of 6-in sides has length at most 12.

2. Split the kite into two congruent triangles

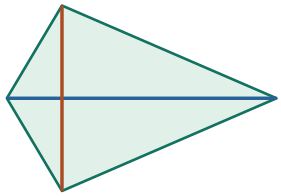
Each half has sides 6,13,15 and semiperimeter $(6+13+15)/2=17$. Heron gives half-area $\sqrt{(17\cdot11\cdot4\cdot2)}=\sqrt{1496}$ in^2. The kite's total area is twice that.



The axis splits the kite into congruent 6–13–15 triangles.

3. Relate area to the other diagonal

The perpendicular diagonals have area product divided by 2. Setting $15d/2=2\sqrt{1496}$ gives $d=4\sqrt{1496}/15\approx10.31$ in.



Perpendicular diagonals give area $\tfrac{1}{2}d_1d_2$.

The remaining diagonal is about 10.31 in.

1. $A=2\sqrt{[17(17-6)(17-13)(17-15)]}=2\sqrt{1496}$
Heron's formula gives one of the two congruent triangular areas.

2. $(15\cdot d)/2=2\sqrt{1496}$
Perpendicular diagonals give this second expression for the kite's entire area.

3. $d=4\sqrt{1496}/15$ in
Double both sides and divide by the known diagonal.

Check. The half-width is the altitude $2\sqrt{1496}/15$ of either triangle to its 15-in side.

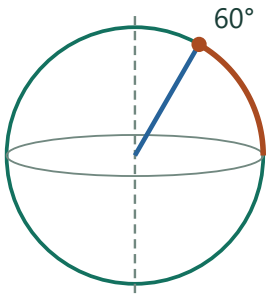
Exercise 230 · Flight around Oslo’s parallel

Printed page 22 · Spherical coordinates / Angular and linear speed

A jet passes Oslo, whose latitude is 60 degrees north of the equator, and flies due west until it returns to Oslo. How far does the jet travel? The radius of the earth is 3960 miles and a jet flies at about 38 000 feet.

1. Move the radius to cruising altitude

The 38,000-ft elevation adds 38000/5280 miles to Earth’s radius. The airplane’s distance from Earth’s center is therefore $R=3960+38000/5280$ mi.

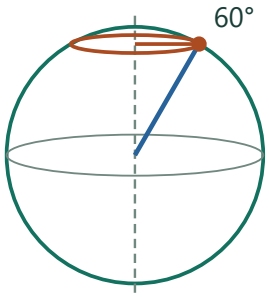


First increase the reference radius from 3960 mi by 38,000 ft converted to miles.

2. Find the radius of the flight circle

At latitude 60° the flight radius about the polar axis is the horizontal component of R, namely R cos 60°=R/2. The plane travels around a parallel of latitude.

Latitude · 60°

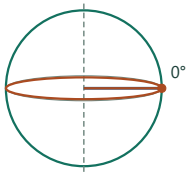


At latitude 60° the radius of the parallel is 1983.598 mi.

Its circumference is 0.5 times the equatorial circumference.

Compare settings · Latitude

0°

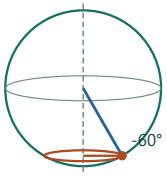


At latitude 0° the radius of the parallel is 3967.197 mi.

Its circumference is 1 times the equatorial circumference.

At latitude 0° the radius of the parallel is 3967.197 mi. Its circumference is 1 times the equatorial circumference.

−60°



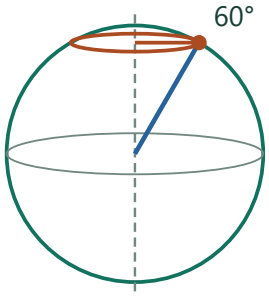
At latitude −60° the radius of the parallel is 1983.598 mi.

Its circumference is 0.5 times the equatorial circumference.

At latitude −60° the radius of the parallel is 1983.598 mi. Its circumference is 0.5 times the equatorial circumference.

3. Compute one circuit

The distance is 2π(R/2)=π(3960+38000/5280)≈12,463 mi. It is half of the equatorial circumference at the same altitude.



The route follows the complete orange parallel.

Its length is approximately 12,463 mi.

Check. At 60° latitude this circumference is half of the equatorial circumference at the same flight altitude.

Exercise 231 · Invert a 3×3 matrix

Printed page 23 · Matrix inverses / Matrix multiplication

Let $\mathbf{M}$ be the matrix $\begin{bmatrix} 1 & 0 & 2 \\ 2 & -1 & 4 \\ 0 & 3 & -1 \end{bmatrix}$. Use a matrix calculator to compute the inverse $\mathbf{M}^{-1}$. You can practice your matrix multiplication skills by checking $\mathbf{M}\mathbf{M}^{-1} = \mathbf{M}^{-1}\mathbf{M} = \mathbf{I}$.

1. Turn inversion into solving linear equations

For the original matrix M, suppose $M(x,y,z)^T=(a,b,c)^T$. Then $x+2z=a$, $2x-y+4z=b$, and $3y-z=c$. Expressing x,y,z in terms of a,b,c will give M^{-1}.

1	0	2
2	-1	4
0	3	-1

Write the input as (x,y,z) and the output as (a,b,c).

2. Eliminate in the order suggested by the rows

Subtract twice the first equation from the second to obtain $-y=b-2a$, so $y=2a-b$. The third gives $z=3y-c=6a-3b-c$. The first then gives $x=a-2z=-11a+6b+2c$.

a	x+2z
b	2x−y+4z
c	3y−z

Subtract twice the first row from the second.

This isolates y; substitute into the third to recover z.

1. $-y=(2x-y+4z)-2(x+2z)=b-2a$
The x terms and the z terms cancel simultaneously.

2. $z=3(2a-b)-c=6a-3b-c$
Use $y=2a-b$ in the third equation.

3. $x=a-2(6a-3b-c)=-11a+6b+2c$
Distribute −2 across all three terms of z.

3. Read and check the inverse

Its rows are $(-11,6,2),(2,-1,0),(6,-3,-1)$. For example row 1 of M times M^{-1} adds the first inverse row to twice the third, giving (1,0,0). Conversely row 1 of M^{-1} times M gives $-11(1,0,2)+6(2,-1,4)+2(0,3,-1)=(1,0,0)$; the other rows similarly restore the other unit coordinates.

-11	6	2
2	-1	0
6	-3	-1

Rows express x,y,z in terms of a,b,c. Together they form M^{-1}.

1. $\mathbf{2R_1-R_2+4R_3=(-22,12,4)-(2,-1,0)+(24,-12,-4)=(0,1,0)}$
Here R labels inverse rows. This checks row two of M times its inverse.

2. $\mathbf{3R_2-R_3=(6,-3,0)-(6,-3,-1)=(0,0,1)}$
The last row of M produces the remaining unit row.

3. $\mathbf{2M_1-M_2=(0,1,0);\ 6M_1-3M_2-M_3=(0,0,1)}$
Reversing the product uses rows of the original M in these combinations.

Check. Both matrix products give the three identity rows (1,0,0), (0,1,0), (0,0,1). For instance the first row in the reverse product is $-11(1,0,2)+6(2,-1,4)+2(0,3,-1)=(1,0,0)$.

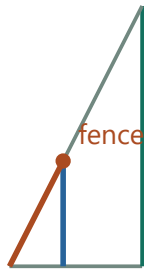
Exercise 232 · Ladder crossing a fence

Printed page 23 · Right-triangle trigonometry / Trigonometric ratios

A ladder leans against a side of a building, making a 63-degree angle with the ground, and reaching over a fence that is 6 feet from the building. The ladder barely touches the top of the fence, which is 8 feet tall. Find the length of the ladder.

1. Split the ladder at the fence top

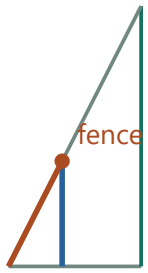
The ladder is straight, so both pieces make the same 63° angle to level ground. Each piece has a different known projection.



The fence top divides one straight ladder into two segments.

2. Find the lower piece

Between ground contact and fence top the vertical rise is 8 ft. If this piece has length L_1, then $L_1 \sin 63° = 8$, so $L_1 = 8/\sin 63°$.

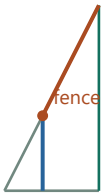


Lower segment: vertical rise 8 ft.

Its length is 8/sin 63°.

3. Find the upper piece and combine

From fence to building the horizontal run is 6 ft, giving $L_2 \cos 63° = 6$. The required ladder length is $L_1 + L_2 = 8/\sin 63° + 6/\cos 63° \approx 22.19$ ft.



Upper segment: horizontal run 6 ft.

Its length is 6/cos 63°; add both.

1. $L_2 = 6/\cos 63° \approx 13.22$ ft
The horizontal projection is 6 ft, so divide by the positive cosine.

2. $L_1 = 8/\sin 63° \approx 8.98$ ft
The lower segment uses its vertical projection.

3. $L = 8/\sin 63° + 6/\cos 63° \approx 22.19$ ft
Add before rounding so rounding two intermediate lengths does not affect the total.

Check. Alternatively the total ground run is $6 + 8\cot 63°$, which divided by $\cos 63°$ gives the same length.

Exercise 233 · Intersection of two spatial lines

Printed page 23 · Parametric lines / Systems of equations

The line defined by $(x, y, z) = (2 + t, 10 - 2t, 6 - t)$ intersects the line defined by $(x, y, z) = (9 + 2u, 2 - u, 5 + u)$. Find coordinates for the point of intersection. To the nearest degree, find the size of the acute angle formed by these lines.

1. Give the lines independent parameters

Call their parameters t and u. The two lines meet only if their three coordinates match at the same spatial point. Equating the x and y coordinates gives t=7+2u and 2t=8+u.

line 1	t
line 2	u

Use separate parameters until all three coordinates agree.

2. Solve and test the remaining coordinate

Substitute t=7+2u into 2t=8+u, giving 14+4u=8+u, hence u=−2 and t=3. Inserting these in the two full parametrizations produces (5,4,3) in each case. The z-coordinates agree too.

x	y	z
5	4	3
5	4	3

Top row: t=3 on the first line. Bottom row: u=−2 on the second.

All three coordinates match.

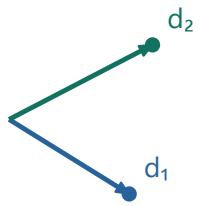
1. 2(7+2u)=8+u
Replace t in the second equation with its value from the first.

2. 14+4u=8+u ⇒ 3u=−6 ⇒ u=−2
Collect parameter terms on the left and constants on the right.

3. t=7+2(−2)=3
Recover the other line's parameter before checking the z coordinates.

3. Find the angle from the direction vectors

The directions are (1,−2,−1) and (2,−1,1), both of length √6. Their dot product is 2+2−1=3. The acute angle has cosine 3/6=1/2, so it is 60°. Different directions ensure the meeting point is unique.



The direction vectors each have length √6 and dot product 3.

Their actual space angle is 60°.

Check. Their direction vectors are nonparallel, so the verified intersection is unique.

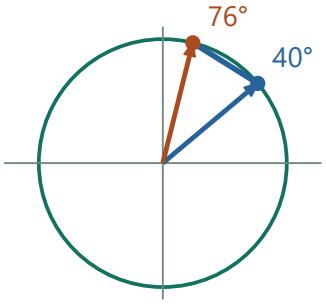
Exercise 234 · Equal angular gaps

Printed page 23 · Chords and inscribed angles / Rotations

Consider the points $A = (\cos 40, \sin 40)$, $B = (\cos 76, \sin 76)$, $C = (\cos 121, \sin 121)$, and $D = (\cos 157, \sin 157)$. Find the lengths of segments AB and CD, then explain what is predictable about the answer.

1. Find the angles subtended by the chords

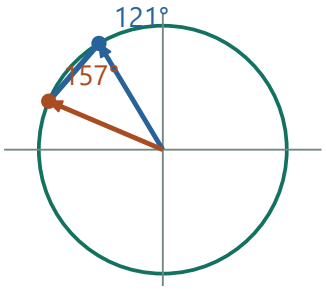
The first endpoints are at 40° and 76°, separated by 36°. The other endpoints are at 121° and 157°, likewise separated by 36°.



The first chord spans a 36° central angle.

2. Relate one chord to the other

A rotation of 81° about the origin takes 40° to 121° and 76° to 157°. Because rotation preserves distances, the two chords must have equal length.

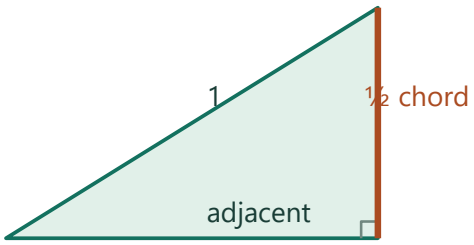


Rotate both ends through the same 81°.

The second chord must have the same length.

3. Calculate the common length

Bisect either central triangle. The half-chord is opposite 18° in a right triangle with unit hypotenuse. Thus the chord is 2 sin 18°≈0.618034.



Schematic section

Half the chord faces half the angle, 18°.

Chord = 2 sin18° ≈ 0.6180.

Check. Squaring a chord using the coordinate formula gives 2−2 cos36°.

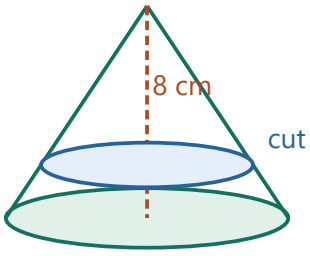
Exercise 235 · Cut a cone parallel to its base

Printed page 23 · Volume / Area and volume scaling

Find the volume of a cone of height 8 centimeters and base radius 6 centimeters. This cone is sliced by a plane that is parallel to the base and 2 centimeters from it. Find the volumes of the two resulting solids. One is a cone, while the other is called a *frustum*.

1. Distinguish the cut height from the upper height

The full cone is 8 cm tall with base radius 6 cm. A slice 2 cm above its base leaves a top cone of height 8−2=6 cm, and the lower portion is a frustum.

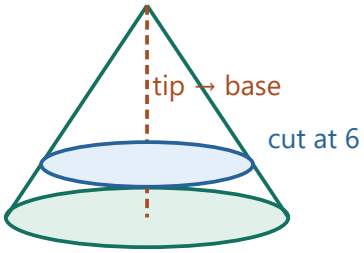


The cut is 2 cm above the base, hence 6 cm below the vertex.

2. Scale the upper cone

The cut is parallel to the base, so all lengths of the upper cone scale by 6/8=3/4. Its volume is $(3/4)^3=27/64$ of the whole.

Fraction measured from the tip · 75%

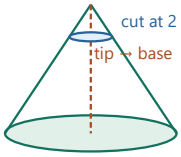


Example cone radius 6, height 8 cm.

Cut height 6 and cut radius 4.5. The upper tip-side fraction of full volume is 42.188%.

Compare settings · Fraction measured from the tip

25%

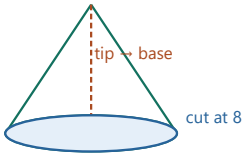


Example cone radius 6, height 8 cm.

Cut height 2 and cut radius 1.5. The upper tip-side fraction of full volume is 1.562%.

Example cone radius 6, height 8 cm. Cut height 2 and cut radius 1.5. The upper tip-side fraction of full volume is 1.562%.

100%



Example cone radius 6, height 8 cm.

Cut height 8 and cut radius 6. The upper tip-side fraction of full volume is 100%.

Example cone radius 6, height 8 cm. Cut height 8 and cut radius 6. The upper tip-side fraction of full volume is 100%.

3. Find both volumes

The full volume is $\pi\cdot6^2\cdot8/3=96\pi$ cm³. The upper cone contains $(27/64)96\pi=81\pi/2$ cm³. The remaining frustum contains $96\pi-81\pi/2=111\pi/2$ cm³.



Of 96π cm³, the top is 81π/2 and the remaining frustum is 111π/2.

1. $V_{top}=(27/64)96\pi=(27\cdot3/2)\pi=81\pi/2$
Multiply whole volume by the cubed height ratio.

2. $V_{frustum}=96\pi-81\pi/2=(192-81)\pi/2=111\pi/2$
Subtract the upper cone from the whole cone.

Check. The upper radius is 4.5 cm; substituting it with height 6 gives the same upper volume.

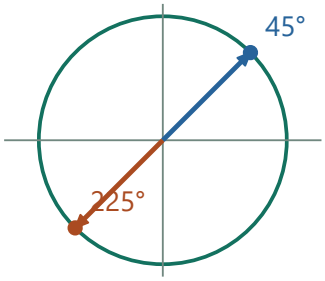
Exercise 236 · Sine equals cosine

Printed page 23 · Trigonometric equations / The unit circle

For what values of θ is it true that $\sin\theta = \cos\theta$?

1. Interpret the equation on the circle

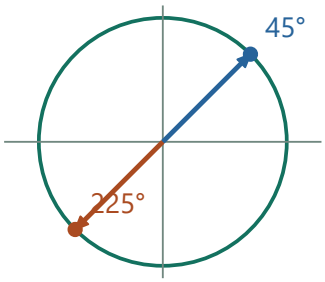
Having $\sin\theta=\cos\theta$ means the vertical and horizontal coordinates of the unit-circle point agree. Both points on the line $y=x$ are candidates.



Equal x and y coordinates lie on the line $y=x$.

2. Recover the two directions

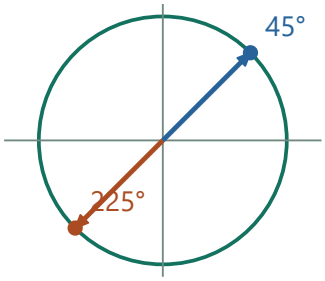
One ray points along $45°$ and the opposite ray points along $225°$. Equivalently, cosine cannot be zero at a solution, so divide to obtain $\tan\theta=1$.



The diagonal meets the unit circle on opposite rays.

3. Include every revolution

The two directions together repeat every $180°$. All solutions are $\theta=45°+180°k$, or $\theta=\pi/4+k\pi$ in radians, for any integer k.



The directions repeat after $180°$.

$\theta=45°+180°k$, for any integer k.

Check. These are both rays of the line $y=x$ through the unit circle.

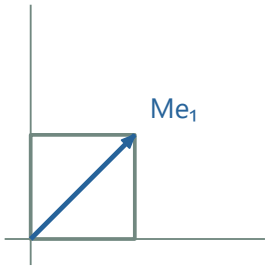
Exercise 237 · Why a matrix is singular

Printed page 23 · Matrix inverses / Determinants and area

Singular matrices. Square matrices that have an inverse are called *invertible* and those that do not have an inverse are called *singular*, or *non-invertible*. For example, consider $\mathbf{L} = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$. Show that there is no matrix $\mathbf{N}$ for which $\mathbf{NL}$ is the 2×2 identity matrix. One way to proceed is to consider $\mathbf{L}$ as a transformation. What is the result of applying $\mathbf{L}$ to the unit square?

1. Apply L to a general point

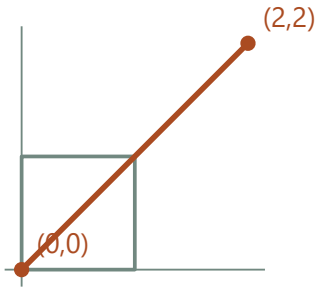
Both rows are (1,1), so L(x,y)=(x+y,x+y). The two output coordinates always agree, constraining every image to the diagonal line y=x.



Look first at the horizontal basis arrow.

2. Trace the whole square

For 0≤x,y≤1, x+y ranges from 0 to 2 and every intermediate value is attained. Thus the unit square collapses to the segment from (0,0) to (2,2).

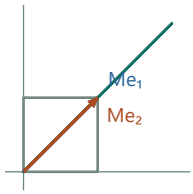


Both output coordinates equal x+y.

As x+y ranges from 0 to 2, the whole square fills this diagonal segment.

3. Explain why the collapse cannot be undone

The distinct inputs (1,0) and (0,1) both produce (1,1). A left inverse would have to send that one image back to both inputs. Equivalently, L's identical columns make NL have identical columns for every candidate N, so NL cannot equal I.



The two-dimensional square collapses onto one segment.

Two independent input coordinates cannot be recovered.

1. N rows=(a,b),(c,d) ⇒ NL rows=(a+b,a+b),(c+d,c+d)
Each column of L is the same vector (1,1), so each dot product repeats.

2. row 1 of I=(1,0) ⇒ a+b=1 and a+b=0
The same sum cannot take both values, precluding a left inverse.

Check. det L=1−1=0.

Exercise 238 · Construct a singular matrix

Printed page 23 · Matrix inverses / Determinants and area

(Continuation) Invent another example of a non-invertible (singular) 2×2 matrix, in which the four entries are all different and nonzero.

1. Choose dependent rows while keeping entries distinct

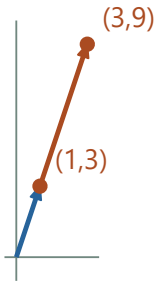
A square matrix loses information when one row is a multiple of another. Choose first row (1,2) and take three times it as the second row, (3,6). Its four entries are nonzero and different.



Look first at the horizontal basis arrow.

2. Find what the matrix does to an input

It maps (x,y) to (x+2y,3x+6y). Once the first output is known, the second is fixed at three times it. The outputs occupy only a line.



Each output satisfies y′=3x′.

Here the two unit basis images add along the same line. Every output remains on that line.

3. Verify noninvertibility

Both the zero vector and (2,−1) map to (0,0), so no inverse can recover every input. The determinant 1·6−2·3=0 confirms the same conclusion.



The two-dimensional square collapses onto one segment.

Two independent input coordinates cannot be recovered.

1. det=1·6−2·3=0
Proportional rows give equal diagonal and off-diagonal products.

2. M(−2,1)=(−2+2,−6+6)=(0,0)
A nonzero input and (0,0) have the same image, so inversion is impossible.

Check. The nonzero vector (2,−1) maps to (0,0), ruling out an inverse.

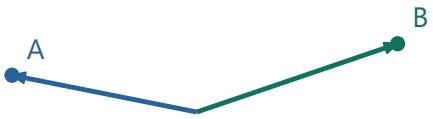
Exercise 239 · Distance on a sphere I

Printed page 23 · Great-circle distances / The dot product

Points $A = (-5, 12, 0)$ and $B = (13, 0, 0)$ are on the sphere $x^2 + y^2 + z^2 = 13^2$. Find the distance that separates A and B, traveling *on* the sphere, not through it.

1. Choose the surface path

On a sphere the shortest surface path between these nonantipodal points follows the shorter arc of their great circle. Its radius is 13; the missing quantity is the angle θ between the two position vectors.

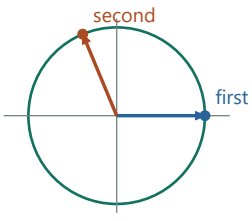


Both radius vectors have length 13.

Their plane through the center cuts out a great circle.

2. Find the central angle

The position vectors (−5,12,0) and (13,0,0) each have length 13. Their dot product is −65, so cos θ=−65/(13·13)=−65/169.



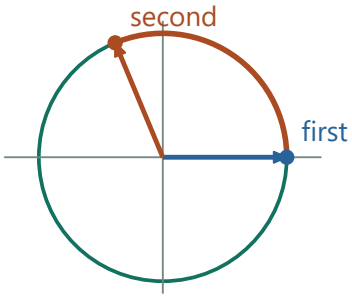
In that plane the central angle has cosine -65/169.

1. OA·OB=(−5)13+12·0+0·0=−65
Only the x-component contributes because B lies on the x-axis.

2. cos θ=−65/(13²)=−5/13
Divide numerator and denominator by 13.

3. Use arc length in radian measure

Choose θ=arccos(−65/169) between 0 and π in radians. The surface distance is 13θ≈25.553 units. The negative cosine agrees with a route longer than a quarter-circle.



Surface distance is 13 times the angle in radians.

Length ≈ 25.553 units.

Check. Both points lie in z=0, so their polar direction difference gives the same equatorial arc.

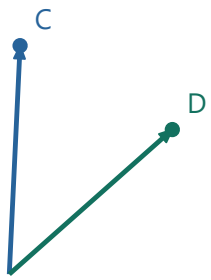
Exercise 240 · Distance on a sphere II

Printed page 23 · Great-circle distances / The dot product

Points $C = (3, 4, 12)$ and $D = (12, 3, 4)$ are on the sphere $x^2 + y^2 + z^2 = 13^2$. Find the distance that separates C and D, traveling *on* the sphere, not through it.

1. Check both radii and choose their great circle

The two position vectors (3,4,12) and (12,3,4) have squared length 169. Their endpoints lie on a common radius-13 great circle in the plane of the vectors and the origin.

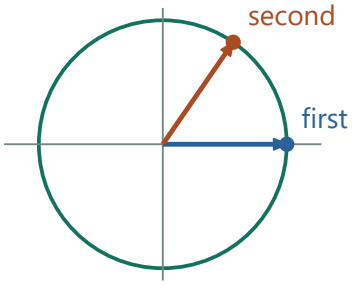


Both radius vectors have length 13.

Their plane through the center cuts out a great circle.

2. Measure the smaller angle between them

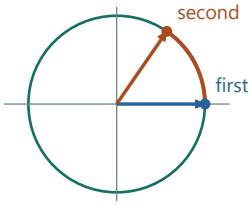
The dot product is 3·12+4·3+12·4=96, giving cos θ=96/169. Choose the angle between 0 and π for the shorter arc.



In that plane the central angle has cosine 96/169.

3. Measure the surface distance

In radians the arc length is 13 arccos(96/169)≈12.567 units. This route follows the curved surface, whereas the direct chord has squared length 146.



Surface distance is 13 times the angle in radians.

Length ≈ 12.567 units.

1. $CD^2=(12-3)^2+(3-4)^2+(4-12)^2=81+1+64=146$
Differences of the endpoints give the straight chord's three components.

2. s=13θ, θ=arccos(96/169) radians
The great circle uses the full sphere radius; convert the dot-product angle to surface length by multiplying by 13.

Check. Their chord squared is 2·169−2·96=146, agreeing with chord 26 sin(θ/2).

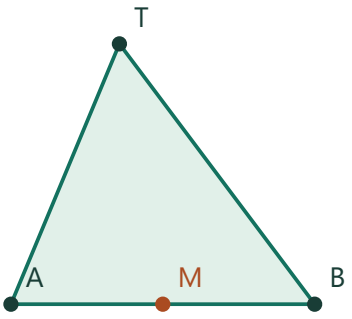
Exercise 241 · Median to side 14

Printed page 23 · Triangle geometry / The cosine rule

A triangle has a 13-inch side, a 14-inch side, and a 15-inch side. To the nearest tenth of an inch, how long is the median drawn to the 14-inch side?

1. Place the required median

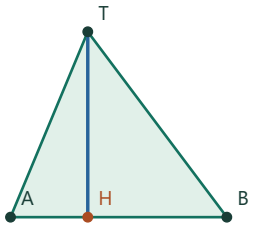
Use the 14-inch side as base AB and mark its midpoint M, seven inches from either end. Let H be the perpendicular foot from the opposite vertex T and set AH=x, TH=h, with AT=13 and BT=15.



The midpoint of the 14-in side is 7 in from either end.

2. Locate the altitude foot

Pythagoras gives $x^2+h^2=169$ and $(14-x)^2+h^2=225$. Subtracting gives $196-28x=56$, so $x=5$. Then $h=\sqrt{(169-25)}=12$ in.



Subtract the two squared distances.

The altitude foot H is at $x=5$ and the altitude measures 12.

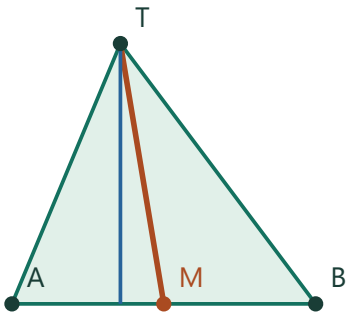
1. $(14-x)^2-x^2=225-169$
Subtract the Pythagorean equations to eliminate h^2.

2. $196-28x+x^2-x^2=56 \Rightarrow 28x=140$
Expand the square and collect terms.

3. $x=5$, $h^2=169-25=144$
Divide by 28, then recover the altitude from the first equation.

3. Find the median TM

The horizontal separation HM is $7-5=2$. Thus $TM^2=12^2+2^2=148$ and $TM=2\sqrt{37}\approx12.2$ in. This derives the median length directly from the three given sides.



HM=2 and TH=12.

The median TM has length $\sqrt{148}$ in.

Check. Drawing altitude 12 onto side 14 divides the base into 5 and 9; the midpoint lies 2 in from the altitude foot, so $m^2=12^2+2^2$.

Exercise 242 · Monthly calculator receipts

Printed page 24 · Matrix multiplication / Data matrices

In January and February, *Edith's* Calculator Shop recorded the sales data shown below in the right-hand 3×2 matrix $\mathbf{S}$. The prices for these models are shown below in the left-hand 1×3 matrix $\mathbf{P}$. What is the meaning of the entries in the matrix product $\mathbf{PS}$? Does the matrix product $\mathbf{SP}$ make sense?

$$\begin{array}{r} \\ \textit{Price} \end{array}\!\!\begin{array}{c} \begin{array}{ccc} \textit{TI-84} & \textit{TI-89} & \textit{TI-Nspire} \end{array} \\ \begin{bmatrix} 85 & 105 & 160 \end{bmatrix} \end{array}$$

$$\begin{array}{r} \\ \textit{TI-84} \\ \textit{TI-89} \\ \textit{TI-Nspire} \end{array}\!\!\begin{array}{c} \begin{array}{cc} \textit{Jan} & \textit{Feb} \end{array} \\ \begin{bmatrix} 12 & 7 \\ 34 & 22 \\ 56 & 39 \end{bmatrix} \end{array}$$

1. Assign meanings to the dimensions

The row P=(85,105,160) lists dollar prices for three calculator types. The two columns of S list counts sold for January and February, in the same type order. Each row-column product is a month's revenue.

Jan	Feb
12	7
34	22
56	39

Sales rows are three calculator types.

Price row = (85,105,160) dollars.



Model prices in row P

The row assigns one unit price per model. Its order is essential when multiplying prices by sales counts.

2. Total sales within each month

January gives 85·12+105·34+160·56=$13,550. February gives 85·7+105·22+160·39=$9,145. Hence PS is the 1-by-2 revenue row (13550,9145).

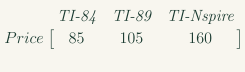
Jan	Feb
12	7
34	22
56	39

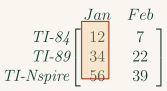
January total = 85·12 + 105·34 + 160·56.

Pairing with February instead changes the sales column.

1. January=1020+3570+8960=13,550 dollars
Multiply each January count by its model's price, then add the three model receipts.

2. February=595+2310+6240=9,145 dollars
Multiply the same price row by the second column of counts.





January sales column

The first count column lists January quantities in the same model order. A price times a count is revenue from that model.

3. Check the reverse order

S is 3-by-2 and P is 1-by-3. In SP the inner dimensions would be 2 and 1, so entries cannot be paired for dot products. That product is undefined.

Jan $	Feb $
13,550	9,145

The output is 1×2.

The reversed product has incompatible inner dimensions.

Check. Each product term has units dollars per calculator times calculators sold.

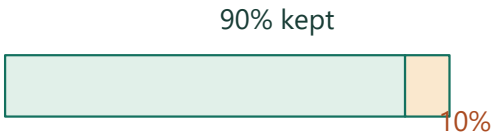
Exercise 243 · Price reduction

Printed page 24 · Matrix multiplication / Percentage change

(Continuation) Edith decides to lower all calculator prices by 10%. Show how this can be done by multiplying one of the matrices above by a suitable scalar.

1. Translate a percentage cut into a factor

A 10% cut leaves 100%−10%=90%=0.90 of each original price. Multiply the entire row P by the scalar 0.9.



A 10% cut leaves a price factor of 0.9.

2. Update the price list

The new row is (0.9·85,0.9·105,0.9·160)=($76.50,$94.50,$144.00). Quantities remain in their original order.

85	105	160
76.5	94.5	144

Apply the same factor to each price.

$$\text{Price}\begin{bmatrix}\textit{TI-84} & \textit{TI-89} & \textit{TI-Nspire}\\ 85 & 105 & 160\end{bmatrix}$$

$$\begin{matrix}\\ \textit{TI-84}\\ \textit{TI-89}\\ \textit{TI-Nspire}\end{matrix}\begin{bmatrix}\textit{Jan} & \textit{Feb}\\ 12 & 7\\ 34 & 22\\ 56 & 39\end{bmatrix}$$

Model prices in row P · Figure from exercise 242

The row assigns one unit price per model. Its order is essential when multiplying prices by sales counts.

3. Update the receipts

Every term in each month's sales total contains exactly one price, so (0.9P)S=0.9(PS). With the earlier quantities the two totals become $12,195.00 and $8,230.50.

Jan $	Feb $
12,195	8,230.50

The unchanged quantities make every receipts total scale by exactly 0.9.

1. 0.9·13,550=13,550−1,355=12,195 dollars
Taking one tenth off all prices removes one tenth of the entire January total.

2. 0.9·9,145=9,145−914.50=8,230.50 dollars
Use February's original sales total with the same retained price fraction.

$$\begin{array}{c} \\ Price \end{array}\begin{array}{c} \begin{array}{cccc} TI\text{-}84 & TI\text{-}89 & TI\text{-}Nspire \end{array} \\ \begin{bmatrix} 85 & 105 & 160 \end{bmatrix} \end{array} \qquad \begin{array}{c} \\ TI\text{-}84 \\ TI\text{-}89 \\ TI\text{-}Nspire \end{array}\begin{array}{c} \begin{array}{cc} Jan & Feb \end{array} \\ \begin{bmatrix} 12 & 7 \\ 34 & 22 \\ 56 & 39 \end{bmatrix} \end{array}$$

January sales column · Figure from exercise 242

The first count column lists January quantities in the same model order. A price times a count is revenue from that model.

Check. Each original price minus its new price is exactly a tenth of the original.

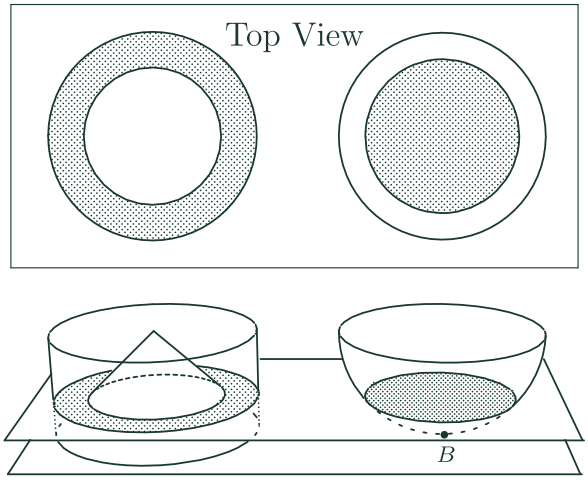
Exercise 244 · Equal slices at height three

Printed page 24 · Cross sections / Cavalieri's principle

A cylinder rests on top of a table, with a cone inscribed within, vertex up. Both heights and radii are 8 cm. A hemispherical bowl of radius 8 cm rests nearby on the same table, its circular rim parallel to the table.

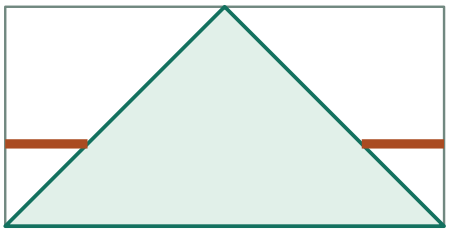
Consider that part of the cylinder that is outside (above) the cone. Slice this region by a plane that is parallel to the table and 3 cm from it. The intersection is a *ring* between two concentric circles. Calculate its area.

The same plane slices the hemisphere, creating a disk. Show that the disk has the same area as the ring. The diagram shows both a top view and a side view of the ring, the disk, and the hemisphere. Note that point B is on the table.



1. Find the cone's obstruction at the chosen height

The cone has radius and height both 8 cm. In a vertical section, similar right triangles show that radius is proportional to distance below the vertex. At the table that ratio is 8/8=1. At height 3 cm, the plane is 8−3=5 cm below the vertex, so its cone radius is 5 cm. The surrounding cylinder keeps radius 8 cm.

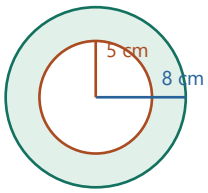


Axial section

At h=3 cm the cone radius narrows from 8 to 5 cm.

2. Calculate the open ring area

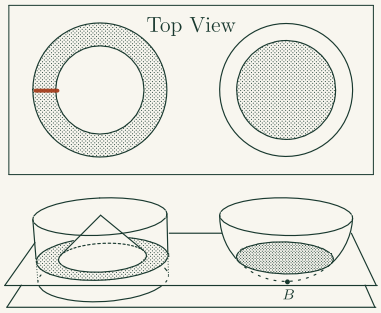
At that height, the available ring inside the cylinder but outside the cone is the outer disk minus the inner disk: $\pi\cdot 8^2-\pi\cdot 5^2=39\pi$ cm^2.



The open horizontal ring has area $\pi(8^2-5^2)=39\pi$ cm^2.

1. $A_{outer}=\pi\cdot 8^2=64\pi$, $A_{hole}=\pi\cdot 5^2=25\pi$
Use the outer and inner radii of the two concentric circles.

2. $A_{ring}=64\pi-25\pi=39\pi$
Subtract the obstructing circular area from the full cylinder disk.



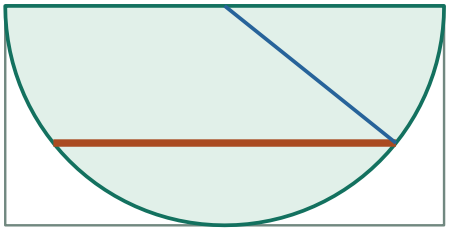
Thickness of the ring in the top view

The highlighted radial segment crosses only the hatched annulus, from the inner cone boundary to the outer cylinder wall. The disk inside the inner circle is excluded.

3. Calculate the hemisphere disk

The sphere center is 8 cm above the table, so the plane is $8-3=5$ cm below it. For slice radius a, $a^2+5^2=8^2$, hence $a^2=39$. Its disk area is $\pi a^2=39\pi$ cm^2, matching the open ring.

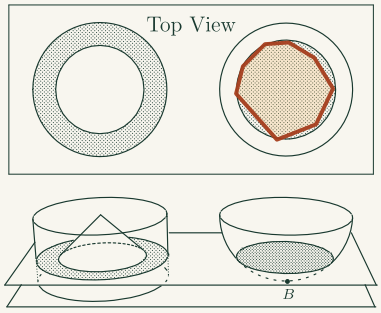
Height above the table or bowl bottom · 3 cm



Axial section

At height 3 cm, plane-to-center distance = 5 cm.

Slice area = $\pi[8^2-5^2] = 39\pi$ cm^2.

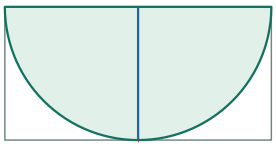


Disk seen from above

The shaded disk at upper right is a complete circular slice of the hemisphere. Compare it with the area of the ring on the left.

Compare settings · Height above the table or bowl bottom

0 cm



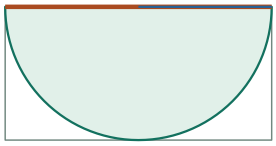
Axial section

At height 0 cm, plane-to-center distance = 8 cm.

Slice area = $\pi[8^2-8^2] = 0\pi$ cm^2.

At height 0 cm, plane-to-center distance = 8 cm. Slice area = $\pi[8^2-8^2] = 0\pi$ cm^2.

8 cm



Axial section

At height 8 cm, plane-to-center distance = 0 cm.

Slice area = $\pi[8^2-0^2] = 64\pi$ cm^2.

At height 8 cm, plane-to-center distance = 0 cm. Slice area = $\pi[8^2-0^2] = 64\pi$ cm^2.

Check. Both areas increase from zero at the table toward 64π at the top.

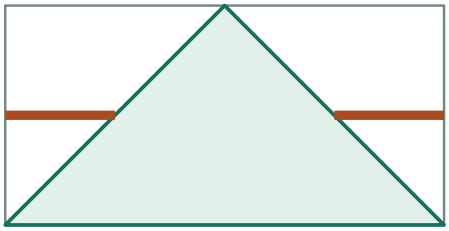
Exercise 245 · Equal slices at every level

Printed page 24 · Cross sections / Cavalieri's principle

(Continuation) Let r be the common radius of the cone, cylinder, and hemisphere, as well as the common height of the cylinder and the cone. Show that the ring and the disk have the same area, for all positions of the slicing plane.

1. Replace the example numbers by r and h

At the table the cone fills the radius-r cylinder cross section. As height h increases toward r, the cone narrows linearly to radius r−h. The outer cylinder radius stays r.

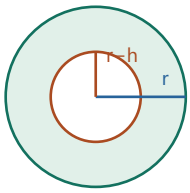


Axial section

At any h above the table, the interior cone's radius is r−h.

2. Calculate each ring

The area available outside the cone is $\pi r^2 - \pi(r-h)^2$. This equals $\pi(2rh - h^2)$.

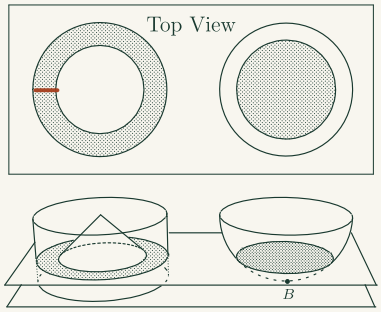


Subtract the excluded disk from the outer disk.

Ring area = $\pi[r^2-(r-h)^2]$.

1. $A_{ring} = \pi[r^2-(r-h)^2]$
The inside cone leaves out a disk of radius r−h.

2. $= \pi[r^2 - r^2 + 2rh - h^2] = \pi h(2r-h)$
Expand the square, distribute the minus sign and factor h.



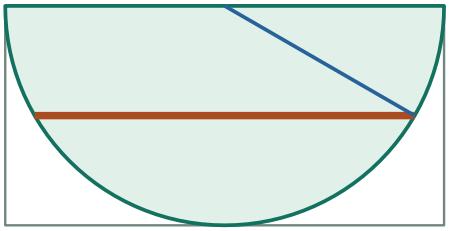
Thickness of the ring in the top view · Figure from exercise 244

The highlighted radial segment crosses only the hatched annulus, from the inner cone boundary to the outer cylinder wall. The disk inside the inner circle is excluded.

3. Compare with the hemisphere at exactly h

The sphere center is r−h above the cutting plane. If its disk radius is a, then $a^2=r^2-(r-h)^2$ by Pythagoras. Thus the hemisphere disk and the open ring have identical areas at every height $0\le h\le r$.

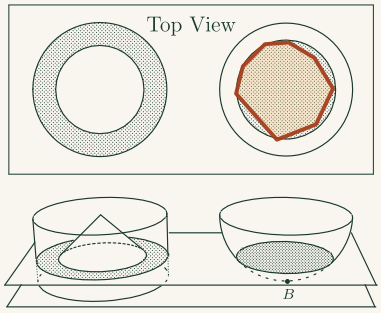
Height above the table or bowl bottom · 4 cm



Axial section

At height 4 cm, plane-to-center distance = 4 cm.

Slice area = $\pi[8^2-4^2] = 48\pi$ cm².

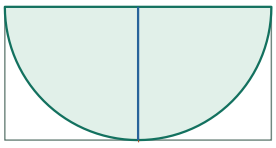


Disk seen from above · Figure from exercise 244

The shaded disk at upper right is a complete circular slice of the hemisphere. Compare it with the area of the ring on the left.

Compare settings · Height above the table or bowl bottom

0 cm



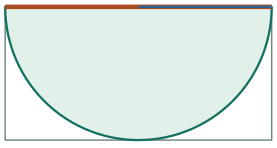
Axial section

At height 0 cm, plane-to-center distance = 8 cm.

Slice area = $\pi[8^2-8^2] = 0\pi$ cm².

At height 0 cm, plane-to-center distance = 8 cm. Slice area = $\pi[8^2-8^2] = 0\pi$ cm².

8 cm



Axial section

At height 8 cm, plane-to-center distance = 0 cm.

Slice area = $\pi[8^2-0^2] = 64\pi$ cm².

At height 8 cm, plane-to-center distance = 0 cm. Slice area = $\pi[8^2-0^2] = 64\pi$ cm².

Check. Both simplify to $\pi(2rh-h^2)$.

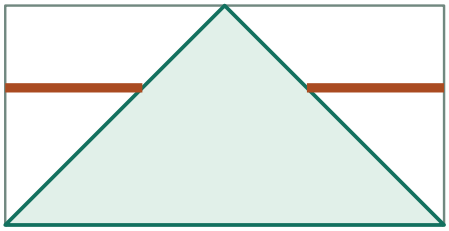
Exercise 246 · Pouring the hemisphere

Printed page 24 · Volume / Cavalieri's principle

(Continuation) If the hemispherical bowl were filled with liquid, it could be poured into the cylinder, which still has the cone inscribed in it. Will all the liquid fit? Expressed in terms of r, what is the volume of the cone? of the empty cylinder? of the hemisphere?

1. Compare matching levels

Exercise 245 showed that every horizontal disk of the hemisphere has the same area as the ring around the cone at the same height. Both regions extend from the table to height r.

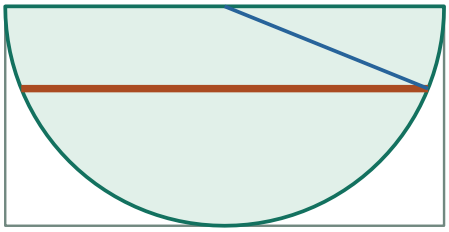


Axial section

Slice the cylinder-minus-cone at a chosen level.

2. Turn area equality into volume equality

A thin layer of either shape has the same area at the same level and the same thickness. Cavalieri's principle states that equal heights and matching cross-sectional areas throughout give equal total volumes. The liquid therefore fits exactly.



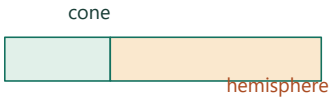
Axial section

The corresponding hemisphere disk has equal area.

Cavalieri therefore gives equal total volumes.

3. Compute the three volumes

The matching cylinder has radius r and height r, so its volume is πr^3. The cone occupies one third, $\pi r^3/3$. The open region, and hence the hemisphere, occupies $\pi r^3 - \pi r^3/3 = 2\pi r^3/3$.



The cone occupies ⅓ of the cylinder.

The hemisphere occupies the other ⅔.

1. $V_{hemisphere} = \pi r^2 \cdot r - \pi r^2 \cdot r/3$
Equal slice areas identify the hemisphere with the cylinder minus cone.

2. $= (3\pi r^3 - \pi r^3)/3 = 2\pi r^3/3$
Combine terms with denominator 3.

Check. Cylinder minus cone plus cone recovers the full cylinder.

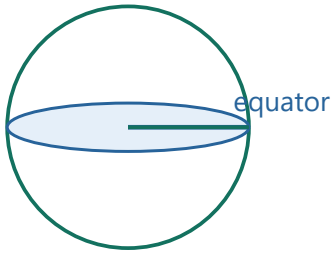
Exercise 247 · Volume of the sphere

Printed page 24 · Volume / Cavalieri's principle

(Continuation) Show that a sphere of radius r encloses a volume of $\frac{4}{3}\pi r^3$.

1. Divide the sphere through its center

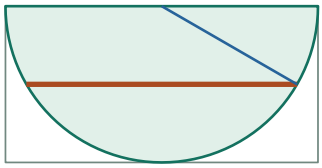
A plane through the center splits the sphere into two congruent hemispheres. Knowing the volume of one is enough because their interiors do not overlap.



The equatorial plane divides the sphere into two congruent hemispheres.

2. Use the matching cross sections

From the cylinder-minus-cone comparison, a hemisphere of radius r has volume $\pi r^3 - \pi r^3/3 = 2\pi r^3/3$. Both comparison solids have radius and height r.



Axial section

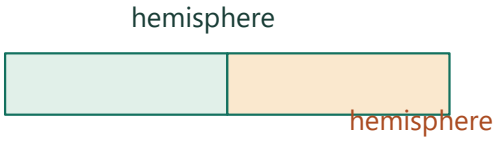
Match horizontal disks within one hemisphere with the rings outside a cone.

1. $V_{hemi} = \pi r^2 \cdot r - (\pi r^2 \cdot r)/3$
Cylinder and cone have the same base and height, and subtracting the cone leaves a solid with the hemisphere's slice areas.

2. $= 3\pi r^3/3 - \pi r^3/3 = (3-1)\pi r^3/3$
Use a common denominator to find one hemisphere's volume.

3. Join the two halves

Twice the hemisphere volume is $2 \cdot (2\pi r^3/3) = 4\pi r^3/3$. The shared flat face contributes zero three-dimensional volume.



Two copies of $2\pi r^3/3$ combine to give $4\pi r^3/3$.

Check. Integrating $\pi(r^2 - z^2)$ over z from $-r$ to r gives the same total.

Exercise 248 · Catalog prices

Printed page 24 · Systems of equations / Matrix multiplication

Three softball teams ordered equipment from the same catalog. The first team spent $285 on 5 shirts, 4 caps, and 8 bats. The second team spent $210 on 12 shirts and 6 caps. The third team spent $250 for 7 shirts, 10 caps, and 3 bats. What were the catalog prices for shirts, caps, and bats?

1. Translate orders into equations

Let s,c,b be unit prices in dollars for a shirt, cap and bat. The three order totals give 5s+4c+8b=285, 12s+6c=210 and 7s+10c+3b=250. Each coefficient is a count of one item.

shirts	caps	bats
5	4	8
12	6	0
7	10	3

Each row gives the counts in one order.

The totals are $285, $210 and $250.

2. Eliminate the cap price

The second order has no bats, so divide it by 6 to get c=35−2s. Substitute this into the other two rows: 8b=145+3s and 3b=13s−100.

shirts	caps	bats
12	6	0

The second order contains no bats.

Divide by 6 and express c as 35−2s.

1. 12s+6c=210 ⇒ 2s+c=35 ⇒ c=35−2s
Divide the second order equation by 6 and isolate the cap price.

2. 5s+4(35−2s)+8b=285 ⇒ 8b=145+3s
Substitute the cap price in the first order and collect bat terms.

3. 7s+10(35−2s)+3b=250 ⇒ 3b=13s−100
Use the same substitution in the third order.

3. Eliminate the bat price

Multiply the two new equations by 3 and 8 respectively to match 24b. Hence 435+9s=104s−800, giving 95s=1235 and s=13.

8b	145+3s
3b	13s−100

Multiply the upper relation by 3 and the lower by 8 to match 24b.

1. 24b=3(145+3s)=435+9s
Multiply the 8b equation by 3.

2. 24b=8(13s−100)=104s−800
Multiply the 3b equation by 8.

3. 435+9s=104s−800 ⇒ 1235=95s ⇒ s=13
The left sides match; subtract 9s and add 800 to isolate s.

4. Recover the other prices

Now c=35−26=9 and b=(145+39)/8=23. The unit prices are \$13, \$9 and \$23; substituting them reproduces all three purchase totals.

shirt \$	cap \$	bat \$
13	9	23

Substitute the recovered prices into all three order totals.

1. c=35−2·13=9, b=(145+3·13)/8=184/8=23
Recover the two eliminated prices using the solved price of a shirt.

2. 5·13+4·9+8·23=65+36+184=285
The first team's shirt, cap and bat totals reconstruct its order.

3. 12·13+6·9=156+54=210; 7·13+10·9+3·23=91+90+69=250
The same prices also reproduce both independent orders.

Check. The totals are 65+36+184=285, 156+54=210, and 91+90+69=250.

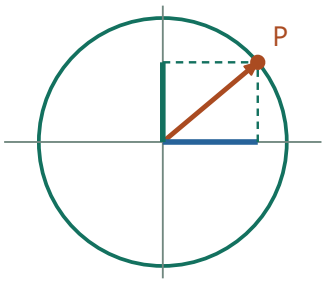
Exercise 249 · Two slope formulas

Printed page 24 · The tangent function / Trigonometric ratios

Find two equivalent ways to express the slope of the vector $[\cos\theta, \sin\theta]$.

1. Read the terminal point's coordinates

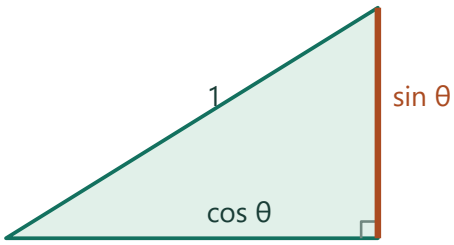
A unit arrow at angle θ ends at (cos θ,sin θ) and starts at (0,0). Its horizontal run is cos θ and its vertical rise is sin θ.



Unit-circle components are x=cos θ and y=sin θ.

2. Calculate the slope from components

For nonzero horizontal run the slope is rise/run=sin θ/cos θ. Those are signed components, so this automatically gives the correct quadrant sign.

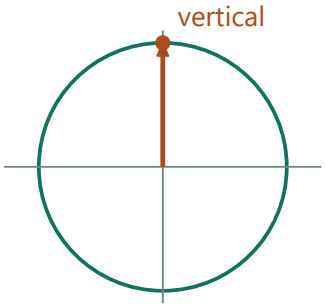


Schematic section

Use signed rise divided by signed run.

3. Identify the trigonometric function

The same ray has slope tan θ by the angle definition, giving tan θ=sin θ/cos θ. At a vertical direction cos θ=0 and the slope and tangent are undefined.



When cos θ = 0, the ray is vertical.

No real slope or tangent is defined there.

Check. At 45° both formulas give 1.

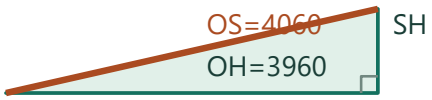
Exercise 250 · Visible polar cap

Printed page 24 · Sphere geometry / Right-triangle trigonometry

Describe the points on the earth's surface that are visible to a viewer who is 100 miles above the North Pole.

1. Build the horizon triangle

Let O be Earth's center, S the observer 100 mi above the North Pole, and H a point on the horizon. The extreme line of sight SH is tangent, giving a right angle at H. Then OS=3960+100 and OH=3960 miles.

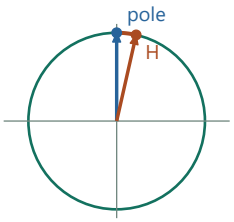


The tangent sight line is perpendicular to OH at H.

Distances in miles, schematic.

2. Measure angular reach from the pole

The angle α between the north polar radius and OH satisfies cos α=3960/4060. Hence α≈12.743°. Points farther around Earth than this angle lie beyond the tangent horizon.



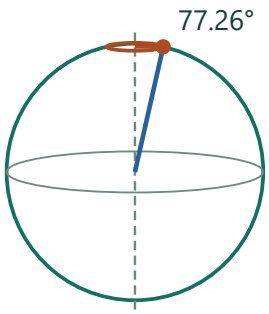
The central angle from the polar axis is arccos(3960/4060) ≈ 12.743°.

1. cos α=OH/OS=3960/(3960+100)=198/203
The adjacent leg ends at the tangent point and the hypotenuse ends at the observer.

2. α=arccos(198/203)≈12.743°
Use an acute angle since the tangent comes before a quarter turn.

3. Describe the entire visible region

Symmetry about the polar axis gives every longitude the same boundary latitude. The visible spherical cap consists of latitudes at least 90°−12.743°≈77.257° N, including its tangent edge in the ideal spherical model.



Every longitude has the same boundary.

The visible cap reaches down to approximately 77.257° N.

Check. Every point at the same latitude is equivalent by rotation about the polar axis.

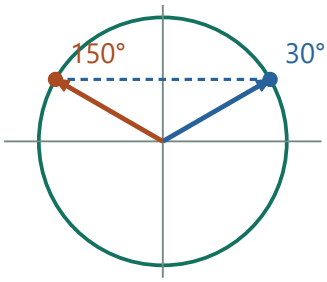
Exercise 251 · Mirror a unit-circle point

Printed page 25 · Reflections / The unit circle

Points P and Q on the unit circle are reflected images of each other, using the y-axis as a mirror. Suppose that P is described by the angle θ; what angle describes Q? In terms of θ, what are the rectangular coordinates of P? Simplify the expressions $\cos(180-\theta)$ and $\sin(180-\theta)$ by finding two different ways of writing the rectangular coordinates for Q.

1. Locate P and its mirror image

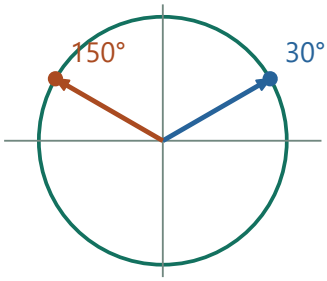
The ray at θ ends at P=(cos θ,sin θ). Reflecting this point in the y-axis reverses its horizontal coordinate and keeps its vertical coordinate, giving (−cos θ,sin θ).



Reflection across the y-axis reverses x while preserving y.

2. Read the direction of the reflected ray

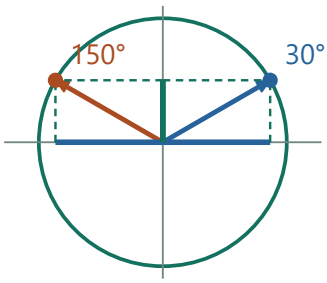
The rays are symmetric about 90°, so the reflected direction measures 180°−θ, up to full revolutions.



The new direction is 180°−θ for the illustrated acute θ.

3. Equate the coordinate descriptions

The reflected unit point is also $(\cos(180°-\theta), \sin(180°-\theta))$. Its x and y coordinates therefore give $\cos(180°-\theta) = -\cos\theta$ and $\sin(180°-\theta) = \sin\theta$.



Thus $\cos(180°-\theta) = -\cos\theta$, while $\sin(180°-\theta) = \sin\theta$.

Check. Reflection preserves length and leaves every point on the vertical axis fixed.

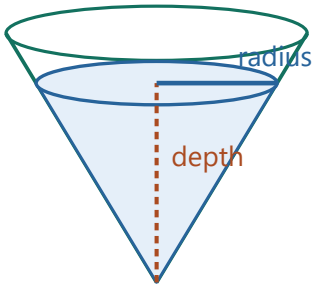
Exercise 252 · Depth at 64/125 capacity

Printed page 25 · Area and volume scaling / Volume

A conical cup is 64/125 full of liquid. What is the ratio of the depth of the liquid to the depth of the cup? Conical cups appear fuller than cylindrical cups — explain why.

1. Relate the depth fraction to all lengths

For a tip-down cone the water is itself a similar cone. If its depth is fraction t of the full cup height, its radius is t times the full radius.



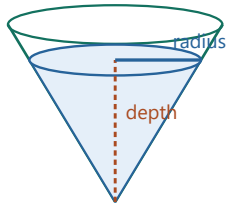
Compare the full cone and the water cone.

Their radii and depths share a single positive scale factor t.

2. Use the given volume fraction

Base area scales by t^2 and height by t, so the volume fraction is t^3. Set $t^3=64/125$ and take the cube root: $t=4/5$.

Water depth as a fraction of the full cup · 80%



Full radius 5 units, depth 10 units (example).

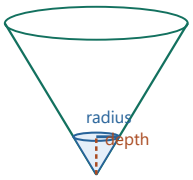
At depth 8 units, radius is 4 units. This is 51.2% of full capacity.

1. $(h/H)^3=64/125=4^3/5^3$
Use the similarity volume ratio for the water cone.

2. $h/H=4/5$
Positive depths select the positive cube root.

Compare settings · Water depth as a fraction of the full cup

25%

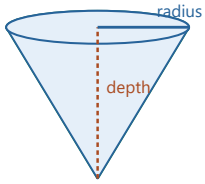


Full radius 5 units, depth 10 units (example).

At depth 2.5 units, radius is 1.25 units. This is 1.562% of full capacity.

Full radius 5 units, depth 10 units (example). At depth 2.5 units, radius is 1.25 units. This is 1.562% of full capacity.

100%



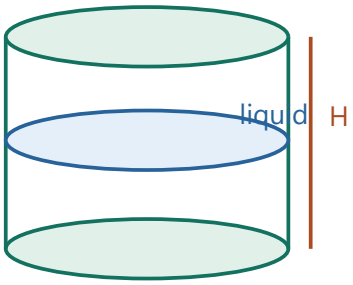
Full radius 5 units, depth 10 units (example).

At depth 10 units, radius is 5 units. This is 100% of full capacity.

Full radius 5 units, depth 10 units (example). At depth 10 units, radius is 5 units. This is 100% of full capacity.

3. Contrast the cylindrical depth calculation

In a cylinder the base area stays constant, so the same volume fraction yields depth fraction 64/125. The cone's narrower lower layers let the liquid reach 4/5 of its height with that amount of water.



In a cylinder the section area is constant.

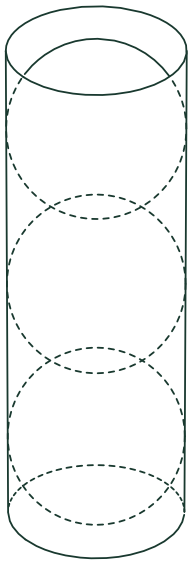
Here the depth fraction equals 64/125 directly.

Check. Cubing 0.8 gives 0.512, the stated fill fraction.

Exercise 253 · Three balls in a can

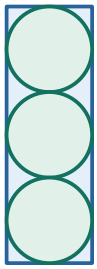
Printed page 25 · Packing / Volume

Three tennis balls fit snugly inside a cylindrical can. What percent of the available space inside the can is occupied by the balls?

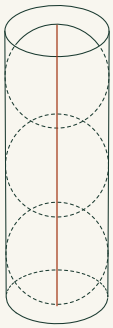


1. Read the cylinder dimensions from contact

Let each ball have radius r. A snug cylindrical container has radius r. Three tangent balls stacked in one column span three diameters, so the container height is 6r.



The can radius is r. Three diameters stacked vertically total 6r.

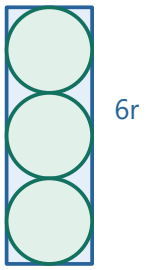


Vertical stack of three diameters

The top-to-bottom length of the stack comprises one full diameter per ball, because neighboring balls are tangent.

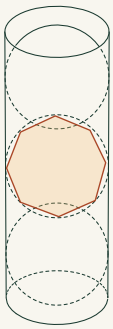
2. Compute occupied and available volume

Each sphere occupies $4\pi r^3/3$, so three occupy $4\pi r^3$. The can holds $\pi r^2 \cdot 6r = 6\pi r^3$.



The three balls contribute three copies of $4\pi r^3/3$.

The can holds $\pi r^2 \cdot 6r$.

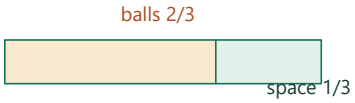


One central sphere

The middle ball occupies only part of the matching cylindrical portion of the can; the space around its curved sides is included in total can volume.

3. Form the percentage

The occupied fraction is $4\pi r^3/(6\pi r^3) = 2/3$. Hence the balls occupy 66⅔% of the container. The size r cancels because every length scales together.



The occupied volume fraction simplifies to 2/3, or 66⅔%.

1. $V_{balls}/V_{can} = [3 \cdot (4\pi r^3/3)]/(6\pi r^3) = 4\pi r^3/(6\pi r^3)$
Three spheres cancel the one-third in each sphere's volume formula.

2. $= 4/6 = 2/3 \Rightarrow$ percent $= 200/3\% = 66\tfrac{2}{3}\%$
Cancel positive πr^3, reduce the fraction and multiply by 100.

Check. Each ball fills the same two-thirds fraction of its own height-2r portion of the can.

Exercise 254 · Two Arctic routes

Printed page 25 · Great-circle distances / Spherical coordinates

Seward, Alaska is 180 degrees due west of St. Petersburg, Russia. Both cities are 60 degrees north of the equator. Calculate the distance from St. Petersburg to Seward, assuming

(a) we travel along the circle of latitude;

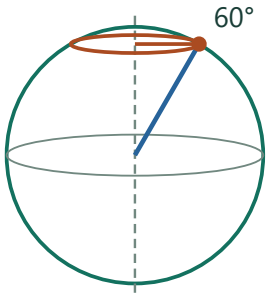
(b) we travel along the circle that passes over the North Pole.

Part (b) is an example of a *great-circle* route. Explain the terminology, and also explain why pilots might prefer to fly along great circles.

1. Follow the parallel first

At latitude 60° N the radius of the parallel is 3960 cos 60°=1980 mi. Opposite longitudes are half a revolution apart, giving route length π·1980≈6220 mi.

Latitude · 60°

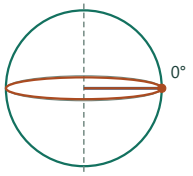


At latitude 60° the radius of the parallel is 1980 mi.

Its circumference is 0.5 times the equatorial circumference.

Compare settings · Latitude

0°

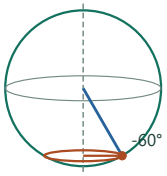


At latitude 0° the radius of the parallel is 3960 mi.

Its circumference is 1 times the equatorial circumference.

At latitude 0° the radius of the parallel is 3960 mi. Its circumference is 1 times the equatorial circumference.

−60°



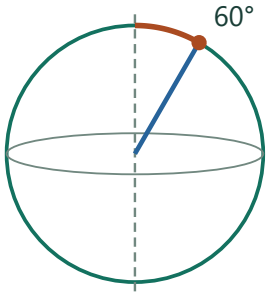
At latitude −60° the radius of the parallel is 1980 mi.

Its circumference is 0.5 times the equatorial circumference.

At latitude −60° the radius of the parallel is 1980 mi. Its circumference is 0.5 times the equatorial circumference.

2. Go by the North Pole

Each city is 90°−60°=30° from the pole. The two meridian arcs total 60°=π/3 radians at Earth's full radius 3960. Their combined length is 1320π≈4147 mi.

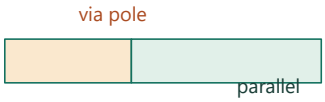


The North Pole is 30° north of either endpoint.

Mirror this orange meridian segment on the opposite longitude.

3. Explain the shorter route

These opposite meridians belong to a common great circle, the section by a plane through Earth's center. Its shorter arc minimizes surface distance. Here the polar route is two thirds as long as the half-parallel route, illustrating the advantage of great-circle routing.



Compare 1320π mi via the pole with 1980π mi along the parallel.

1. Lpolar/Lparallel=(1320π)/(1980π)=1320/1980
The common factor π cancels between the two routes in miles.

2. 1320/1980=(1320÷660)/(1980÷660)=2/3
The great circle path covers two thirds of the parallel route's distance.

Check. The polar route is two-thirds the length of the parallel route.

Exercise 255 · Zero included angle

Printed page 25 · The cosine rule / Triangle geometry

What does the SAS version of the Law of Cosines have to say about the "triangle" whose sides p and q form a 0-degree angle?

1. Interpret a zero included angle

Place the sides of lengths p and q on the same ray from the common vertex. Their endpoints may occur in either order depending on which length is larger.



Start the two sides at O. Their rays point in the same direction.

2. Simplify the cosine rule

With cos 0°=1, the squared separation is $p^2+q^2-2pq=(p-q)^2$.



With cos 0°=1 the squared distance is $(p-q)^2$.

3. Take an unsigned distance

The positive square root is $|p-q|$. This agrees with simply subtracting the smaller distance along the ray from the larger. If $p=q$, the endpoints coincide and the separation is zero.



The distance between endpoints is $p-q$, including zero if the two ends coincide.

1. $c=\sqrt{[(p-q)^2]}=\text{abs}(p-q)$
Taking a nonnegative square root cannot give a signed length.

2. $c=p-q$ if $p\geq q$; $c=q-p$ if $q>p$
The farther endpoint determines which difference is positive.

Check. The endpoints are on the same ray, so ordinary subtraction gives their separation.

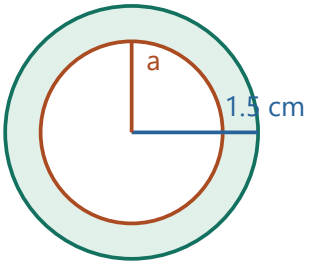
Exercise 256 · A hollow chocolate shell

Printed page 25 · Volume / Ring and shell geometry

In an effort to make their product seem like a better bargain, the Chock-a-Lot candy company increased the size of their chocolate balls, from a 2-cm diameter to a 3-cm diameter, without increasing the price. In fact, the new balls still contain the same amount of chocolate, because they are *hollow spherical shells*, while the 2-cm balls are solid chocolate. How thick are the spherical chocolate shells that Chock-a-Lot is now selling?

1. Distinguish outer radius, cavity radius and thickness

The large ball's outer radius is $3/2$ cm. Let a be its empty inner radius, so its chocolate thickness is $3/2-a$. The original solid ball has radius 1 cm.



An axial cut shows the chocolate shell and empty cavity.

The unknown thickness is $1.5-a$.

2. Conserve the chocolate volume

Outer sphere volume minus cavity volume equals the original solid volume. Cancel $4\pi/3$ from both sides: $(3/2)^3-a^3=1^3$. Hence $a^3=27/8-1=19/8$.

outer^3	inner^3
27/8	19/8

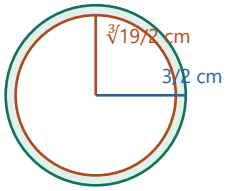
The difference of cubed radii is the original 1-cm ball's cubed radius, 1.

1. $(4\pi/3)[(3/2)^3-a^3]=(4\pi/3)\cdot 1^3$
Outer sphere minus cavity equals the solid chocolate volume.

2. $27/8-a^3=1 \Rightarrow a^3=27/8-8/8=19/8$
Cancel the positive sphere constant and isolate the cubed cavity radius.

3. Find the thickness rather than just the cavity size

The inner radius is $\sqrt[3]{19}/2$ cm. Subtract this from the outer radius to obtain $(3-\sqrt[3]{19})/2 \approx 0.166$ cm.



Subtract inner radius from outer.

Chocolate thickness ≈ 0.166 cm.

1. $a=\sqrt[3]{19}/2$
The cube root of the denominator 8 is 2.

2. $\text{thickness}=3/2-a=(3-\sqrt[3]{19})/2$ cm
Measure the difference of radii along the same ray.

Check. Subtracting cavity volume from the larger sphere recovers exactly $4\pi/3$ cm³ of chocolate.

Exercise 257 · Write three systems as matrices

Printed page 26 · Matrix multiplication / Systems of equations

You have perhaps noticed that any linear equation can be interpreted as a dot-product equation. For example, $3x - 5y = 17$ is equivalent to $[3, -5] \bullet [x, y] = 17$. In this way, any system of linear equations can be written as a single matrix equation. For example,

$$\begin{cases} 2x - 5y + 4z = -7 \\ 6x - y + 3z = 12 \\ -5x + 4y + z = 3 \end{cases} \quad \text{can be written as} \quad \begin{bmatrix} 2 & -5 & 4 \\ 6 & -1 & 3 \\ -5 & 4 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -7 \\ 12 \\ 3 \end{bmatrix}$$

Rewrite each of the following systems in matrix form:

(a) $\begin{cases} 5x + 3y = 15 \\ 4x - 2y = 1 \end{cases}$ (b) $\begin{cases} 2x - 5z = -8 \\ 3y + 4z = 17 \\ x + y + z = 6 \end{cases}$ (c) $\begin{cases} 5 + 2t = 4 - u \\ 1 - 5t = 2 + 3u \end{cases}$

1. Choose one fixed variable order per system

A matrix row holds coefficients in the same order as the unknown column. The right-hand constants form another column in equation order. A missing variable contributes coefficient zero.

x	y	z
2	-5	4

Write variables in one fixed order.

A coefficient row corresponds to one equation.

2. Encode (a) and (b)

In (a), rows (5,3),(4,−2) multiply $(x,y)^{\intercal}$ to give $(15,1)^{\intercal}$. In (b), rows (2,0,−5),(0,3,4),(1,1,1) multiply $(x,y,z)^{\intercal}$ to give $(-8,17,6)^{\intercal}$.

x	y	z
2	0	-5
0	3	4
1	1	1

Missing terms give zero entries.

These are the rows of system (b).

3. Rearrange before encoding (c)

Move the t and u terms to the left and the constants to the right. The resulting rows are (2,1),(−5,−3) multiplying $(t,u)^{\intercal}$ to give $(-1,1)^{\intercal}$. Each row's ordinary dot product rebuilds its equation.

t	u
2	1
-5	-3

Collect t and u on the left before forming matrix (c).

1. $5+2t=4-u \Rightarrow 2t+u=4-5=-1$
Move u left and both constants right.

2. $1-5t=2+3u \Rightarrow -5t-3u=2-1=1$
Subtract 3u and 1 from both sides before reading coefficients.

Check. Expanding each row restores its original scalar equation, including zero coefficients for absent variables.

Exercise 258 · Test candidate solutions

Printed page 26 · Systems of equations / Verifying equations

(Continuation) Calculate the following matrix products:

(a) $\begin{bmatrix} 5 & 3 \\ 4 & -2 \end{bmatrix} \begin{bmatrix} 1.5 \\ 2.5 \end{bmatrix}$ (b) $\begin{bmatrix} 2 & 0 & -5 \\ 0 & 3 & 4 \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 3 \\ 2 \end{bmatrix}$ (c) $\begin{bmatrix} 2 & 1 \\ -5 & -3 \end{bmatrix} \begin{bmatrix} -2 \\ 3 \end{bmatrix}$ (d) $\begin{bmatrix} 2 & -5 & 4 \\ 6 & -1 & 3 \\ -5 & 4 & 1 \end{bmatrix} \begin{bmatrix} 2 \\ 3 \\ 1 \end{bmatrix}$

What do your results tell you about the solutions to the systems in the preceding question?

1. Understand what each matrix product checks

Each row of the coefficient matrix contains one equation’s coefficients. Multiplying that row by the proposed unknown column evaluates the left side of that equation. Collect the results from top to bottom, then compare the entire output column with the constants in exercise 257.

row·input
row·input

Each output component evaluates one whole equation.

2. Calculate the two rows in (a)

Insert (x,y)=(1.5,2.5). The top row gives 5(1.5)+3(2.5)=7.5+7.5=15. The bottom row gives 4(1.5)−2(2.5)=6−5=1. Hence the product is $(15,1)^{\mathsf{T}}$ and the input solves 257(a).

5	3
4	-2
1.5	2.5

Use input (1.5,2.5) with each of the two rows.

Output = (15,1).

3. Calculate the three rows in (b)

For (x,y,z)=(1,3,2), the first row gives 2·1+0·3−5·2=−8. The second gives 0·1+3·3+4·2=17. The third adds all three entries, giving 1+3+2=6. The output $(-8,17,6)^{\mathsf{T}}$ matches all three constants of 257(b).

2	0	-5
0	3	4
1	1	1

Dot each row with (1,3,2).

The highlighted row gives 17.

4. Follow the variable names in (c)

Here the column entries stand for t and u, with t=−2 and u=3. The first row gives 2(−2)+3=−1. The second gives (−5)(−2)−3·3=10−9=1. Thus the product is $(-1,1)^{\mathsf{T}}$. As a check in the original, unrearranged equations of 257(c), the two sides equal 1 in the first and 11 in the second.

2	1
-5	-3

With input (−2,3), the second equation gives 10−9=1.

5. Identify and test the introductory system in (d)

This matrix comes from the three-equation example in the introduction to 257. Its candidate is (x,y,z)=(2,3,1). The first row gives 2·2−5·3+4·1=−7, the second gives 6·2−3+3·1=12, and the third gives −5·2+4·3+1=3. The product $(-7,12,3)^{\mathsf{T}}$ verifies all three equations simultaneously.

2	-5	4
6	-1	3
-5	4	1

Use (2,3,1) for the introductory system.

The three results are −7,12,3.

Check. For instance the last product has rows 4−15+4=−7, 12−3+3=12, and −10+12+1=3.

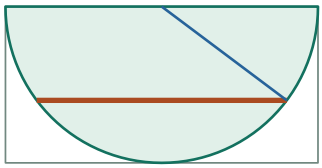
Exercise 259 · Water in a spherical bowl

Printed page 26 · Sphere geometry / Cross sections

A hemispherical bowl with a 30-centimeter radius contains some water, which is 12 centimeters deep. Find the volume of the water, to the nearest cubic centimeter.

1. Choose a vertical coordinate from the bowl bottom

At water height z the sphere center is 30−z cm above the slice. A radius to a rim point, that vertical offset and the slice radius a form a right triangle. Thus $a^2=30^2-(30-z)^2=60z-z^2$.



Axial section

At the bowl bottom z=0. The top of the water is z=12 cm; the sphere center is at z=30.

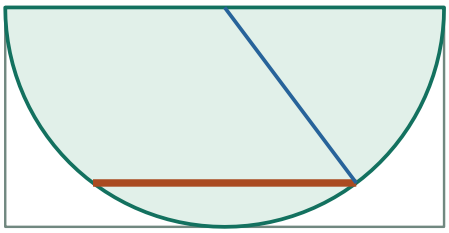
1. $a^2=900-(900-60z+z^2)$
Expand the squared center-to-slice distance.

2. $a^2=60z-z^2$
The outer square cancels and both inner signs reverse.

2. Measure one horizontal layer

Its area is $\pi a^2=\pi(60z-z^2)$. A thin layer of thickness dz contributes approximately that area times dz. Summing infinitesimal layers from z=0 to the water surface z=12 gives the volume integral.

Height of a water slice above the bottom · 6 cm



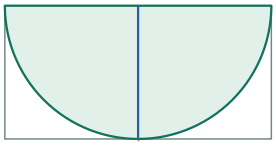
Axial section

The water reaches z=12 cm, so $0 \le z \le 12$.

At z=6 cm the slice area is $\pi[30^2-24^2] = 324\pi$ cm^2.

Compare settings · Height of a water slice above the bottom

0 cm



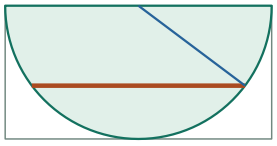
Axial section

The water reaches z=12 cm, so $0 \le z \le 12$.

At z=0 cm the slice area is $\pi[30^2-30^2] = 0\pi$ cm^2.

The water reaches z=12 cm, so $0 \le z \le 12$. At z=0 cm the slice area is $\pi[30^2-30^2] = 0\pi$ cm^2.

12 cm



Axial section

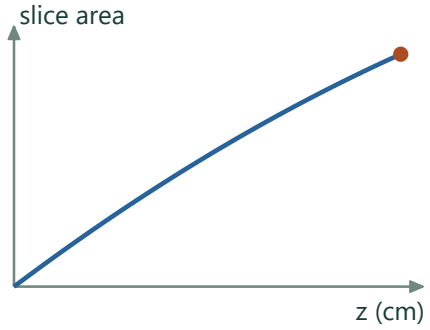
The water reaches z=12 cm, so $0 \le z \le 12$.

At z=12 cm the slice area is $\pi[30^2-18^2] = 576\pi$ cm^2.

The water reaches z=12 cm, so $0 \le z \le 12$. At z=12 cm the slice area is $\pi[30^2-18^2] = 576\pi$ cm^2.

3. Evaluate the sum

Integrating gives $\pi[30z^2 - z^3/3]$ from 0 to 12, or $\pi(4320 - 576) = 3744\pi \approx 11{,}762$ cm³.

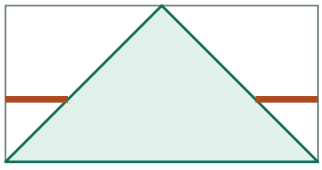


The area under the slice-area curve gives $\int_0^{12} \pi(60z - z^2)\,dz = 3744\pi$ cm³.

1. $\int (60z - z^2)\,dz = 30z^2 - z^3/3 + C$	Integrate each power separately, keeping the minus sign.
2. $V = \pi[(30 \cdot 12^2 - 12^3/3) - (30 \cdot 0^2 - 0^3/3)]$	Evaluate the antiderivative at the water level and bottom.
3. $V = \pi(4320 - 576) = 3744\pi$ cm³	Subtract the lower endpoint contribution and simplify.

4. Recover the answer with ordinary solids

Equal-area ring slices give the same volume as a 12-cm slab of radius-30 cylinder minus a conical frustum of height 12 and radii 30 and 18. Subtracting the radius-18, height-18 cone from the radius-30, height-30 cone gives that frustum. The difference of cylinder and frustum again equals 3744π cm³.



Axial section

The matching volume is a cylindrical slab minus the conical frustum occupying its middle.

1. $V_{slab} = \pi \cdot 30^2 \cdot 12 = 10{,}800\pi$
First use the full cylinder slab over the water depth.

2. $V_{frustum} = (\pi/3)(30^3 - 18^3) = 7056\pi$
The frustum is the difference between two similar upright cones.

3. $V_{water} = 10{,}800\pi - 7056\pi = 3744\pi$ cm³
Subtract the cone obstruction, whose cross sections match the removed disks.

Check. The cap formula $\pi h^2(R - h/3)$ gives $\pi \cdot 144 \cdot 26$.

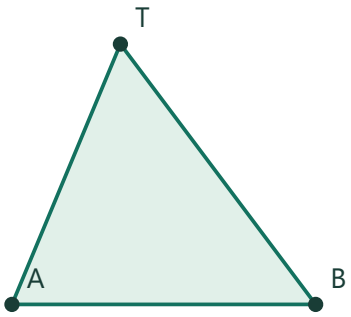
Exercise 260 · Circumradius of a 13-14-15 triangle

Printed page 26 · Incircles and circumcircles / Triangle area

A triangle has sides 13, 14, and 15. What is the radius of its circumscribed circle?

1. Identify the circle that is being sought

A circumscribed circle goes through all three vertices. Its radius R runs from the circle center to a vertex. To connect R to the known side lengths, first find the triangle's area, then use a chord of that same circle.



Choose the 13–14–15 triangle.

Its circumcircle must pass through all three vertices.

2. Find the area from the three side lengths

The semiperimeter is half the sum of the sides: $s=(13+14+15)/2=21$. Heron's formula gives $A^2=s(s-13)(s-14)(s-15)=21\cdot8\cdot7\cdot6=7056$. Taking the positive square root gives $A=84$ square units.

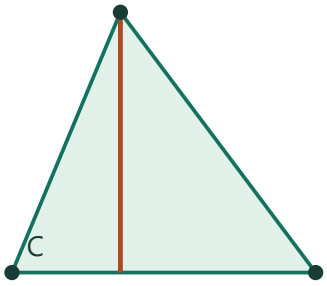
21	8	7	6

Heron factors multiply to 7056.

Their square root is area 84.

3. Find a sine from the area

Name C the angle between the sides of lengths 13 and 14; the side opposite C has length 15. Using 14 as a base, the corresponding altitude is $13\sin C$. Thus $84=\tfrac{1}{2}\cdot14\cdot13\sin C$, so $\sin C=168/182=12/13$.

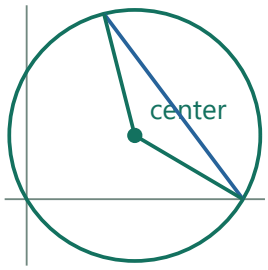


The angle C between the 13 and 14 sides has sine 12/13.

It faces the side of length 15.

4. Connect that angle to the circle radius

The 15-unit side is a chord. The inscribed angle C subtends the same arc as a central angle of twice its measure. Bisecting the central isosceles triangle gives a right triangle with hypotenuse R and opposite leg half the chord. Here C is acute because $15^2<13^2+14^2$. Thus $15/2=R \sin C$.

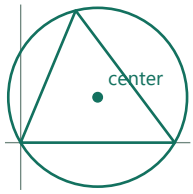


The side of length 15 is a circle chord.

Half its length equals R sin A.

5. Solve and check the radius

Substitute $\sin C=12/13$ into $15=2R \sin C$. Dividing gives $R=15\cdot13/24=65/8=8.125$ length units. Combining the area and chord equations gives the equivalent familiar formula $R=abc/(4A)$.



$2R = 15/(12/13)$.

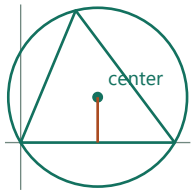
$R = 65/8$ length units.

1. $A=(ab/2)\sin C \Rightarrow \sin C=2A/(ab)$
Both adjacent sides are positive, so divide after doubling the area.

2. $c=2R \sin C=4RA/(ab) \Rightarrow R=abc/(4A)$
Substitute the sine expression into the chord equation and use the positive area of a nondegenerate triangle.

6. Check the result with distances

Place the 14-unit side on a coordinate axis, from (0,0) to (14,0). The third vertex can be (5,12), since its distances from the endpoints are 13 and 15. A circle through both endpoints has center (7,k) on their perpendicular bisector. Equating squared distances to (0,0) and (5,12) gives $49+k^2=4+(12-k)^2$, hence $k=33/8$. The radius is $\sqrt{[49+(33/8)^2]}=65/8$ again.



The circumcenter lies at (7,33/8) on the perpendicular bisector of the 14-unit base.

1. $49+k^2=4+144-24k+k^2$
Expand $(12-k)^2$ in the center-to-vertex distance condition.

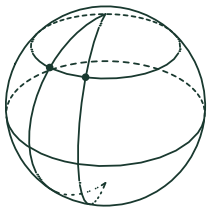
2. $24k=148-49=99 \Rightarrow k=33/8$
Cancel the common k^2 and isolate the center height.

3. $R^2=49+1089/64=(3136+1089)/64=4225/64$
Insert the center coordinate before taking the positive root.

Check. The altitude to side 14 is 12. The sine of an angle adjacent to that base is 12/13 opposite side 15, giving diameter $15/(12/13)=65/4$.

Exercise 261 · Distances at 45° north

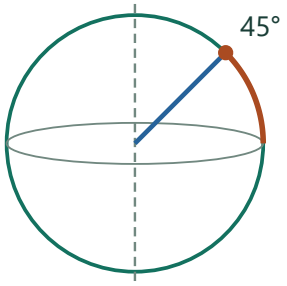
Printed page 26 · Great-circle distances / Spherical coordinates



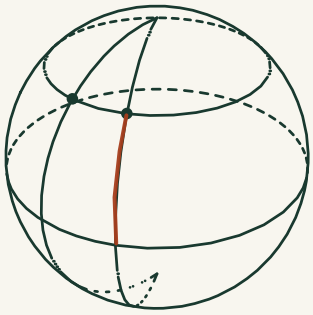
Salem, Oregon is 30 degrees due west of St. Paul, Minnesota, the latitude of both cities being 45° north of the equator. How far is it
(a) from Salem to the equator?
(b) from St. Paul to Salem, traveling due west along the circle of latitude?

1. Use the meridian to the equator

Salem is at 45° latitude. Along a meridian the radius is Earth's full 3960 mi. The 45° angle is π/4 radians, so the shortest surface distance to the equator is $3960\pi/4 = 990\pi \approx 3110$ mi.



A 45° meridian arc to the equator measures 3960π/4 mi.



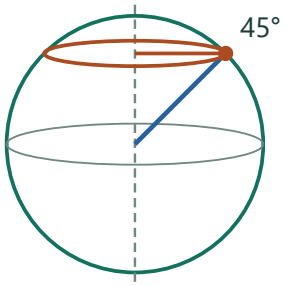
Route toward the equator

The meridian is a great-circle direction that changes latitude. Moving down it to the equator spans the latitude angle at Earth's center.

2. Choose the parallel radius for the city-to-city route

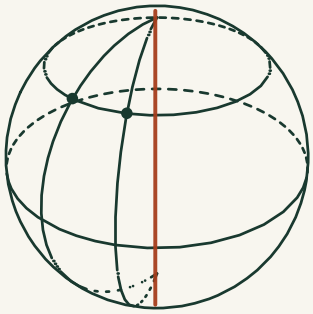
Both cities lie on the 45° parallel, whose radius about the polar axis is $3960\cos 45° = 3960/\sqrt{2}$. The specified parallel route uses this smaller radius.

Latitude · 45°



At latitude 45° the radius of the parallel is 2800.143 mi.

Its circumference is 0.707 times the equatorial circumference.

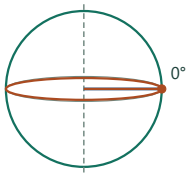


Polar axis

Rotation around this axis carries a point around a parallel. The radius of that parallel is its perpendicular distance to the axis.

Compare settings · Latitude

0°

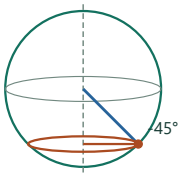


At latitude 0° the radius of the parallel is 3960 mi.

Its circumference is 1 times the equatorial circumference.

At latitude 0° the radius of the parallel is 3960 mi. Its circumference is 1 times the equatorial circumference.

−45°



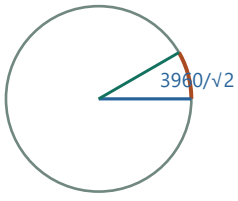
At latitude −45° the radius of the parallel is 2800.143 mi.

Its circumference is 0.707 times the equatorial circumference.

At latitude −45° the radius of the parallel is 2800.143 mi. Its circumference is 0.707 times the equatorial circumference.

3. Take the 30° longitudinal arc

The longitude separation is $30° = \pi/6$ radians, so the parallel-route distance is $(3960/\sqrt{2})(\pi/6) = 330\pi\sqrt{2} \approx 1466$ mi.



Plan view of the parallel

The 30° longitude change corresponds to π/6 radians of this smaller circle.

1. $s = (3960/\sqrt{2})(\pi/6) = 660\pi/\sqrt{2}$
The radius of the parallel is smaller than Earth's radius and $30° = \pi/6$ radians.

2. $s = (660\pi\sqrt{2})/2 = 330\pi\sqrt{2}$ mi
Multiply numerator and denominator by $\sqrt{2}$.

Check. The parallel trip is one twelfth of that latitude circle's circumference.

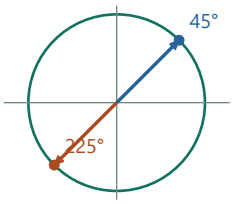
Exercise 262 · Sine/cosine intersections

Printed page 26 · Trigonometric equations / The unit circle

If the equations $y = \sin x$ and $y = \cos x$ are both graphed on the same xy-axis system, the curves will intersect many times. Find coordinates for at least two intersection points.

1. Translate graph intersection into an equation

At an intersection the same input x gives the same output from both functions. Thus sin x=cos x, so the unit-circle endpoint lies on y=x.



Sine equals cosine when the horizontal and vertical unit-circle coordinates match.

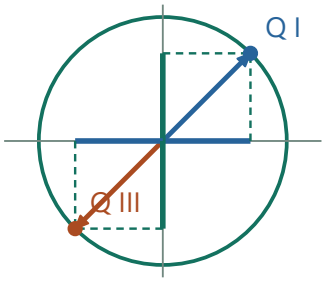
1. cos x=0 ⇒ sin x=±1
These points cannot solve sin x=cos x.

2. sin x=cos x ⇒ tan x=1
Divide by the nonzero cosine on the solution set.

3. x=45°+180°k
The slope-one radius repeats after a half-turn.

2. Find two such inputs

The diagonal meets the unit circle in the upper-right and lower-left quadrants, at 45° and 225° in one turn. Their common coordinate values are √2/2 and −√2/2.



The two matching coordinates have magnitude √2/2.

3. Write graph points, not only directions

The graph's horizontal coordinate is the input angle itself. Two intersections are (45°,√2/2) and (225°,−√2/2), or use π/4 and 5π/4 as inputs in radians.

x	y
45°	√2/2
225°	−√2/2

Each graph intersection needs both the input angle and shared function value.

Check. Both are unit-circle directions on the line y=x.

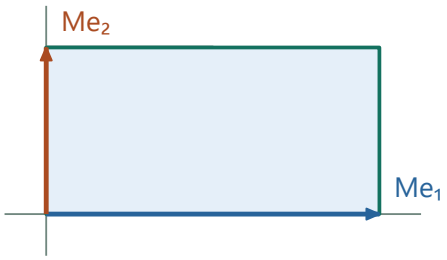
Exercise 263 · Order of matrix products

Printed page 26 · Matrix multiplication

Is it generally true that matrix products $\mathbf{MN}$ and $\mathbf{NM}$ are the same? Explain.

1. Choose transformations whose order can matter

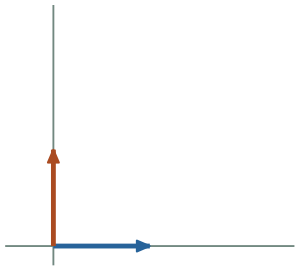
Let M scale the horizontal coordinate by 2 and leave the vertical one alone, and let N swap the coordinates. Thus M has rows (2,0),(0,1) and N rows (0,1),(1,0).



M doubles only x; N swaps x and y.

2. Follow a single test vector through MN

Starting from (1,0), first apply the rightmost N, giving (0,1). Applying M still gives (0,1).

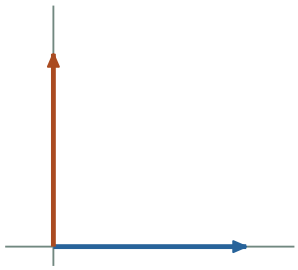


For MN, swap (1,0) to (0,1) before scaling.

The result is (0,1).

3. Reverse the order in NM

First M sends (1,0) to (2,0); N sends that to (0,2). Different outputs on a single input prove MN≠NM, so matrices do not commute in general. With rectangular factors even the existence of a product can depend on order.



For NM, stretch (1,0) to (2,0) before swapping.

The result is (0,2).

Check. Multiplying explicitly gives rows (0,2),(1,0) versus (0,1),(2,0).

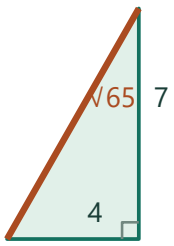
Exercise 264 · Polar coordinates of (4,−7)

Printed page 26 · Polar coordinates / Trigonometric ratios

Find polar coordinates for the point described by $x = 4$ and $y = -7$.

1. Calculate distance to the origin

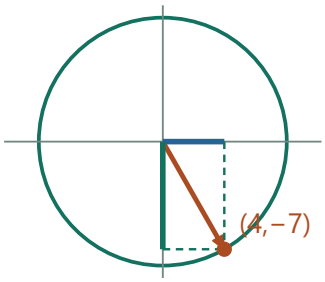
For the Cartesian point (4,−7) the positive polar radius is $\sqrt{(4^2+7^2)}=\sqrt{65}$.



Absolute run 4 and drop 7 give radius √65.

2. Orient the radius

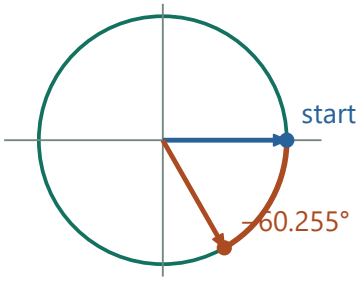
The point is right of the vertical axis and below the horizontal axis. Its ray is in quadrant IV, with acute reference angle arctan(7/4).



Positive x, negative y: turn clockwise from positive x.

3. Give equivalent polar pairs

A clockwise turn from the positive x-axis gives $(\sqrt{65},-\arctan(7/4))\approx(\sqrt{65},-60.255°)$. Adding 360° gives the same point at about 299.745°.



Equivalent positive angle ≈ 299.745°.

Either ray with radius √65 reaches the point.

Check. The unit direction is (4,−7)/√65.

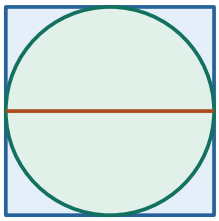
Exercise 265 · Sphere in a cube

Printed page 26 · Sphere geometry / Volume

A squash ball fits snugly inside a cubical box whose edges are 4 cm long. Guess the percentage of the box's volume that the ball occupies, and then calculate that percentage. (This is an example of a *sphere inscribed in a cube*.)

1. Estimate before computing

The ball spans the box from face to face while leaving space near all the corners. A rough starting estimate is about one half of the box.

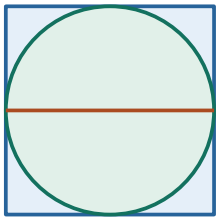


Central section of the 4-cm cube.

The sphere touches the faces and leaves the corners empty.

2. Calculate actual volumes

The cube edge is 4 cm, so its volume is $4^3=64$ cm^3. The ball radius is 2 cm, so its volume is $(4/3)\pi\cdot 2^3=32\pi/3$ cm^3.

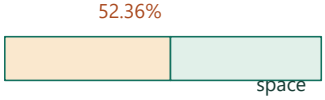


The inscribed circle spans the full 4-cm edge.

Sphere radius is 2 cm; cube volume 64 cm^3.

3. Compare with the estimate

The fraction occupied is $(32\pi/3)/64=\pi/6\approx 0.5236$, or 52.36%. It exceeds the half-box estimate by about 2.36 percentage points.



The occupied fraction is $\pi/6$.

Compare with the initial half-box estimate.

1. fraction=$(32\pi/3)/64=32\pi/(3\cdot 64)=\pi/6$
Cancelling 32 gives the occupied fraction of the cube.

2. percent=$100\pi/6\approx 52.36\%$
Convert the fraction to a percentage.

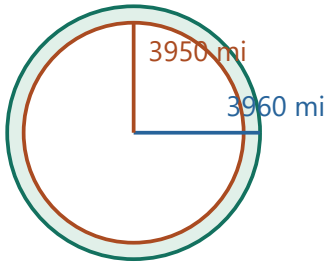
Check. For any edge $2r$ the same ratio follows from dividing $4\pi r^3/3$ by $8r^3$.

Exercise 266 · Earth's crust as a thin shell

Find the volume of material that makes up the earth's crust, which is ten miles thick. Knowing this volume should make it fairly easy to estimate the surface area of the earth. (In fact, it is an especially simple calculation for members of the Flat Earth Society.) Your estimate is either larger or smaller than the exact area. Which? How do you know?

1. Describe the shell with two radii

The outer Earth radius is 3960 mi. A crust 10 mi thick leaves inner radius 3950 mi. Its volume is the difference between the corresponding sphere volumes.

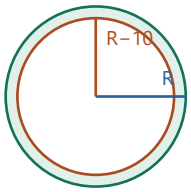


Schematic shell thickness exaggerated.

The 10-mi crust lies between concentric spheres.

2. Compute a thickness-based area estimate

The crust volume is $(4\pi/3)(3960^3-3950^3)\approx 1{,}965{,}635{,}880$ mi^3. Treating it as thin material of equal thickness gives shell volume divided by 10 mi, about 196,563,588 mi^2.



Compute the difference of two sphere volumes, then divide by 10 mi.

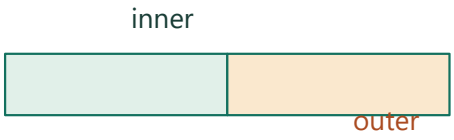
1. $R^3-r^3=(R-r)(R^2+Rr+r^2)$
Factor the shell's difference of cubed radii.

2. $V_{shell}/(R-r)=(4\pi/3)(R^2+Rr+r^2)$
Cancel the positive thickness, which is 10 miles.

3. $=(4\pi/3)(3960^2+3960\cdot3950+3950^2)$
Use the two actual radii for the mean shell area.

3. Determine the direction of error

Inner layers have smaller radius and hence smaller area than the outside surface. Dividing total shell volume by its thickness gives an average of those layer areas, below the actual outer value $4\pi\cdot3960^2$.



Layers at larger radius have greater surface area.

An average across the thickness is below the outermost area.

Check. Exact outer area is $4\pi \cdot 3960^2$; expanding the shell quotient gives $4\pi(R^2 - Rh + h^2/3)$ with $h = 10$, strictly smaller.

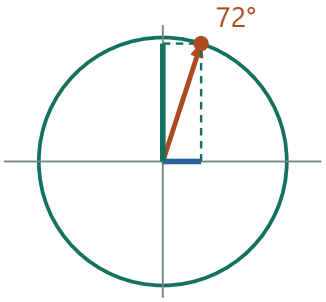
Exercise 267 · Negative-angle identities

Printed page 26 · Trigonometric identities / The unit circle

Calculate $\sin 72$ and $\sin(-72)$. Explain why $\sin(-\theta)$ is always the same as $-\sin\theta$. What can be said about $\cos(-\theta)$?

1. Locate the positive angle

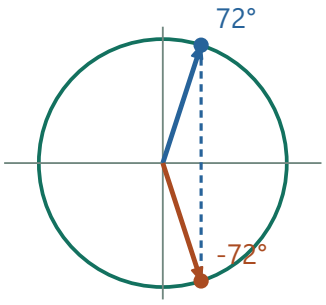
The 72° ray ends high in quadrant I. In degree mode its vertical coordinate is $\sin 72° \approx 0.951057$.



Start with the radius at 72°.

2. Reflect to the negative angle

Turning clockwise through the same angle gives the x-axis reflection of the first point. Its height changes sign, so $\sin(-72°) \approx -0.951057$.

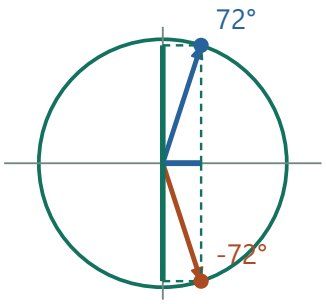


Reflect across the horizontal axis.

The horizontal coordinate stays fixed; the vertical sign reverses.

3. Generalize to every angle

The horizontal coordinate is unchanged in this reflection and the vertical coordinate reverses. Thus $\cos(-\theta)=\cos\theta$ and $\sin(-\theta)=-\sin\theta$.



Reflect across the horizontal axis.

The horizontal coordinate stays fixed; the vertical sign reverses.

Check. At $-90°$ the terminal point is $(0,-1)$, showing the same reflection directly.

Exercise 268 · Solve an invertible matrix equation

Printed page 27 · Matrix inverses / Systems of equations

A system of simultaneous linear equations can always be written in matrix form $\mathbf{CV}=\mathbf{B}$, where $\mathbf{C}$ is the *coefficient matrix* and $\mathbf{V}$ is the matrix of (unknown) variables.

(a) If $\mathbf{C}$ is invertible, explain why the equation $\mathbf{V}=\mathbf{C}^{-1}\mathbf{B}$ finds the solution to the system.

(b) It follows that

$$\begin{bmatrix}2&-5&4\\6&-1&3\\-5&4&1\end{bmatrix}\begin{bmatrix}x\\y\\z\end{bmatrix}=\begin{bmatrix}-7\\12\\3\end{bmatrix} \quad \text{is solved by computing} \quad \begin{bmatrix}x\\y\\z\end{bmatrix}=\begin{bmatrix}2&-5&4\\6&-1&3\\-5&4&1\end{bmatrix}^{-1}\begin{bmatrix}-7\\12\\3\end{bmatrix}.$$

Use a matrix calculator to finish the solution to this system of equations.

1. Keep the matrix order straight

The original system has coefficient matrix C, unknown column V and right column B, with CV=B. Left-multiply both sides by C^{-1}: $(C^{-1}C)V=C^{-1}B$. The left factor becomes I, so $V=C^{-1}B$.

$C^{-1}C$	V
I	V

Multiplying CV=B on the left by C^{-1} cancels the coefficient matrix.

2. Use the inverse on the actual right side

Here C^{-1} has rows $(-13,21,-11),(-21,22,18),(19,17,28)$, all divided by 155. Multiply them by $(-7,12,3)^{T}$: the numerators are $91+252-33=310$, $147+264+54=465$ and $-133+204+84=155$.

-13	21	-11
-21	22	18
19	17	28

Dot these rows with $(-7,12,3)$, then divide every result by 155.

3. Interpret and verify the solution

Dividing the column by 155 gives $(x,y,z)=(2,3,1)$. Substitution yields $-7,12,3$ in the original three rows. Because C has an inverse, the solution is unique.

310	465	155
2	3	1

Top: numerators. Bottom: recovered coordinates.

The original equations return $-7,12,3$.

1. $2\cdot2-5\cdot3+4\cdot1=-7$
Use the first coefficient row with the recovered unknowns.

2. $6\cdot2-1\cdot3+3\cdot1=12$
The second row reproduces its own right-hand constant.

3. $-5\cdot2+4\cdot3+1\cdot1=3$
The last row confirms the solution satisfies all equations simultaneously.

Check. Substitute: $4-15+4=-7$, $12-3+3=12$, and $-10+12+1=3$.

Exercise 269 · Reflections in different orders

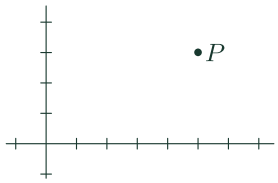
Printed page 27 · Matrix multiplication / Reflections

Apply each of the following transformations to $P = (5, 3)$:

(a) Reflect across the x-axis, then reflect across the line $y = x$.

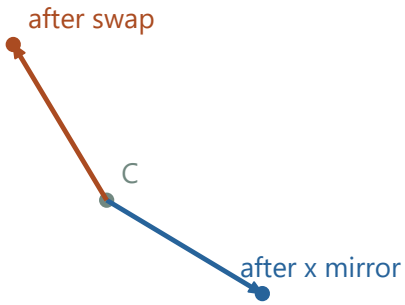
(b) Reflect across the line $y = x$, then reflect across the x-axis.

(c) Rotate 90 degrees counterclockwise around the origin.

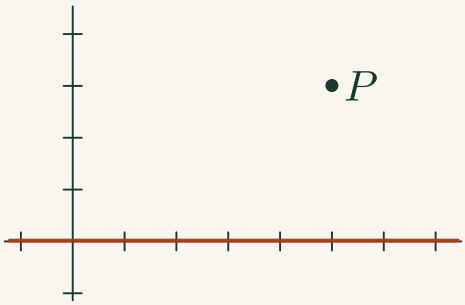


1. Apply the x-axis reflection before the diagonal mirror

In (a), (5,3) reflects to (5,−3). Reflection in y=x swaps these coordinates, giving (−3,5).



Reflect (5,3) in the x-axis first; then reflect in y=x.

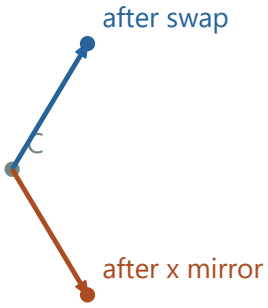


Horizontal reflection axis

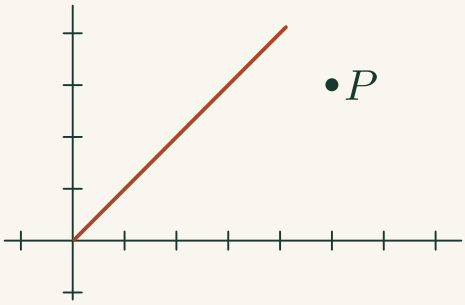
Reflection in the x-axis preserves the horizontal coordinate while reversing the sign of the vertical one.

2. Reverse the mirrors

In (b) swapping first gives (3,5), and then reflecting in the x-axis gives (3,−5). The same two mirrors in a different order reach another point.



Swapping first gives a different endpoint (3,−5).

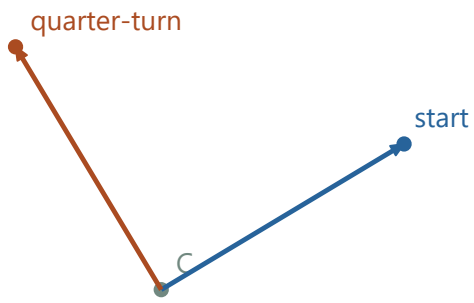


Diagonal mirror y=x

This constructed diagonal through the origin illustrates the other mirror. It exchanges horizontal and vertical coordinates.

3. Compare with a quarter-turn

In (c) a counterclockwise 90° rotation uses (x,y)↦(−y,x), giving (−3,5), the same as (a). The composition in (b) gives the clockwise quarter-turn instead.



The first composition matches a 90° counterclockwise turn.

Check. The compositions are opposite quarter-turns, preserving squared radius 34.

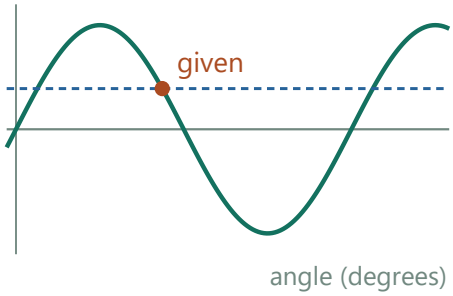
Exercise 270 · Nearby sine intersections

Printed page 27 · Trigonometric equations / Periodic functions

On the graph of $y = \sin x$, many points have 0.39073 as the y-coordinate, among them $P = (157, 0.39073)$. Without relying on a calculator or graphing tool, show how you can determine the coordinates of the two closest to P.

1. Interpret the height reported to five decimals

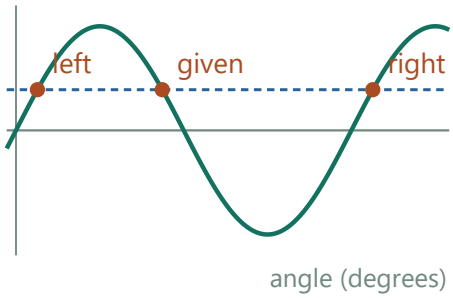
The printed point uses x=157° and a rounded value 0.39073 for sin 157°. For that reading, seek other exact solutions to sin x=sin 157°.



Read 0.39073 as the printed rounded height.

2. Reflect, then translate

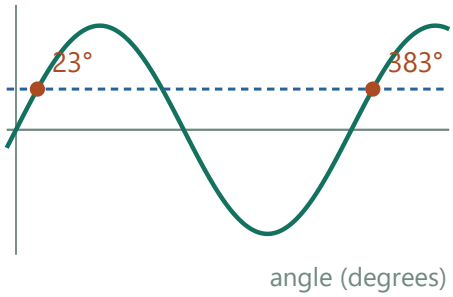
Reflection about x=90° gives x=180°−157°=23°. Translate this reflected solution by 360° to get 383°. These neighbors are 134° to the left and 226° to the right, closer than shifting 157° by a whole turn.



Use reflection and a full period to locate neighboring crossings.

3. Write the two graph points

They are (23°,sin 23°) and (383°,sin 23°). Both have y≈0.39073.



The two neighbors of the given crossing are at 23° and 383°.

4. Distinguish an exact-decimal reading

If the target height is exactly 0.39073, let a=arcsin(0.39073) in degrees. The complete solution set is a+360°k or 180°−a+360°k. The printed point is slightly off the curve, so the two closest intersections to that printed coordinate have abscissas 180°−a and a.

printed	0.39073
true value	0.39073113

Treating the decimal as exact changes the equation slightly.

One nearby solution then lies almost exactly at the printed point.

Check. Their horizontal distances from 157° are 134° and 226°; repeats of 157° are 360° away.

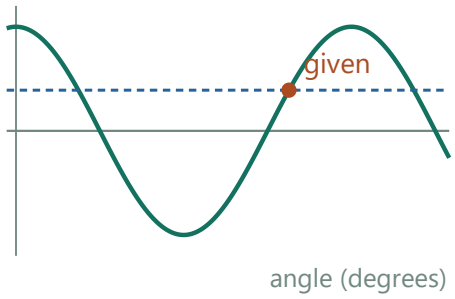
Exercise 271 · Nearby cosine intersections

Printed page 27 · Trigonometric equations / Periodic functions

(Continuation) On the graph of $y = \cos x$, many points have 0.39073 as their y-coordinate, among them, $Q = (293, 0.39073)$. Without relying on a calculator or graphing tool, show how you can determine the coordinates of the two nearest Q. Could you have used your answer to #270 to answer this?

1. Use cosine's reflection symmetry

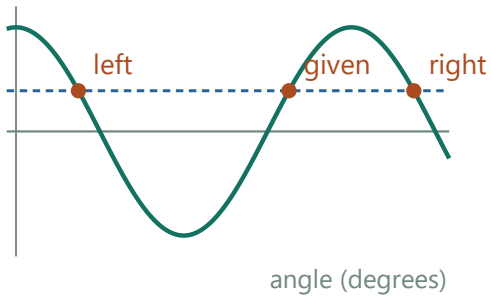
Treat the reported height as cos 293°, rounded to 0.39073. Since 293°=360°−67°, the same horizontal coordinate is reached at 67°. The next such crossing is 67°+360°=427°.



Read 0.39073 as the printed rounded height.

2. Select the nearest other points

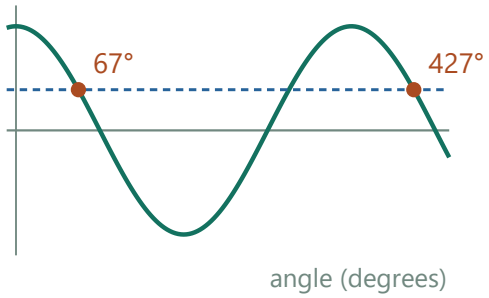
Their distances in x from 293° are 226° and 134°, smaller than any full-period repeat. The required points are approximately (67°,0.39073) and (427°,0.39073), with exact y-coordinate cos 67°.



Use reflection and a full period to locate neighboring crossings.

3. Compare with the sine graph

The identity cos x=sin(x+90°) shifts the sine graph 90° left. Applying this shift to the complete solution set in exercise 270 produces the cosine set here.



The two neighbors of the given crossing are at 67° and 427°.

4. Check the exact-decimal alternative

If 0.39073 is exact, put b=arccos(0.39073) in degrees. Solutions have x=±b+360°k. Since the printed Q is slightly off that exact graph height, its two closest horizontal intersections have abscissas 360°−b and 360°+b.

printed	0.39073
true value	0.39073113

Treating the decimal as exact changes the equation slightly.

One nearby solution then lies almost exactly at the printed point.

Check. The first neighbors are 226° to the left and 134° to the right of 293°.

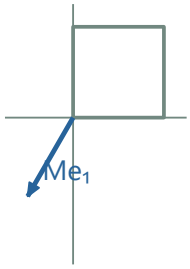
Exercise 272 · Recognizing a rotation

Printed page 27 · Matrix transformations / Rotations

Describe the transformation whose matrix is given by $\begin{bmatrix} -\frac{1}{2} & \frac{1}{2}\sqrt{3} \\ -\frac{1}{2}\sqrt{3} & -\frac{1}{2} \end{bmatrix}$.

1. Read what happens to the positive x-axis

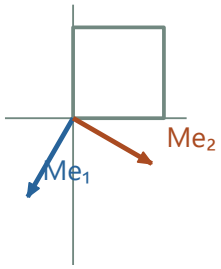
The first column, $(-1/2, -\sqrt{3}/2)$, is the image of $(1,0)$. It is a unit vector in the third quadrant, making a $60°$ reference angle with the negative x-axis.



The first column gives the new horizontal unit arrow for $240°$.

2. Check that the entire map is a rotation

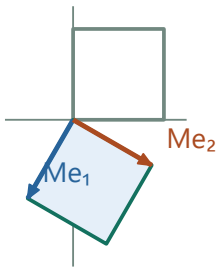
The second column is $(\sqrt{3}/2, -1/2)$, precisely the first column turned counterclockwise by $90°$. Thus the two axis images retain lengths, perpendicularity and orientation.



The other column stays a quarter turn ahead of the first; the columns have equal unit length.

3. Name the angle

The first axis moved counterclockwise through $180° + 60° = 240°$. Equivalently it moved clockwise through $120°$. The map is this rotation about the origin.



The two columns together specify the rotation. Both first-column entries are negative, locating $240°$.

Check. The first unit vector maps to the quadrant-III point $(-1/2, -\sqrt{3}/2)$.

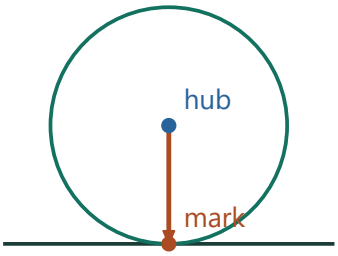
Exercise 273 · Paint spot on a rolling wheel

Printed page 27 · Circular motion / Radian measure

A wheel of radius 5 is placed in the corner of the first quadrant of the xy-plane so that it is tangent to both axes. A paint mark is made on the wheel at the point where it touches the x-axis. What is the position of this paint mark after the wheel has rolled 10 units along the x-axis in the positive direction?

1. Track the moving center

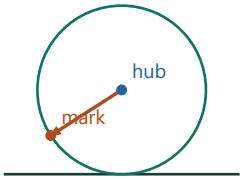
A radius-5 circle initially tangent to both positive axes has hub (5,5). After rolling 10 units right on the x-axis, the hub is (15,5). The originally lowest point starts 5 units directly below its hub.



The wheel begins with hub (5,5) and the mark touching the ground.

2. Find the amount and direction of rotation

With no slip, travel distance equals radius times angle. The clockwise angle is 10/5=2 radians. Rotating the initial offset (0,−5) by −2 radians yields (−5 sin 2,−5 cos 2).



Five units of travel give one clockwise radian.

Ten units give two.

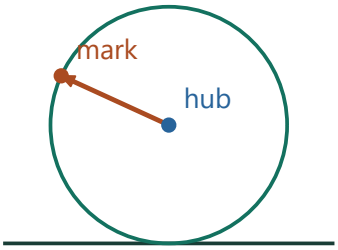
1. $\theta = -s/r = -10/5 = -2$ radians
Travel to the right turns the wheel clockwise; the negative sign encodes this convention.

2. $x' = 0\cos\theta - (-5)\sin\theta = 5\sin(-2) = -5\sin 2$
Rotate the downward displacement about the hub.

3. $y' = 0\sin\theta + (-5)\cos\theta = -5\cos 2$
Cosine is unchanged under angle negation.

3. Combine rotation and translation

Add the new hub to get (15−5 sin 2,5−5 cos 2)≈(10.454,7.081). Subtracting the new hub recovers a radius vector of length 5.



Hub after rolling: (15,5).

Adding the rotated offset gives approximately (10.454,7.081).

Check. Its distance to the new center is exactly 5. The sign of the horizontal offset is leftward for a clockwise rotation from the bottom.

Exercise 274 · Solve three systems by matrices

Printed page 27 · Systems of equations / Matrix inverses

Use matrix methods to solve the following systems of linear equations:

(a) $\begin{cases} 3x + y = 11 \\ 4x + 3y = -7 \end{cases}$ **(b)** $\begin{cases} 5x + 2y + z = 9 \\ x - 4y - 3z = 1 \\ -3x + 4z = -2 \end{cases}$ **(c)** $\begin{cases} 1.69x + 9.61y - 2.56z = 19.36 \\ 3.61x - 7.84y + 1.21z = 11.56 \\ 4.41x - 5.29y - 6.76z = 12.25 \end{cases}$

1. Represent each system as Ax=b

Keep unknowns in the same column order as the printed matrix. One matrix method is Cramer's rule: replace column j of A by b to form A_j. When det A≠0, the jth coordinate is det A_j/det A.

a	b
c	d

For Cramer's rule replace one coefficient column at a time with the constants.

2. Work the 2-by-2 example (a)

The coefficient determinant is $3\cdot3-1\cdot4=5$. Replacing its first column gives $11\cdot3-1\cdot(-7)=40$, replacing the second gives $3\cdot(-7)-11\cdot4=-65$. Dividing by 5 gives $(8,-13)$.

det A	5
det A_x	40
det A_y	−65

Divide each replaced-column determinant by 5.

The result is $(8,-13)$.

3. Apply the same operation in (b)

Using rows $(5,2,1),(1,-4,-3),(-3,0,4)$, det $A=-82$. Replacing the x, y and z columns in turn by $(9,1,-2)^{T}$ gives determinants $-148,36,-70$. Their quotients yield $(74/41,-18/41,35/41)$.

det A_x	det A_y	det Az
−148	36	−70

Divide by −82 in part (b).

Obtain $(74/41,-18/41,35/41)$.

1. $-16=(-4)(4)-(-3)(0)$
Delete row 1 and column 1. Compute the surviving 2 by 2 determinant as diagonal product minus off-diagonal product.

2. $D=(5)(-16)-(2)(-5)+(1)(-12)=-82$
Expand the coefficient matrix along its first row.

3. $D_x=(9)(-16)-(2)(-2)+(1)(-8)=-148$
Replace the x column by the right-hand side, then expand along the first row.

4. $D_y=(5)(-2)-(9)(-5)+(1)(1)=36$
Replace the y column of the original matrix and expand with signs +, −, +.

5. $Dz=(5)(8)-(2)(1)+(9)(-12)=-70$
Replace the z column of the original matrix and expand with the same signs.

4. Preserve precision in (c)

Multiply each full equation by 100 first, including its right side. The new right column is $(1936,1156,1225)^{\mathsf{T}}$. The coefficient determinant is 346560934. Replacing its columns gives 1954081633,435942679,305620087. Dividing by the common determinant yields (5.638494,1.257911,0.881865) to six decimal places.

1936
1156
1225

After multiplying all equations by 100, use this integer constants column.

The determinant arithmetic stays exact until the final division.

1. $593993=(-784)(-676)-(121)(-529)$
Delete row 1 and column 1. Compute the surviving 2 by 2 determinant as diagonal product minus off-diagonal product.

2. $D=(169)(593993)-(961)(-297397)+(-256)(154775)=346560934$
Expand the coefficient matrix along its first row.

3. $D_x=(1936)(593993)-(961)(-929681)+(-256)(348876)=1954081633$
Replace the x column by the right-hand side, then expand along the first row.

4. $D_y=(169)(-929681)-(1936)(-297397)+(-256)(-67571)=435942679$
Replace the y column of the original matrix and expand with signs +, −, +.

5. $Dz=(169)(-348876)-(961)(-67571)+(1936)(154775)=305620087$
Replace the z column of the original matrix and expand with the same signs.

Check. For (b), the three numerators on substitution are 369,41,−82, exactly 41 times the targets. In (c), multiplying by the original coefficient matrix reproduces (19.36,11.56,12.25) to five decimal places.

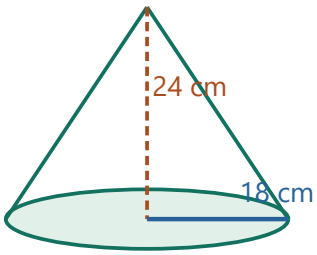
Exercise 275 · Shortest path along a cone

Printed page 27 · Nets and surface development / Distance and loci

A spider is on the rim of an empty conical cup when it spies a fly one third of the way around the rim. The cone is 36 cm in diameter and 24 cm deep. In a hurry for lunch, the spider chooses the shortest path to the fly. How long is this path?

1. Open the cone without stretching it

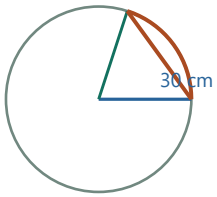
Cut along a generator and flatten the lateral surface. The cone has base radius 18 cm and depth 24 cm, so the flattened sector radius is $\sqrt{(18^2+24^2)}=30$ cm. Surface curves retain their lengths in this development.



Slant distance $\sqrt{(18^2+24^2)} = 30$ cm becomes the sector radius.

2. Place the spider and fly on the arc

The whole rim length 36π is $36\pi/(60\pi)=3/5$ of a full radius-30 circle, so the sector angle is 216°. One third of that rim corresponds to 72°. A straight chord in that sector has length $2\cdot30\sin 36°\approx35.27$ cm and stays inside the surface.



The full rim spans a 216° sector; one third spans 72°.

Its straight chord gives about 35.27 cm.

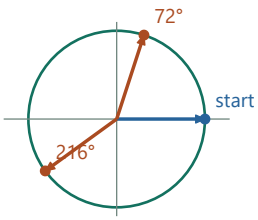
1. angle for full rim=$(36\pi)/30=6\pi/5$
Divide the rim arc by the development radius to obtain radians.

2. one-third angle=$(6\pi/5)/3=2\pi/5=72°$
The fly and spider are only a third of the way around.

3. chord=$2\cdot30\cdot\sin(72°/2)=60\sin36°$
The midpoint of the flat chord bisects the isosceles triangle.

3. Check alternative ways around the seam

For a route avoiding the apex, continue unwrapping into additional copies each time the seam is crossed. The net developed angle differs from 72° by an integer multiple of 216°. The only absolute angles under 180° are 72° and 144°. The 144° chord is longer, since its half-angle has larger sine.



A second route crosses the seam in the other direction.

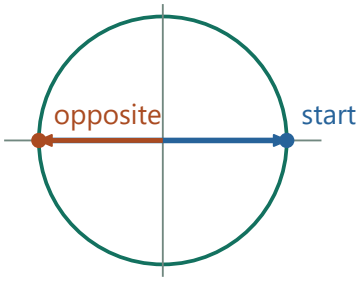
Its absolute developed angle is 144°, giving a longer chord.

1. $72°+216°(0)=72°$; $72°+216°(-1)=-144°$
These are the only net changes strictly between −180° and 180°.

2. $2\cdot30\sin72°>2\cdot30\sin36°$
The chord for 144° is longer because sine increases between 36° and 72°.

4. Rule out longer windings and the apex

Any route changing angle by at least 180° must cross the ray opposite its start. If it crosses at radius s, reaching that point costs at least 30+s by straight-line distance; finishing costs at least 30−s by radial distance. Total ≥60 cm. Going through the apex also costs at least 60 cm. Thus the 72° chord realizes the global shortest length.



A route winding at least a half-turn crosses the opposite ray.

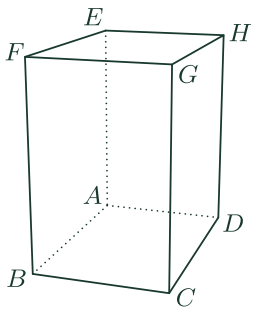
Its two distance lower bounds sum to at least 60 cm.

Check. The shorter rim path is 12π≈37.70 cm, longer than the verified shortest chord on the cone.

Exercise 276 · Spherical direction in a framework

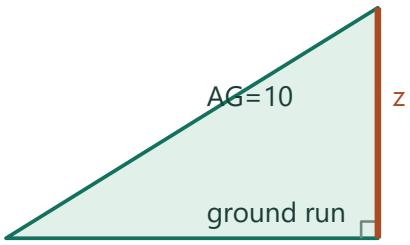
Printed page 27 · Vectors and components / Trigonometric ratios

In the rectangular framework shown at right, GAC is a 40-degree angle, CAB is a 33-degree angle, segments AB and AD lie on the x-axis and y-axis, respectively, and $AG = 10$. Find the coordinates of G.



1. Use the vertical triangle first

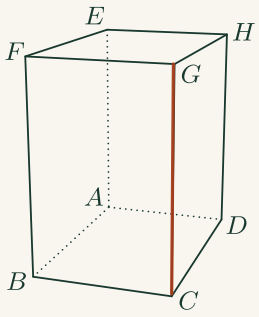
The space diagonal AG is 10 units long. Its elevation of 40° is measured above its horizontal projection AC, so vertical height z=10 sin 40° and horizontal length AC=10 cos 40°.



Schematic section

The elevation angle is 40°.

z = 10 sin 40°, ground run = 10 cos 40°.

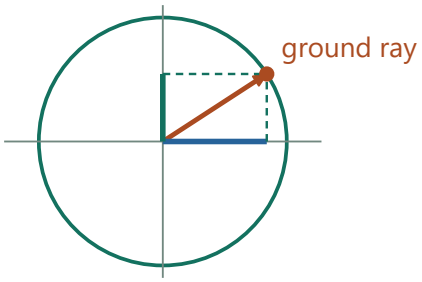


Vertical component CG

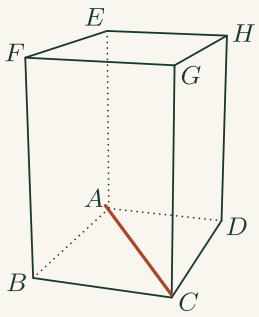
The vertical edge from C to G is the rise above the base, complementary to the horizontal run AC.

2. Use the horizontal triangle next

The projected segment AC makes angle 33° with the positive x-axis. Its x-component is AC cos 33° and its y-component is AC sin 33°.



In the horizontal plane the run makes 33° with the positive x-axis.



Horizontal projection AC

Drop G vertically to C. The base diagonal AC is the horizontal component of AG, and its length is then resolved along the two base axes.

3. Assemble the three coordinates

With A as origin this gives G=(10 cos 40° cos 33°,10 cos 40° sin 33°,10 sin 40°)≈(6.42,4.17,6.43). Squaring the coordinates and adding recovers 10^2 before rounding.



Combine the two horizontal projections and the height.

G≈(6.42,4.17,6.43).

1. $x^2+y^2=(10\cos40°)^2(\cos^2 33°+\sin^2 33°)=100\cos^2 40°$
The azimuth splits AC into perpendicular components whose squared sum gives AC^2.

2. $x^2+y^2+z^2=100(\cos^2 40°+\sin^2 40°)=100$
The elevation splits the space diagonal into AC and z, independently confirming its original length.

Check. Squaring and summing gives $AG^2=100$; y/x=tan33° recovers the azimuth.

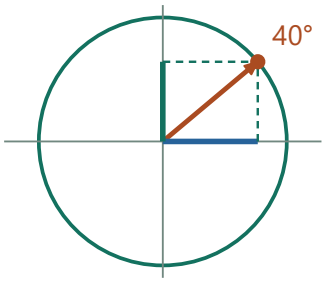
Exercise 277 · Comparing 40° and 50°

Printed page 27 · Reference angles and symmetry / The unit circle

Without using a calculator, choose **(a)** the larger of $\cos 40$ and $\cos 50$; **(b)** the larger of $\sin 40$ and $\sin 50$. Explain your reasoning.

1. Trace the arc in the first quadrant

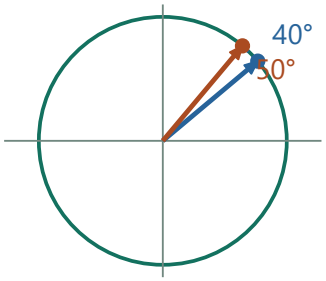
Increasing the angle from 40° to 50° moves the unit-circle point upward toward the y-axis. Both points still lie in the first quadrant.



Start with the radius at 40°.

2. Compare horizontal positions

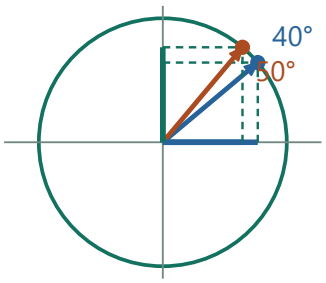
Cosine is the x-coordinate. The later point is farther left, hence cos 40°>cos 50°.



Turning in quadrant I from 40° to 50° moves left and upward.

3. Compare heights

Sine is the y-coordinate, which increases, so sin 50°>sin 40°. Since the two angles sum to 90°, these comparisons also agree with cos 40°=sin 50°.



The cosine at 40° is larger and the sine at 50° is larger.

Check. The angles are complementary, so cos40°=sin50° and cos50°=sin40°.

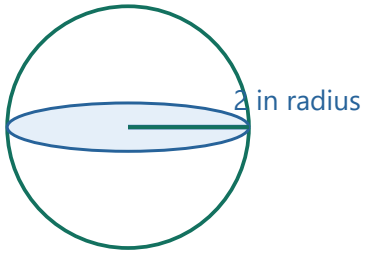
Exercise 278 · Ice cream fills a cone

Printed page 28 · Volume / Sphere geometry

As a spherical glob of ice cream that once had a 2-inch radius melts, it drips into a cone of the same radius. The melted ice cream exactly fills the cone. What is the height of the cone?

1. Identify what melting preserves

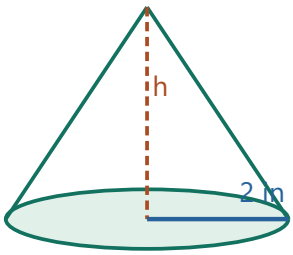
Assuming no material loss or change of volume, the melted sphere and the filled cone must occupy equal volumes. The common radius is 2 inches.



Melting transfers the whole volume of the radius-2-in sphere.

2. Equate the correct volumes

The sphere occupies $(4/3)\pi\cdot 2^3$ in^3, and the cone occupies $(1/3)\pi\cdot 2^2 h$. Multiplying both sides by 3 and canceling π gives $4\cdot 8=4h$.

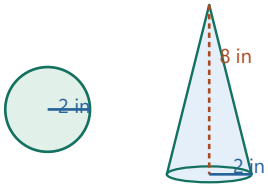


The cone's base has the same 2-in radius.

Use $(1/3)\pi\cdot 2^2 h$ for the filled volume.

3. Solve for the vertical height

Divide by 4 to get h=8 in. In general the same cancellation at radius r gives h=4r.



The two radii both measure 2 in.

Equal volume requires the cone to have height 4r = 8 in.

1. $(4/3)\pi\cdot 8=(1/3)\pi\cdot 4h$
Cube the ball radius and square the cone radius before simplifying.

2. $32=4h \Rightarrow h=8$ in
Cancel $\pi/3$ and divide by the positive base-radius square.

Check. Generally a sphere and cone of equal radius have the same volume when h=4r.

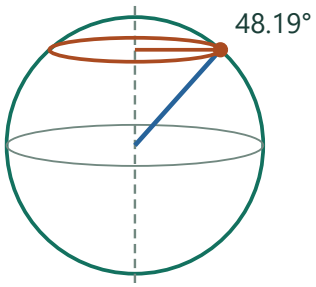
Exercise 279 · Latitude from circumference

Printed page 28 · Spherical coordinates / Circumference

The circumference of a circle of latitude is two thirds of the circumference of the equator. What is the latitude?

1. Express the radius of a latitude circle

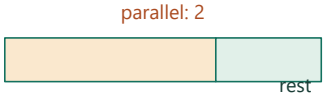
In a vertical meridian section, the radius from Earth's center makes latitude angle φ with the equatorial plane. Its horizontal component $R\cos\varphi$ is the parallel's radius about the polar axis.



The radius of a latitude circle is the horizontal component $R\cos\varphi$.

2. Compare the two circumferences

The latitude parallel has circumference $2\pi R\cos\varphi$, while the equator has circumference $2\pi R$. Their ratio is $\cos\varphi$, independent of Earth's radius.



Parallel/equator circumference = 2/3.

The same ratio holds for the radii.

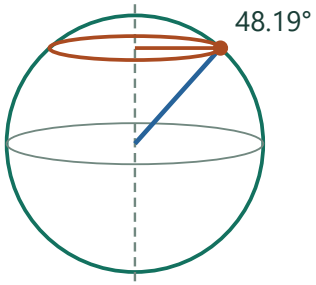
1. $C_{parallel}/C_{equator}=[2\pi R\cos\varphi]/[2\pi R]$
Compare the two full-circle distances over the same day.

2. $=\cos\varphi=2/3$
Cancel their common positive equatorial circumference.

3. Solve and include both hemispheres

Set $\cos\varphi=2/3$. The latitude magnitude is $\arccos(2/3)\approx 48.19°$. Both 48.19° N and 48.19° S meet the condition because reflection about the equator preserves circumference.

Latitude · 48.19°

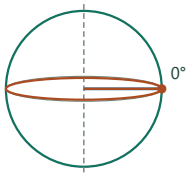


At latitude 48.19° the radius of the parallel is 2640 mi.

Its circumference is 0.667 times the equatorial circumference.

Compare settings · Latitude

0°

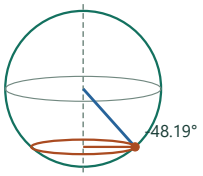


At latitude 0° the radius of the parallel is 3960 mi.

Its circumference is 1 times the equatorial circumference.

At latitude 0° the radius of the parallel is 3960 mi. Its circumference is 1 times the equatorial circumference.

−48.19°



At latitude −48.19° the radius of the parallel is 2640 mi.

Its circumference is 0.667 times the equatorial circumference.

At latitude −48.19° the radius of the parallel is 2640 mi. Its circumference is 0.667 times the equatorial circumference.

Check. The two latitude circles are mirror images across the equator and have the same radius.

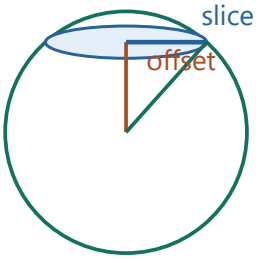
Exercise 280 · Matching circular fruit sections

Printed page 28 · Sphere geometry / Cross sections

How far from its center should a grapefruit with a 6-inch diameter be sliced, in order that both circular sections have the same radius as the two halves of a perfectly sliced orange with a 4-inch diameter?

1. Read the desired circle size

A central slice of a 4-inch-diameter orange is a disk of radius 2 in. The larger grapefruit has radius 3 in. Its slice must be displaced from the center to reduce its radius to 2.

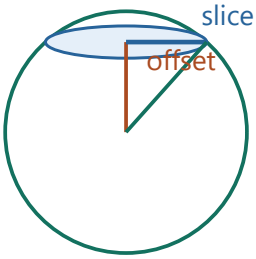


Cut the sphere with a plane. The circular cross section shrinks away from the center.

2. Use the cross-section right triangle

Drop a perpendicular of length k from the grapefruit center to the slicing plane. From its foot to the rim is a 2-in slice radius, and the sphere radius to that rim point is 3 in.

Offset from the sphere center · 2.236 in

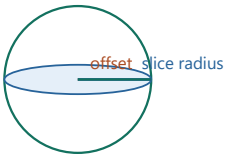


Sphere radius 3; offset 2.236.

Pythagoras gives slice radius $\sqrt{(3^2-2.236^2)} \approx 2$.

Compare settings · Offset from the sphere center

0 in



Sphere radius 3; offset 0.

Pythagoras gives slice radius $\sqrt{(3^2-0^2)} \approx 3$.

Sphere radius 3; offset 0. Pythagoras gives slice radius $\sqrt{(3^2-0^2)} \approx 3$.

3 in



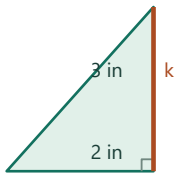
Sphere radius 3; offset 3.

Pythagoras gives slice radius $\sqrt{(3^2-3^2)} \approx 0$.

Sphere radius 3; offset 3. Pythagoras gives slice radius $\sqrt{(3^2-3^2)} \approx 0$.

3. Solve for the distance of the cut

Pythagoras gives $k^2+2^2=3^2$, so $k=\sqrt{5}\approx 2.24$ in. A plane this far from the center on either side yields the desired circle.



The offset is √5 in for a 2-in slice radius.

1. $k^2+4=9 \Rightarrow k^2=9-4=5$
Remove the orange's squared slice radius from the grapefruit's squared sphere radius.

2. $k=\sqrt{5}$ in
The offset is a distance. Two opposite planes can have this same positive distance from the center.

Check. The circle radius reconstructed from $\sqrt{(9-5)}$ is exactly 2 inches.

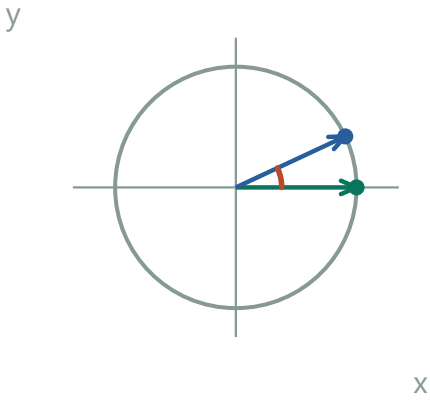
Exercise 281 · Compose two turns

Printed page 28 · Rotations / Matrix multiplication / Angle addition formulas

Simplify the matrix product $\begin{bmatrix}\cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha\end{bmatrix}\begin{bmatrix}\cos\beta & -\sin\beta \\ \sin\beta & \cos\beta\end{bmatrix}$.

1. Follow the two turns

Treat each column as the image of one coordinate unit vector. The matrix on the right acts first, so (1,0) turns through β before the α rotation acts.

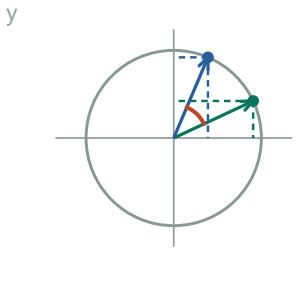


First turn β

Track the image of the horizontal basis vector. The right matrix acts first.

2. Multiply a row by a column

For the top-left entry, pair the first row $(\cos\alpha, -\sin\alpha)$ with the first column $(\cos\beta, \sin\beta)$. Add the products. The minus sign comes from the horizontal component of the rotated vertical axis.



Then turn by α

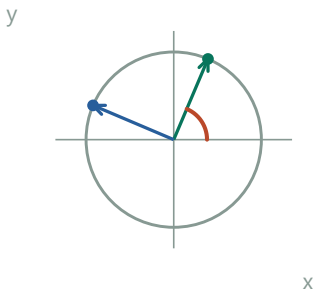
The second turn acts on both components of the first image.

1. $c\alpha c\beta+(-s\alpha)s\beta=c\alpha c\beta-s\alpha s\beta$
Write $c\alpha=\cos\alpha$ and $s\alpha=\sin\alpha$. Pair row 1 with column 1.

2. $s\alpha c\beta+c\alpha s\beta=\sin(\alpha+\beta)$
Pair row 2 with that same column.

3. Recognize the combined rotation

Each first-column entry is a row–column dot product. The first column is (cos α cos β − sin α sin β, sin α cos β + cos α sin β). The second is the first rotated 90°, so the result has rows (cos(α+β), −sin(α+β)) and (sin(α+β), cos(α+β)).



Angle α + β

The two image columns still point a quarter-turn apart.

1. cα(−sβ)−sαcβ=−sin(α+β)
Use the second column of the right matrix for the top entry.

2. sα(−sβ)+cαcβ=cos(α+β)
Use it again for the bottom entry.

Check. Applying the transformations successively turns every vector through β then α.

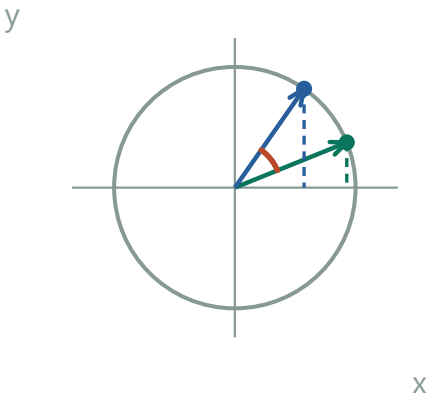
Exercise 282 · Cosine of a sum

Printed page 28 · Angle addition formulas / Trigonometric identities

(Continuation) Asked to find an expression that is equivalent to $\cos(\alpha + \beta)$, a student responded $\cos\alpha + \cos\beta$. What do you think of this answer, and why?

1. Interpret the claim

An identity must work for every allowed pair of angles. One failed pair therefore settles whether the proposed expression is an identity.

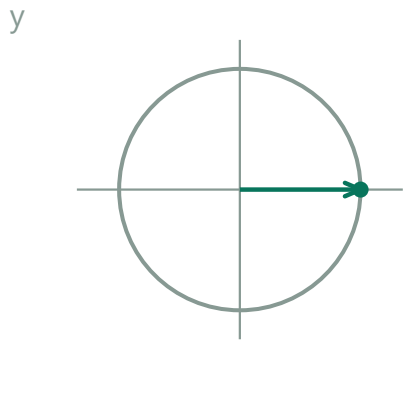


A cosine reads horizontal position

The claim concerns the final x-coordinate of a point rotated twice.

2. Design a simple test

Use two zero rotations: adding the angles leaves the direction (1,0) exactly where it began. Its horizontal component stays 1. Now evaluate the two separate cosine terms.

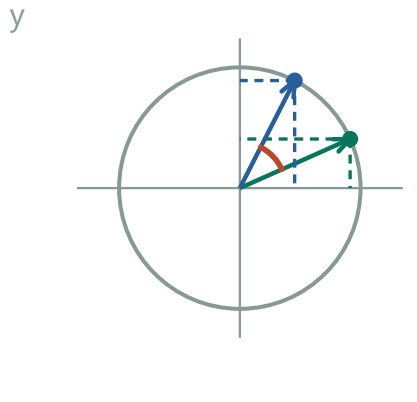


Set α = β = 0

Both starting radii stay at (1, 0). The final horizontal position is 1, while 1 + 1 = 2.

3. Recover the correct identity

At α=β=0 the sides are 1 and 2. The rotation product instead gives cos α cos β−sin α sin β.



Rotate both components

The horizontal part of the second rotation is cos α cos β − sin α sin β.

Check. For two 45° angles the correct expression is ½−½=0=cos 90°.

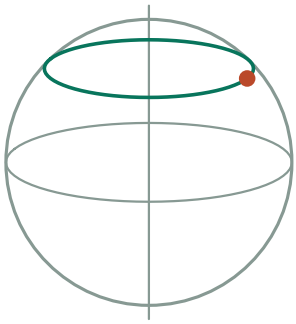
Exercise 283 · Speed along a parallel

Printed page 28 · Spherical coordinates / Angular and linear speed

The earth rotates, so a person living in Quito (on the equator) travels 24881 miles in a day — nearly 1037 miles per hour. To the nearest mph, what rotational speed applies in Exeter, which is 43 degrees north of the equator? What rotational speed applies in your home town?

1. Find the axis of rotation

The point moves east in a circle around the polar axis. In a north–south cross section the radius from Earth's center has length R and makes angle φ above the equator.

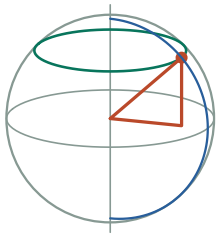


Track the spin axis

A fixed-latitude point circles the polar axis once in 24 hours.

2. Project onto the equator

The horizontal leg adjacent to φ extends from the rotation axis to the town. Its circumference is reduced by the same factor as that leg. Every latitude still completes one turn in 24 hours.



Use the horizontal projection

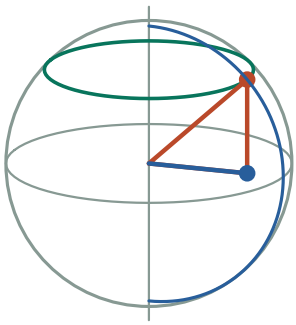
At latitude 43° the rotating radius is R cos 43°, shorter than the equator's radius.

1. $C_\varphi = 2\pi(R\cos\varphi) = (2\pi R)\cos\varphi$
The orange parallel has radius R cos φ; its period is still one day.

2. $v_\varphi = C_\varphi/24 = (24881/24)\cos\varphi$
Replace 2πR by the equatorial circumference, then divide miles by hours.

3. Evaluate and adapt to another town

At latitude φ the rotation radius is R cos φ. Speed is (24,881/24)cos φ mph. At 43° N that is 758 mph to the nearest mile per hour. Substitute your own town's latitude for φ. For a town at 30° north or south, for example, the same calculation gives about 898 mph. For the personal answer, insert the latitude of the town you choose.



Compare speeds on the same clock

The Exeter parallel travels cos 43° of 24,881 miles per day, approximately 758 mph.

Check. At a pole the formula gives zero; at the equator it gives the full equatorial speed.

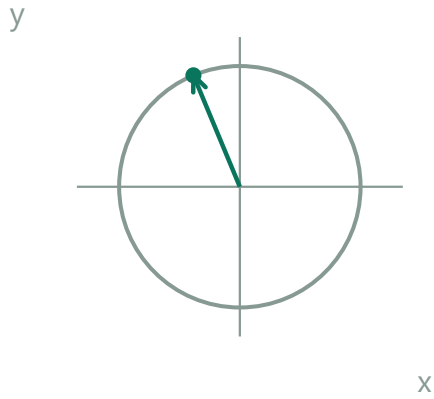
Exercise 284 · Second-quadrant ratios

Printed page 28 · Trigonometric ratios / Reference angles and symmetry / The Pythagorean theorem

Given that $\sin\theta$ is $12/13$, with $90 < \theta < 180$, find the values of $\cos\theta$ and $\tan\theta$. Try to do this without finding θ first.

1. Locate the point on a circle

Use a circle of radius 13. Sine is vertical coordinate divided by radius, so the indicated height is 12. The angle puts the point left of the vertical axis.

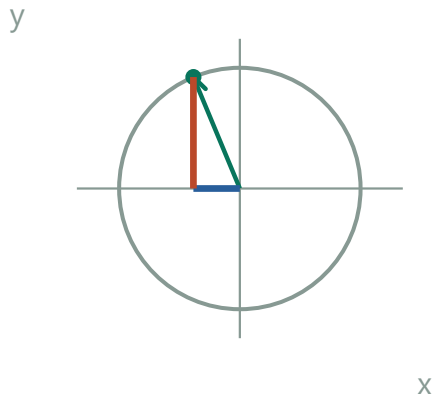


Quadrant II

The tip is left of the vertical axis and above the horizontal axis.

2. Choose signs before calculating

If x is the horizontal coordinate, $x^2+12^2=13^2$. Select the negative root for quadrant II. Cosine is x/13, and tangent is height divided by signed horizontal position.

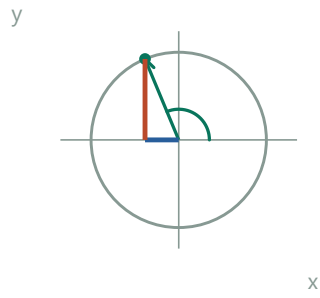


A 13-unit radius

Its vertical rise is 12. The horizontal leg points left.

3. Compute the two ratios

The missing leg has magnitude $\sqrt{(13^2-12^2)}=5$. Cos θ=−5/13; tan θ=(12/13)/(−5/13)=−12/5.



Horizontal component −5

The length is $\sqrt{(13^2-12^2)}=5$. The signed ratios are −5/13 and −12/5.

1. $x^2=13^2-12^2=169-144=25$
Rearrange the right-triangle relation for the unsigned horizontal length.

2. x=−5; tan θ=12/(−5)=−12/5
Quadrant II selects the negative coordinate; the slope divides rise by run.

Check. The squared sine and cosine add to 1 and both requested signs match quadrant II.

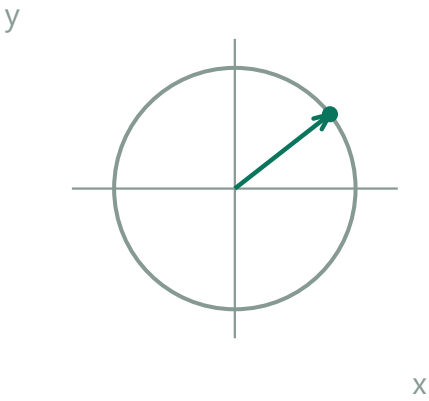
Exercise 285 · A symbolic right triangle

Printed page 28 · Trigonometric ratios / The Pythagorean theorem

Given that $\sin\theta$ is a/b, with $0 < \theta < 90$, $a > 0$, and $b > 0$, find expressions for $\cos\theta$ and $\tan\theta$.

1. Translate the ratio into a triangle

Sine equals opposite over hypotenuse. A triangle with sides proportional to a and b has the same ratios as the original, so take these lengths literally.

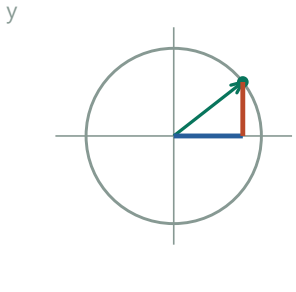


A first-quadrant radius b

The opposite component is a because sin θ=a/b.

2. Identify the missing length

Call the horizontal leg c. The right triangle gives $c^2+a^2=b^2$. An acute angle makes c positive and forces a<b.



Complete the right triangle

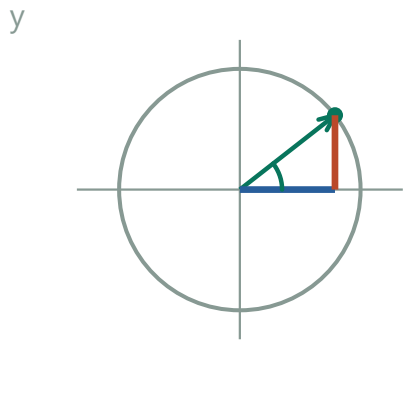
The adjacent leg squares to b^2-a^2 and is positive.

1. $a^2+c^2=b^2 \Rightarrow c^2=b^2-a^2$
Let c be the adjacent leg, and subtract the opposite-leg square.

2. $c=\sqrt{(b^2-a^2)}>0$
The given acute triangle has 0<a<b, so use the positive root.

3. Express cosine and tangent

Use opposite leg a and hypotenuse b. The adjacent leg is $\sqrt{(b^2-a^2)}$, so cosine is $\sqrt{(b^2-a^2)}/b$ and tangent is $a/\sqrt{(b^2-a^2)}$.



Two other ratios

Cosine divides $\sqrt{(b^2-a^2)}$ by b.
Tangent divides a by $\sqrt{(b^2-a^2)}$.

Check. The acute-angle premise ensures $0<a<b$, making both results positive and finite.

Exercise 286 · Counting factors

Printed page 28 · Exponent rules

Faced with the problem of multiplying 5^6 times 5^3, Brook is having trouble deciding which of these four answers is correct: 5^{18}, 5^9, 25^{18}, or 25^9. Your assistance is needed. Once you have answered Brook's question, experiment with other examples of this type until you are able to formulate the *common-base principle for multiplication* of exponential expressions.

1. Read the exponent as a count

Write 5^6 as six factors of 5 and 5^3 as three more. Multiplication joins the two lists without grouping them into pairs.



Expand the first factor

Each oval stands for one factor of 5. Six of them make the factor 5^6.

2. Test why the base stays fixed

With a smaller example, $5^2 \cdot 5$ is $(5 \cdot 5) \cdot 5$. Each individual factor is still 5. The exponent records how many times it occurs.



Check the shorter example

Two green factors followed by one blue factor reproduce $5^2 \cdot 5$.

3. State the product rule

Six factors of five followed by three more yield nine factors, or 5^9. In general $a^m a^n = a^{m+n}$ wherever the powers are defined.



Concatenate both lists

The six factors of 5^6 and three factors of 5^3 give nine factors in all.

Check. For example $2^2 \cdot 2^3 = 4 \cdot 8 = 32 = 2^5$.

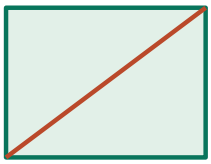
Exercise 287 · A plate in a bowl

Printed page 28 · Sphere geometry / Cross sections / The Pythagorean theorem

A 6×8 metal plate is resting inside a hemispherical bowl, whose radius is 13. The plate is parallel to the rim of the bowl, which is parallel to the tabletop on which the bowl is sitting. How far is it from the plate to the bottom of the bowl?

1. Look at the plate from above

A horizontal cross section of a hemisphere is a circle. Its radius must reach both opposite corners of the plate. Pythagoras gives the corner-to-corner length $\sqrt{(6^2+8^2)}=10$.

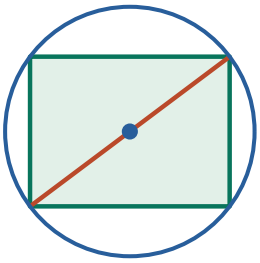


A top view of the 6-by-8 plate

Opposite corners are 10 apart by Pythagoras.

2. Find the lowest possible slice

A horizontal slice of the bowl must contain a rectangle with a 10-unit diagonal. A disk containing both ends of a 10-unit diagonal needs radius at least 5. Centering the plate realizes that bound since all four corners are then 5 from the slice center. As a freely supported level plate settles, the lowest fitting slice therefore has radius 5.

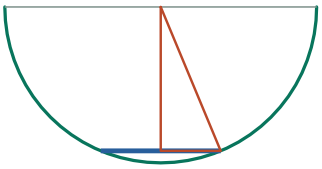


Fit all four corners

A circle of radius 5 exactly circumscribes the centered rectangle, attaining the smallest possible fitting section.

3. Return to the vertical section

Its distance below the sphere center is $\sqrt{(13^2-5^2)}=12$. Its height above the bottom is $13-12=1$.



Take a vertical section

The depth below the center is $\sqrt{(13^2-5^2)}=12$. The bottom is 13 below the center, so the clearance is 1.

1. $h^2+5^2=13^2 \Rightarrow h^2=169-25=144$
The slice radius and depth from the sphere center are perpendicular.

2. $h=12 \Rightarrow$ clearance $=13-12=1$
Take the positive depth and subtract it from the depth of the bottom.

Check. A vertex at height 1 above the bottom is 12 vertically and 5 horizontally from the center, a distance of 13.

Exercise 288 · Products in scientific notation

Printed page 28 · Scientific notation / Dimensional analysis

Exponents are routinely encountered in scientific work, where they help invesigators deal with large numbers:
(a) The human population of earth is nearly 8000000000, which is usually expressed in *scientific notation* as 8×10^9. The average number of hairs on a human head is 5×10^5. Use scientific notation to estimate the number of human head hairs on earth.
(b) Light moves very fast — approximately 3×10^8 meters every second. At that rate, how many meters does light travel in one year, which is about 3×10^7 seconds long? This so-called *light-year* is used in astronomy as a yardstick for measuring even greater distances.

1. (a) Count hairs using the given average

Treat the stated population and average as estimates. Each person contributes one average share.

1 **hairs ≈ (8 × 10^9 people)(5 × 10^5 hairs/person)**
Multiplying cancels the unit “people” and leaves hairs.

2 **≈ (8 × 5) × 10^(9 + 5) hairs**
Group the coefficients. Powers of the same base multiply by adding exponents.

3 **= 40 × 10^{14} hairs**
Eight times five is forty, and nine plus five is fourteen.

4 **= 4 × 10^{15} hairs**
Write 40 as 4 × 10. The leading coefficient is now between 1 and 10.

2. (b) Convert a year of travel to a distance

The speed and the year length are approximate. Their product gives the one-year distance in meters.

1 **distance ≈ (3 × 10^8 m/s)(3 × 10^7 s)**
Use distance = speed × elapsed time. Seconds cancel.

2 **≈ (3 × 3) × 10^(8 + 7) m**
Collect the ordinary factors and then the powers of ten.

3 **= 9 × 10^{15} m**
The coefficient 9 already has the required scientific-notation form.

Check. Multiplying counts per person by people leaves hairs, and multiplying speed by seconds leaves meters.

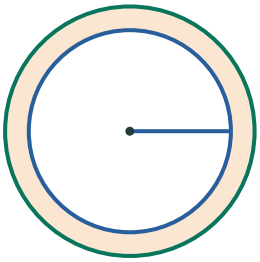
Exercise 289 · Thin spherical shell

Printed page 29 · Volume / Surface area / Limits and convergence

Find the volume of material that is needed to form a spherical shell whose outer radius is 6.0 inches and whose thickness is 0.01 inch. Use your answer to estimate the *surface area* of the 6-inch sphere.

1. Place both radii on one ray

At the outer surface the distance to the center is 6 inches. Moving inward by one shell thickness leaves a radius of 5.99 inches. Only the material between these two surfaces counts.

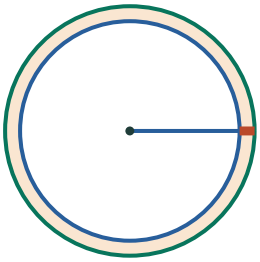


Outer radius 6.00 in

The gap between the two radii is exaggerated here. Its actual thickness is only 0.01 in.

2. Relate a layer to its area

A thin layer of approximately constant area A and thickness t has volume approximately At. Solve that relation for area by dividing the measured layer volume by its thickness.

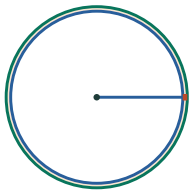


Material lies in the narrow gap

Subtract the volume inside 5.99 in from the volume inside 6.00 in.

3. Subtract and estimate

Subtract radii-cubed volumes: $V=(4\pi/3)(6^3-5.99^3)\approx 4.51636$ in^3. Divide by 0.01 to estimate area ≈ 451.636 in^2.



Volume divided by thickness

The quotient 4.51636/0.01 estimates the area as about 451.636 in^2.

1. $6^3-5.99^3=(6-5.99)(36+35.94+35.8801)$
Factor the difference of cubes before subtracting two nearly equal numbers.

2. $\Delta V/0.01=(4\pi/3)(107.8201)$
Cancel the shell thickness. The quotient estimates surface area.

Check. The exact outer area $144\pi\approx 452.389$ in^2 is slightly larger because inner layers have smaller areas.

Exercise 290 · Earth-centered coordinates

Printed page 29 · Spherical coordinates / Coordinate geometry

An xyz-coordinate system is placed with its origin at the center of the earth, so that the equator (consisting of points with 0-degree latitude) is in the xy-plane, the North Pole (the only point with 90-degree latitude) has coordinates $(0, 0, 3960)$, and the *prime meridian* (see the next paragraph) is in the xz-plane. Where the prime meridian crosses the equator, the positive x-axis emerges from the South Atlantic Ocean, near the coast of Ghana.

The prime meridian is the *great semicircle* that runs through Greenwich, England on its way from the North Pole to the South Pole. Points on this meridian are all said to have *longitude* 0 degrees. The point $(0, 3960, 0)$ has longitude 90 degrees east, and the point $(0, -3960, 0)$ has longitude 90 degrees west. Thus the positive y-axis points east, into the Indian Ocean.

(a) Make a large diagram of this coordinate system.

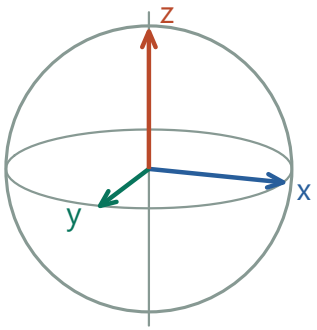
(b) The latitude of Greenwich is approximately 51 degrees north. What are its xyz-coordinates?

(c) There is a point on the equator whose longitude is 33 degrees east. What are its xyz-coordinates?

(d) The latitude of Ankara, Turkey, is approximately 40 degrees north. What is its z-coordinate? The longitude of Ankara is approximately 33 degrees east. What are its xy-coordinates?

1. Mark the axes of the globe

Sketch north as +z, the equator in the xy-plane, +x at 0° longitude and +y at 90°E.

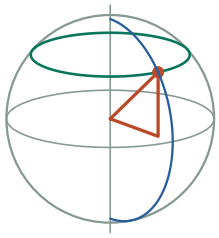


Orient the globe

The equator is the xy-plane, and the north polar direction is +z.

2. Build two right triangles

First project the radius onto the equatorial plane. Its vertical component is 3960 sin φ and its horizontal projection is 3960 cos φ. Then turn that horizontal projection through longitude λ to split it into east-plane components.



Start at Ankara

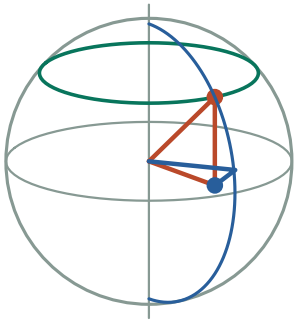
Its height and horizontal projection form the vertical right triangle.

1. $\rho^2+z^2=3960^2$; $z=3960 \sin \varphi$
In the meridian section the vertical leg is opposite latitude φ.

2. $\rho=3960 \cos \varphi$
The radius of the parallel is the adjacent leg.

3. Write the coordinates

For latitude φ and east longitude λ, use 3960(cos φ cos λ, cos φ sin λ, sin φ).

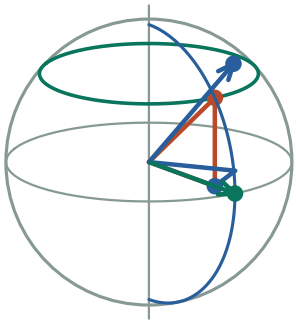


Resolve the equatorial projection

A second right triangle in the xy-plane gives x and y from longitude.

4. Place the three cities or points

Greenwich gives 3960(cos 51°,0,sin 51°). The equatorial point gives 3960(cos 33°,sin 33°,0). Ankara gives 3960(cos 40°cos 33°,cos 40°sin 33°,sin 40°).

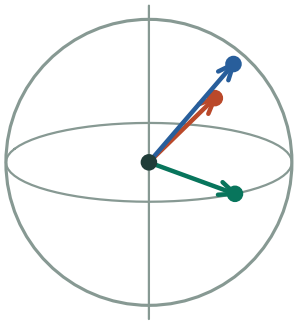


Compare three locations

Greenwich lies in the xz-plane. The equatorial point has z=0. Ankara generally has three nonzero components.

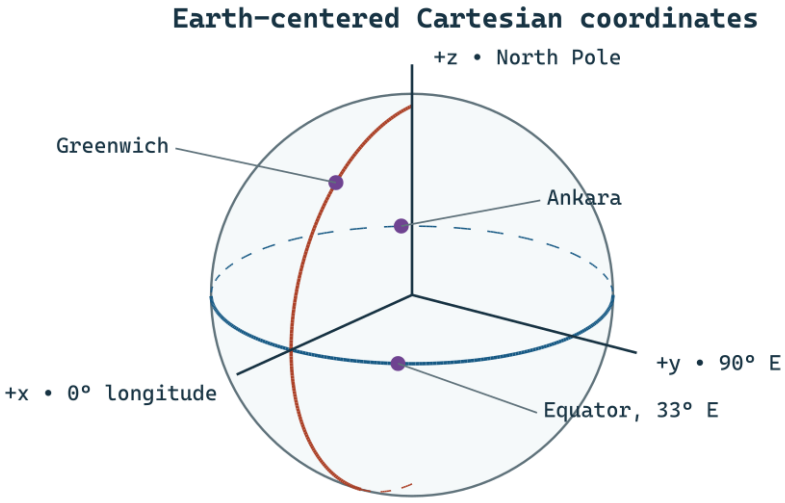
5. Read the distances in miles

In miles, to the nearest mile, Greenwich is (2492, 0, 3077), the equatorial point is (3321, 2157, 0), and Ankara is (2544, 1652, 2545). The first two coordinates of Ankara are the horizontal projection of its radius vector.



Read miles from the common center

All three position vectors have length 3960 miles; the individual projections differ.



Check. Each coordinate vector has squared length 3960^2; the height depends only on latitude.

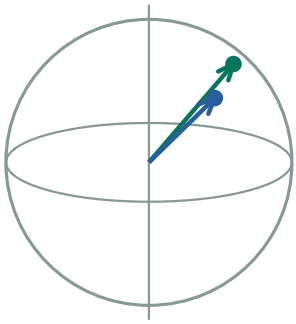
Exercise 291 · Ankara to Greenwich

Printed page 29 · Great-circle distances / The dot product

(Continuation) To the nearest mile, how far is it from Ankara to Greenwich? Travel along the surface of the earth, instead of tunneling through it.

1. Connect both cities to Earth's center

The two radii determine a plane through the center. The shorter great-circle arc in that plane has length radius times central angle, measured in radians.

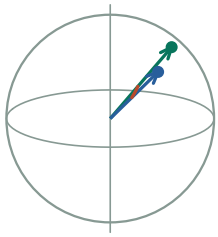


Join both cities to Earth’s center

Two surface points determine a plane containing both 3960-mile radii.

2. Extract the angle without rounding the cities

For vectors u and v, their dot product is ||u||||v||cos θ. Both lengths are 3960. Inserting the sine and cosine coordinate expressions keeps more precision than using rounded mile coordinates.



Use a normalized dot product

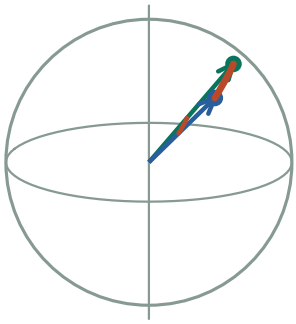
Dividing by the squared radius gives the cosine of the included angle.

1. $u \cdot v = (\cos 40° \cos 33°)\cos 51° + \sin 40° \sin 51°$
Use unit position vectors. Greenwich has zero y-component.

2. $\theta = \arccos(u \cdot v)$
The vectors have length one; take the angle in $[0,\pi]$ for the shorter route.

3. Compute the surface distance

Divide the position-vector dot product by 3960^2. The central angle in radians is $\arccos(\sin 40° \sin 51° + \cos 40° \cos 51° \cos 33°)$. Multiply by 3960 to get 1750.74 miles, or 1751 miles.



Follow the smaller great-circle arc

The arc length is Rθ, about 1,751 miles. A straight chord would cut through the Earth.

Check. The haversine quantity $\sin^2(11°/2) + \cos 40° \cos 51° \sin^2(33°/2)$ gives the same angle via 2 arcsin of its square root.

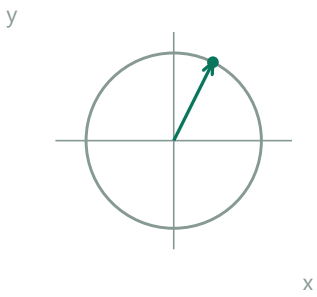
Exercise 292 · A ratio between sine and cosine

Printed page 29 · Trigonometric ratios / Trigonometric equations

Is it possible for $\sin\theta$ to be exactly twice the size of $\cos\theta$? If so, find such an angle θ. If not, explain why not.

1. Convert the condition to a ratio

Divide sin θ=2 cos θ by cos θ. A zero cosine cannot solve the original equation because the corresponding sine is ±1. That leaves tan θ=2.



sin θ = 2 cos θ

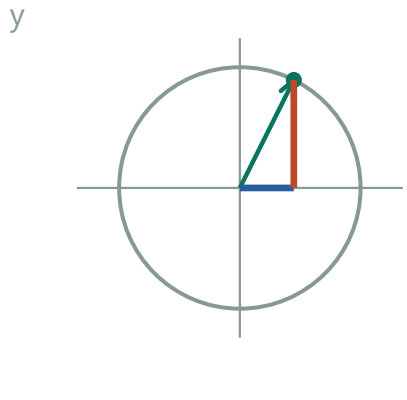
The vertical coordinate must be twice the horizontal coordinate.

1. cos θ=0 ⇒ sin θ=±1
At a vertical radius the proposed equality would require ±1=0.

2. cos θ≠0 ⇒ tan θ=sin θ/cos θ=2
Division is valid for every possible solution.

2. Give the ratio a shape

Draw a horizontal leg and make its vertical partner twice as long. The acute angle at the horizontal leg is one possible solution. Pythagoras supplies the denominator for checking its two coordinates.

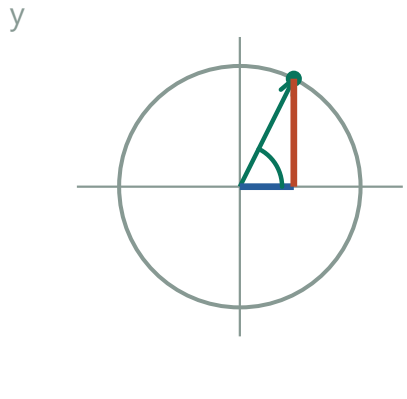


Rise 2 for run 1

A radius along this ray provides a concrete example.

3. Calculate a working angle

Take a right triangle with adjacent leg 1 and opposite leg 2. Then θ=arctan 2≈63.435°, and sin θ=2/√5 while cos θ=1/√5.



Hypotenuse √5

The normalized components 1/√5 and 2/√5 retain the factor of two.

Check. All angles θ+180°k also satisfy the relation because both signs change together.

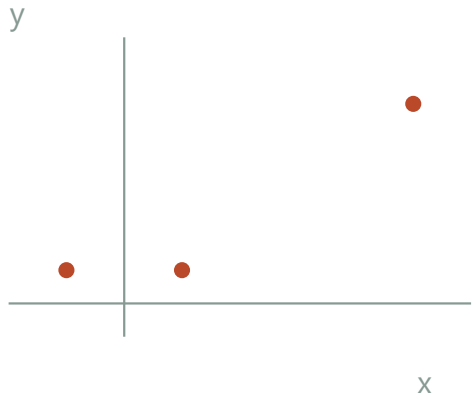
Exercise 293 · Interpolate three points

Printed page 29 · Polynomial interpolation / Systems of equations

There is a unique parabola that goes through the points $P = (-1, 1)$, $Q = (1, 1)$, and $R = (5, 6)$ that can be described by $y = ax^2 + bx + c$. Find coefficients a, b, and c.

1. Use every given point

Substituting P gives a−b+c=1. Substituting Q gives a+b+c=1. Only the sign of the b term has changed, because squaring −1 and 1 gives the same result.

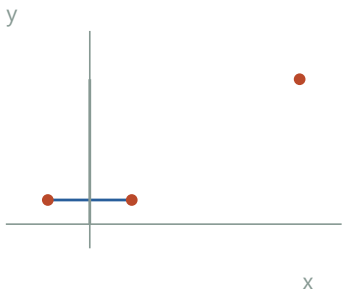


Three separate constraints

A fitted parabola must pass through all three marked data points.

2. Eliminate the linear term

Subtract the first equation from the second. Then insert R=(5,6), remembering that the coefficient multiplying a is 5^2.



Match the heights at −1 and 1

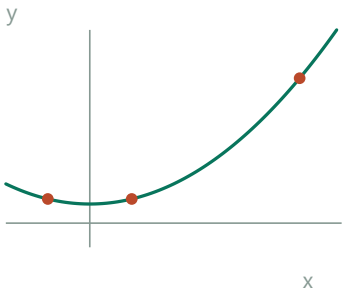
The equal-height pair is symmetric about x=0, forcing the linear coefficient b to vanish.

1. a−b+c=1; a+b+c=1
Substitute x=−1 and x=1 into ax^2+bx+c.

2. 2b=0 ⇒ b=0
Subtract the first equation from the second to eliminate a and c.

3. Solve the remaining equations

The first two equations imply b=0 and a+c=1. The third gives 25a+c=6. Subtract to get a=5/24 and c=19/24.



Use the third point

The remaining scale and offset give $y=(5x^2+19)/24$ through all three points.

1. (25a+c)−(a+c)=6−1 ⇒ 24a=5
Subtract the shared equation at x=1 from the equation at x=5.

2. a=5/24; c=1−5/24=19/24
Divide by 24 and substitute back.

Check. At x=±1 the numerator $5x^2+19$ is 24, and at x=5 it is 144.

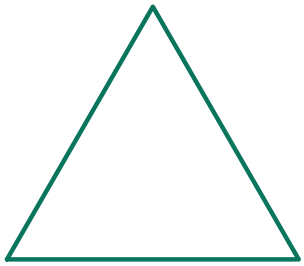
Exercise 294 · Sphere inside a cone

Printed page 29 · Incircles and circumcircles / Volume / Area and volume scaling

The diameter and the slant height of a cone are both 24 cm. Find the radius of the largest sphere that can be placed inside the cone. (The sphere is therefore tangent to the base of the cone.) The sphere occupies a certain percentage of the cone's volume. First estimate this percentage, then calculate it.

1. Estimate and draw the axial section

As an advance estimate, roughly half the volume is plausible. The axial triangle is equilateral with side 24, altitude $12\sqrt{3}$, area $144\sqrt{3}$ and semiperimeter 36.

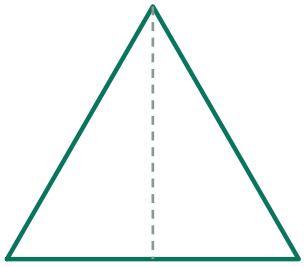


24-unit axial triangle

The side-view plane contains both the cone axis and the candidate sphere center.

2. Explain the height and area

The centerline halves the 24-cm base. In the resulting right triangle the hypotenuse is 24 and the short leg is 12, so the height is $\sqrt{(24^2-12^2)}$. The triangle area is half its base times this height.

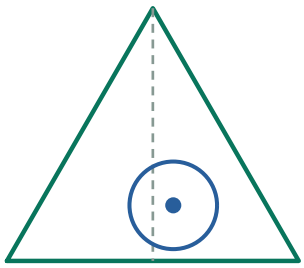


Drop the cone altitude

Half the base is 12; the altitude is $\sqrt{(24^2-12^2)}=12\sqrt{3}$.

3. Allow the center to move

For any sphere inside the cone, take a plane through the cone axis and the sphere center. The section contains a disk of the sphere’s full radius r inside the axial triangle.

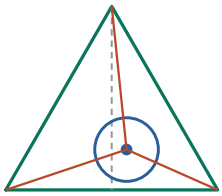


Allow an off-axis center

The disk section must fit inside all three sides of the triangle.

4. Prove a radius bound

Join the disk center to the three triangle vertices. The three perpendicular distances to the sides are each at least r, so summing base times height gives triangle area $A \geq r \cdot s$, where s is its semiperimeter.



Compare triangle area and radius

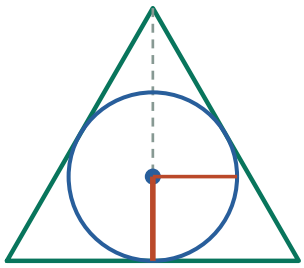
The center-to-side distances are each at least r. Their three triangles give $A \geq r \cdot s$.

1. $A = (24/2)(d_1 + d_2 + d_3) \geq 12(3r)$
Use each side as a base; each center-to-side distance is at least r.

2. $r \leq 144\sqrt{3}/36 = 4\sqrt{3}$
Divide by 36, a positive number. Equality needs all three distances equal.

5. Show the bound is attainable

Thus $r \leq A/s$. The triangle’s incircle is centered on the cone axis and rotating it produces a sphere attaining this bound.

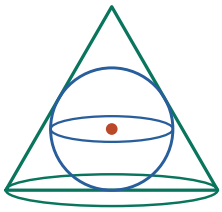


Attain the bound

The centered incircle touches every side. Its radius A/s is $4\sqrt{3}$ cm.

6. Compare the two solid volumes

The maximum radius is A/s=4√3 cm. Dividing sphere volume by cone volume gives $4(4\sqrt{3})^3/(12^2\cdot 12\sqrt{3})=4/9$, or about 44.44%.



Return to the two solids

Rotating the section makes the sphere. Its volume occupies 4/9 of the cone.

1. Vball/Vcone=4(4√3)³/[12²(12√3)]
Cancel π/3 in the two volume formulas.

2. =4·64·3√3/(144·12√3)=4/9
Cube both factors in the radius and cancel the common √3.

Check. Area divided by semiperimeter of the axial triangle gives (144√3)/36=4√3 again.

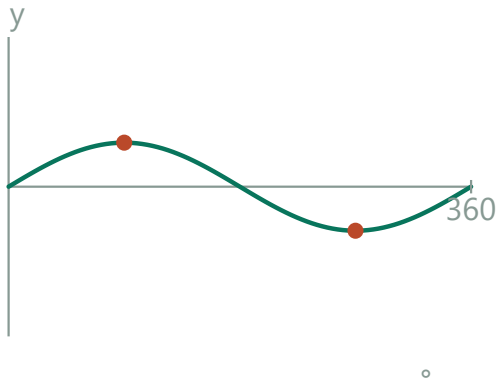
Exercise 295 · Scale the sine wave

Printed page 29 · Sinusoidal functions / Graph transformations / Amplitude and period

Graph each pair of equations on a separate system of coordinate axes:
(a) $y = \sin x$ and $y = 2\sin x$ **(b)** $y = \sin x$ and $y = 3\sin x$ **(c)** $y = \sin x$ and $y = 0.5\sin x$
In general, what do the graphs of $y = a\sin x$ and $y = \sin x$ have in common, and how do they differ? What if the coefficient a is negative? The value of $|a|$ is called the *amplitude*.

1. Keep the input angles fixed

Pick one full cycle and use five inputs a quarter-turn apart. Horizontal positions belong to x, so multiplying sin x by a coefficient changes only the heights.

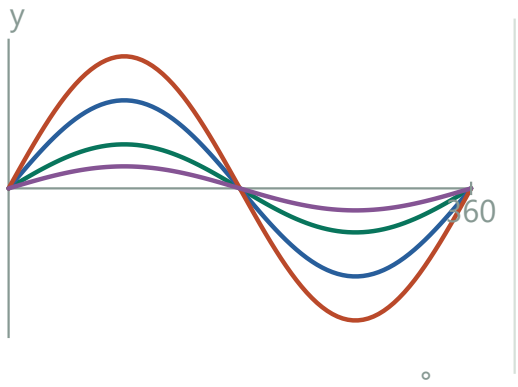


Start with the original sine

All comparisons use the same horizontal angles.

2. Construct the three overlays

At x=0°,90°,180°,270°,360° the base values are 0,1,0,−1,0. Multiply these values by 2, 3, or ½ to sketch each pair.

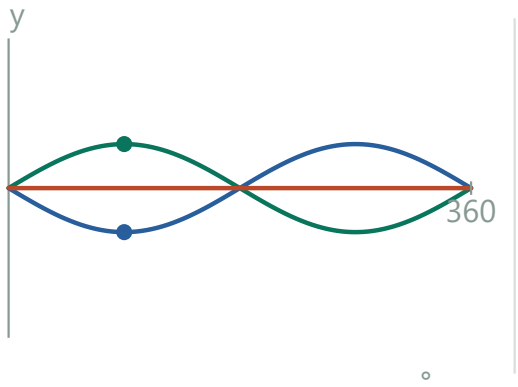


Change only the height

Green is sin x; the other three amplitudes are 2, 3 and ½. Each crest stays at 90°.

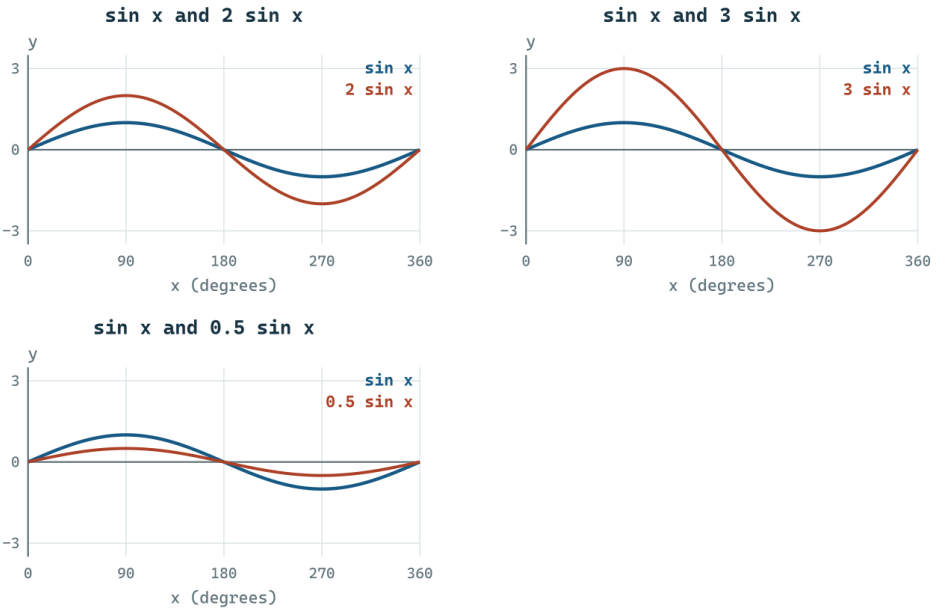
3. Explain what the pictures keep and change

For nonzero a the zeros and period stay fixed and amplitude becomes the absolute value of a. A negative coefficient also reflects the curve vertically. When a=0 the graph is the horizontal axis.



Reverse or erase the heights

For a=−1 each rise becomes a fall. At a=0 every output is zero.



Check. The peak-to-trough distance is twice the amplitude in each case.

Exercise 296 · Difference of cubes

Printed page 29 · Polynomial identities / Algebraic manipulation

Multiply the trinomial $x^2 + xy + y^2$ times the binomial $x - y$. The result is very simple.

1. Identify the two copies of the trinomial

In $(x-y)(x^2+xy+y^2)$, the x multiplies every term in one copy. The $-y$ multiplies every term in a second copy, including the sign on each term.

	x^2	xy	y^2
x	x^3	x^2y	xy^2
$-y$	$-x^2y$	$-xy^2$	$-y^3$

Distribute each row factor

Each column comes from one term of x^2+xy+y^2. Its two entries multiply that term by x and by $-y$.

2. Look for equal powers

Products can cancel only when the powers of x and y match. Track x^2y separately from xy^2 while combining the two rows.

	x^2	xy	y^2
x	x^3	x^2y	xy^2
$-y$	$-x^2y$	$-xy^2$	$-y^3$

Match like powers

The red entries pair x^2y with $-x^2y$, and xy^2 with $-xy^2$.

3. Distribute and simplify

Distribute x to obtain $x^3+x^2y+xy^2$ and $-y$ to obtain $-x^2y-xy^2-y^3$. Cancellation leaves x^3-y^3.

	x^2	xy	y^2
x	x^3	~~x^2y~~	~~xy^2~~
$-y$	~~$-x^2y$~~	~~$-xy^2$~~	$-y^3$

Cross out opposite pairs

The terms left in green are x^3 and $-y^3$. Together they form the difference of cubes.

1. $(x-y)(x^2+xy+y^2) = x^3+x^2y+xy^2-x^2y-xy^2-y^3$
Copy the first row of the diagram and add the signed terms in the second row.

2. $= x^3+(x^2y-x^2y)+(xy^2-xy^2)-y^3 = x^3-y^3$
Reorder the six terms so opposite coefficients cancel in pairs.

Check. Setting $x=2, y=1$ gives 7 on both sides.

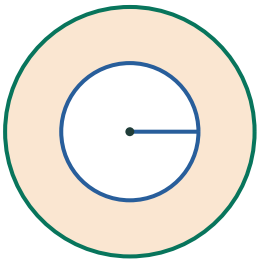
Exercise 297 · Shell area from volume

Printed page 30 · Volume / Surface area / Limits and convergence

Write an expression for the volume of the spherical shell formed between two concentric spheres, the inner one of radius r, the outer one of radius R. Factor your answer so that it has the form $\frac{4}{3}\pi \cdot (trinomial) \cdot (binomial)$. In this situation, what is the meaning of the *binomial*? What can be said about the value of the *trinomial* when the *binomial* has a very small value? Make a conjecture concerning the surface area of a sphere of radius R.

1. Assign the two radii

Both spheres have one center. The shell occupies the part inside the larger sphere and outside the smaller one, so begin with $(4\pi/3)(R^3 - r^3)$.

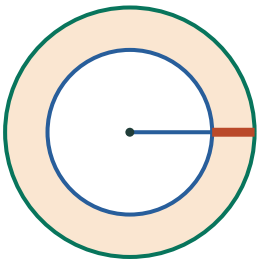


Two concentric spherical boundaries

The cross section cuts the outer radius R and the inner radius r.

2. Factor the difference of cubes

The difference of volumes factors as $(4\pi/3)(R^2 + Rr + r^2)(R - r)$. The binomial is the thickness.

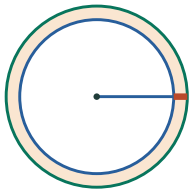


The radial gap is R−r

It appears as the linear factor in $R^3 - r^3$.

3. Shrink the layer

On a ray from the center the shell begins at r and ends at R. If its thickness tends to zero with R fixed, every radius in the shell approaches R. Volume divided by thickness measures the average area of its successive spherical layers.



Hold R fixed and enlarge r

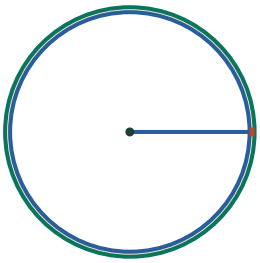
The shell narrows as the inner sphere approaches the outer one.

1. $\Delta V/(R-r)=(4\pi/3)(R^2+Rr+r^2)$
Cancel the thickness only while $r<R$.

2. $r\to R \Rightarrow R^2+Rr+r^2\to 3R^2$
Take the limit after cancellation to avoid division by zero.

4. Take the limiting area

Holding R fixed as the thickness approaches zero makes the trinomial approach $3R^2$. Thus volume per thickness tends to $4\pi R^2$, the sphere's area.



Limit of volume per thickness

When r approaches R, R^2+Rr+r^2 approaches $3R^2$, giving outer area $4\pi R^2$.

Check. Differentiating $(4\pi/3)R^3$ with respect to R yields the same area.

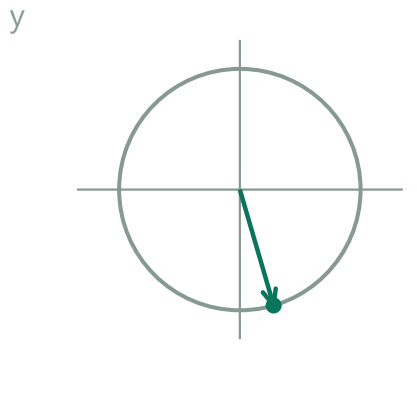
Exercise 298 · Fourth-quadrant ratios

Printed page 30 · Trigonometric ratios / Reference angles and symmetry / The Pythagorean theorem

Given that $\cos\theta$ is $7/25$, with $270 < \theta < 360$, find $\sin\theta$ and $\tan\theta$, without finding θ.

1. Draw the quadrant-IV radius

Cosine fixes the horizontal leg at 7 for a radius of 25. Quadrant IV places the endpoint below the horizontal axis, giving its vertical coordinate a minus sign.

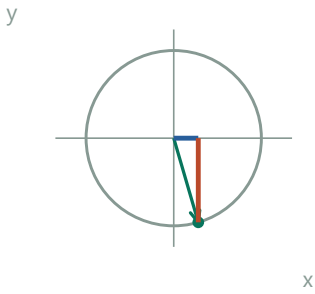


Quadrant IV

Cosine is positive and the vertical component points downward.

2. Use Pythagoras for the magnitude

If y is the vertical coordinate then $7^2+y^2=25^2$. The positive square root gives a length; the quadrant supplies the sign for the coordinate used in sine and tangent.



Horizontal leg 7, radius 25

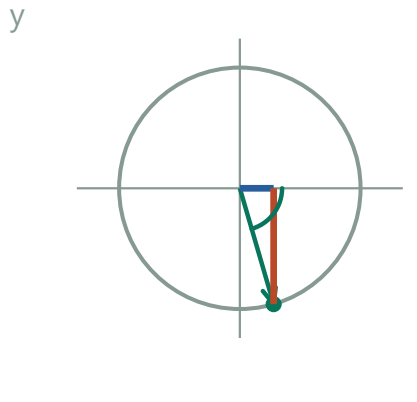
The missing vertical length is $\sqrt{(25^2-7^2)}=24$.

1. $y^2=25^2-7^2=625-49=576$
Subtract the horizontal-leg square from the radius squared.

2. $y=-\sqrt{576}=-24$
The point is below the horizontal axis.

3. Calculate both ratios

Opposite magnitude is $\sqrt{(25^2-7^2)}=24$. Sine is $-24/25$ and tangent is $-24/7$.



Use the downward sign

Sine = $-24/25$ and tangent = $-24/7$.

Check. The triple 7,24,25 satisfies the Pythagorean theorem.

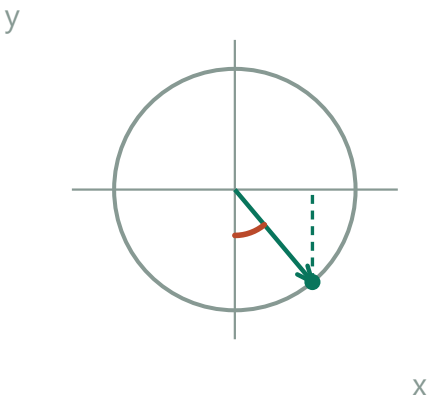
Exercise 299 · Cosine near a full turn

Printed page 30 · The unit circle / Reference angles and symmetry

Without using a calculator, choose the larger of $\cos 310$ and $\cos 311$. Explain.

1. Picture the last quarter turn

The unit-circle point moves from (0,−1) toward (1,0) as its angle grows from 270° to 360°. Both candidate angles are on this arc.

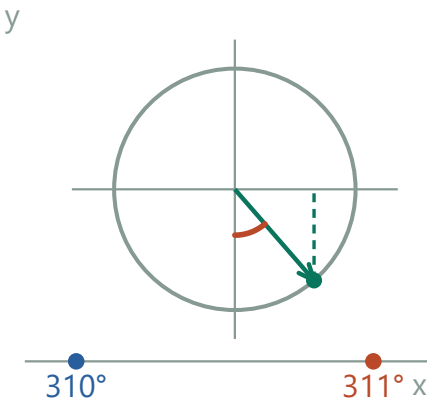


Locate 310°

The radius is in the last quarter of a turn.

2. Compare horizontal position

Cosine is the signed horizontal coordinate. The ray for 311° is one degree farther around toward the positive x-axis.

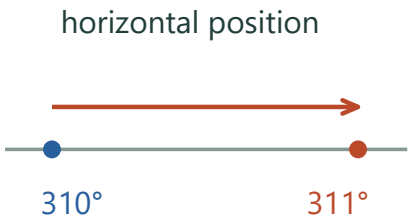


Move toward the positive x-axis

The vertical foot moves right as the angle increases from 310° to 311°.

3. Identify the larger value

Both rays are in quadrant IV. Turning from 310° toward 360° increases the unit-circle x-coordinate, so cos 311° is larger.



Read the larger projection

At 311° more of the same unit radius projects horizontally.

Check. Using complements to a full turn compares cos 49° to cos 50°, which decreases on the first quadrant.

Exercise 300 · Does multiplication commute?

Printed page 30 · Matrix multiplication

For each of the following pairs of matrices, calculate $\mathbf{MN}$ and $\mathbf{NM}$:

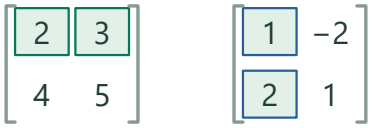
(a) $\mathbf{M} = \begin{bmatrix} 2 & 3 \\ 4 & 5 \end{bmatrix}$ and $\mathbf{N} = \begin{bmatrix} 1 & -2 \\ 2 & 1 \end{bmatrix}$

(b) $\mathbf{M} = \begin{bmatrix} 3 & -4 \end{bmatrix}$ and $\mathbf{N} = \begin{bmatrix} 12 \\ 5 \end{bmatrix}$

(c) $\mathbf{M} = \begin{bmatrix} 1 & 2 \\ -6 & -3 \end{bmatrix}$ and $\mathbf{N} = \begin{bmatrix} -4 & 1 \\ -3 & -6 \end{bmatrix}$

1. Check product dimensions

If the inner dimensions match, each entry of a product is a row dotted with a column. The result takes its row count from the left factor and its column count from the right.



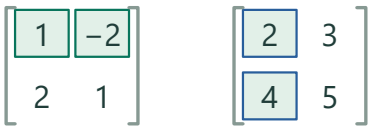
row 1 column 1

Match a row to a column

For MN, the highlighted first row (2,3) pairs with the highlighted column (1,2).

2. Compute an entry in both orders

In (a), the upper-left entry of MN is 2·1+3·2=8, whereas for NM it is 1·2+(−2)·4=−6. For (b), reversing a row and a column even changes the output shape.



1·2 + (−2)·4 = −6

Reverse the factors

For NM, the selected row is (1,−2) and the selected column is (2,4). Their dot product is −6.

3. Complete the first pair

In (a), row-by-column multiplication gives MN=[[8,−1],[14,−3]] and NM=[[−6,−7],[8,11]].

$$\begin{bmatrix} 8 & -1 \\ 14 & -3 \end{bmatrix}$$ MN

$$\begin{bmatrix} -6 & -7 \\ 8 & 11 \end{bmatrix}$$ NM

Compare the completed arrays

Even with identical 2×2 dimensions, the corresponding entries differ.

1. MN, row 1: (2·1+3·2, 2·(−2)+3·1)=(8,−1)
Pair the first row of M with each column of N.

2. MN, row 2: (4·1+5·2, 4·(−2)+5·1)=(14,−3)
Move down one row of M and repeat with the same two columns.

3. NM, row 1: (1·2−2·4, 1·3−2·5)=(−6,−7)
Switch the matrices before selecting the rows and columns.

4. NM, row 2: (2·2+1·4, 2·3+1·5)=(8,11)
The second row of N supplies both dot products.

4. Compare row and column products

In (b), the row times column gives [16]; the column times row gives [[36,−48],[15,−20]].

$$\begin{bmatrix} 16 \end{bmatrix}$$ MN

$$\begin{bmatrix} 36 & -48 \\ 15 & -20 \end{bmatrix}$$ NM

Observe the changed shape

The row times the column produces one entry, 16. Reversal produces a 2×2 outer product.

1. 36+(−20)=16
The row–column product adds its two component products.

2. 36+(−20)=tr(NM)
In the reverse order these two products occupy different diagonal cells of the 2×2 matrix.

5. Recognize a commuting pair

In (c), both products give [[−10,−11],[33,12]].

$$\begin{bmatrix} -10 & -11 \\ 33 & 12 \end{bmatrix}$$ MN

$$\begin{bmatrix} -10 & -11 \\ 33 & 12 \end{bmatrix}$$ NM

Check all four positions

For the third pair, reversing the factors leaves each of the four entries equal.

Check. In (b), the product dimensions are 1×1 and 2×2; in (c), checking all four entries is necessary.

Exercise 301 · A commuting pair

Printed page 30 · Matrix multiplication

(Continuation) Find two new matrices $\mathbf{M}$ and $\mathbf{N}$ that have the so-called *commutative property* $\mathbf{MN} = \mathbf{NM}$ that was noticed in part (c) of the preceding.

1. Choose actions whose order you can predict

Use diagonal matrices. They independently multiply the horizontal and vertical components, so a horizontal input never contributes to a vertical output.



Stretch components separately

M independently multiplies the horizontal coordinate by 2 and the vertical one by 3.

2. Track a general point

If the scale factors are (a,b) then (c,d), the point (x,y) ends at (acx,bdy). Doing the scalings in the opposite order gives the same coordinates because ordinary numbers commute.



Interchange the two diagonal stretches

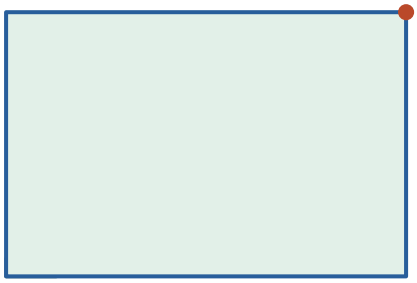
N independently scales by 4 and 5. Each component remains on its own axis.

1. N(x,y)=(4x,5y) ⇒ MN(x,y)=(8x,15y)
Scale x by 2 and y by 3 after applying N.

2. M(x,y)=(2x,3y) ⇒ NM(x,y)=(8x,15y)
Reversing the two independent scale factors gives the same image.

3. Supply two explicit matrices

Choose M=diag(2,3) and N=diag(4,5). Multiplication gives diag(8,15) in both orders.



Reach the same final corner

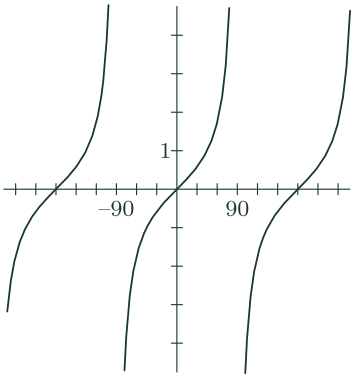
In either order the unit square maps to width 8 and height 15.

Check. Each matrix independently scales the x and y components, so their order does not affect either coordinate.

Exercise 302 · Tangent as slope

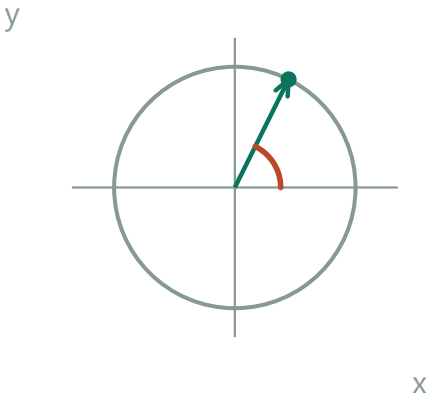
Printed page 30 · The tangent function / Asymptotes

A graph of $y = \tan x$ is shown at right, drawn in degree mode. Confirm that one of the points on this graph is $(63.56, 2.011)$. Recall that the number 2.011 can be interpreted as the *slope* of a certain ray drawn from the origin. What ray? In contrast to a sine graph, which is a connected curve, this graph is in pieces. Explain why.



1. Read the two different pictures

On the tangent graph the horizontal coordinate is an angle in degrees and the vertical coordinate is a slope. On the unit circle, the same angle determines a ray through the origin.

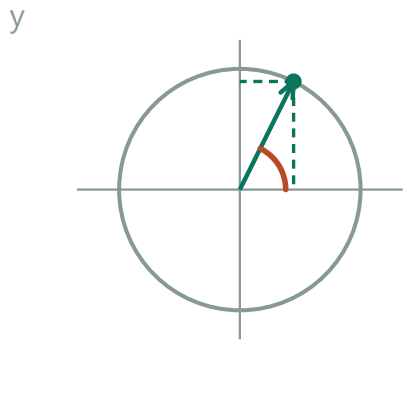


Input angle 63.56°

The graph uses the turn as its horizontal input. The circle locates the corresponding ray.

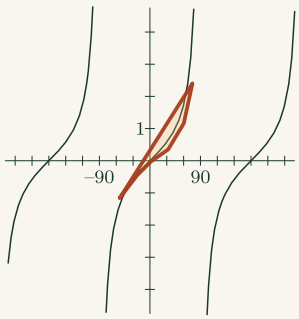
2. Measure the ray's rise and run

The point on the ray has coordinates (cos 63.56°, sin 63.56°). Dividing its height by its horizontal displacement gives the slope.



Rise divided by run

The rise is sin θ and the run is cos θ on a unit radius.

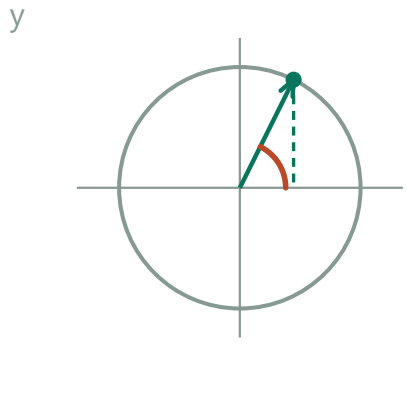


The central tangent branch

The curve's horizontal coordinate measures the angle in degrees. Moving up the branch changes the vertical value, which represents the slope of the rotating ray.

3. Verify the plotted point

The ray at 63.56° from +x has slope sin 63.56°/cos 63.56°≈2.011.

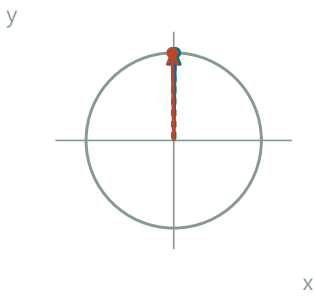


tan 63.56° ≈ 2.011

The vertical component is just over twice the horizontal one.

4. Explain each break in the graph

Vertical rays have zero x-component, so tangent is undefined at 90°+180°k and the graph separates into branches.

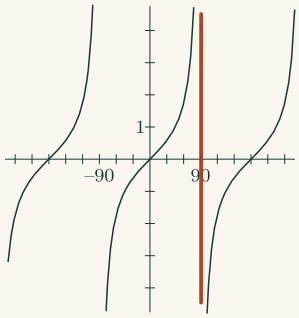


At the vertical direction

The run reaches zero at 90°. Rays on opposite sides have slopes of opposite signs.

1. cos θ=0 at θ=90°+180°k, k an integer
The ray then has zero horizontal component.

2. tan θ=sin θ/cos θ is undefined there
Sine is ±1; division by zero creates the breaks.



A vertical-direction break

Trace the vertical through 90°. The curve climbs on one side and restarts below the axis on the other as the ray's horizontal run passes through zero.

Check. As a ray approaches a vertical direction its slope grows arbitrarily large in magnitude.

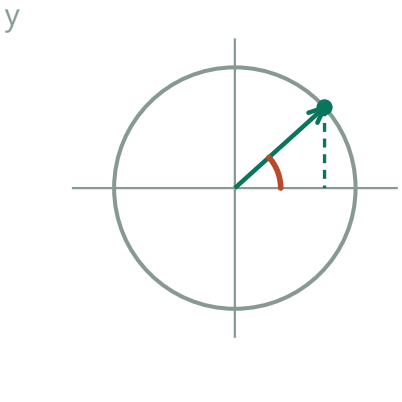
Exercise 303 · All tangent solutions

Printed page 30 · Trigonometric equations / The tangent function

The equation $\tan\theta = 0.9004$ has infinitely many solutions. Find a way of describing all these values of θ.

1. Find the first ray

A tangent of 0.9004 means a rise of 0.9004 for a run of 1. Use inverse tangent in degree mode to find the acute inclination.

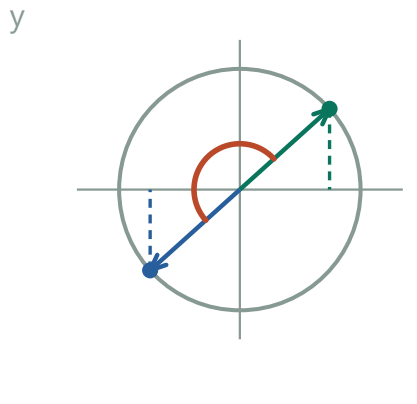


Principal ray

Set rise/run to 0.9004; a ray with positive components gives the first solution.

2. Follow the same line through the origin

A half-turn reverses both rise and run. Their quotient stays the same. A whole family is needed since positive and negative rotations are both allowed.

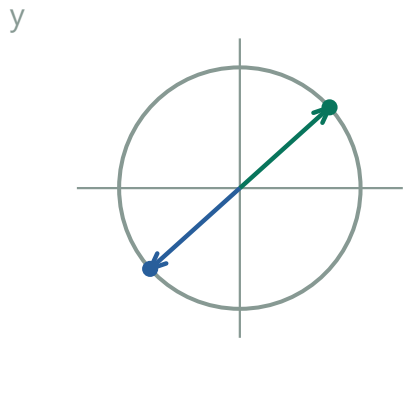


Same line, opposite rays

Changing the signs of both components preserves their ratio.

3. Describe all the angles

The principal solution is arctan(0.9004)≈41.99987°. All solutions are arctan(0.9004)+180°k, for any integer k.



Repeat every 180°

Adding any integer number of half-turns preserves this line and its slope.

Check. Opposite rays have the same slope, and each 180° interval has one solution.

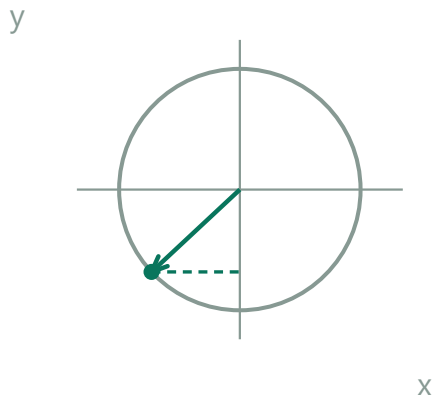
Exercise 304 · Three positive sine solutions

Printed page 30 · Trigonometric equations / Reference angles and symmetry

Find the three smallest positive solutions to $2\sin\theta = -1.364$.

1. Isolate the vertical coordinate

Divide by two to get -0.682. Dividing both sides of $2\sin\theta = -1.364$ by 2 isolates sine. Interpreted on the unit circle, this value is the height both selected rays must reach.

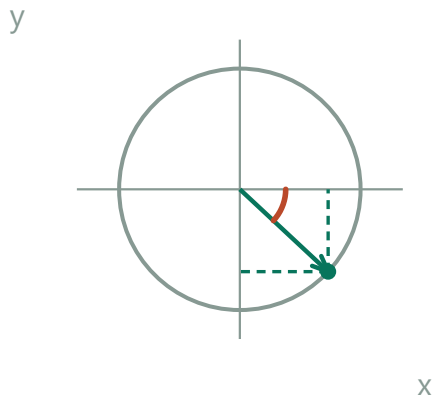


The required height is -0.682

Divide the original equation by 2; the ray tip must be below the horizontal axis.

2. Find the reference angle

Let $\alpha = \arcsin(0.682) \approx 43.00013°$. The reference angle is the acute angle with the same sine magnitude. Taking the inverse sine of the absolute value locates that acute angle before deciding the quadrant.

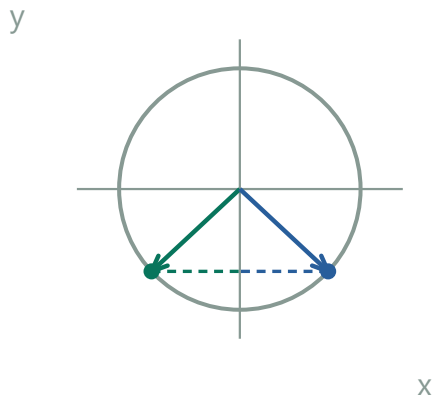


Use a positive reference angle

The acute triangle has opposite magnitude 0.682 on a unit hypotenuse.

3. Place the two rays below the axis

A negative sine puts both rays in the lower half-plane. One lies just after $180°$ and the other just before $360°$. Another full turn of the earlier ray gives the next positive solution.

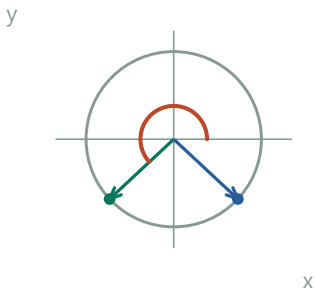


Two points at one height

Reflection in the vertical axis gives the two rays below the axis.

4. Order the first three answers

The first two are 180°+α and 360°−α; the third is 540°+α, approximately 223°, 317°, 583°.



Continue one revolution

The first two angles are about 223° and 317°. Returning to the first ray adds 360°, giving about 583°.

1. α≈43° ⇒ 180°+α≈223°, 360°−α≈317°
Sine is negative in quadrants III and IV.

2. 223°+360°≈583°
The next positive solution is the earlier ray one full turn later.

Check. The first pair is in quadrants III and IV, and adding a full turn to the earlier one preserves sine.

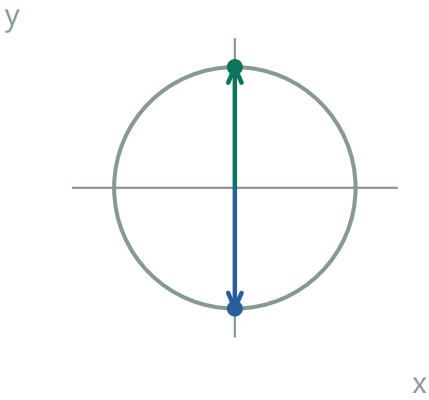
Exercise 305 · Possible trigonometric values

Printed page 30 · Trigonometric ratios / Domain and range

Explain why equation $\tan\theta = -2$ has solutions, but equation $\sin\theta = -2$ does not.

1. Interpret sine geometrically

No point on a unit circle can have a vertical coordinate below −1, since its entire distance from the origin is only 1.

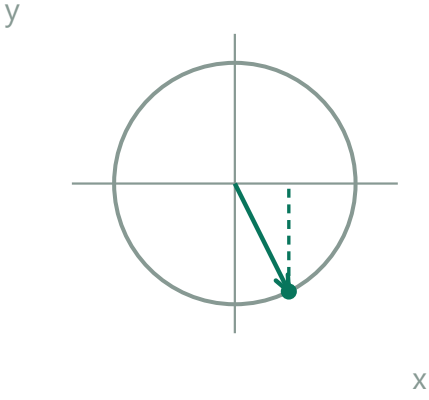


Sine is bounded

The top and bottom of a radius-1 circle have heights +1 and −1, so −2 lies out of reach.

2. Interpret tangent geometrically

The vertical drop of a ray can be twice its horizontal run. Its slope depends on a ratio of two lengths, with no requirement that the ratio fit inside a unit interval.

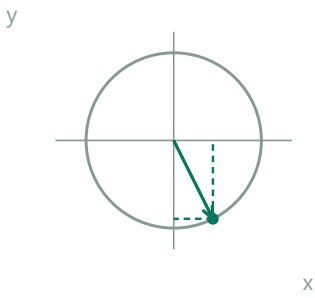


Tangent is a slope

A steep downward ray is allowed; its rise can exceed its run in magnitude.

3. Construct the example and compare

Tangent is slope: a ray through $(1,-2)$ has tangent -2. Sine is a unit-circle y-coordinate and always lies between -1 and 1.



A ratio of -2

Scale the ray to pass through $(1,-2)$. Its slope is -2 even though its unit-circle height exceeds -1.

1. $r=\sqrt{(1^2+(-2)^2)}=\sqrt{5}$
Normalize the vector through $(1,-2)$.

2. $\sin\theta=-2/\sqrt{5}$; $\tan\theta=(-2)/1=-2$
Sine divides by the radius, whereas tangent divides by the horizontal coordinate.

Check. Normalizing that ray gives sine $-2/\sqrt{5}$, safely inside the required interval.

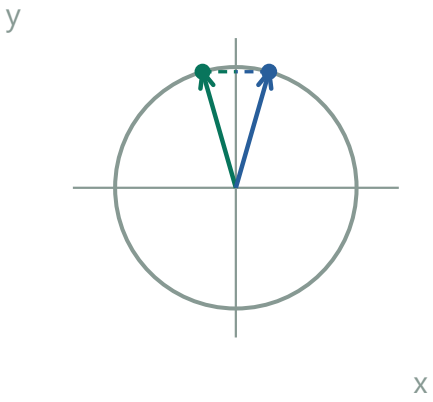
Exercise 306 · Compare sines

Printed page 30 · The unit circle / Reference angles and symmetry

Without using a calculator, choose the larger of $\sin 76$ and $\sin 106$. Explain.

1. Reflect one ray

The angle $106°$ is $16°$ past the top of the circle. Reflect it across the vertical axis to get a first-quadrant ray, keeping its vertical coordinate unchanged.

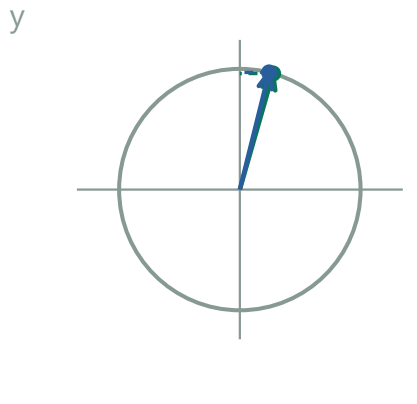


Reflect $106°$ in the y-axis

Its partner makes $74°$ with the positive x-axis. Both tips have the same height.

2. Compare the two heights

In the first quadrant the height grows up to the top. Compare 76° to the reflected 180°−106°, then translate the comparison back to the original two sines.

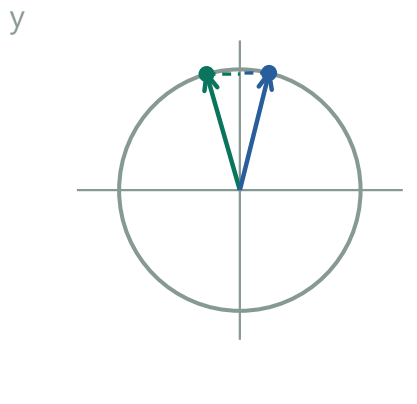


Compare heights before 90°

The 76° ray lies closer to the top, where the y-coordinate is larger.

3. State the inequality

The supplementary-angle identity gives sin 106°=sin 74°. Sine increases between 0° and 90°, so sin 76° is larger.



sin 76° > sin 106°

The second-quadrant comparison reduces to 76° versus 74° in the first quadrant.

Check. The 76° point is 14° from the top of the circle; the 106° point is 16° from it.

Exercise 307 · Power of a power

Printed page 30 · Exponent rules

Faced with the problem of calculating $(5^4)^3$, Brook is having trouble deciding which of these three answers is correct: 5^{64}, 5^{12}, or 5^7. Once you have answered Brook's question, experiment with other examples of this type until you can formulate the principle that applies when exponential expressions are raised to powers.

1. Unpack the outside exponent

Parentheses make 5^4 the whole factor that is repeated. Write three copies of that four-factor group side by side.



Open one set of parentheses

A single factor 5^4 contains four individual fives.

2. Count the inner factors

Every copy contributes four 5s. Counting equal-size groups uses multiplication, so three groups give 3·4 factors of the same base.



Repeat the complete group

Two copies supply eight fives. A third copy contributes another four.

3. Express the general rule

Three copies of a four-factor product contain twelve factors. Thus $(5^4)^3=5^{12}$. In general $(a^m)^n=a^{mn}$ for integer exponents where defined; positive bases extend to real exponents.



Count all three groups

Three groups of four give twelve factors, hence $(5^4)^3=5^{12}$.

Check. For instance $(2^3)^2=64=2^6$.

Exercise 308 · Atoms in a cube

Printed page 30 · Scientific notation / Exponent rules

The diameter of a typical atom is so small that it would take about 10^8 of them, arranged in a line, to reach just one centimeter. It is therefore a plausible estimate that a cubic centimeter could contain about $10^8 \times 10^8 \times 10^8 = (10^8)^3$ atoms. Write this huge number as a power of 10.

1. Build the first row

One edge of the centimeter cube holds roughly 10^8 atoms. Copying that row 10^8 times makes a square layer.



Start with a line of atoms

Each modeled edge contains 10^8 atoms. The dots represent only a few positions from the row.

2. Stack equal layers

Another 10^8 layers fill the third direction in the stated cubic model. This explains why the linear estimate appears as three factors.

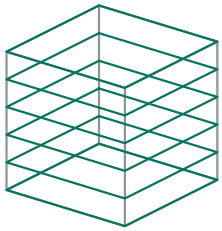


Repeat the line across a layer

There are 10^8 rows of 10^8 atoms, giving 10^{16} per layer.

3. Express the cube as one power

In a cubic arrangement multiply the count along each of the three axes. Adding 8+8+8 gives 10^{24} atoms.



Stack the equal layers

The third factor 10^{8} gives a total of 10^{24} atoms in the cubic arrangement.

Check. A centimeter contains 10^{8} atomic widths per axis, so cubing the linear count gives the volume count.

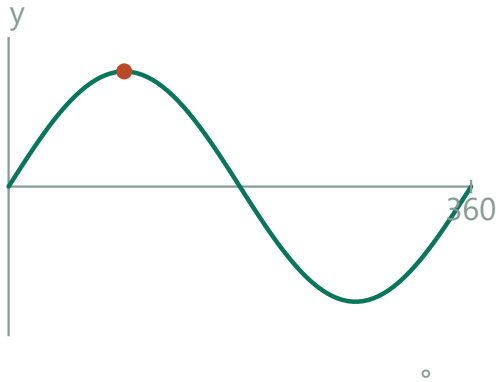
Exercise 309 · Flip a sine graph

Printed page 30 · Sinusoidal functions / Reflections

Reflect the graph $y = 2\sin x$ across the x-axis. Find an equation to describe the curve that results. Use a graphing tool to check your answer.

1. Define the reflection point by point

A point 2 units above the x-axis is carried to a point 2 units below it at the same x. Points on the axis stay fixed.

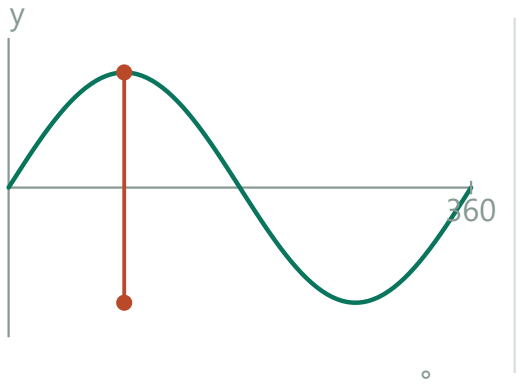


Begin with y=2 sin x

A reflection keeps every input fixed.

2. Test one crest

At 90° the original graph has height 2. Its reflected partner must make a trough at that same input. Use the same sign change for every input.

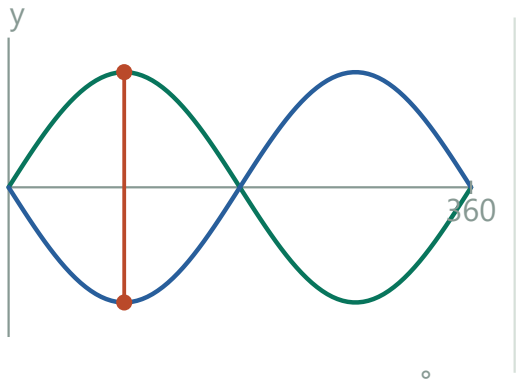


Follow a crest

The point (90°,2) goes straight down to (90°,−2).

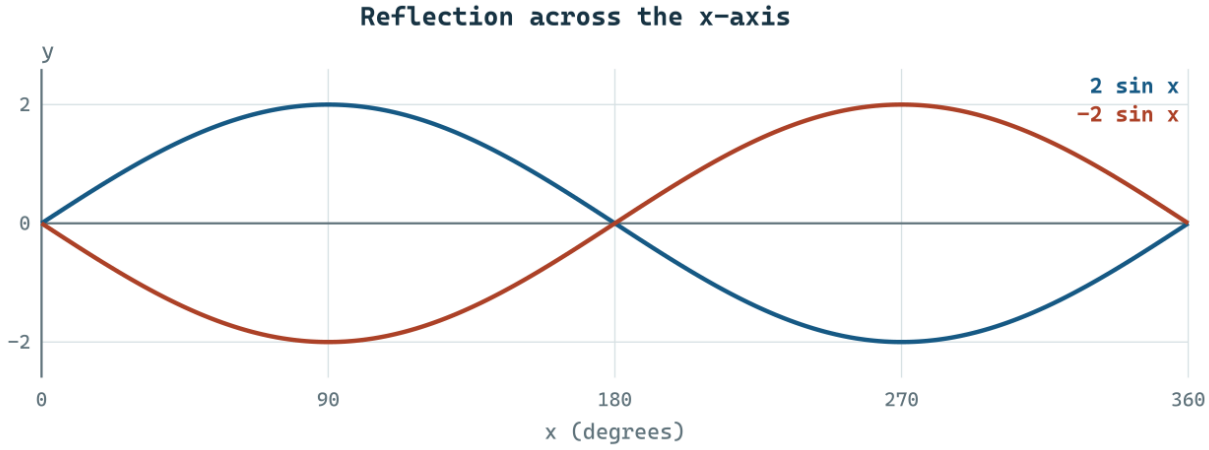
3. Write and graph the reflected function

Reflection takes (x,y) to (x,−y), hence the new equation is y=−2 sin x. Plot cardinal points at 0°,90°,180°,270°,360° with heights 0,−2,0,2,0.



Trace the entire reflection

The blue curve has the opposite height at every angle, including the original zero crossings.



Check. The same curve is 2 sin(x+180°), confirming the reflection.

Exercise 310 · A 24-second Ferris wheel

Printed page 31 · Circular motion / Sinusoidal functions / Trigonometric equations

Avery is riding a Ferris wheel that turns once every 24 seconds, and whose radius is 8 meters. The function h defined by $h(t) = 9 - 8\cos(15t)$ describes Avery's distance from the ground (in meters) after t seconds of riding. For example, $h(8) = 13$ means that Avery is 13 meters above the ground after 8 seconds of riding. By the way, "h of 8" or "h at 8" are two common ways to say $h(8)$.

(a) Evaluate $h(0)$, and explain its significance.

(b) Explain why $h(16) = h(8)$.

(c) Find a value for t that fits the equation $h(t) = 10$. Interpret this t-value in the story.

(d) Explain why $h(t + 24) = h(t)$ is true, no matter what value t has.

(e) What is the complete range of values that $h(t)$ can have?

It is not unusual to see the equation $h = 9 - 8\cos 15t$ used to define the function h, even though h is written instead of $h(t)$ and the parentheses around $15t$ are missing.

1. Decode the wheel model

The center is 9 m above the ground. The coefficient 8 is the radius, and 15 degrees per second converts time into rotation angle. The minus sign starts the rider below the center.

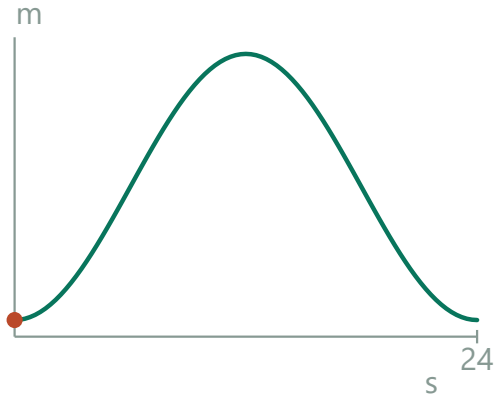


Read the center and radius

The midline is 9 m above ground; the radius gives 8 m of vertical travel on either side.

2. Locate the rider at the start

At t=0 the rider begins at h=1 m. In the height formula cos(15·0°)=1, so h(0)=9−8. The one-meter clearance is the height of the lowest seat position above the ground.

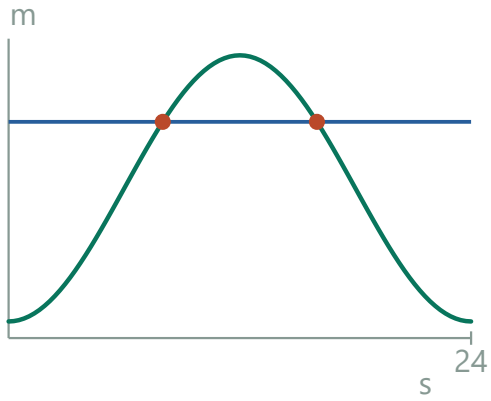


Start at the bottom

At t=0 the cosine is 1, giving the lowest height 1 m.

3. Explain the repeated height

At 8 s and 16 s the angles are 120° and 240°, both with cosine −½, so both heights are 13 m.

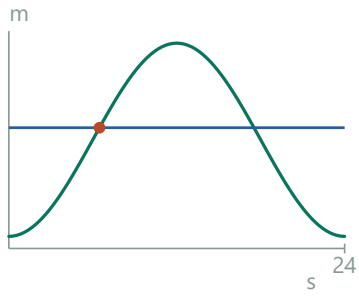


Equal heights on opposite sides

At 8 and 16 seconds the same horizontal level meets a rising and a falling part of the graph.

4. Find when the rider reaches 10 m

At 10 m solve $\cos(15t) = -1/8$; the first such time is $\arccos(-1/8)/15 \approx 6.479$ s.



First arrival at 10 m

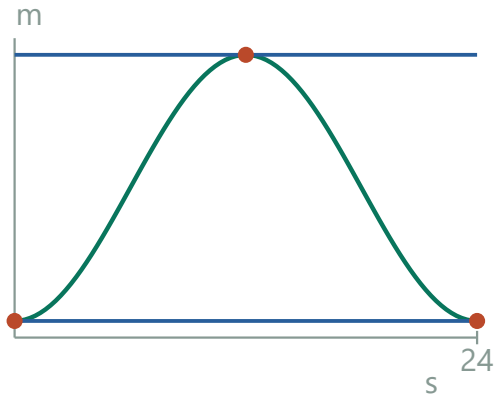
Find the first intersection with the 10 m line at approximately 6.479 seconds.

1. $10 = 9 - 8\cos(15t) \Rightarrow 1 = -8\cos(15t)$
Subtract the midline height.

2. $\cos(15t) = -1/8 \Rightarrow t = \arccos(-1/8)/15$
Divide by −8, then take the first positive angle in degrees.

5. Get the period and extremes

Adding 24 s adds 360°, so the ride repeats. The full range is [1,17] m.



Read one complete cycle

The maximum and minimum are 17 and 1 m. At 24 seconds the motion repeats.

Check. The bottom and top at 0 and 12 s sit 8 m below and above the 9 m center.

Exercise 311 · Projection coefficient

Printed page 31 · Vector projections / The dot product

Given vectors $\mathbf{u} = [5, 12]$ and $\mathbf{v} = [4, 3]$, Calculate $\mathbf{u}\bullet\mathbf{v}$ and $\mathbf{u}\bullet\mathbf{u}$. Where have you seen the ratio $\frac{\mathbf{u}\bullet\mathbf{v}}{\mathbf{u}\bullet\mathbf{u}}$ before?

1. Pair like coordinates

The dot product multiplies horizontal components together and vertical components together, then adds. Dotting u with itself gives its squared length.



Place u and v at a common origin

The two arrows represent (5,12) and (4,3).

2. Why divide by u·u?

Write the projection as ku, a multiple of the direction vector. The leftover v−ku must be perpendicular to u. Setting u·(v−ku)=0 gives a linear equation for k.



Drop v onto the direction of u

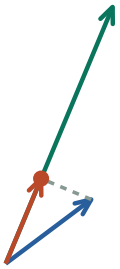
The remainder from the foot to v must be perpendicular to u.

1. (v−ku)·u=0 ⇒ v·u−k(u·u)=0
The residual is perpendicular to u, and the dot product distributes.

2. k=(v·u)/(u·u)=56/169
The squared direction length is positive because u is nonzero.

3. Compute the scale factor

The dot products are 5·4+12·3=56 and 25+144=169. The ratio 56/169 scales u to give the projection of v onto u.



Scale the long direction

The projected arrow is (56/169)u. Its coefficient uses (u·v)/(u·u).

Check. Subtracting (56/169)u from v leaves a vector with zero dot product with u.

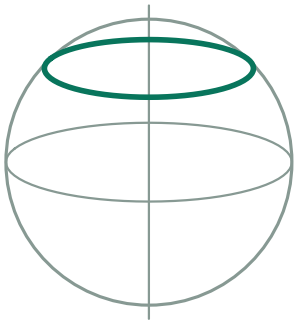
Exercise 312 · Two geographic loci

Printed page 31 · Spherical coordinates / Distance and loci

Describe the configuration of all points on the earth's surface
(a) whose latitude is 43° N;
(b) whose longitude is 71° W.

1. Fix the first angle

Latitude measures the tilt above the equator. At the same latitude every radius has the same vertical component. Vary longitude to sweep out all directions at that height.

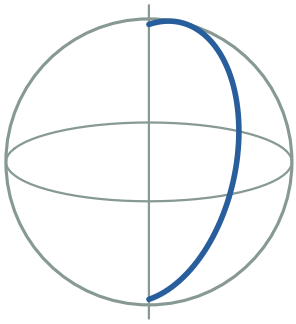


Hold latitude constant

All points at 43° N form a parallel, a circle in a horizontal plane.

2. Fix the second angle instead

West longitude fixes the direction of the horizontal projection. Vary latitude from south to north without crossing to the opposite side of the Earth.

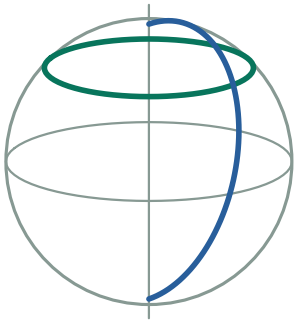


Hold longitude constant

West longitude 71° picks one half of a great circle from pole to pole.

3. Identify the two loci

Fixed latitude gives a horizontal small circle at height R sin 43° and radius R cos 43°. Fixed longitude gives the meridian semicircle through the poles, with constant west bearing 71° from Greenwich in the equatorial projection.



Compare the two loci

The parallel and meridian share the position with both the prescribed latitude and longitude.

Check. A parallel avoids the poles; all meridians approach the same poles, where longitude itself is undefined.

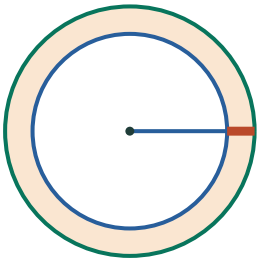
Exercise 313 · Changing track lanes

Printed page 31 · Circumference / Arc length

The lanes on a circular outdoor track are one meter wide. By running in the lane next to the innermost lane, Corey ran extra distance in an eight-lap race. How much?

1. Compare matching paths

Measure both runners along the same relative position inside each lane, for example the centerlines. The second path lies 1 m farther from the common center at every point.

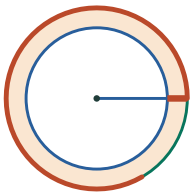


Compare two concentric running paths

Neighboring lanes differ by one meter of radius; the drawn gap is schematic.

2. Subtract circumferences first

Call the inner path radius r. Use radius r+1 for the adjacent path. Over eight laps the same one-lap difference is repeated eight times.



Subtract one lap from the other

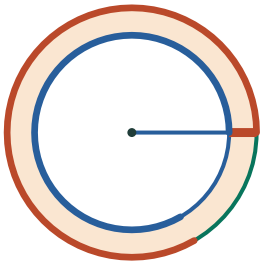
The shared 2πr term cancels and one additional lap length contributes 2π meters.

1. $2\pi(r+1)-2\pi r=2\pi r+2\pi-2\pi r=2\pi$
Expand the larger circumference and cancel the shared part.

2. $8\cdot 2\pi=16\pi$
The same extra length occurs on each lap.

3. Find the extra distance

One lap grows by $2\pi(r+1)-2\pi r=2\pi$ m. Over eight laps the difference is 16π m, about 50.27 m.



Repeat the excess for eight laps

Eight repetitions of the same extra distance add to 16π meters.

Check. The original radius cancels, so the result applies to any adjacent lanes run along matching paths.

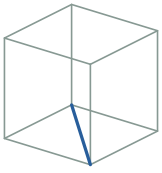
Exercise 314 · Cube in a sphere

Printed page 31 · Sphere geometry / The Pythagorean theorem

Find the volume of the largest cube that fits in a sphere of radius 8 cm.

1. Build the longest cube segment

Across one square face the diagonal is $a\sqrt{2}$. Pair this length with a perpendicular edge a to reach the opposite corner in space. Pythagoras gives $\sqrt{(2a^2+a^2)}$.



Trace a face diagonal

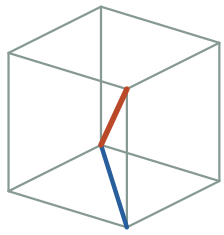
Two perpendicular edges of length a give a face diagonal of $a\sqrt{2}$.

1. $d_{face}^2=a^2+a^2=2a^2$
The two edges in one face are perpendicular.

2. $d_{space}^2=d_{face}^2+a^2=3a^2 \Rightarrow d_{space}=a\sqrt{3}$
Use the third perpendicular edge. The diagonal is a length.

2. Compute the candidate cube

The longest cube diagonal is $a\sqrt{3}$ and spans the diameter 16 when inscribed. Thus side length is $16/\sqrt{3}$ and volume is $4096/(3\sqrt{3})$ cm^3.

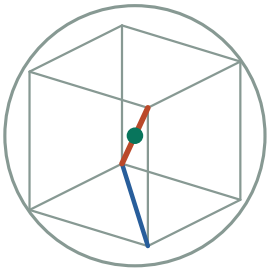


Extend to the opposite corner

Including the third perpendicular edge makes the space diagonal $a\sqrt{3}$.

3. Test every possible placement

For any placement, two opposite vertices lie in the sphere, so their separation is at most $8+8=16$ by the triangle inequality. The centered construction in the check attains this limit, and volume increases with side length.

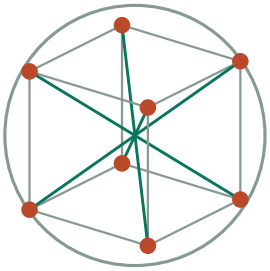


Compare the farthest vertices

In a radius-8 sphere, no pair of points can be more than 16 apart.

4. Explain equality

Place the cube center at the sphere center. Every vertex then lies half a space diagonal from it. With diagonal 16, all vertices are exactly on the surface and convexity keeps the whole cube inside.



Center the cube to attain the bound

All eight vertices lie exactly 8 from the center when $a\sqrt{3}=16$. Convexity keeps the cube inside.

Check. Centering a cube with vertices $(\pm 8/\sqrt{3}, \pm 8/\sqrt{3}, \pm 8/\sqrt{3})$ puts all eight vertices exactly 8 cm from the center.

Exercise 315 · Difference-angle identity

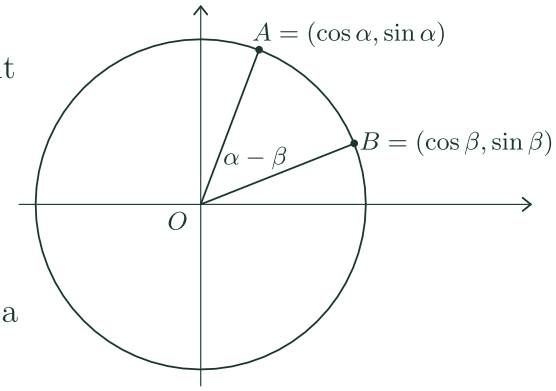
Printed page 31 · Angle addition formulas / The dot product / The unit circle

In the diagram, points A and B are on the unit circle, and O is the origin.

(a) Explain why the central angle BOA is labeled $\alpha - \beta$.

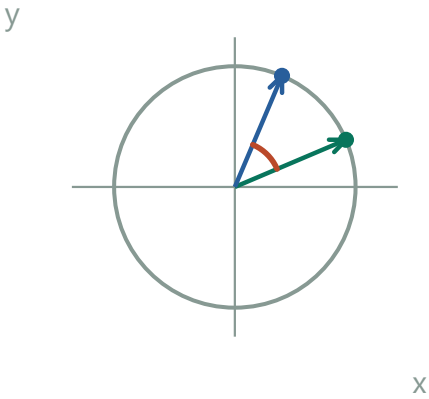
(b) Obtain a formula for $\cos(\alpha - \beta)$ by applying the Law of Cosines in its vector form.

(c) By replacing β by $-\beta$ in your formula, obtain a familiar formula for $\cos(\alpha + \beta)$.



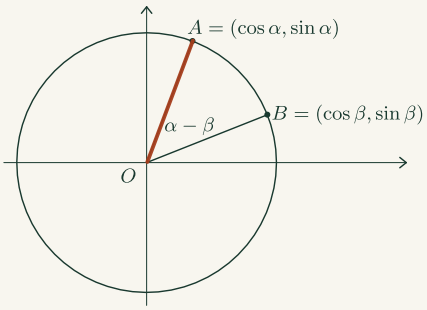
1. Read the two radius directions

Their directions differ by α−β in the indicated order.



Read β and α from the same baseline

The piece between the rays is left after subtracting the smaller angle.

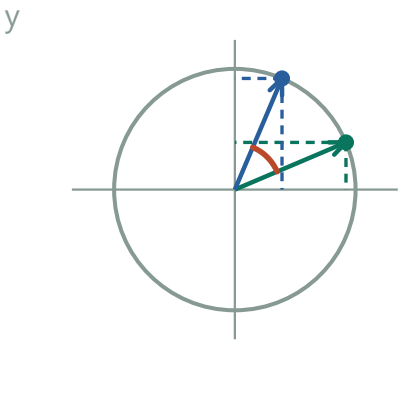


Radius OA

The longer upward ray joins O to the point A labeled with cosine and sine of α. Since the circle has unit radius, those coordinates are also OA's components.

2. Compare two descriptions of the same scalar

OA and OB each have length 1. The dot product can be computed from components or from lengths times cosine of the included angle. The two calculations must agree.

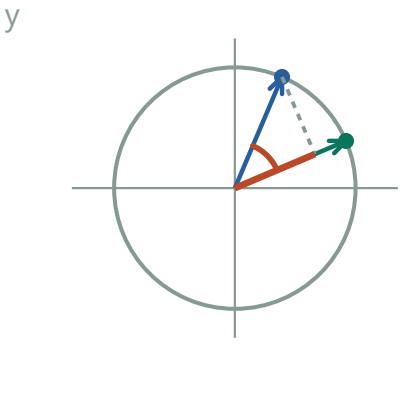


Match corresponding components

Each unit vector contributes a horizontal and a vertical component to the dot product.

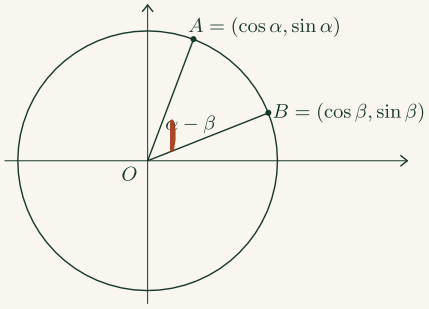
3. Derive the difference identity

Dotting their coordinates gives cos α cos β+sin α sin β; dotting by lengths and included angle gives cos(α−β).



Dot product = cos(α−β)

Both radii are 1. The dot product also equals cos α cos β + sin α sin β.

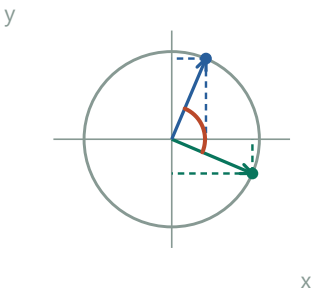


The angle between the radii

Both angles use the same horizontal reference. Sweeping from OB toward OA measures the gap between their directions.

4. Change the sign of the second angle

Replace β by −β, using even cosine and odd sine, to get cos(α+β)=cos α cos β−sin α sin β.



Reflect the β-ray

Its cosine stays fixed and its sine changes sign, producing the sum-angle identity.

1. cos(α+β)=cos(α−(−β))
Substitute −β for β in the difference identity.

2. =cos α cos β−sin α sin β
Cosine is even; sine changes sign at −β.

Check. At α=β the difference formula reduces to $\cos^2\alpha+\sin^2\alpha=1$.

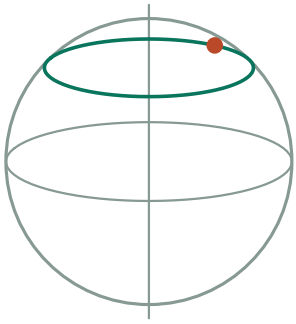
Exercise 316 · Exeter on the globe

Printed page 31 · Spherical coordinates / Coordinate geometry

An xyz-coordinate system is placed with its origin at the center of the earth, so that the equator is in the xy-plane, the North Pole has coordinates $(0, 0, 3960)$, and the xz-plane contains the prime meridian, which is the great semicircle that runs through Greenwich, England on its way from the North Pole to the South Pole. Recall that the y-axis is oriented so that it points east, into the Indian Ocean. Find the coordinates (x, y, z) of Exeter NH, whose latitude and longitude are 43 degrees north and 71 degrees west.

1. Set the sign conventions

Exeter lies north of the equator and west of Greenwich. Its height z is positive and its longitude angle is negative because the y-axis points east.

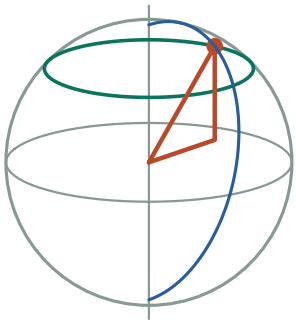


Locate Exeter

North latitude puts the point above the equatorial plane; west longitude makes y negative.

2. Compute height before longitude

The radius and its projection form a vertical right triangle. The opposite leg at latitude 43° is $z=3960\sin 43°$. The adjacent leg is the length to resolve in the xy-plane.

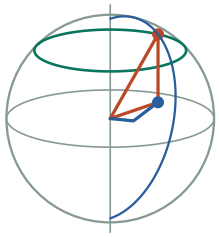


Project the 3,960-mile radius

The vertical component is R sin 43°; the equatorial projection is R cos 43°.

3. Resolve the projected radius

The horizontal radius is $3960\cos 43°$. Resolve it with longitude $-71°$, giving $(3960\cos 43°\cos 71°, -3960\cos 43°\sin 71°, 3960\sin 43°)$ miles.



Split the horizontal radius

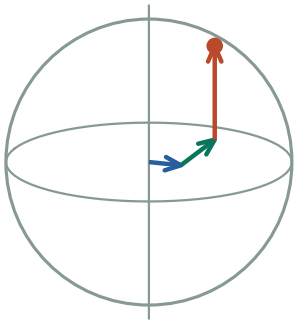
Resolve its projection at longitude $-71°$ into positive x and negative y.

1. $\lambda=-71° \Rightarrow \sin\lambda=-\sin 71°, \cos\lambda=\cos 71°$
East is positive so the west longitude has a negative sine.

2. $(x,y)=(\rho\cos 71°, -\rho\sin 71°)$
Multiply the equatorial unit direction by horizontal radius ρ.

4. Evaluate in miles

Evaluating the three components gives approximately $(943, -2738, 2701)$ miles.



Read three signed components

From the center the displacement is about $(943, -2738, 2701)$ miles.

Check. West longitude gives negative y, north latitude gives positive z, and the norm is 3960.

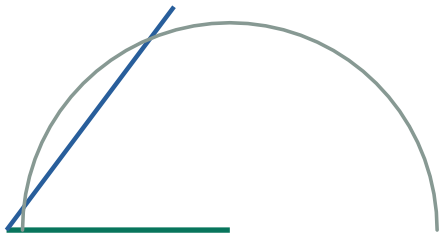
Exercise 317 · Two triangles from SSA

Printed page 31 · The ambiguous SSA case / The sine rule

There is more than one triangle PQR that can be described using the data $p = 13$, $r = 14$, $\sin P = 4/5$. For each triangle, find q (the length of the third side), the sizes of the angles, and make a sketch.

1. Why two triangles can appear

Start with the 14-unit base and a ray leaving P. The opposite side forces the remaining vertex onto a circle of radius 13 about the other base endpoint. A circle can cut a ray twice.

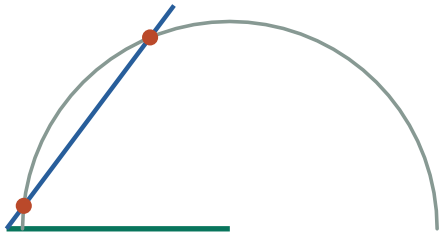


Construct the prescribed ray and circle

Place P at the origin and the other known endpoint 14 units to its right. The remaining opposite side must reach a radius-13 circle.

2. Recover the possible angles

Since r>p, P must be acute, so P≈53.130°. Law of sines gives sin R=56/65. Thus R≈59.490° or 120.510°, with Q≈67.380° or 6.360°, respectively.

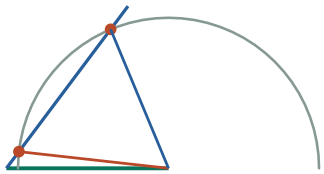


Find two intersections on that ray

The ray with sin P=4/5 cuts the circle at two positive distances, creating two possible shapes.

3. Solve for the paired side lengths

The cosine rule $p^2=q^2+r^2-2qr\cos P$ gives $q^2-16.8q+27=0$; $q=15$ (larger Q) or 1.8.



Trace the two triangles

Both have the same given opposite side 13 and base-side length 14, but different remaining sides.

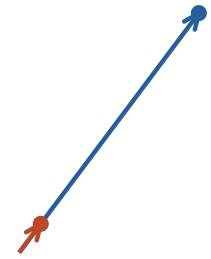
1. $169=q^2+196-28q(3/5)$
Positive $\cos P=\sqrt{(1-16/25)}=3/5$, since P must be acute.

2. $0=q^2-16.8q+27=(q-15)(q-1.8)$
Subtract 169 and factor the quadratic.

3. $q=15$ or $q=1.8$
One factor must vanish; both positive lengths fit their respective sketches.

4. Locate both sketches

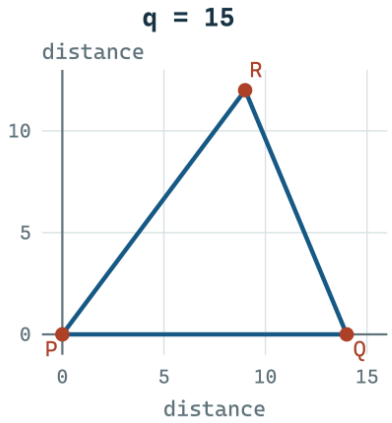
To sketch, draw a 14-unit base from P, emit the ray at angle P, and intersect it twice with the radius-13 circle about the base endpoint.

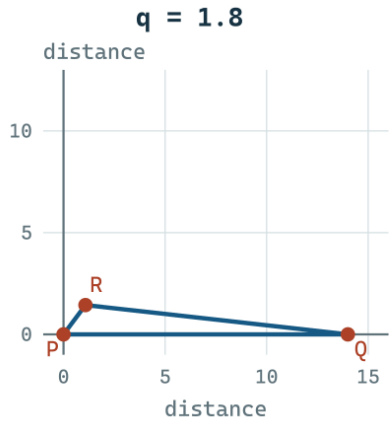


1.8 and 15 units from P

Read the two lengths along the ray

The near and far vertices are 1.8 and 15 units from P, respectively.





Check. The roots sum to 16.8 and multiply to 27. Both triangles have positive angles summing to 180°.

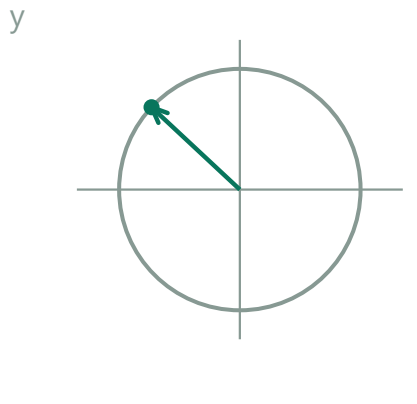
Exercise 318 · Second-quadrant parameter

Printed page 31 · Trigonometric ratios / Trigonometric identities

Given that $\sin\theta = k$, and that $90 < \theta < 180$, find expressions for $\cos\theta$ and $\tan\theta$.

1. Give k a physical meaning

On the unit circle, k is the positive height in quadrant II. Let x denote the horizontal coordinate of the same point. Then $x^2+k^2=1$.

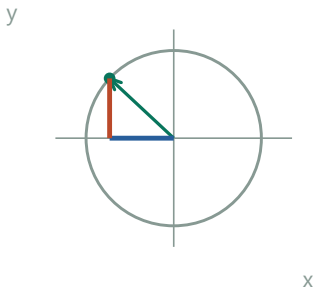


A height k in quadrant II

The point lies above the horizontal axis and to the left of the origin.

2. Select the correct square root

Solving for x^2 gives $1-k^2$. There are generally two possible signs for x, but only the negative root puts the point left of the vertical axis.



Unit radius

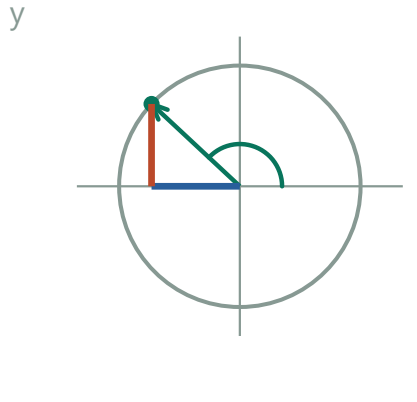
Pythagoras gives horizontal magnitude $\sqrt{(1-k^2)}$. Its coordinate has a minus sign.

1. $\cos^2\theta=1-\sin^2\theta=1-k^2$
Subtract the squared sine from both sides of the unit-circle identity.

2. $\cos\theta=-\sqrt{(1-k^2)}$
Quadrant II requires negative cosine. Strictly inside it, $0<k<1$.

3. Write the two ratios

Since cosine is negative, $\cos\theta=-\sqrt{(1-k^2)}$. Divide sine by this to get $\tan\theta=-k/\sqrt{(1-k^2)}$, with $0<k<1$.



Use the signed adjacent leg

Cos θ = $-\sqrt{(1-k^2)}$. Dividing the height k by it gives tangent.

Check. The signs follow from $x<0, y>0$ on the unit circle.

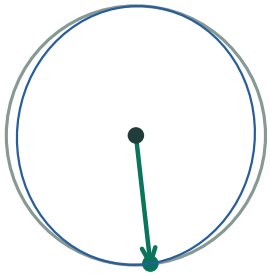
Exercise 319 · Other great-circle intersection

Printed page 32 · Great-circle distances / Sphere geometry

On the sphere $x^2+y^2+z^2 = 13^2$, there are many great circles that intersect at $(3, 4, -12)$. Find coordinates for the *other* point where these circles all intersect.

1. Use the defining plane

A great circle is cut by a plane through the sphere center. Every such plane through P contains the entire line through O and P.

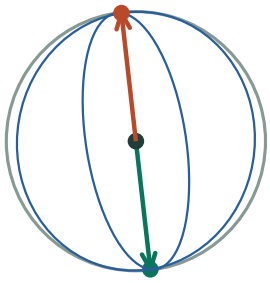


Choose a great-circle plane

Each such plane contains both the origin and the supplied point (3,4,−12).

2. Look on the opposite side of the center

That line meets the sphere twice, at equal distances from the origin. Reversing the radius flips all three coordinate signs while preserving its length.

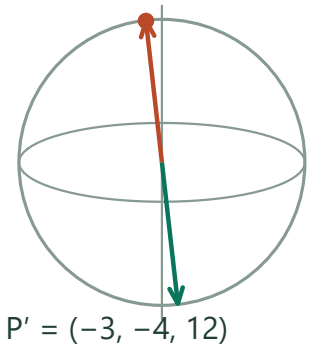


Follow the same diameter past O

Every possible great circle through that point contains its antipode.

3. Identify the shared antipode

A great-circle plane contains both the origin and the given point. Its opposite radius reaches (−3,−4,12), the other common point.



Mark the second common point

Negating all center-based coordinates gives (−3,−4,12).

Check. The new coordinates have squared norm 9+16+144=169 and sum with the original vector to zero.

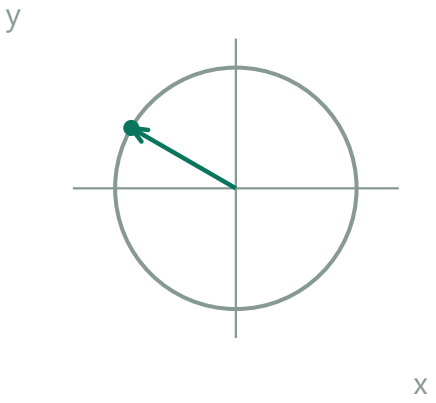
Exercise 320 · Polar to Cartesian

Printed page 32 · Polar coordinates / Trigonometric ratios

Convert the polar pair $(r = 8; \theta = 150)$ to an equivalent Cartesian pair (x, y).

1. Project the polar radius

The radius 8 is the hypotenuse. The horizontal and vertical signed legs are obtained by multiplying it by the cosine and sine of the direction angle.

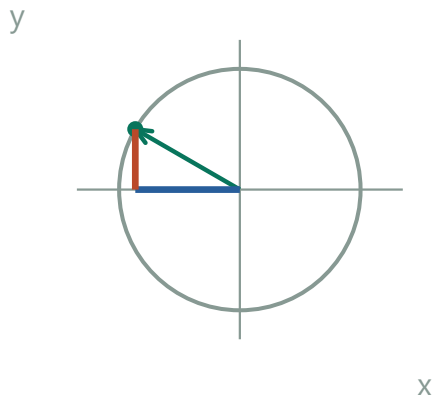


Radius 8 at 150°

Start on the positive x-axis and turn counterclockwise into quadrant II.

2. Use the 30° reference triangle

The radius at 150° points left and up, 30° above the negative horizontal axis. The adjacent magnitude uses √3/2 and the opposite magnitude uses 1/2.

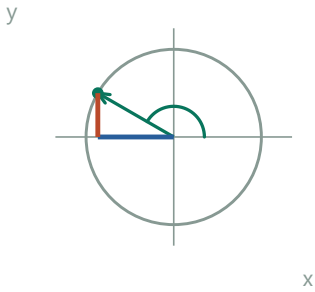


A 30° reference triangle

The leftward length is 8 cos 30°.
The rise is 8 sin 30°.

3. Convert to Cartesian coordinates

Compute x=8 cos 150°=−4√3, y=8 sin 150°=4. The point is (−4√3,4).



Cartesian pair (−4√3, 4)

Moving left and then upward reaches the same point as the polar radius.

1. cos 150°=−√3/2; sin 150°=1/2
Use the 30° reference values and quadrant-II signs.

2. (x,y)=(−8√3/2,8/2)=(−4√3,4)
Scale both unit-circle coordinates by 8.

Check. Its distance from the origin is √(48+16)=8 and it lies in quadrant II.

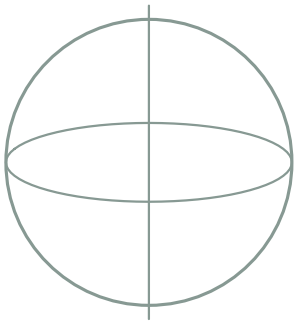
Exercise 321 · Area between meridians

Printed page 32 · Spherical coordinates / Surface area

To the nearest square mile, what is the surface area of the earth? How much area is found between the meridians 40 degrees west and 75 degrees west?

1. Choose the relevant surface

The boundary of the Earth is modeled by a sphere of radius 3960 miles. Use radius squared in the area formula so that the result is in square miles.

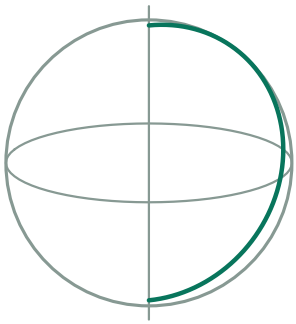


Use the spherical surface

Latitude bands cover a two-dimensional surface, measured in square miles.

2. Calculate the entire surface

The total area $4\pi(3960^2)$ rounds to 197,060,797 mi^2.

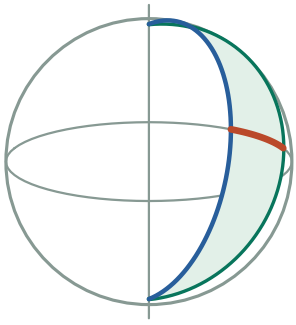


Start with the entire sphere

Its full area is $4\pi \cdot 3960^2$ square miles.

3. Measure the wedge

Both meridians lie west of the prime meridian, so their smaller angular separation is the difference of the two west-longitude numbers. Equal-angle wedges can be rotated onto one another.

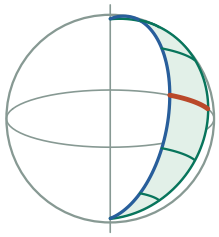


Select the longitude wedge

The meridians at 75° W and 40° W enclose a 35° angle at every latitude circle.

4. Scale by the longitude fraction

The smaller strip spans 35° of longitude, so its area is 35/360 of the total, or 19,158,689 mi^2.



Take the same fraction at each latitude

35/360 of the full surface lies between these meridians, about 19,158,689 square miles.

1. Astrip/Aearth=35°/360°=7/72
Equal longitude widths enclose equal areas by rotation of the sphere.

2. Astrip=(7/72)·4π·3960² mi²
Multiply by the unrounded sphere area before rounding the answer.

Check. Rotating strips of equal longitude width around the axis tiles the whole globe.

Exercise 322 · Cancel equal bases

Printed page 32 · Exponent rules / Scientific notation

Faced with the problem of dividing 5^{24} by 5^8, Brook is having trouble deciding which of these four answers is correct: 5^{16}, 5^3, 1^{16}, or 1^3. Your assistance is needed. Once you have answered Brook's question, experiment with other examples of this type until you can formulate the *common-base principle for division* of exponential expressions. Then apply this principle to the following situations:

(a) As of 2021, Earth's human population is roughly 7.9×10^9, and its total land area excluding the polar caps is roughly 5×10^7 square miles. If the human population were distributed uniformly over all available land, approximately how many persons would be found per square mile?

(b) At the speed of light, which is 3×10^8 meters per second, how many seconds does it take for the sun's light to travel the 1.5×10^{11} meters to Earth?

1. Cancel like factors

Cancel eight factors to get 5^{16}. Write the numerator as 24 factors of 5 and the denominator as eight such factors. Pair them one for one; sixteen unpaired factors remain in the numerator.



8 cancelled

Match factors in the quotient

Cross out eight of the numerator's 24 fives with the eight denominator factors.

2. Interpret the quotient rule

Generally $b^m/b^n=b^{m-n}$ for $b\neq0$. Subtracting exponents counts the remaining factors after cancellation. Division requires a nonzero denominator, and the common base must support the powers used.



24 − 8 = 16

Count what survives

Sixteen numerator factors remain. Cancelling n factors from m explains the difference m−n for nonzero base.

3. Compute population per unit land area

Density is $(7.9\times10^9)/(5\times10^7)=158$ persons/mi^2. The coefficient ratio is $7.9/5=1.58$ and the power-of-ten ratio is 10^2. Their product is the estimated number of people in each square mile.

$$\frac{7.9\times10^9 \text{ people}}{5\times10^7 \text{ mi}^2}$$

exponent: 9 − 7

Divide the population by land area

The bars compare the given coefficients after factoring the separate powers of ten. The quotient is 158 persons per square mile.

1. $7.9\times10^9/(5\times10^7)=(7.9/5)\times10^2$
Divide the coefficients and subtract 7 from 9.

2. $1.58\times100=158$ people/mi^2
The power of ten restores the scale of the density.

4. Convert distance to elapsed time

Sunlight takes $(1.5\times10^{11})/(3\times10^8)=500$ s, or 8 min 20 s.



Divide distance by speed

The time is $5\times10^2=500$ seconds. Its common-log position is between 2 and 3, a little under three decades.

1. t=(1.5/3)×10^(11−8) s=0.5×1000 s
Divide the distance by speed; meters cancel.

2. 500=8·60+20
Eight full minutes leave 20 seconds.

5. Test division in both directions

Other cancellations give $2^5/2^2=2^3$ and $3^2/3^4=1/3^2$. The latter illustrates why the difference can be negative.



Allow a remainder below the bar

In the four denominator threes, cancel the two supplied by the numerator. Two denominator threes remain.

Check. Multiplying the two quotients by land area and speed respectively recovers the original quantities.

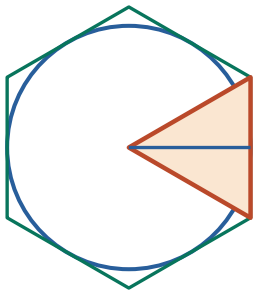
Exercise 323 · Disk area from its boundary

Printed page 32 · Circle geometry / Triangle area / Limits and convergence

The area of a circle of radius r is πr^2, which happens to be $\frac{1}{2}\,r$ times the circumference $2\pi r$. Explain why this relationship should be expected. One way is to apply your knowledge of circular sectors. Another way is to consider a billion-sided regular polygon that is circumscribed around a circle of radius r; how are its area and perimeter related?

1. Draw one triangular piece

Choose one side of the circumscribed polygon and connect its endpoints to the circle center. The radius to the contact point is perpendicular to the side, so it is the triangle's altitude.

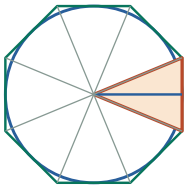


Isolate one triangle

The radius to the point of tangency is perpendicular to the polygon side, giving altitude r.

2. Add up the triangles

Split a circumscribed regular polygon into triangles with apex at the center. Each triangle has height r, and the bases sum to the perimeter. The area equals rP/2.



Add the side-based triangles

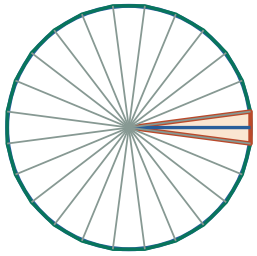
The bases sum to the polygon perimeter P. Combining the triangle areas gives rP/2.

1. $A = \tfrac{1}{2}rb_1 + \tfrac{1}{2}rb_2 + \cdots + \tfrac{1}{2}rb_n$
Each triangular piece has the same perpendicular height r.

2. $A = \tfrac{1}{2}r(b_1 + \cdots + b_n) = \tfrac{1}{2}rP$
Factor out the height; the bases make the perimeter.

3. Pass from polygon to circle

Increasing the number of sides takes the polygon to the circle.



Increase the number of sides

The circumscribed perimeter approaches 2πr; the polygon area approaches the disk area πr^2.

Check. Substituting $C = 2\pi r$ recovers πr^2.

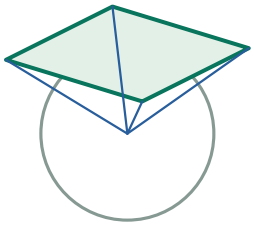
Exercise 324 · Sphere volume from surface

Printed page 32 · Volume / Surface area / Limits and convergence

(Continuation) The volume enclosed by a sphere of radius r is $\frac{4}{3}\pi r^3$. The surface area of the same sphere is $4\pi r^2$. You may already have noticed that the volume is exactly $\frac{1}{3}r$ times the surface area. Explain why this relationship should be expected. One way is to consider a billion-faceted polyhedron that is circumscribed about a sphere of radius r; how are its volume and surface area related?

1. Extend the flat argument into space

Take one flat face of a solid circumscribed about the sphere. Join all its boundary points to the sphere center, producing a pyramid whose base is that face.

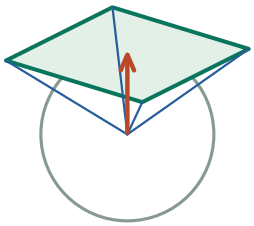


Join a face to the sphere center

A face of a circumscribed cube is the base of one pyramid. Its tip is the sphere center.

2. Recognize the common altitude

The radius drawn to the tangent face meets its plane perpendicularly. All the pyramids have this same height, even if their bases have different shapes.

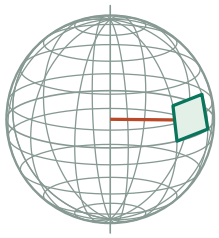


Drop the radius to the face

The face is tangent to the sphere, so the perpendicular altitude is exactly r.

3. Add the volumes and refine the surface

Join the center of a circumscribed polyhedron to its faces. Each resulting pyramid has altitude r and volume equal to a third of its base area times r. Sum the pyramids and take increasingly fine faces to reach the sphere.



Accumulate all face pyramids

Using many small tangent faces makes their total base area approach the sphere surface, while the common height stays r.

1. $V_{poly} = \tfrac{1}{3}rA_1 + \cdots + \tfrac{1}{3}rA_n = (r/3)\Sigma A_i$
The pyramids have disjoint interiors and share altitude r.

2. $\Sigma A_i \to S \Rightarrow V_{poly} \to rS/3$
Refine the circumscribed surface with radius fixed.

Check. Putting $S = 4\pi r^2$ into the relation gives $4\pi r^3/3$.

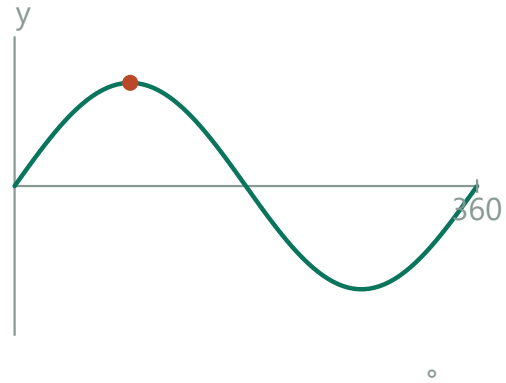
Exercise 325 · Compress a sine wave

Printed page 32 · Sinusoidal functions / Amplitude and period

With a graphing tool, graph each pair of functions on separate coordinate axes:
(a) $y = \sin x$ and $y = \sin 2x$ **(b)** $y = \sin x$ and $y = \sin 3x$ **(c)** $y = \sin x$ and $y = \sin 0.5x$
What do the graphs of $y = \sin mx$ and $y = \sin x$ have in common, and how do they differ?
Use what you have learned to graph $y = \sin 4x$ by hand.

1. Watch the angle inside the function

The sine of mx completes one cycle when its input mx has advanced 360°. Thus x needs to advance only 360°/m for positive m.

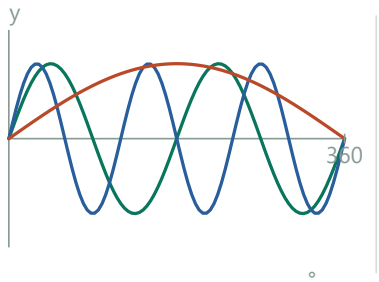


Measure a full baseline cycle

Follow the original curve until both the height and the direction of motion repeat.

2. Find amplitudes and periods

The range stays [−1,1]. Each positive m divides horizontal distances by m, so the periods are 180°,120°,720°,90°, respectively.



Change the input speed

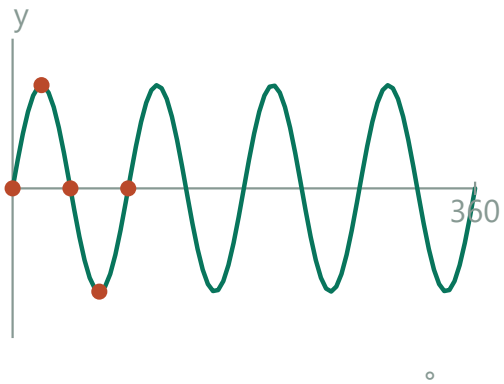
The periods for multipliers 2, 3 and ½ are 180°, 120° and 720°. The vertical range stays fixed.

1. m(x+T)−mx=mT
Determine how much the inside angle advances in one period.

2. abs(m)T=360° ⇒ T=360°/abs(m)
Use a full turn in either direction; this requires m≠0.

3. Construct the curve by quarters

Sketch one period using heights 0,1,0,−1,0 at successive quarter-periods, then repeat; base sine has period 360°.

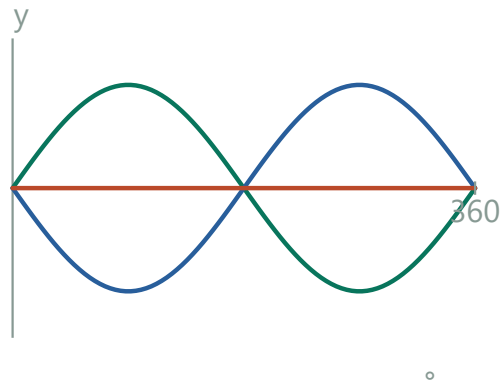


Construct one compressed cycle

The 4x curve completes a cycle in 90°. Mark its quarter-period landmarks.

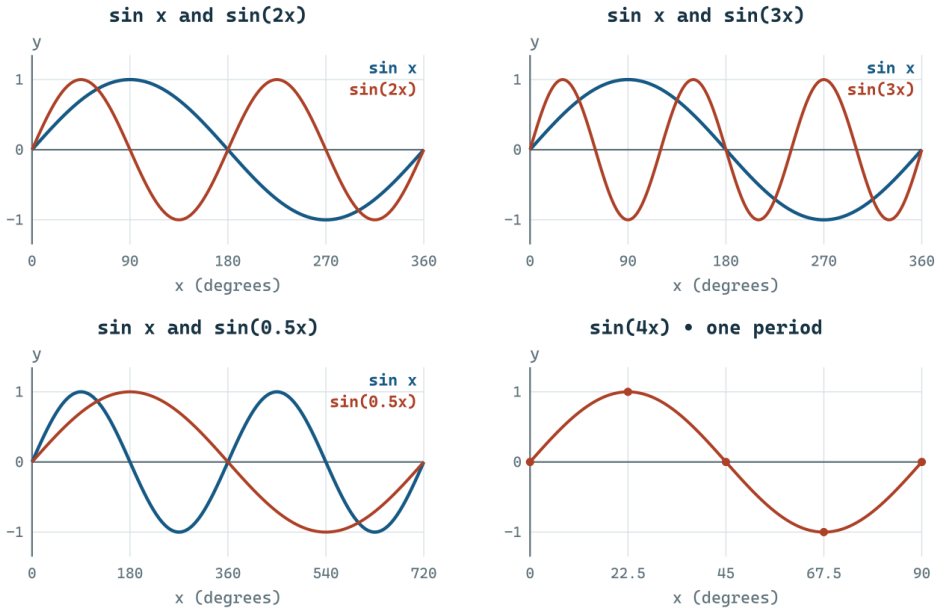
4. Handle zero and negative multipliers

For negative m, the period is 360°/|m| and the range is still [−1,1]. Relative to |m| the signs of all heights reverse because sine is odd. At m=0 the function is identically zero and has no least positive period.



Inspect the sign and zero cases

Negative m reverses sine heights. At m=0 there is only the horizontal zero line.



Check. For m=4 the first peak occurs at 22.5°, giving sin(4·22.5°)=1.

Exercise 326 · Recover ticket prices

Printed page 32 · Systems of equations / Matrix multiplication

Tickets for a concert were sold in three categories: adult, child, and senior citizen. For each type, the number of tickets sold for the three performances is shown in the matrix. The box office receipts were \$2715 for Friday, \$2613 for Saturday, and \$2412 for Sunday. Find the cost of each type of ticket.

$$\begin{array}{c} \\ Fri \\ Sat \\ Sun \end{array} \begin{array}{c} \begin{array}{ccc} Child & Adult & Senior \end{array} \\ \begin{bmatrix} 35 & 120 & 15 \\ 22 & 124 & 12 \\ 58 & 96 & 6 \end{bmatrix} \end{array}$$

1. Translate each receipt row

Let c, a, s be the child, adult, and senior ticket prices. The three performances give 35c+120a+15s=2715, 22c+124a+12s=2613, and 58c+96a+6s=2412.

c	a	s	receipts
35	120	15	2715
22	124	12	2613
58	96	6	2412

Read each performance as one row

Each row lists child, adult and senior ticket counts, followed by that performance's total receipts.

$$\begin{array}{c} \\ Fri \\ Sat \\ Sun \end{array} \begin{array}{c} \begin{array}{ccc} Child & Adult & Senior \end{array} \\ \begin{bmatrix} 35 & 120 & 15 \\ 22 & 124 & 12 \\ 58 & 96 & 6 \end{bmatrix} \end{array}$$

The Friday performance row

Read across Friday to see how many tickets of each price type contribute to that evening's receipts.

2. Eliminate the senior price

Choose multiples whose senior-ticket coefficients match before subtraction. The first and third rows have coefficients 15 and 6, which can both be scaled to 30.

c	a	s	receipts
70	240	30	5430
290	480	30	12060

Match the senior coefficients

Twice the first row and five times the third both use 30 seniors, so subtraction removes that price.

1. $2E_1 - 5E_3$:
$(70-290)c+(240-480)a=5430-12060$
The senior coefficients both become 30, so they cancel.

2. $-220c-240a=-6630 \Rightarrow 22c+24a=663$
Divide both sides by -10.

$$\begin{array}{c} \\ Fri \\ Sat \\ Sun \end{array} \begin{array}{c} \begin{array}{ccc} Child & Adult & Senior \end{array} \\ \begin{bmatrix} 35 & 120 & 15 \\ 22 & 124 & 12 \\ 58 & 96 & 6 \end{bmatrix} \end{array}$$

Senior column

The vertical column groups senior-ticket counts by night. Compare its coefficients to choose multipliers that eliminate one ticket price.

3. Reduce to two unknowns

Double the first equation, then subtract five times the third. After division by -10 this gives $22c+24a=663$. Subtract twice the third equation from the second and reverse the signs to get $94c+68a=2211$.

c	a	s	receipts
22	24	0	663
94	68	0	2211

Retain the two reduced equations

Both rows now involve only the child and adult prices. The zero column records the eliminated senior term.

4. Solve and back-substitute

Multiply those two reduced equations by 47 and 11 respectively and subtract. This gives $380a=6840$, so $a=\$18$. Back-substitution gives $c=\$10.50$, then $s=\$12.50$.

c	a	s	receipts
1034	1128	0	31161
1034	748	0	24321
0	380	0	6840

Eliminate the child price

Scale the reduced rows by 47 and 11. Their difference isolates 380 adult tickets for \$6840, or \$18 each.

1. $47(22c+24a)-11(94c+68a)=47\cdot663-11\cdot2211$
The two scaled child coefficients both equal 1034.

2. $380a=6840 \Rightarrow a=18$; $22c=663-432=231$
Divide to find a, then substitute in the first reduced equation.

3. $s=(2715-35\cdot10.5-120\cdot18)/15=12.5$
Divide the remaining receipt total by the number of senior tickets.

Check. Substitution gives \$2715, \$2613, and \$2412 in order.

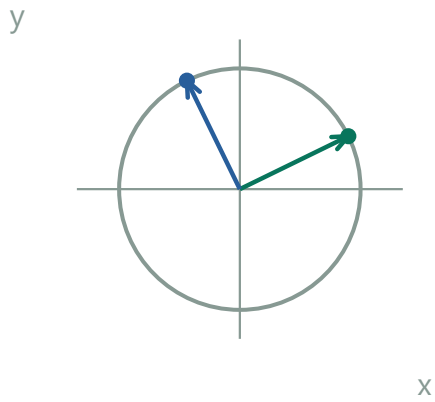
Exercise 327 · A perpendicular dot product

Printed page 32 · The dot product / Orthogonality

Explain why the value of $[\cos\theta, \sin\theta] \bullet [\cos(90+\theta), \sin(90+\theta)]$ is independent of θ.

1. Recognize radius vectors

The pair (cos θ,sin θ) ends on the unit circle in direction θ. The second pair does the same thing for an angle a quarter-turn farther around.

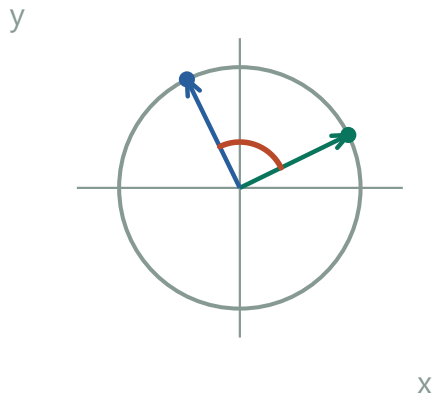


Two unit directions

The second angle includes an extra 90°.

2. Keep track of relative angle

Rotating both radii together leaves their lengths and their included angle unchanged. The dot product depends only on these three quantities.

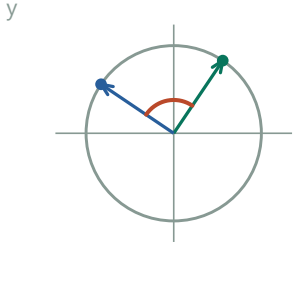


A fixed right angle

Rotating the two rays together preserves their separation.

3. Evaluate the scalar

These directions are perpendicular, so the dot product is 1·1·cos 90°=0 for every θ.



Zero projection

The dot product is 1·1·cos 90° = 0 for every common starting turn.

1. (cos θ,sin θ)·(−sin θ,cos θ)
The quarter turn maps (x,y) to (−y,x).

2. =−cos θ sin θ+sin θ cos θ=0
The products have equal magnitudes and opposite signs.

Check. Their coordinates are (c,s) and (−s,c); −cs+sc cancels.

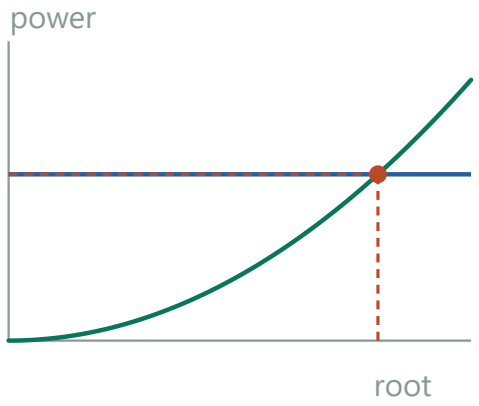
Exercise 328 · Fractional powers as roots

Printed page 33 · Exponent rules / Radicals and rational powers

The common-base principle for multiplication predicts that $5^{1/2}$ times $5^{1/2}$ should be 5. Explain this logic, then conclude that $5^{1/2}$ is just another name for a familiar number. Use a calculator to check your prediction. How would you describe the number $6^{1/3}$, given that $6^{1/3} \cdot 6^{1/3} \cdot 6^{1/3}$ equals 6? Formulate a general meaning of expressions like $b^{1/n}$, and use a calculator to test your interpretation on simple examples like $8^{1/3}$ and $32^{1/5}$.

1. Interpret two half powers

Adding two half exponents gives 1, so 5^(1/2) is the positive square root of 5, about 2.236.

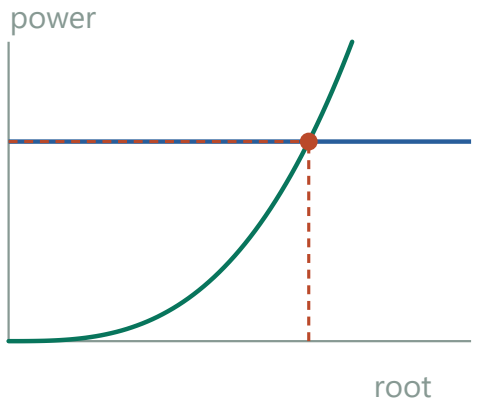


Read a square root from a power curve

On the nonnegative branch, y=5 meets $y=x^2$ at $x=\sqrt{5}$.

2. Interpret three third powers

Three factors 6^(1/3) give 6, hence the cube root, about 1.817.

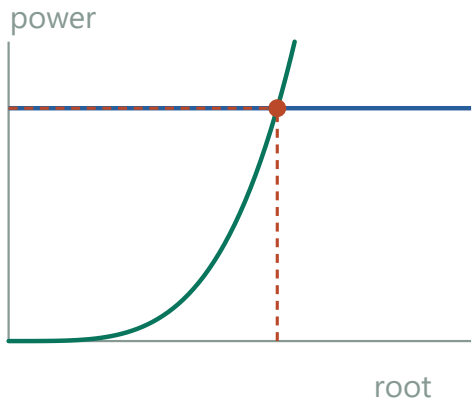


Use the cube instead

The level y=6 crosses $y=x^3$ at $x\approx1.817$, the cube root of 6.

3. Generalize the equal-factor argument

Positive b raised to 1/n is its positive nth root for positive integer n.

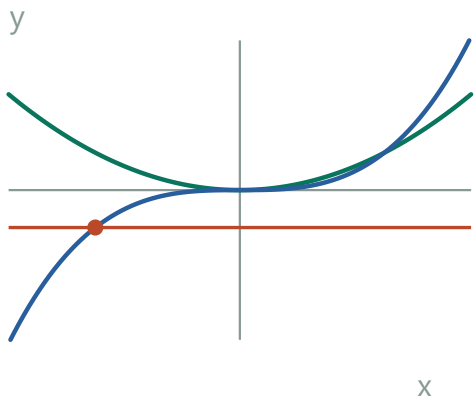


Generalize the root index

For an illustrative n=4 and b=7, the positive crossing is 7^(1/4). An nth root reverses an nth power.

4. Distinguish real roots by sign

For b=0 the real nth root is 0. With b<0 a real nth root exists for odd n and is negative. For even n no real number has a negative nth power.



Try a negative level

At y=−1 the blue cube has a crossing but the green square has none. Odd and even powers admit different real targets.

Check. 8^(1/3)=2 and 32^(1/5)=2 because $2^3=8$ and $2^5=32$.

Exercise 329 · Water in a cone

Printed page 33 · Area and volume scaling / Volume

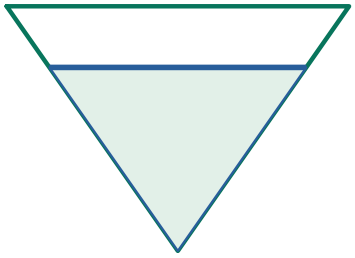
Water is being poured into a conical cup that is 12 cm tall.

(a) When the water in the cup is 9 cm deep, what percentage of the cup is filled?

(b) When the cup is 75 percent filled, how deep is the water?

1. Relate the water surface to the rim

A cross section through the cone axis shows two right triangles with the same angle at the tip. If the water depth is h, its surface radius is h/12 times the full radius.

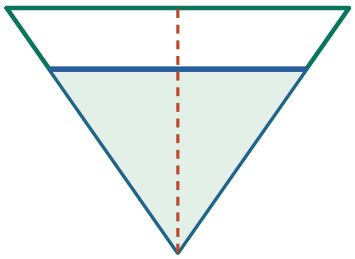


Match a smaller cone inside the cup

The disk width shrinks in the same ratio as the depth, measured from the tip.

2. Scale in three directions

In the volume formula $\pi r^2 h/3$, the radius contributes two length factors and the height one. All three shrink by the same factor for the smaller water cone.

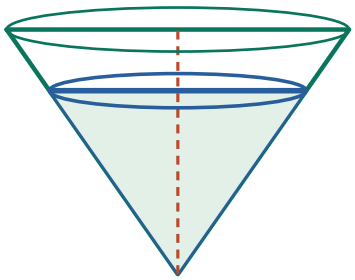


Depth fraction 9/12

Volume scales in three dimensions, so the liquid fraction is $(9/12)^3$.

3. Compute the filled fraction

With the tip down, similar water and cup cones have volume fraction $(h/12)^3$. At h=9 this is 27/64=42.1875%.

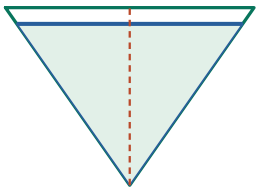


Compare 9 cm of water with the full cup

The marked water region occupies 27/64 of the full cone.

4. Solve the inverse question

At 75% the depth is 12·(3/4)^(1/3)≈10.90 cm. The target 3/4 refers to volume, so set $(h/12)^3=3/4$. A cube root undoes the three equal length factors.



Now prescribe the volume

A 75% volume fraction requires a height fraction $\sqrt[3]{0.75}$, about 10.90 cm in the 12 cm cup.

1. $(h/12)^3=3/4 \Rightarrow h/12=\sqrt[3]{(3/4)}$
Cube roots undo cubing and preserve the positive sign of depth.

2. $h=12\sqrt[3]{(3/4)}$ cm
Multiply by the full height to return from a ratio to centimeters.

Check. Cubing the recovered height ratio returns 0.75.

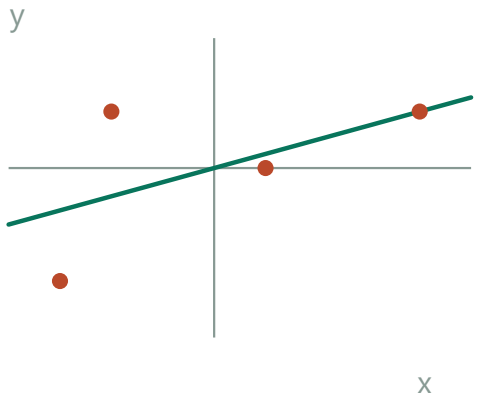
Exercise 330 · Fit a line through the mean

Printed page 33 · Linear regression / Least squares and residuals

Plot the four data points $(-3,-2)$, $(-2,1)$, $(1,0)$, and $(4,1)$.
(a) Verify that the centroid of the data is at the origin. Draw a line through the origin that looks like it does a good job of fitting this data. Let m be its slope.
(b) For each of the four points (x,y), the *residual* is $y-mx$, which depends on the variable slope m. For example, the residual for the first point is $-2-(-3m)$, or just $3m-2$. Calculate the other three residuals, then square each of the four residuals and simplify the *sum* of the four squares. The result should be a quadratic polynomial, in which m is the variable.
(c) The method of *least squares* seeks the m-value that *minimizes* this sum of squared residuals. Find this value of m. If you use a calculator, you may need to use the symbol x in place of m. Compare your m-value with the slope of the line you drew in part (a).
(d) Confirm your answer by using a calculator's least-squares capability.

1. Plot the four observations and locate their center

Put x horizontally and y vertically. Mark (−3,−2), (−2,1), (1,0), and (4,1). The mean x-coordinate is (−3−2+1+4)/4=0. The mean y-coordinate is (−2+1+0+1)/4=0. Their centroid is therefore (0,0). For part (a), draw a trial line through it. A slope near ¼ is a reasonable visual estimate.

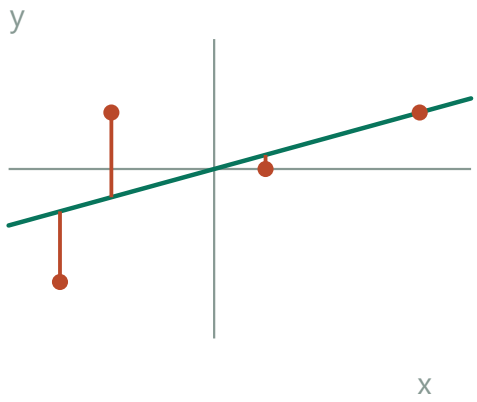


Center the four data points

Their coordinate averages both equal zero. A trial line can pivot through this centroid.

2. Read the signed errors vertically

For a trial line $y=mx$, the predicted height at $x=-3$ is $-3m$. Subtract it from the observed height -2 to get $3m-2$. The same observed-minus-predicted calculation gives $1-(-2m)=1+2m$ at $x=-2$, $0-m=-m$ at $x=1$, and $1-4m$ at $x=4$.

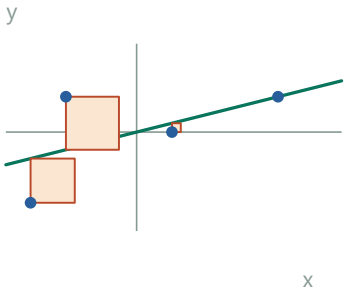


Measure four vertical residuals

The marked segments connect each observed height to its fitted height at the same x.

3. Combine the four squared errors

Square each residual so a point above the line cannot cancel the error of one below it. The four squares are $9m^2-12m+4$, $4m^2+4m+1$, m^2, and $16m^2-8m+1$. Collect like terms to obtain $S(m)=30m^2-16m+6$.



Square each residual

The total fits the upward quadratic shown in the next frame; upward and downward errors both count.

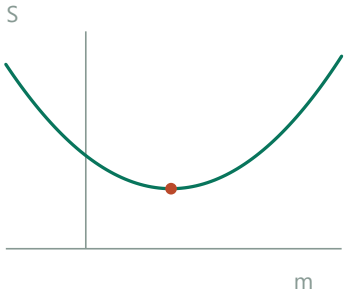
1. $E=(3m-2)^2+(1+2m)^2+m^2+(1-4m)^2$
Square each vertical residual so opposite errors cannot cancel.

2. $E=(9+4+1+16)m^2+(-12+4-8)m+(4+1+1)$
Expand each square and collect matching powers of m.

3. $E=30m^2-16m+6$
Add the coefficients of each power separately.

4. Find the least possible error

The coefficient 30 is positive, so the graph of S opens upward. Its vertex occurs at $m=-(-16)/(2\cdot30)=4/15\approx0.267$. The fitted line $y=(4/15)x$ is a little steeper than the trial slope ¼. A different hand-drawn estimate can be compared the same way.



Minimize the squared-error curve

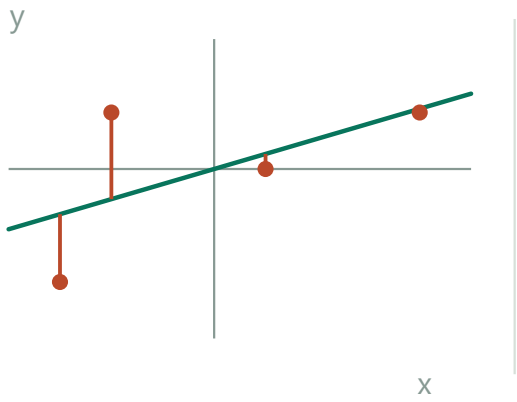
Its lowest point occurs at $m=4/15$. The vertical coordinate is the sum of four squares.

1. $E=30(m-4/15)^2+58/15$
Complete the square; $30\cdot(4/15)^2=32/15$.

2. $E\geq58/15$, equality at $m=4/15$
The squared term is nonnegative and vanishes only at the minimizing slope.

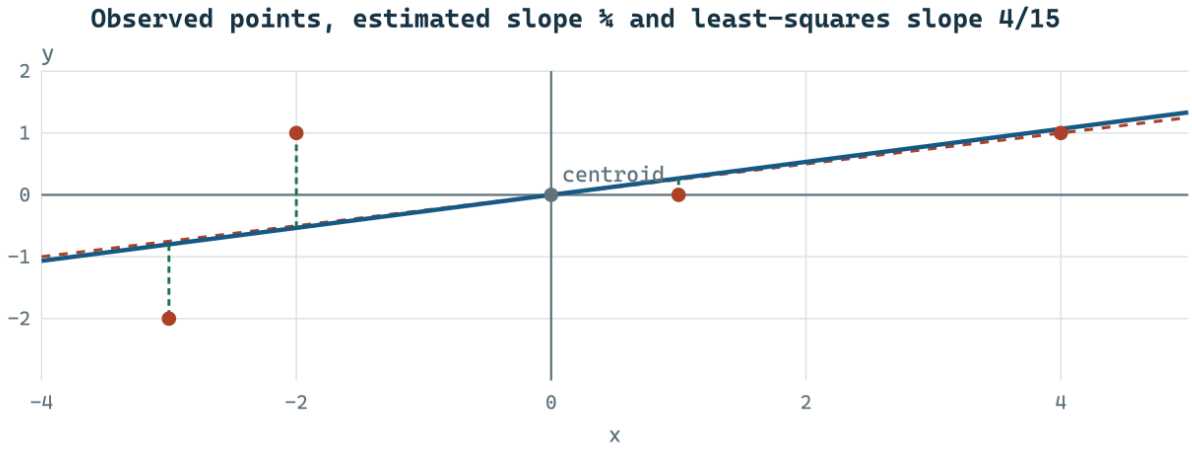
5. Confirm with a linear regression

Enter the four x-values in one calculator list and their y-values in a second list in the same order. Run a linear regression of y on x. It gives slope 0.2666… and intercept 0. The usual regression line passes through the centroid, so allowing an intercept here recovers the same line.



Least-squares line through the centroid

The green line has slope 4/15. Its residuals give the minimum total square.



Check. The regression formula $\Sigma xy/\Sigma x^2$ gives 8/30=4/15, with zero intercept because both means vanish.

Exercise 331 · Recognize a reflection

Printed page 33 · Matrix transformations / Reflections / Isometries

The matrix $\mathbf{M} = \begin{bmatrix} -3/5 & 4/5 \\ 4/5 & 3/5 \end{bmatrix}$ defines an isometry of the xy-plane.

(a) What special properties do the column vectors of this matrix have?

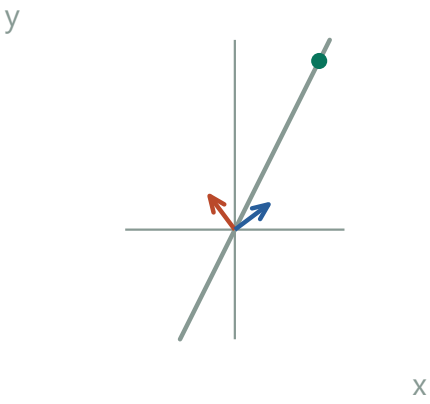
(b) Verify that the point $(2, 4)$ remains stationary when $\mathbf{M}$ is applied to it.

(c) What is the significance of the stationary point $(2, 4)$? What does it tell you about the possible isometries that $\mathbf{M}$ could be? Do other points invite examination?

(d) Show that $\mathbf{MM}$ is the 2×2 identity matrix. What does this suggest about the geometric transformation that $\mathbf{M}$ represents? Confirm your suspicions.

1. Inspect the transformed basis

The columns are perpendicular unit vectors. For the two columns, the squared lengths are (9+16)/25 and (16+9)/25. Their dot product is (−12+12)/25. This calculation shows how M treats the original unit basis.

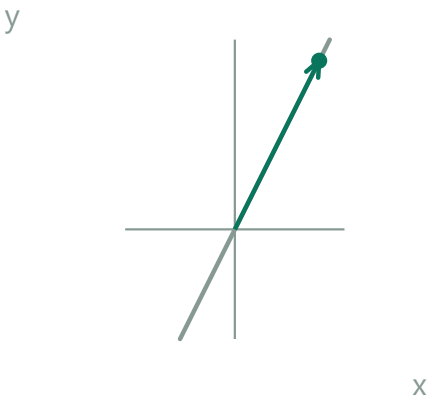


Examine the image basis

The image columns (−3/5,4/5) and (4/5,3/5) are perpendicular unit vectors.

2. Identify the fixed line

Multiplication fixes (2,4), hence the whole line through (1,2) and the origin. A nonidentity rotation could not fix that entire line.

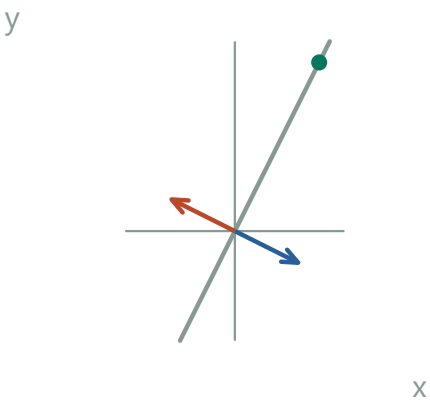


Mark a fixed point

The point (2,4) stays put, as does the entire line y=2x.

3. Interpret the matrix square

Squaring the matrix gives I, since both diagonal sums are 1 and the off-diagonal sums cancel. It is reflection in y=2x.

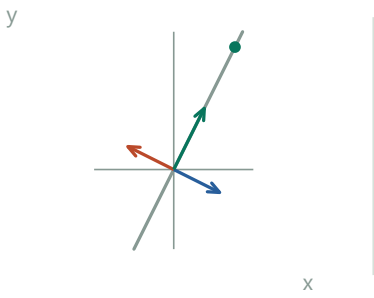


Test the effect twice

A vector perpendicular to the fixed line reverses direction under the transformation.

4. Prove the reflection using two directions

On the axis direction (1,2) the multiplier is +1. Perpendicular to it, (−2,1) goes to (2,−1), giving multiplier −1. Every vector is a sum of these two components, which is exactly how a reflection acts.



Compare fixed and negated directions

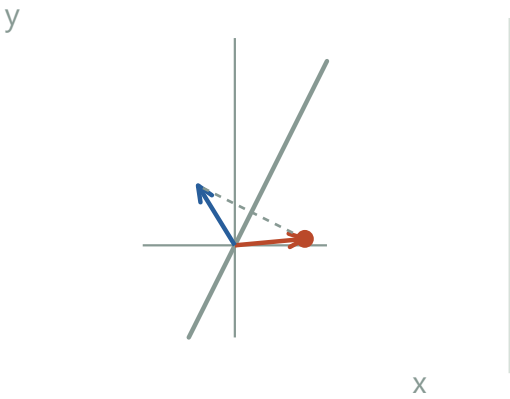
Parallel components remain fixed and perpendicular components change sign.

1. M(1,2)=((−3+8)/5,(4+6)/5)=(1,2)
The line direction stays fixed.

2. M(−2,1)=((6+4)/5,(−8+3)/5)=(2,−1)
The perpendicular direction reverses; together these directions span the plane.

5. Check by multiplying entries

For example, the image of (2,4) is ((−6+16)/5,(8+12)/5)=(2,4). In the square of the matrix the diagonal entries are (9+16)/25; off the diagonal the two terms are +12/25 and −12/25.



Return after a second reflection

Reversing the perpendicular component twice restores every original point, so $M^2=I$.

Check. The perpendicular vector (−2,1) maps to (2,−1), while the axis direction stays fixed.

Exercise 332 · Vertical cosine shifts

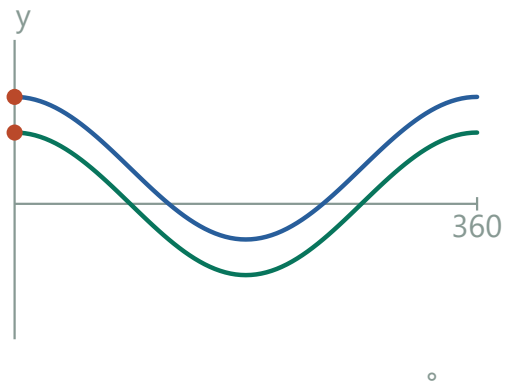
Printed page 33 · Sinusoidal functions / Graph transformations

Graph each of the following pairs of functions on a separate system of coordinate axes. What does each pair of graphs have in common? How do the graphs differ?

(a) $y = 2\cos x$ and $y = 1 + 2\cos x$ **(b)** $y = -3\cos x$ and $y = 1 - 3\cos x$

1. Follow one coordinate pair

At the same x-value the first curve has height y and the second has height y+1. The vertical separation stays one unit at every point of each pair.

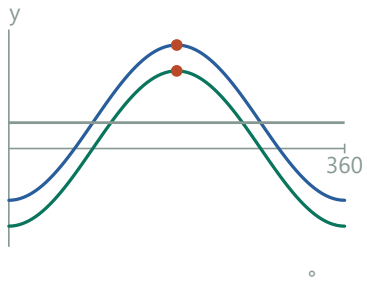


Lift a point one unit

The original crest at 2 rises to 3 while its x-coordinate is unchanged.

2. Read the shifted midline and crests

Both translated curves move up exactly one unit. The amplitudes remain 2 and 3, the periods 360°, and the midline moves from y=0 to y=1.



Follow the inverted cosine

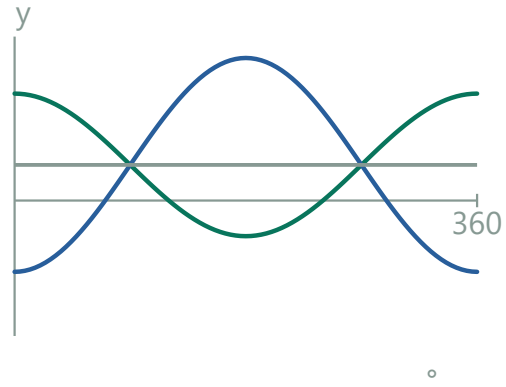
The original midline 0 becomes 1. Its crest rises from 3 to 4.

1. $-2 \le 2\cos x \le 2 \Rightarrow -1 \le 1+2\cos x \le 3$
Add one to all three terms of the range.

2. $-3 \le -3\cos x \le 3 \Rightarrow -2 \le 1-3\cos x \le 4$
The other curve shifts up by the same amount.

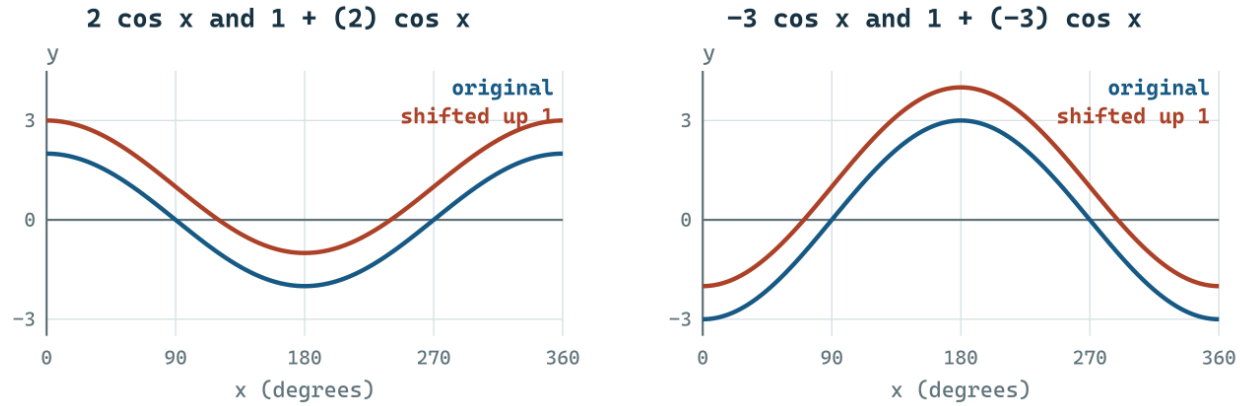
3. Construct each graph from landmarks

To sketch, use cosine values 1,0,−1,0,1 at quarter turns, then apply the given multiplier and offset.



Reconstruct from quarter turns

Both translated graphs oscillate around y=1; their amplitudes are 2 and 3.



Check. The translated ranges are [−1,3] and [−2,4], each with the same width as before.

Exercise 333 · Two points per trig value

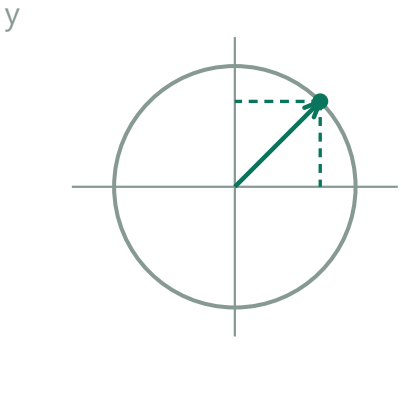
Printed page 33 · The unit circle / Symmetry

For each of the following, there are two points on the unit circle that fit the given description. Without finding θ, describe how the two points are related to each other.

(a) $\cos\theta = -0.4540$ **(b)** $\sin\theta = 0.6820$ **(c)** $\tan\theta = -1.280$

1. Convert a trigonometric condition into a coordinate

A point on the unit circle has coordinates (cos θ,sin θ). A fixed first coordinate describes a vertical line and a fixed second coordinate a horizontal line.

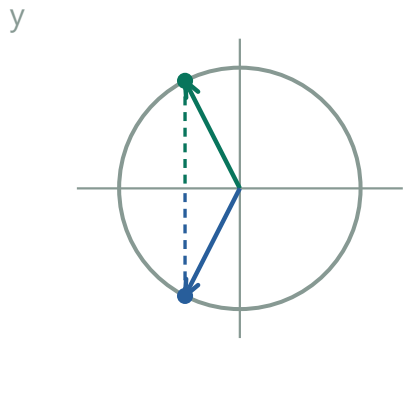


Read coordinates from a radius

Cosine fixes a horizontal coordinate, sine a height, and tangent a slope.

2. Locate the equal cosines

Fixed cosine gives equal x-coordinates: the two points reflect across the x-axis.

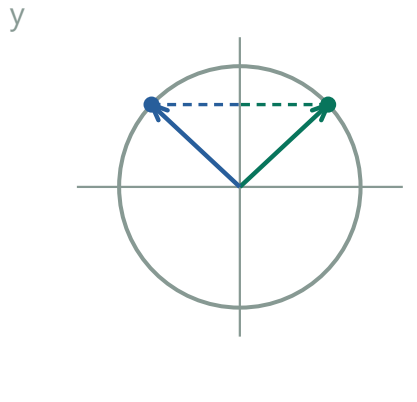


$\cos\theta = -0.4540$

A vertical line at that x-coordinate meets the circle at two reflected heights.

3. Locate the equal sines

Fixed sine gives equal y-coordinates: reflect across the y-axis.

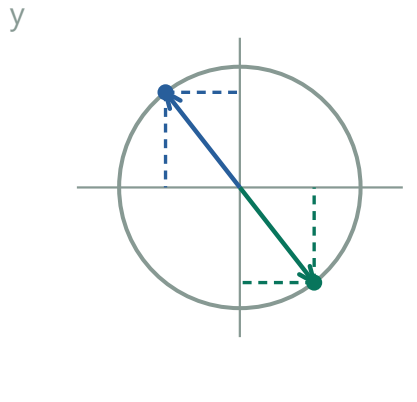


$\sin\theta = 0.6820$

The two tips share a horizontal line and reflect in the y-axis.

4. Locate the equal tangents

Fixed tangent gives equal slopes: the two points are opposite ends of a diameter.



$\tan\theta = -1.280$

Extending a line with that slope through the center locates two antipodal tips.

Check. Reflecting or negating coordinates preserves the specified x, y, or ratio y/x.

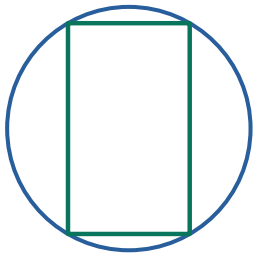
Exercise 334 · Cylinder in a sphere

Printed page 33 · Sphere geometry / The Pythagorean theorem

A cylinder of radius 4 and height h is inscribed in a sphere of radius 8. Find h.

1. Cut through the central axis

In this section the sphere appears as a radius-8 circle. The cylinder becomes a rectangle with two vertical sides; its corners are on that circle.

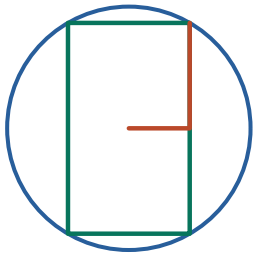


Slice through the cylinder axis

The sphere cuts to a circle and the inscribed cylinder to a rectangle.

2. Identify the half-dimensions

From the sphere center move horizontally to the cylinder side (distance 4) and vertically to its rim (distance h/2). The straight segment to that rim point is a sphere radius.



Read the two half-dimensions

The horizontal leg to the rim is 4. Height h is shared equally above and below the center.

3. Solve the right triangle

At a rim point the axial right triangle gives $4^2+(h/2)^2=8^2$. Hence $h=8\sqrt{3}$.



Use the rim-point radius

The 8-unit radius is the hypotenuse, giving $h=2\sqrt{(8^2-4^2)}=8\sqrt{3}$.

1. $(h/2)^2=64-16=48$
Subtract the squared horizontal radius.

2. $h/2=\sqrt{48}=4\sqrt{3} \Rightarrow h=8\sqrt{3}$
Use the positive root for a length and double the half-height.

Check. The section through the axis is an 8-by-$8\sqrt{3}$ rectangle with diagonal 16.

Exercise 335 · Sphere in a 5–12–13 cone

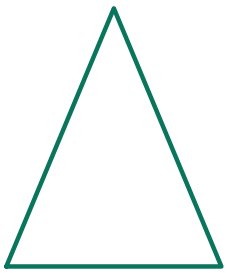
A cone with 10-inch diameter is 12 inches tall.

(a) Find the radius of the largest sphere that can be inscribed in the cone.

(b) The volume of this sphere is what percentage of the volume of the cone?

1. Construct the axial triangle

A vertical section through the axis gives base 10 and altitude 12. Each half-base is 5. The sloping side is $\sqrt{(5^2+12^2)}=13$.

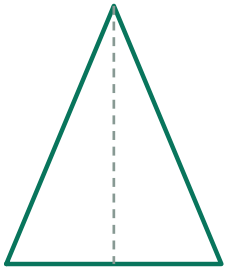


Diameter 10, height 12

A section through the axis gives the 10-inch base triangle.

2. Find the triangle measures

The axial triangle has sides 13,13,10, area 60 and semiperimeter 18.

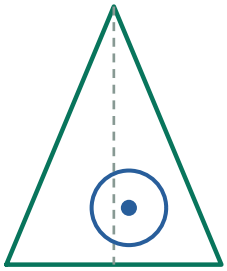


Calculate side and semiperimeter

The slant edge is $\sqrt{(5^2+12^2)}=13$. Area 60 and semiperimeter 18 constrain its incircle.

3. Permit any proposed sphere center

For any sphere inside the cone, take a plane through the cone axis and the sphere center. The section contains a disk of the sphere’s full radius r inside the axial triangle.

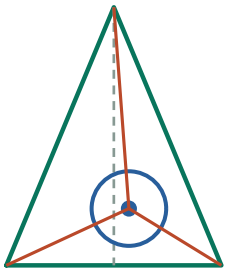


Try an off-center sphere

A full-radius disk is visible in the plane through the sphere center and the cone axis.

4. Compare area to disk radius

Join the disk center to the three triangle vertices. The three perpendicular distances to the sides are each at least r, so summing base times height gives triangle area $A \geq r \cdot s$, where s is its semiperimeter.

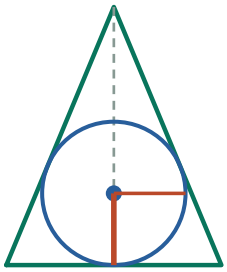


Bound every disk radius

Sum three triangle areas from the center. Each altitude is at least r, giving $60 \geq 18r$.

5. Establish that equality is possible

Thus $r \leq A/s$. The triangle’s incircle is centered on the cone axis and rotating it produces a sphere attaining this bound.

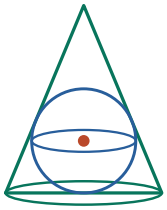


Place the tangent sphere

The incircle has radius 10/3. Rotation about the cone axis attains the three-dimensional bound.

6. Calculate the sphere and its volume share

The maximum radius is 60/18=10/3 inches. The volume fraction is $4(10/3)^3/(5^2\cdot 12)=40/81$, about 49.38%.



Compute the occupied fraction

The inscribed sphere occupies 40/81 of the cone's volume.

1. r=60/18=10/3
Tangency makes all three distances equal, attaining the area bound.

2. Vball/Vcone=4(1000/27)/(25·12)=4000/8100
Cancel π/3 and cube 10/3.

3. 4000/8100=40/81
Reduce both numerator and denominator by 100.

Check. The distance from (0,10/3) to the side line 12x+5y−60=0 is 10/3.

Exercise 336 · Fall time and height

Printed page 34 · Linear regression / Data linearization

The table at right shows how many seconds are needed for a stone to fall to earth from various heights (measured in meters). Make a scatter plot of this data. Explain how the data suggests that the underlying relationship is not linear.

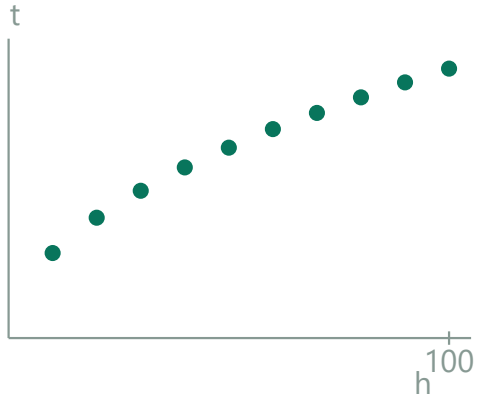
(a) Calculate the squares of the times and enter them in a third column. A scatter plot of the relation between the first and third columns does suggest a linear relationship. Use linear regression to find it, letting x stand for height and y stand for the square of the time.

(b) It is now easy to write a nonlinear relation between h and t by expressing t^2 in terms of h. Use this equation to *extrapolate* how long it will take for a stone to fall from a height of 300 meters. Is this a reasonable prediction?

height	*time*
10	1.42
20	2.01
30	2.46
40	2.85
50	3.18
60	3.49
70	3.76
80	4.02
90	4.27
100	4.50

1. Look at the first scatter plot

Equal 10-meter height increments give shrinking increases in time, so the original scatter bends.



Plot the recorded falls

A fixed 10 m increase in height gives smaller and smaller increases in seconds.

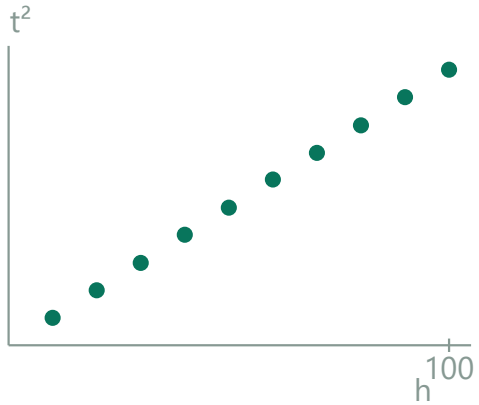
height	*time*
10	1.42
20	2.01
30	2.46
40	2.85
50	3.18
60	3.49
70	3.76
80	4.02
90	4.27
100	4.50

The height measurements

The left column lists drop heights in meters. Consecutive rows change this input by the same amount.

2. Square each measured time

Squared times in order are 2.0164,4.0401,6.0516,8.1225,10.1124,12.1801,14.1376,16.1604,18.2329,20.25 s^2.



Square the time in each row

The measured heights stay fixed. The vertical entries become 1.42^2, 2.01^2 and so on.

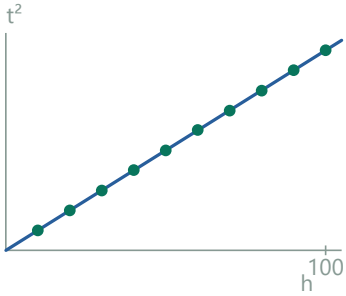
height	*time*
10	1.42
20	2.01
30	2.46
40	2.85
50	3.18
60	3.49
70	3.76
80	4.02
90	4.27
100	4.50

The measured times

Each right-hand measurement belongs to the height on the very same row. Keep this pairing when squaring the time.

3. Fit a line to the transformed data

Least squares gives $t^2 \approx 0.2024903h - 0.0065667$. With height as input and the squared times as response, linear regression chooses the slope and intercept that minimize the sum of squared vertical errors in this transformed plot.



Fit t^2 against height

The fitted slope is about 0.20249 s^2 per meter. The plotted data now follow a line.

1. x=h, y=t²; mean x=55, mean y=11.1304
Average the ten entries in each transformed column.

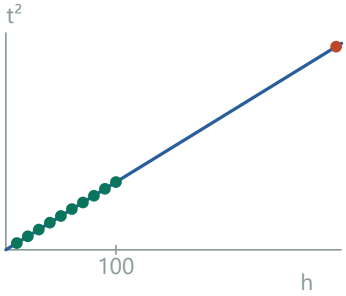
2. Σ(x−55)²=8250; Σ(x−55)(y−11.1304)=1670.545
Subtract the means row by row, then sum squared horizontal deviations and paired products.

3. m=1670.545/8250≈0.2024903
The least-squares slope is the paired-product sum divided by the horizontal sum of squares.

4. b=11.1304−55m≈−0.0065667
The fitted line passes through the two means. Keep the full slope until the final rounding.

4. Predict beyond the plotted heights

At 300 m take the positive square root, about 7.79 s. This extrapolation assumes comparable free-fall conditions; air resistance can affect a real drop.

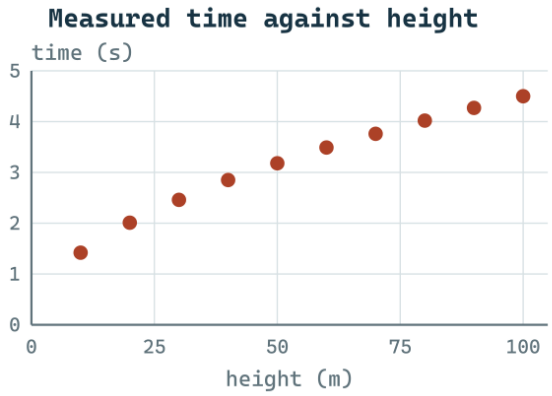


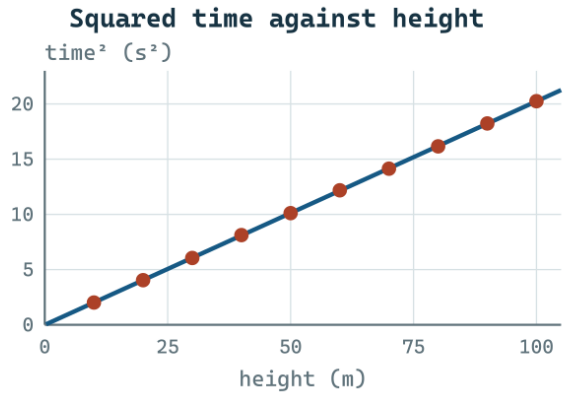
Extend the fitted trend

At 300 m the line predicts time squared. Taking the positive square root gives about 7.79 s.

1. $t^2 \approx 0.2024903 \cdot 300 - 0.0065667 \approx 60.74\ s^2$
The fitted line predicts squared time from height.

2. $t \approx \sqrt{60.74}\ s \approx 7.79\ s$
Time elapsed cannot be the negative square root.





Check. The fitted slope is near 2/9.8, the constant-gravity prediction.

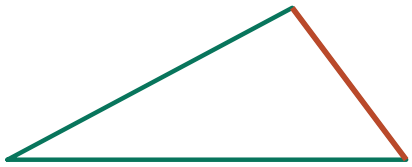
Exercise 337 · Find a circumdiameter

Printed page 34 · The sine rule / Incircles and circumcircles

The sides of a triangle are 10, 17, and 21 inches long. Find **(a)** the smallest angle of the triangle; **(b)** the diameter of the circumscribed circle.

1. Pair each angle with its opposite side

The smallest angle is opposite the shortest side. The two sides touching that angle have lengths 17 and 21, so use them in the cosine rule for the opposite length 10.

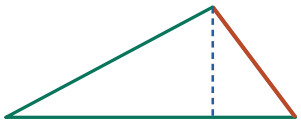


Locate the shortest side

The 10-unit side is opposite the angle formed by sides 17 and 21.

2. Work from cosine to sine

Once cos A is known, obtain sin A from $\sin^2 A+\cos^2 A=1$. An interior triangle angle has positive sine.



Read the remaining lengths

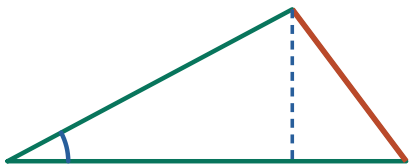
The altitude forms a right reference triangle with horizontal component 15 and rise 8.

1. $\cos A=(289+441-100)/714=630/714=15/17$
Use the cosine rule with A opposite side 10.

2. $\sin A=\sqrt{(1-225/289)}=\sqrt{(64/289)}=8/17$
Interior triangle angles have positive sine.

3. Find the angle

The smallest angle A faces 10. $\cos A=(17^2+21^2-10^2)/(2\cdot17\cdot21)=15/17$, so $A\approx28.07°$ and $\sin A=8/17$.

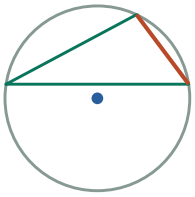


Determine the small angle

The marked angle has cosine 15/17 and sine 8/17.

4. Use the extended sine rule

The extended sine rule gives diameter 10/sin A=21.25 inches.



Pass the circle through all three vertices

The extended sine law gives diameter 10/(8/17)=21.25 inches.

1. 2R=10/(8/17)=10·17/8=21.25 in

The extended sine rule gives the diameter. Dividing by a fraction multiplies by its reciprocal.

Check. Heron's formula gives area 84 and diameter abc/(2 area)=21.25.

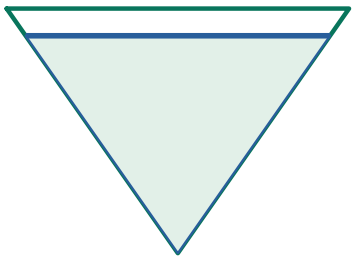
Exercise 338 · Invert a partly filled cone

Printed page 34 · Area and volume scaling / Volume

A clear plastic cone is 9 inches tall, with some liquid sealed inside. When the cone is held point down, the liquid is 8 inches deep. When the cone is inverted and held point up, the liquid is d inches deep. Find d, to the nearest hundredth of an inch.

1. Choose the similar cone in each orientation

With the tip down it is the water that forms a smaller cone. With the tip up it is the air above the water that forms a smaller cone. The height of that air cone is measured from the top tip.

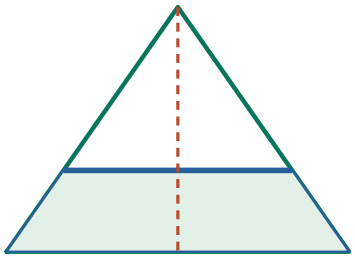


Start with a tip-down container

A depth of 8 in a height of 9 gives a filled volume fraction $8^3/9^3$.

2. Use complementary volume fractions

Water occupies $8^3/9^3$ of the volume. When inverted, the empty top cone has height $9-d$ and occupies $(9-d)^3/9^3$.

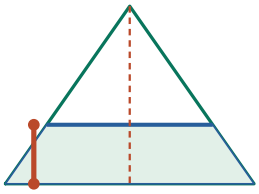


Turn the cone over

The empty volume above the water is now the similar tip-cone. Its fraction is 217/729.

3. Recover the bottom-up depth

Thus $(9-d)^3=729-512=217$ and $d\approx2.99$ inches. First take the cube root of 217 to find the air-cone height. Subtract that height from the nine-inch total to find the liquid depth measured from the broad bottom.



Measure from the bottom

Subtract the empty-cone height $\sqrt[3]{217}$ from the full 9 inches to obtain about 2.99 inches.

1. $(9-d)^3/729=1-512/729=217/729$
The empty part is the complement of the water fraction.

2. $9-d=\sqrt[3]{217} \Rightarrow d=9-\sqrt[3]{217}$
Multiply by 729 and take a real cube root before isolating depth.

Check. The empty and wet volume fractions sum to one.

Exercise 339 · Negative exponents

Printed page 34 · Exponent rules

The result of dividing 5^7 by 5^3 is 5^4. What is the result of dividing 5^3 by 5^7, however? By considering such examples, decide what it means to put a *negative* exponent on a base.

1. Reverse the division

In $5^3/5^7$ the denominator contains four factors of 5 left after matching the numerator. What remains sits below the fraction bar.



Pair off numerator and denominator

Seven denominator fives are shown. The three crossed-out factors cancel the numerator.

2. Compare the exponent rule

Subtracting exponents in the same calculation gives $3-7=-4$. For the rule to stay consistent, the negative power has to denote the reciprocal.



Track where the survivors remain

Four factors survive below the fraction bar. Their location supplies the reciprocal.

3. Write and generalize the value

Cancel three common factors to get $1/5^4=1/625$; the exponent rule calls this 5^{-4}. In general $b^{-n}=1/b^n$ for nonzero b.



Interpret a negative exponent

The difference $3-7$ is -4. The same quotient equals one divided by four uncancelled fives.

Check. Multiplying the two opposite quotients gives 1.

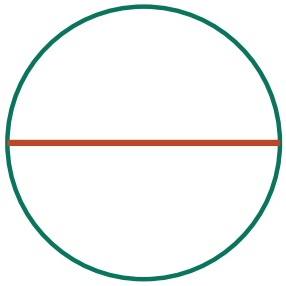
Exercise 340 · Scale of a proton

Printed page 34 · Scientific notation / Surface area / Volume

Exponents are routinely encountered in science, where they help to deal with small numbers. For example, the diameter of a proton is 0.0000000000003 cm. Explain why it is logical to express this number in scientific notation as 3×10^{-13}. Calculate the surface area and the volume of a proton.

1. Read a negative exponent as a decimal shift

Moving the decimal point 13 places left makes 3×10^{-13}.

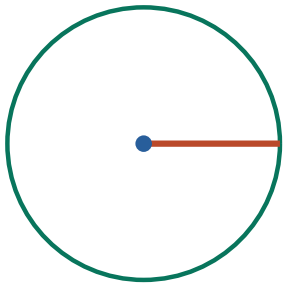


Locate the stated diameter

The line crosses the entire modeled proton: 3×10^{-13} cm.

2. Identify the model's radius

Radius is 1.5×10^{-13} cm. The given diameter stretches from one surface to the other, through the center. The radius is half that entire distance.

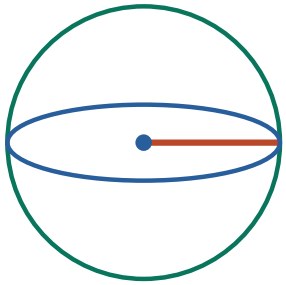


Halve the diameter

From center to surface the distance is 1.5×10^{-13} cm.

3. Track the units and powers

The diameter is a length across the whole sphere, so halve it before using the sphere formulas. Squaring 10^{-13} gives 10^{-26}, and cubing it gives 10^{-39}.

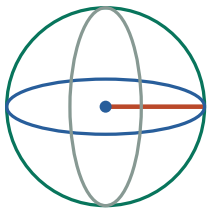


Distinguish the surface from the interior

Every surface point lies one radius from the center. Squaring the radius gives a factor of 10^{-26}.

4. Compute area and volume

Area $4\pi r^2 = 9\pi \times 10^{-26}$ cm^2 and volume $4\pi r^3/3 = 4.5\pi \times 10^{-39}$ cm^3. These area and volume calculations use the exercise's spherical model and its stated diameter.



Apply the sphere formulas

The area is $9\pi \times 10^{-26}$ cm^2, and the volume is $4.5\pi \times 10^{-39}$ cm^3, under the stated spherical model.

1. $r^2 = 1.5^2 \cdot 10^{-26}$; $r^3 = 1.5^3 \cdot 10^{-39}$
Apply the power to the coefficient and exponent separately.

2. $4 \cdot 2.25 = 9$; $(4/3) \cdot 3.375 = 4.5$
These coefficients multiply π in area and volume respectively.

Check. Powers of length square and cube, explaining exponents −26 and −39.

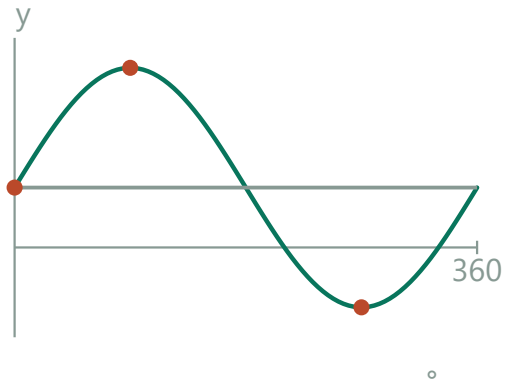
Exercise 341 · Zeros of shifted sine

Printed page 34 · Sinusoidal functions / Trigonometric equations

By hand, graph the equation $y = 1 + 2\sin x$. This curve crosses the x-axis in several places. Without the use of a calculator, solve for the x-intercepts with $0 < x < 360$.

1. Construct the shifted sine wave

Quarter-turn heights are 1,3,1,−1,1. For x=0°, 90°, 180°, 270°, 360°, start with sine heights 0, 1, 0, −1, 0. Double them and add 1 to each before joining the points smoothly.

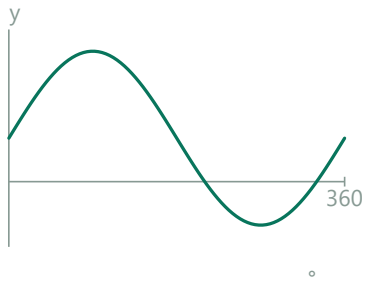


Plot the shifted landmarks

The midline is 1 and the amplitude is 2.

2. Locate a zero algebraically

At an x-intercept y=0. Undo the shift by subtracting 1 and undo the vertical stretch by dividing by 2. The resulting unit-circle height must be negative.



Find where the height vanishes

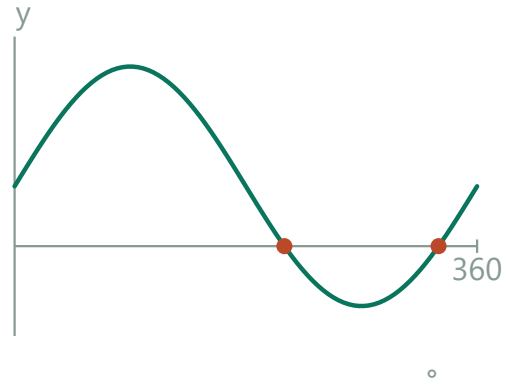
At a zero, sine itself must have value −½.

1. $1+2\sin x=0 \Rightarrow 2\sin x=-1$
At an intercept the plotted height is zero; subtract one.

2. $\sin x=-1/2$
Divide by two before choosing the lower-half-circle rays.

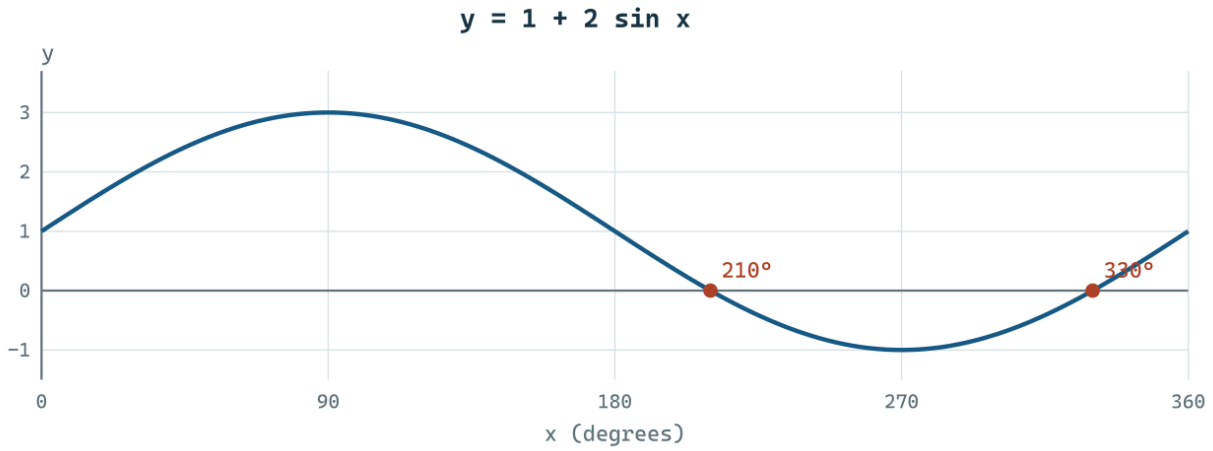
3. Apply the reference triangle

The zeros have sin x=−½, reference angle 30° in quadrants III and IV. They occur at x=210°,330°.



Mark the two crossings

The reference angle 30° gives 210° and 330° below the centerline.



Check. Substitution gives 1+2(−½)=0 at each.

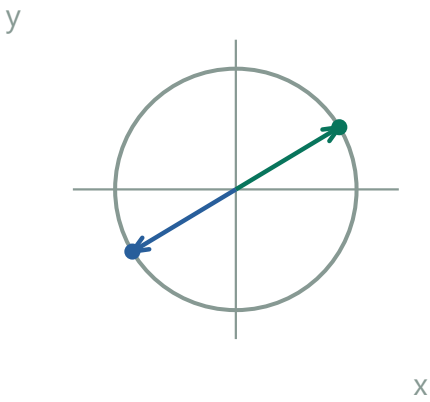
Exercise 342 · Opposite unit vectors

Printed page 34 · The dot product / The unit circle

The value of $[\cos\theta, \sin\theta]\bullet[\cos(180+\theta), \sin(180+\theta)]$ does not depend on the value of θ. Explain why.

1. Put both arrows on the unit circle

One vector starts at angle θ. The second is a half-turn farther around. Their tips are opposite ends of a diameter, so both coordinates change sign.

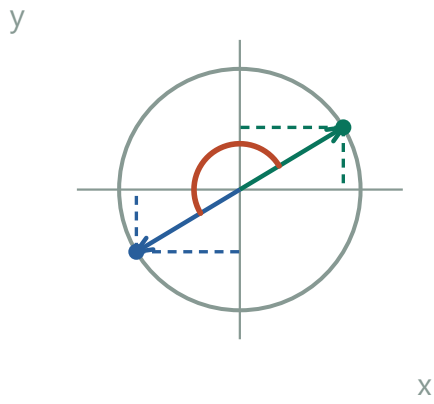


One radius and its opposite

Add 180° to move to the other end of the diameter.

2. Translate the half-turn into a product

Multiply the two horizontal components and then the two vertical ones. The minus signs factor out of a familiar squared-length expression.

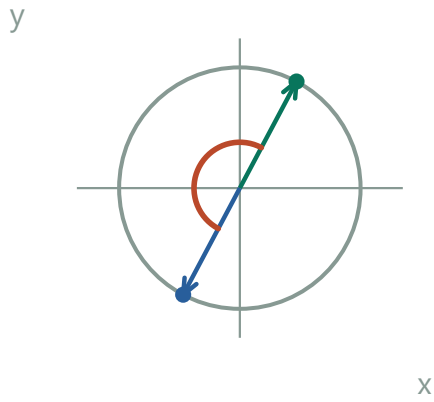


Both components change sign

Dotting (x,y) with $(-x,-y)$ produces $-x^2-y^2$.

3. Evaluate for every θ

The second vector is the negative of the first. The product is $-\cos^2\theta-\sin^2\theta=-1$.



Every diameter works

The squared components of a unit radius sum to 1, leaving a dot product of -1.

Check. The geometric dot-product formula gives $\cos 180° = -1$.

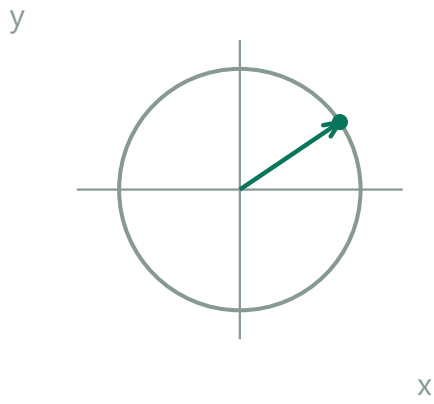
Exercise 343 · Resolve a polar radius

Printed page 34 · Polar coordinates / Trigonometric ratios

Given polar coordinates r and θ for a point, how do you calculate the Cartesian coordinates x and y for the same point?

1. Draw the radius and its two shadows

Join the origin to the polar point. Drop a perpendicular from the point to the horizontal axis. The horizontal leg is x and the directed vertical leg is y.

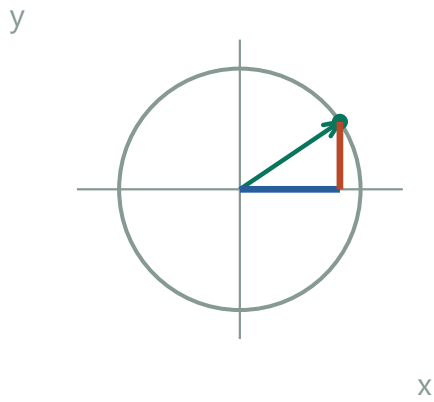


Place radius r at angle θ

The tip is the point whose Cartesian coordinates are sought.

2. Use ratios, including their signs

For a positive radius, cosine gives horizontal displacement per unit radius and sine gives vertical displacement per unit radius. Multiplying each ratio by r restores the actual displacement.

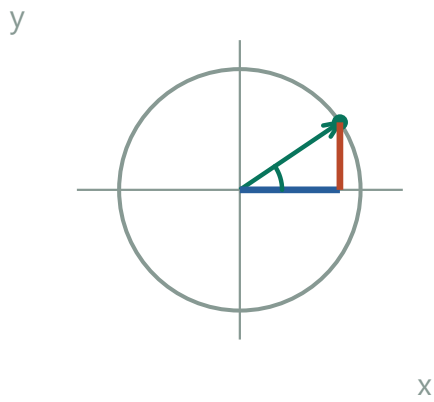


Project to the axes

The horizontal component equals r cos θ and the vertical component r sin θ.

3. State the coordinate rule

Project the radius onto the coordinate axes. The x-component is r cos θ; the y-component is r sin θ.



One point, two descriptions

The ordered Cartesian pair is (r cos θ, r sin θ), with the trigonometric signs set by the quadrant.

Check. Their squared sum is r^2.

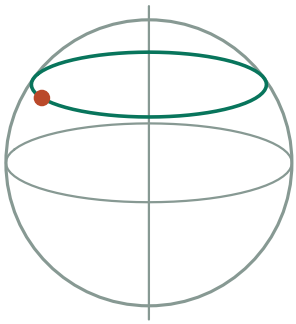
Exercise 344 · Osaka in space

Printed page 34 · Spherical coordinates / Coordinate geometry

An xyz-coordinate system is placed with its origin at the center of the earth, so that the equator is in the xy-plane, the North Pole has coordinates $(0, 0, 3960)$, and the xz-plane contains the prime meridian. Find the coordinates (x, y, z) of Osaka, Japan, whose latitude and longitude are 34.7 degrees north and 135.5 degrees east.

1. Interpret the latitude and longitude

The positive latitude places Osaka above the xy-plane. An east longitude between 90° and 180° points into quadrant II of that plane. Expect x negative and y positive.

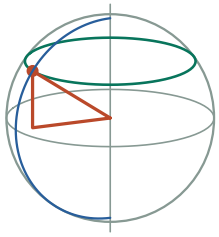


Locate Osaka

The point lies north of the equator at 135.5° east longitude.

2. Compute two projections

Project the radius horizontally, then resolve that part in longitude. Coordinates in miles are 3960(cos 34.7°cos 135.5°, cos 34.7°sin 135.5°, sin 34.7°).



Make the two projections

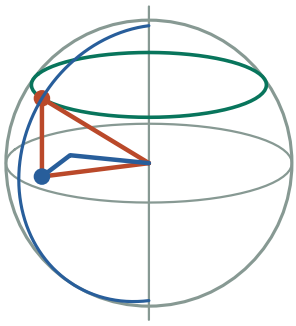
First find R cos 34.7° horizontally; then use longitude to split that distance into x and y.

1. ρ=3960 cos 34.7°; z=3960 sin 34.7°
Resolve Earth's radius into its horizontal and vertical components.

2. x=ρ cos 135.5°<0; y=ρ sin 135.5°>0
The east longitude lies in quadrant II of the equatorial plane.

3. Evaluate the position

The numerical coordinates, to the nearest mile, are (−2322, 2282, 2254).



Read the octant

The signed Cartesian coordinates are approximately (−2322,2282,2254) miles.

Check. The signs are (−,+,+), agreeing with northern latitude and east longitude between 90° and 180°.

Exercise 345 · Price of fries

Printed page 34 · Systems of equations / Matrix multiplication

The results of three PEA dorm outings to Burger Palace: Dutch spent \$54.39 for 14 burgers, 12 shakes, and 15 fries; Wentworth spent \$291.95 for 81 burgers, 62 shakes, and 72 fries; Lamont spent \$111.93 for 25 burgers, 33 shakes, and 29 fries. How much does an order of fries cost at Burger Palace? It may be helpful to label the rows and columns of your matrices.

1. Enter the receipt totals as equations

Let b, s, and f denote the prices in dollars of a burger, shake, and order of fries. The three equations are 14b+12s+15f=54.39, 81b+62s+72f=291.95, and 25b+33s+29f=111.93.

b	s	f	bill (\$)
14	12	15	54.39
81	62	72	291.95
25	33	29	111.93

Read the receipts

Burger, shake and fries counts form the three columns of coefficients. Each total is the measured bill.

2. Produce the first reduced equation

Eliminate f twice. Compute 15 times the second equation minus 72 times the first to get 207b+66s=463.17.

b	s	f	bill ($)
1215	930	1080	4379.25
1008	864	1080	3916.08
207	66	0	463.17

Cancel the fries in receipts two and one

Multiply receipt two by 15 and receipt one by 72. Subtract the matched fries column.

3. Produce the second reduced equation

Compute 15 times the third minus 29 times the first to get −31b+147s=101.64.

b	s	f	bill ($)
375	495	435	1678.95
406	348	435	1577.31
−31	147	0	101.64

Cancel fries in the other pair

Multiply receipt three by 15 and receipt one by 29. The new reduced equation is independent of the first.

4. Separate the remaining two prices

For instance, multiply the first reduced equation by 147 and the second by 66. Subtract to eliminate the shake price, solve for b, and substitute back for s.

b	s	f	bill ($)
207	66	0	463.17
−31	147	0	101.64

Prepare to cancel shakes

Scale the first reduced row by 147 and the second by 66. Both shake coefficients then equal 9702.

1. 49(207b+66s)−22(−31b+147s)=49·463.17−22·101.64
The two shake coefficients become 3234.

2. 10825b=20459.25 ⇒ b=1.89
Collect burger terms and divide by their nonzero coefficient.

3. s=(463.17−207·1.89)/66=1.09
Back-substitute the burger price.

5. Recover the fries price

Solve the two equations for b=$1.89 and s=$1.09. The first receipt leaves $14.85 for fifteen orders of fries, so one order costs $0.99.

b	s	f	bill ($)
14	12	15	54.39
26.46	13.08	14.85	54.39

Check the item totals

The first receipt splits into $26.46, $13.08, and $14.85. Dividing the fries subtotal by 15 gives $0.99.

Check. For example 25(1.89)+33(1.09)+29(.99)=111.93; the first two totals likewise match.

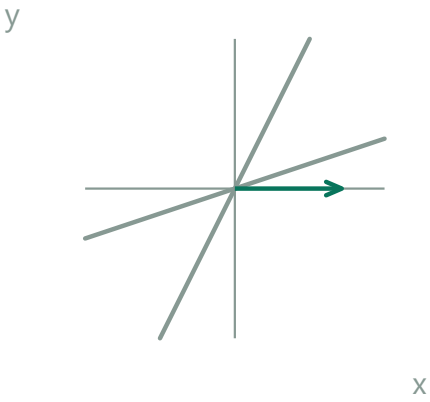
Exercise 346 · Reverse two reflections

Printed page 35 · Reflections / Rotations / Matrix multiplication

Matrices $\mathbf{M} = \begin{bmatrix} -0.6 & 0.8 \\ 0.8 & 0.6 \end{bmatrix}$ and $\mathbf{N} = \begin{bmatrix} 0.8 & 0.6 \\ 0.6 & -0.8 \end{bmatrix}$ represent reflections in the lines $y = 2x$ and $3y = x$, respectively. Verify that $\mathbf{MN}$ is not equal to $\mathbf{NM}$, and explain why this should have been expected. What transformations do the two products represent?

1. Interpret multiplication order

In MN the point reflects in the axis of N and then the axis of M. In NM the same mirrors are visited in the reverse order. Follow the unit arrow (1,0) to see the turn direction.

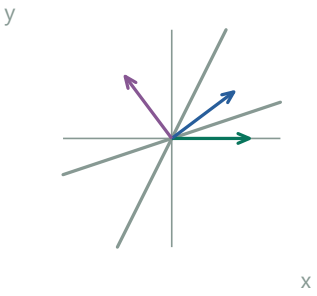


Fix the two mirror axes

The axes y=2x and y=x/3 meet at 45°. Both orders act on the same starting vector (1,0).

2. Compute a representative column

N sends (1,0) to (0.8,0.6). M sends that to (−0.48+0.48,0.64+0.36)=(0,1). Reversing the steps instead sends (1,0) to (0,−1).



Change which reflection goes first

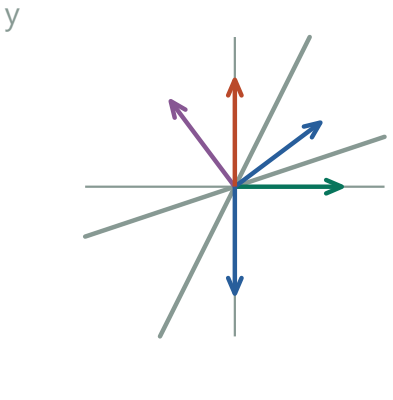
Reflecting first in the shallow line sends (1,0) to (0.8,0.6). Reflecting first in the steep line gives (−0.6,0.8).

1. N(0,1)=(0.6,−0.8); M(0.6,−0.8)=(−0.36−0.64,0.48−0.48)=(−1,0)
The image of the second basis vector is the second column of MN.

2. M(0,1)=(0.8,0.6); N(0.8,0.6)=(0.64+0.36,0.48−0.48)=(1,0)
Reversing the applications gives the second column of NM.

3. Identify both transformations

Multiplication gives MN=[[0,−1],[1,0]], a counterclockwise quarter-turn, and NM=[[0,1],[−1,0]], its clockwise inverse. Each reflecting axis is through the origin; their angles differ by 45°.



Read the two final rotations

The two orders send (1,0) to opposite vertical directions: counterclockwise and clockwise quarter-turns.

Check. The composition turns through twice the directed angle between axes, reversing sign when the order is reversed.

Exercise 347 · Start at the bottom

Printed page 35 · Sinusoidal functions / Circular motion

Centered 7 meters above the ground, a Ferris wheel of radius 6 meters is rotating with angular speed 24 degrees per minute. Assuming that Harley's ride began at time $t = 0$ minutes at the lowest point on the wheel, write a formula for the function that describes the distance $h(t)$ from Harley to the ground (in meters) after t minutes of riding.

1. Identify the midline and amplitude

The center height of 7 m is the middle of the motion. The radius 6 m is the maximum vertical distance above or below this height.

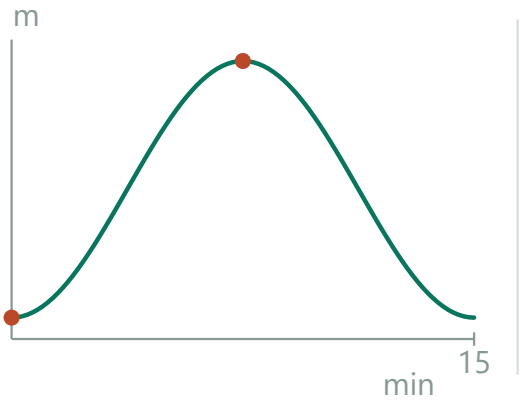


Center 7 m above the ground

The radius is 6 m, so the rider travels between 1 and 13 m.

2. Select the phase from the starting point

At t=0 the rider is at 7−6. Cosine is 1 at angle zero, so a negative cosine coefficient puts the rider at the bottom. The angle after t minutes is 24t degrees.

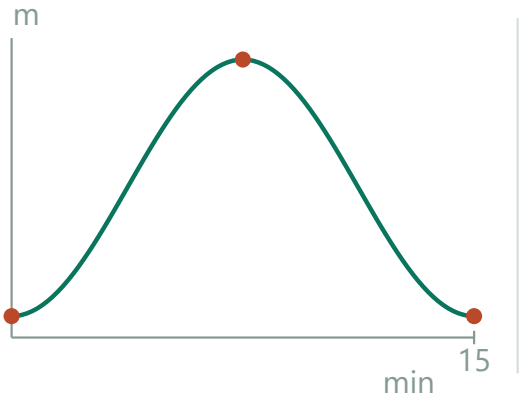


Start at the bottom

A negative cosine begins a full radius below the midline and reaches the top halfway around.

3. Write the height function

Starting at the bottom requires h(0)=1, so subtract a cosine of amplitude 6: h(t)=7−6 cos(24t°), with t in minutes.



Use 24 degrees per minute

h=7−6 cos(24t°) completes a cycle in 15 minutes.

Check. At t=7.5 the rider reaches the top at 13 m; at t=15 the rider returns to 1 m.

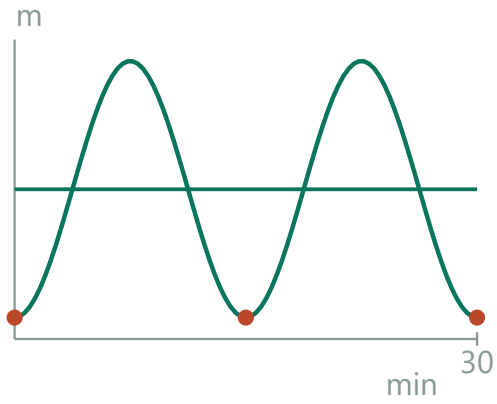
Exercise 348 · Twice climbing past 10 m

Printed page 35 · Sinusoidal functions / Trigonometric equations

(Continuation) Draw a graph of $h(t)$ for the restricted *domain* $0 \le t \le 30$, and find coordinates for two points on your graph that both represent the situation when Harley is 10 meters above the ground and climbing. Interpret the domain restriction in context.

1. Construct the two-cycle picture

A cycle of 15 min has heights 1,7,13,7,1 at quarter periods. Repeat twice.

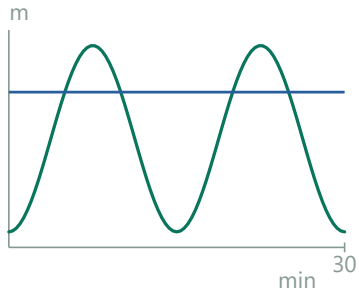


Restrict the time window

The 0–30 minute domain contains exactly two complete revolutions.

2. Isolate the target height

Height 10 requires cos(24t°)=−½; the ascending crossing is at angle 120° mod 360°.



Set the target height

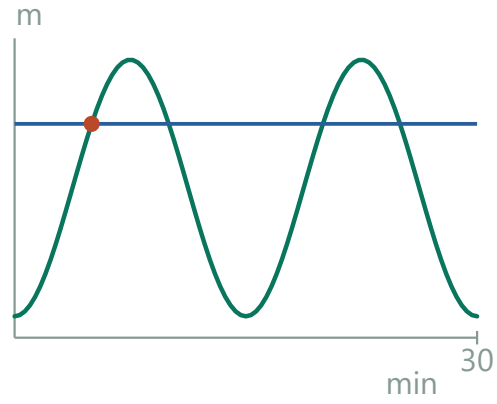
The horizontal line at 10 m crosses each cycle twice.

1. $7-6\cos(24t°)=10 \Rightarrow -6\cos(24t°)=3$
Subtract the center height.

2. $\cos(24t°)=-1/2$
Divide by −6; the height has crossed above the midline.

3. Pick the upward crossing

During the first half-turn the seat climbs from 1 m to 13 m. Of the angles 120° and 240° with cosine −1/2, only 120° is in that climbing half. Divide its angle by 24 degrees per minute, then add a full period.

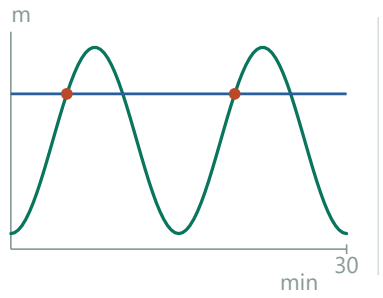


Choose the climbing branch

At the first rising crossing t=5 minutes; the falling crossing has the wrong direction.

4. Give the two ordered pairs

So the requested points are (5,10),(20,10). Each coordinate pair uses minutes first and meters second. Adding the full 15-minute revolution to the first crossing gives the second ascending crossing within the first 30 minutes.

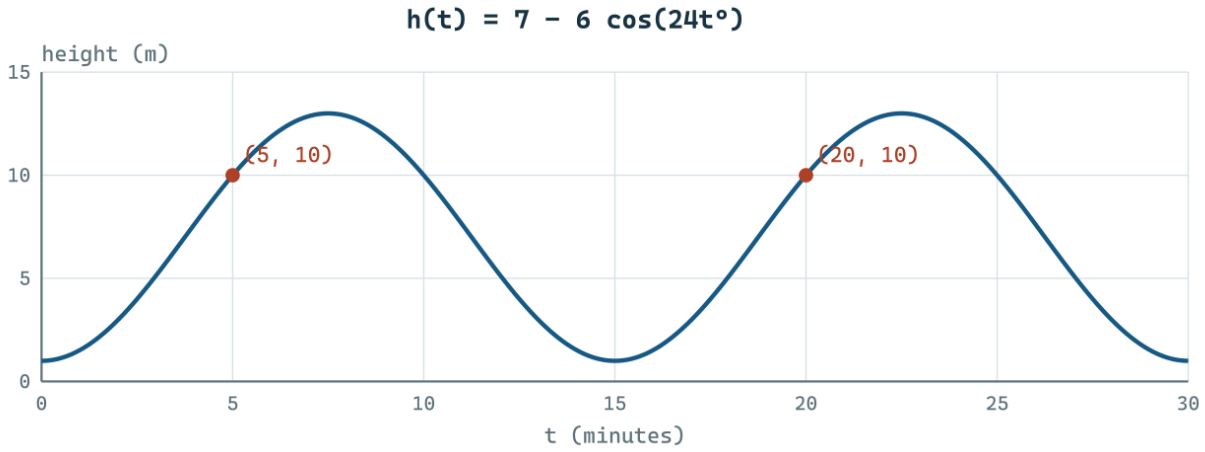


Repeat one cycle later

The two climbing points are (5,10) and (20,10), with time measured in minutes.

1. 24t=120+360k ⇒ t=5+15k
Use the ascending crossing, then divide degrees by degrees per minute.

2. 0≤t≤30 ⇒ k=0 or 1
Only 5 and 20 minutes lie in the requested interval.

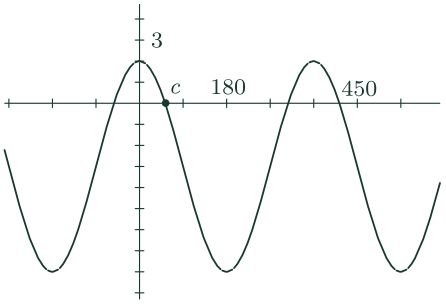


Check. The restriction gives two complete revolutions, each with exactly one ascending crossing.

Exercise 349 · Other cosine intercepts

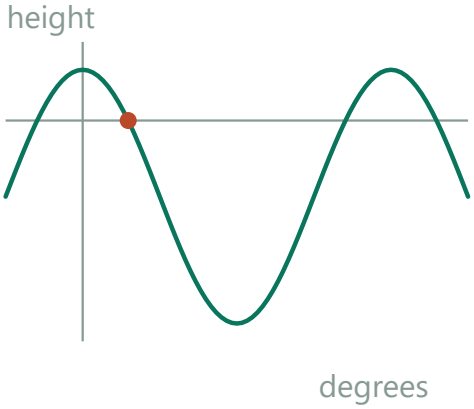
Printed page 35 · Sinusoidal functions / Symmetry

As shown, the graph $y = -3 + 5\cos x$ intersects the x-axis repeatedly, one of the x-intercepts is $(c, 0)$. Without a graphing tool or calculator, determine the values of the other x-intercepts that appear in the figure. Your answers should be in terms of c.



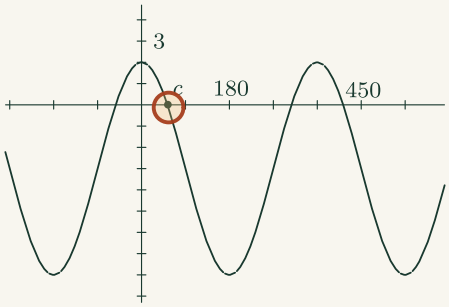
1. Read the marked intercept

The height at c is zero, hence −3+5 cos c=0. Another angle gives a crossing whenever its cosine has that same value.



Trace the root at c

The orange point is the positive root near the first crest. At this height, 5 cos c = 3.

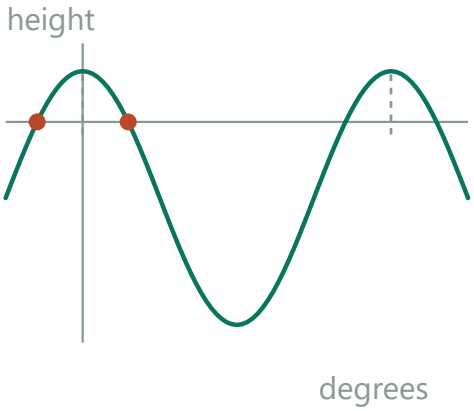


The marked crossing at c

At the marked point the curve meets the horizontal axis. Its y-value is zero, even while its horizontal coordinate remains symbolic.

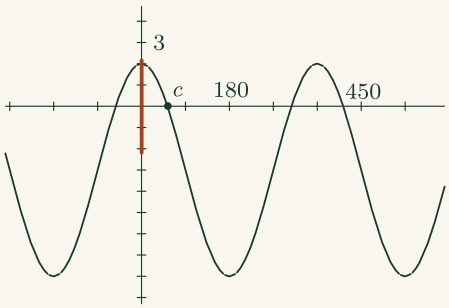
2. Use the axes of symmetry

A cosine crest is centered on 0° and the next one on 360°. Equal horizontal offsets from either crest produce the same height. Read which of those offsets lie inside the shown window.



Reflect the horizontal offset

The first two roots sit an equal distance on either side of the crest at zero degrees.

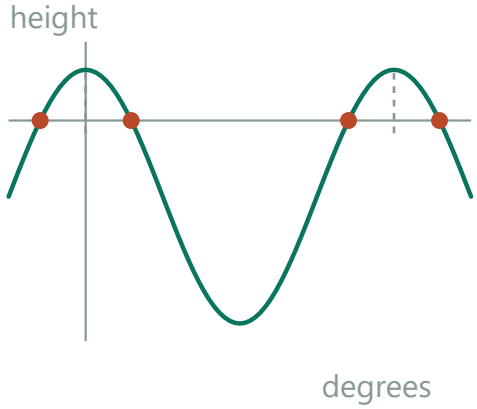


A crest's symmetry line

Just to the left and right of this crest, equal horizontal offsets give equal heights. Use this symmetry to pair crossings.

3. Name the other crossings

Cosine is even and repeats every 360°. Reflect c across 0 to get −c, and across the next peak at 360° to get the pair 360°−c and 360°+c. These are the three other intercepts visible in the figure, each with y=0.



Repeat around the second crest

Across the displayed window, the four positions are −c, c, 360°−c and 360°+c.

Check. Every candidate satisfies cos x=cos c=3/5.

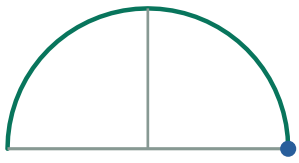
Exercise 350 · Shortest route on a cone

Printed page 35 · Nets and surface development / Distance and loci

Both the slant height and the base diameter of a cone are 12 inches. How far is it from a point on the base circle to the diametrically opposite point on the circle, if it is required that the path must lie on the lateral surface of the cone?

1. Turn a surface into a plane

Unroll the lateral surface to a radius-12 sector. The full rim length 12π occupies a half-circle.

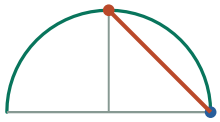


Open the cone along a seam

A slant radius 12 and rim length 12π produce a semicircular sector.

2. Find a concrete route

Opposite rim points then subtend a 90° central angle on an unrolling with no seam between them. The straight chord has length 12√2 inches.



Join the two rim points

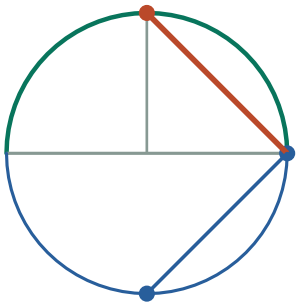
Opposite points on the original rim are 90° apart on this development. Their chord stays inside.

1. $\theta_{full} = 12\pi/12 = \pi$ radians
Use the unrolled sector radius 12 and the whole base circumference.

2. $\theta_{half} = \pi/2 \Rightarrow \text{chord}^2 = 12^2 + 12^2 = 288$
Opposite rim points correspond to half the rim arc and a right angle.

3. Account for paths that cross the seam

To rule out a shorter winding route, keep unrolling past a seam whenever the path crosses it. Each full turn around the cone adds π to the developed polar angle, so every image of the other endpoint is at angle $\pi/2 + k\pi$ from the first.

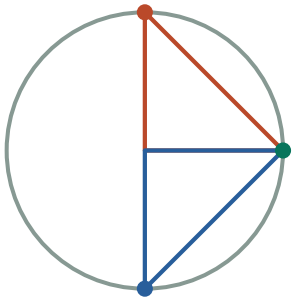


Unroll across another seam

Further unrollings place copies of the target at angles $90° + 180°k$ from the start.

4. Bound every unwrapped route

Its straight-line distance is always 12√2. A developed path has at least that length.

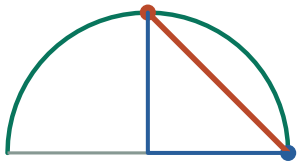


Compare all endpoint images

Every target copy has straight-line distance 12√2; a developed path cannot be shorter.

5. Check the apex and attainment

A path through the apex has length at least 12+12=24. The quarter-sector chord stays inside the sector, so the lower bound is attained.



Compare the apex route

Going through the apex requires 12+12=24. The red chord attains the smaller bound 12√2.

Check. Pythagoras gives the same chord from perpendicular radii; the other way around is symmetric.

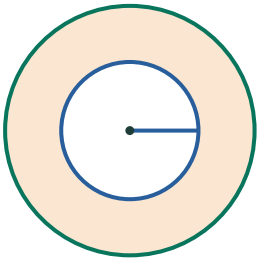
Exercise 351 · A hollow ball

Printed page 35 · Volume / Ring and shell geometry

A spherical ball weighs three times as much as another ball of identical appearance and composition. The second ball weighs less because it is actually hollow inside. Find the radius of the hollow cavity in the second ball, given that each ball has a 5-inch radius.

1. Translate mass into material volume

The balls have the same material density and outer radius. Equal amounts of that material have equal weight. If the hollow ball weighs one-third as much, one-third of its outer volume contains material.

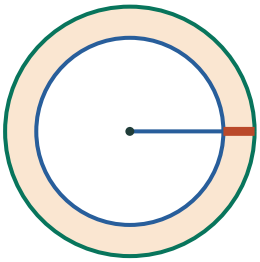


Distinguish the ball from its material

The outer radius is 5 inches; the cavity contributes no iron mass.

2. Distinguish the cavity from the shell

The shell contains one-third of the full volume; the cavity is two-thirds.

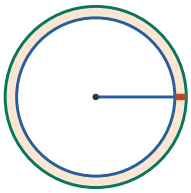


Subtract the occupied fraction

One-third of the full volume is iron, so two-thirds is empty.

3. Recover its radius

Thus $r^3=(2/3)\cdot 5^3$, so r=5(2/3)^(1/3) inches. Both the cavity sphere and the full sphere contain the same factor 4π/3 in their volume formulas. Cancelling it leaves a comparison of cubed radii.



Convert cavity volume to radius

The inner radius is $5\sqrt[3]{(2/3)}$, larger than two-thirds of 5 because volume scales cubically.

1. $(4\pi/3)r^3=(2/3)(4\pi/3)5^3$
The cavity accounts for the missing two thirds of the material.

2. $r^3=(2/3)5^3 \Rightarrow r=5\sqrt[3]{(2/3)}$
Cancel the sphere constant and take the positive cube root.

Check. Substituting this radius into the shell volume gives exactly one-third of the solid volume.

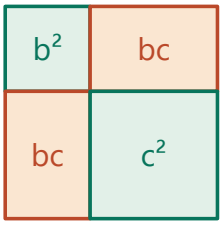
Exercise 352 · Which operations distribute?

Printed page 35 · Exponent rules / Algebraic manipulation

It is well known that multiplication can be *distributed* over addition or subtraction, meaning that $a \cdot (b + c)$ is equivalent to $a \cdot b + a \cdot c$, and that $a \cdot (b - c)$ is equivalent to $a \cdot b - a \cdot c$. It is *not* true that multiplication distributes over multiplication, however, for $a \cdot (b \cdot c)$ is not the same as $a \cdot b \cdot a \cdot c$. Now consider distributive questions about exponents: Is $(b + c)^n$ equivalent to $b^n + c^n$? Explore this question by choosing some numerical examples. Is $(b \cdot c)^n$ equivalent to $b^n \cdot c^n$? Look at more examples.

1. Test a sum before claiming a rule

For n=2 expand (b+c)(b+c). In addition to two squares, multiplication produces bc twice, so a numerical counterexample should be easy to find.



Tile a square of side b+c

The two cross-rectangles each have area bc; these areas must be included with the two squares.

2. Test a product differently

For positive integer n, writing (bc) n times leads to n separate factors b and n factors c. Rearranging them explains the product property.

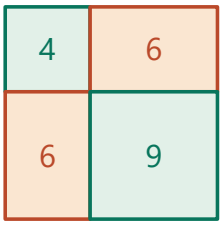


Compare multiplication of factors

Three copies of bc contain three b factors and three c factors. Rearrangement gives b^3c^3.

3. Compare both claims and their domains

For n=2, $(2+3)^2=25$ while $2^2+3^2=13$. A power of a product does distribute for positive integers n: group n copies of bc to obtain b^nc^n. For real exponents positive bases also suffice.



Substitute b=2 and c=3

The full square has area 4+6+6+9=25. Keeping only its two square tiles gives 13.

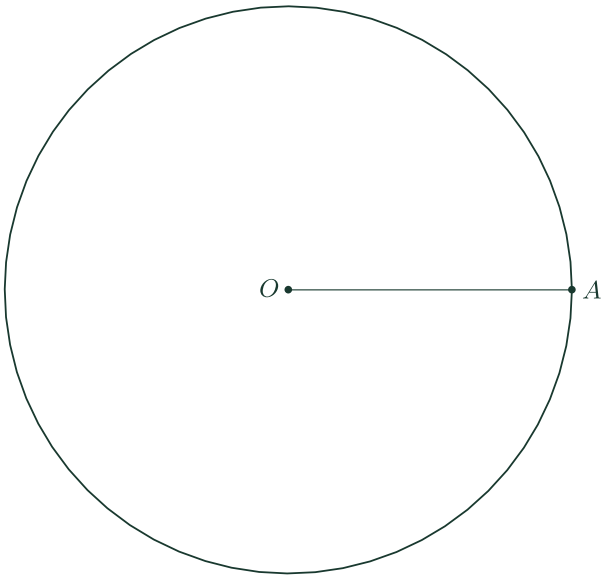
Check. Taking b=2,c=3,n=3 gives $6^3=216=8\cdot27$.

Exercise 353 · One radian

Printed page 36 · Radian measure / Arc length

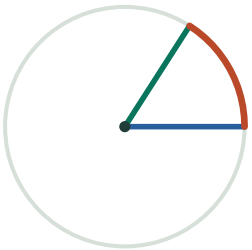
The circle shown at right is centered at O. Use a string or thin cord to find a point B on this circle for which minor arc AB has the *same length* as the radius OA. Draw radius OB and use a protractor to measure the size of angle AOB. Your answer should be close to 60 degrees. By considering triangle AOB and the relation between the arc AB and its chord, explain why angle AOB must in fact be smaller than 60 degrees.

Angle AOB is an example of a *radian* — a central angle whose arc has the same length as the radius of the circle in which it is drawn.



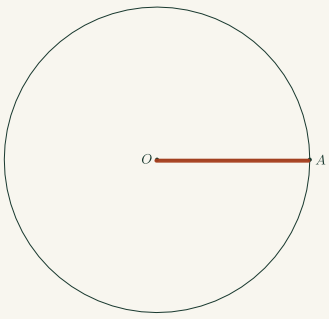
1. Transfer the radius length to the circle

Mark a length OA on a cord. Hold one mark at A, lay the marked length along the circumference and mark its other end B. Draw OB.



Transfer one radius onto the rim

The radius and the marked curved length have equal length.

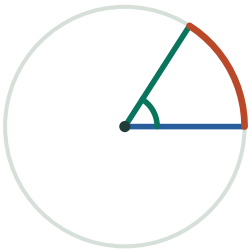


The reference radius OA

OA is the straight center-to-rim length to transfer to the cord. It is also the starting arm when measuring a central angle.

2. Measure at the center

Center a protractor at O with baseline OA; the expected reading is about 57°. The arc length is r, so its angle is 1 radian (about 57.3°).

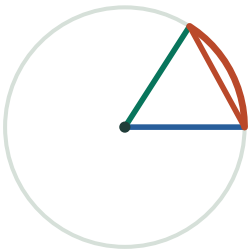


Join the arc's two ends to O

The angle between the radii is one radian, about 57.3°.

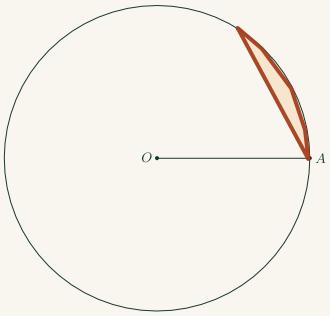
3. Why one radian falls below 60°

Chord AB is shorter than that arc. A 60° central angle gives an equilateral triangle and a chord of length r.



Compare arc and chord

The straight chord between endpoints is shorter than the radius-length arc.

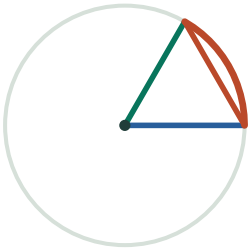


The circular path for the cord

Start at A and lay the cord around the circumference. The endpoint of the measured arc sets the direction of the second radius.

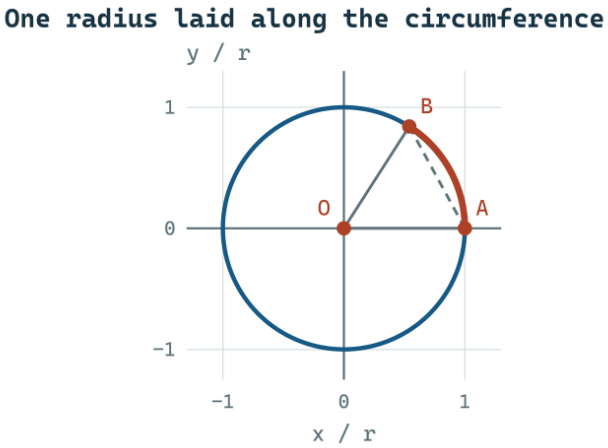
4. Compare the possible central angles

On the minor arc the chord increases with the central angle, so angle AOB is smaller than 60°.



A 60° angle makes equal sides

At 60° the chord would be a full radius. The shorter observed chord requires a smaller angle.



Check. The formula s=rθ gives θ=1 in radians.

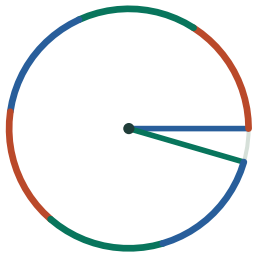
Exercise 354 · Radians in a circle

Printed page 36 · Radian measure / Circumference

(Continuation) How many 1-radian arcs does it take to fill a complete circle? First make an estimate using the string or thin-chord approach, then look for a theoretically exact answer. Do any of your answers depend on the size of the circle used?

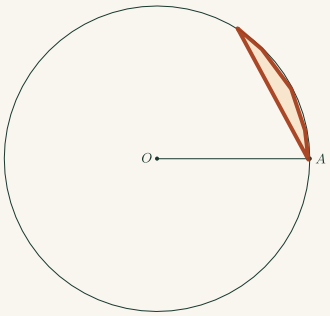
1. Estimate with an actual arc unit

Step a cord marked with one radius at a time around the circumference. Six full lengths fit and a little over a quarter of another remains, giving a visual estimate near 6.3.



Step off radius-length arcs

Each colored piece spans one radian. Six of them leave a small gap.

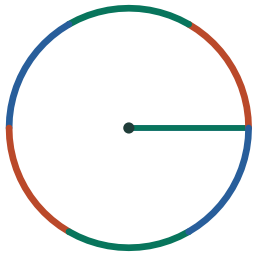


The circular path for the cord · Figure from exercise 353

Start at A and lay the cord around the circumference. The endpoint of the measured arc sets the direction of the second radius.

2. Replace the count by a division

The exact count is $2\pi r/r = 2\pi \approx 6.283$. Circumference is the distance traveled in one turn. Dividing that distance by the chosen unit of arc, one radius, counts the number of radian-size pieces including the fractional piece.

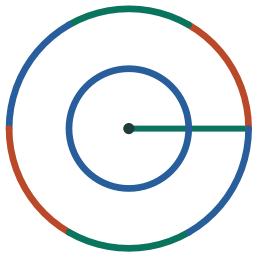


One circumference ÷ one radius

The exact count is $2\pi r/r = 2\pi$ radians. Each sixth of the circle is slightly longer than one radian.

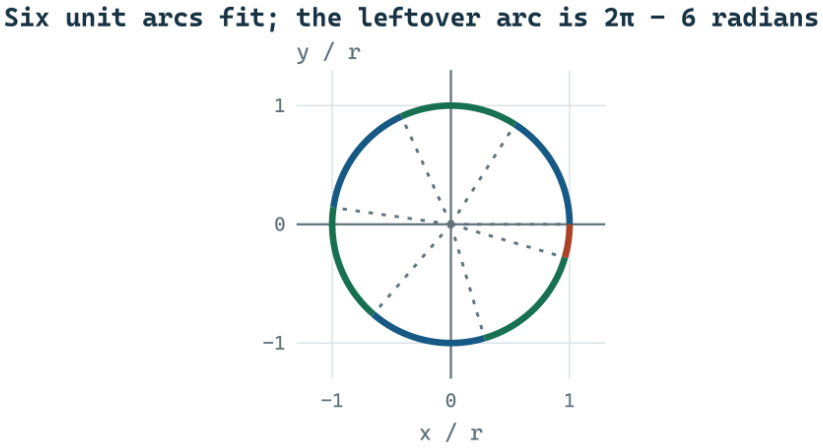
3. Explain independence of radius

Scaling the circle multiplies both the arc length and circumference by the same factor, leaving the count unchanged.



Scale both the circle and radius

The inner and outer circles complete a turn with the same 2π-radian total.



Check. Scaling the circle multiplies both lengths equally.

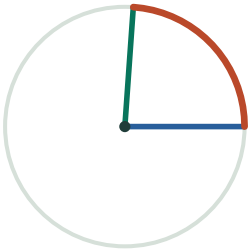
Exercise 355 · Six-inch arc

Printed page 36 · Radian measure / Arc length

A 6-inch arc is drawn using a 4-inch radius. Describe the angular size of the arc **(a)** using radians; **(b)** using degrees.

1. Read one radian as one radius of arc

Imagine marking successive 4-inch lengths on the curved arc. Its angle in radians is the number of such lengths that make up the 6 inches, including a fractional last length.

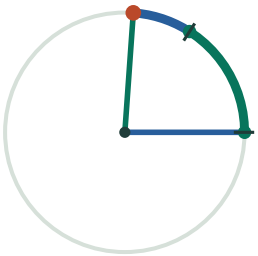


Arc 6 in, radius 4 in

Divide the curved length by the straight radius.

2. Compute the radius count

$\theta = s/r = 1.5$ rad. The 6-cm arc contains 6/4 radius-lengths, or one and a half. The centimeter labels cancel because the numerator and denominator measure lengths in the same units.

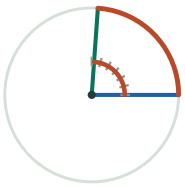


One and a half radii

The arc measures 6/4 = 1.5 radians.

3. Convert to degrees

Multiply by $180°/\pi$ to obtain $270°/\pi \approx 85.94°$. One radian corresponds to $180°/\pi$. Keep the exact factor 3/2 from the first part until the very last evaluation.



Same turn in degrees

Multiply 1.5 by $180°/\pi$ to obtain about 85.94°.

1. $\theta = 6/4 = 3/2$ radians
Divide arc length by radius in the same length units.

2. $(3/2)(180°/\pi) = 270°/\pi$
π radians corresponds to 180 degrees.

Check. The arc is just shorter than a quarter-circle of length 2π in.

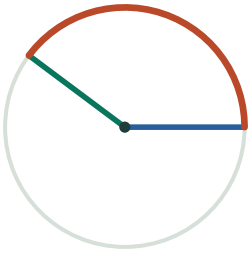
Exercise 356 · From radians to length

Printed page 36 · Radian measure / Arc length

A 2.5-radian arc is drawn using a radius of 6 inches. How long is the arc?

1. Interpret the given angle

An angle of one radian cuts an arc one radius long. Here each unit of radian measure represents 6 inches along the circle.

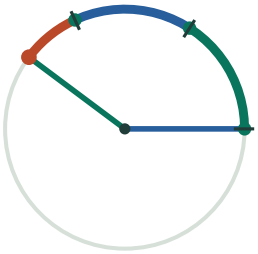


Turn through 2.5 radians

The rim spans two and a half radius-length units.

2. Scale one arc unit

Two radians use two full radius-lengths along the rim. A further half radian adds another half-radius. Multiplying the full count by the radius combines these contributions.

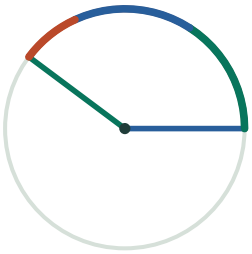


One radius-length arc is 6 in

Every unit radian contributes one more 6-inch curved length.

3. Calculate the arc length

Use s=rθ=6·2.5=15 inches. Two full radius lengths account for 12 inches, and half a radius contributes 3 more inches around the curve.



6 in + 6 in + 3 in

Total arc length

Scaling 6 inches by 2.5 gives 15 inches.

Check. It spans 2.5/(2π) of a circumference of 12π in.

Exercise 357 · Exponent rules in action

Printed page 36 · Exponent rules / Algebraic manipulation

Find equivalent ways to rewrite (without using a calculator) the following expressions:

(a) $\frac{6a^8}{3a^4}$ (b) $(3p^3q^4)^2$ (c) $b^{1/2}b^{1/3}b^{1/6}$ (d) $\left(\frac{2x^3}{3y^2}\right)^2$ (e) $\left(d^{1/2}\right)^6$

1. (a) Cancel the common factors

The original denominator is $3a^4$. It vanishes exactly at a = 0, so that input stays excluded.

1 **$6a^8/(3a^4) = (6/3)(a^8/a^4)$**

Separate the numerical factors from the powers of a.

2 **= 2a^(8 − 4)**

Cancel four matching factors of nonzero a.

3 **$= 2a^4,\ a \neq 0$**

Subtract the exponents. The simplified formula retains the original exclusion.

2. (b) Square every factor

All exponents here are positive integers. Every real p and q is allowed, including zero.

1 **$(3p^3q^4)^2 = (3p^3q^4)(3p^3q^4)$**
A square means two copies of the whole product.

2 **$= (3\cdot3)(p^3\cdot p^3)(q^4\cdot q^4)$**
Regroup equal bases.

3 **$= 9p^6q^8$**
Add 3 + 3 for p and 4 + 4 for q.

3. (c) Put the exponents in sixths

The square root and sixth root are real only for b ≥ 0. At b = 0 every factor is zero, and the equality still holds.

1 **b^(1/2) · b^(1/3) · b^(1/6)**
All three factors have base b.

2 **= b^(3/6 + 2/6 + 1/6)**
Rewrite one half as three sixths and one third as two sixths, then add.

3 **= b^(6/6) = b, b ≥ 0**
The exponents sum to 1.

4. (d) Square both parts of the fraction

The denominator $3y^2$ rules out y = 0. The numerator imposes no restriction on x.

1 **$(2x^3/(3y^2))^2 = (2x^3)^2/(3y^2)^2$**
Squaring a fraction squares both its numerator and its nonzero denominator.

2 **$= (2^2\ x^{(3\cdot2)})/(3^2\ y^{(2\cdot2)})$**
Apply the outer power to each factor and multiply nested exponents.

3 **$= 4x^6/(9y^4),\ y \neq 0$**
Evaluate the integer powers and products.

5. (e) Group six roots into pairs

A real square root requires d ≥ 0. Raising it to the sixth power leaves that requirement in place.

1 **$(d^{(1/2)})^6 = (\sqrt{d})^6$**

The exponent one half denotes the nonnegative square root.

2 **$= ((\sqrt{d})^2)^3$**

Six copies make three pairs.

3 **$= d^3,\ d \geq 0$**

Each pair equals d, including at d = 0.

Check. Coefficients and exponents follow different rules: squaring 3 gives 9, while squaring q^4 doubles its exponent.

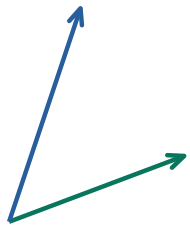
Exercise 358 · Triangle area in 3D

Printed page 36 · Vectors and components / Triangle area / Cross products

Given the points $A = (4, 7, 1)$, $B = (12, -1, 5)$, and $C = (2, -3, 12)$, find the area of triangle ABC.

1. Build side vectors from one endpoint

Subtract A from B and A from C, coordinate by coordinate. Starting both arrows at A ensures that they span the parallelogram containing two copies of triangle ABC.

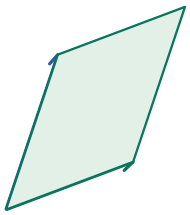


Use one triangle vertex as origin

The edge vectors AB=(8,−8,4) and AC=(−2,−10,11) share their tails.

2. Compute the normal vector

AB=(8,−8,4), AC=(−2,−10,11). Their cross product is (−48,−96,−96), magnitude 144.



Complete a parallelogram

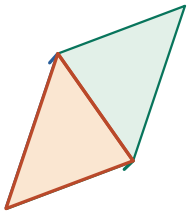
The cross product magnitude 144 measures the full parallelogram area.

1. $n_1=(-8)\cdot 11-4(-10)=-48$
Form the first cross-product component from y and z.

2. $n_2=4(-2)-8\cdot 11=-96$; $n_3=8(-10)-(-8)(-2)=-96$
Cycle the coordinates, keeping subtraction order fixed.

3. Move from the parallelogram to the triangle

The triangle area is half of this, 72 square units.



Halve along its diagonal

The original triangle occupies exactly half, or 72 square units.

1. $\|n\|=\sqrt{(48^2+96^2+96^2)}=\sqrt{20736}=144$
Cross-product magnitude gives parallelogram area.

2. $A=144/2=72$
The diagonal splits the parallelogram into equal-area triangles.

Check. The dot product is 108. The Gram determinant $(144)(225)-108^2=20736$ equals 144^2.

Exercise 359 · Why a zero exponent is one

Printed page 36 · Exponent rules

Invent a division problem whose answer is b^0, and thereby discover the meaning of b^0.

1. Choose a quotient with equal exponents

Choose b^3 as both numerator and denominator. Division is defined only when $b^3 \neq 0$, which means $b \neq 0$.

1 **$Q = b^3/b^3,\ b \neq 0$**
Use the same nonzero power twice.

2 **$Q = 1$**
Every nonzero quantity divided by itself is one.

2. Evaluate the same quotient by the exponent law

Both calculations describe the same quotient. This construction gives no value for 0^0, because the proposed division would be 0/0.

1 **Q = b^(3 − 3)**

In a quotient of nonzero powers of one base, subtract the exponents.

2 **Q = b^0**

Equal exponents have difference zero.

3 **b^0 = 1 for every b ≠ 0**

Equate the two values found for Q.

Check. The pattern b^2, b^1, b^0 divides by b at each step.

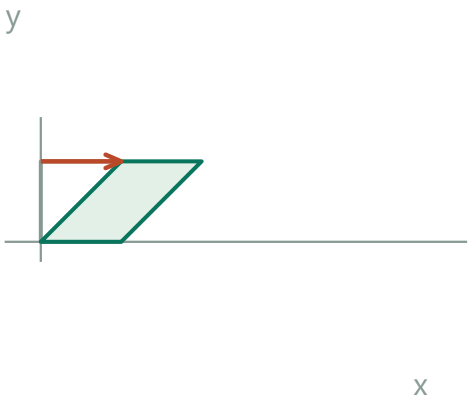
Exercise 360 · Powers of a shear

Printed page 36 · Matrix multiplication / Matrix transformations

Let $\mathbf{J}$ be the matrix $\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$, and calculate $\mathbf{J}^2$, $\mathbf{J}^3$, and $\mathbf{J}^4$. Do you detect a pattern?

1. Follow a general coordinate pair

Multiply J by the column (x,y). Its top entry gains one copy of y and the bottom entry stays unchanged. Repeat on the new point, keeping the original y available at every step.

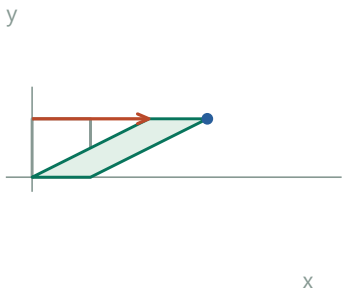


Apply the shear once

The top edge at height one slides one unit to the right; the bottom edge stays put.

2. Recognize repeated shear

Each application changes (x,y) into (x+y,y). After n applications this is (x+ny,y).



Apply it twice

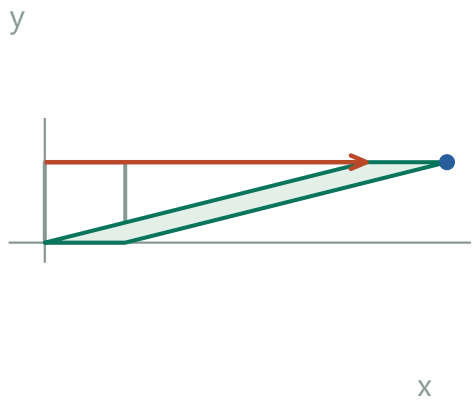
At height y the second application adds another y to the horizontal coordinate.

1. $J(x+ny,y)=(x+ny+y,y)=(x+(n+1)y,y)$
One more application adds the unchanged y-coordinate once.

2. $J^n(1,0)=(1,0)$; $J^n(0,1)=(n,1)$
These images form the two columns of the nth power.

3. Read off the three powers

Thus J^2,J^3,J^4 have rows (1,n),(0,1) for n=2,3,4, respectively.



Repeat n times

The top-left corner at height 1 is now (n,1); for n=4 the highlighted top edge begins at x=4.

Check. Multiplying J^n by J adds exactly 1 to the upper-right entry.

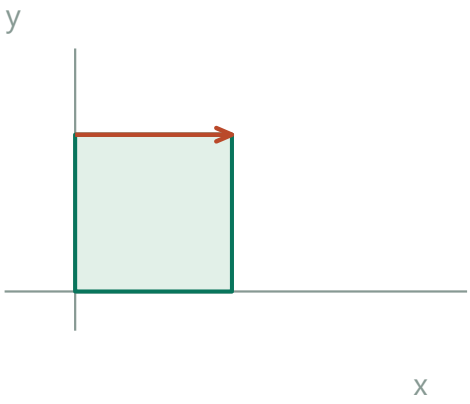
Exercise 361 · A sheared square

Printed page 36 · Matrix transformations / Determinants and area

(Continuation) Viewed as a geometric transformation, what effect does **J** have? To find out, apply **J** to a variety of points, including those in the *unit square*. Is the image of the unit square another square?

1. Predict which points move

Apply J to (x,0), then to (x,1). The whole bottom edge of the square stays fixed while the top edge moves to the right. All points at the same height move equally far.

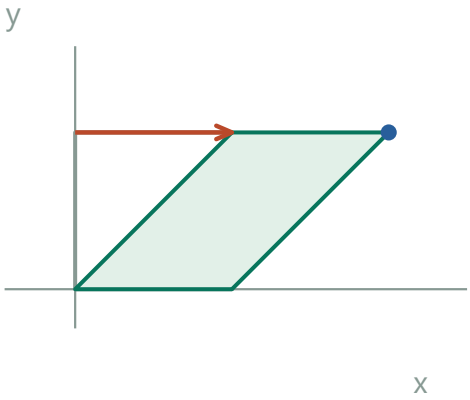


Predict top-edge motion

At y=1 every point will move one unit right. At y=0 no point moves.

2. Plot the transformed corners

Horizontal lines stay at their original heights; each point shifts horizontally by y. The unit-square corners map to (0,0),(1,0),(2,1),(1,1), a parallelogram with adjacent lengths 1 and √2.

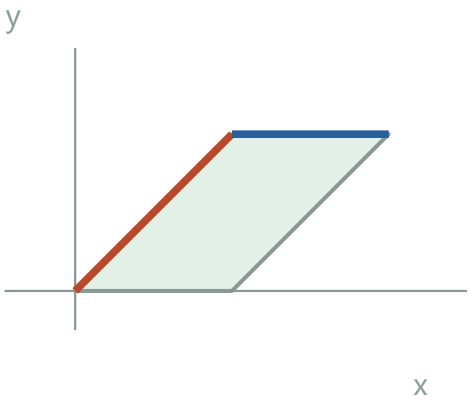


Plot the output vertices

The new top corners are (1,1) and (2,1).

3. Classify the quadrilateral

Two opposite horizontal edges are still parallel, and the other two have the same slope. Their neighboring lengths differ, so the image fails the equal-side test for a square.



Compare neighboring edges

The sloping sides have length √2 while the horizontal sides have length 1. The result is a parallelogram.



Check. Its area remains 1, in agreement with det J=1, while adjacent sides cease to be perpendicular.

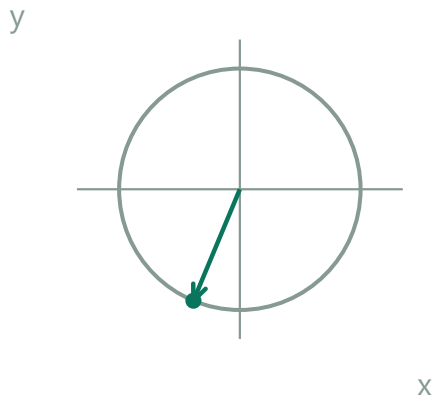
Exercise 362 · Third-quadrant tangent

Printed page 36 · Trigonometric ratios / The Pythagorean theorem

Given that $\tan\theta = 2.4$, with $180 < \theta < 270$, without a calculator, find the exact values of $\sin\theta$ and $\cos\theta$. Are your answers *rational numbers*?

1. Convert the decimal slope to a ratio

Write 2.4=24/10=12/5. Tangent compares vertical and horizontal components, and a common scale factor will cancel from all three trigonometric ratios.

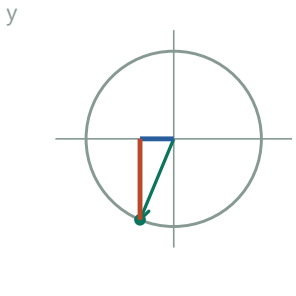


Reference slope 2.4 = 12/5

Both components are negative in the third quadrant, so their ratio is positive.

2. Place the reference triangle

Quadrant III places both signed coordinates below zero. Choose x=−5 and y=−12. The radius length is positive; find it by adding the coordinate squares.



A 5–12–13 triangle

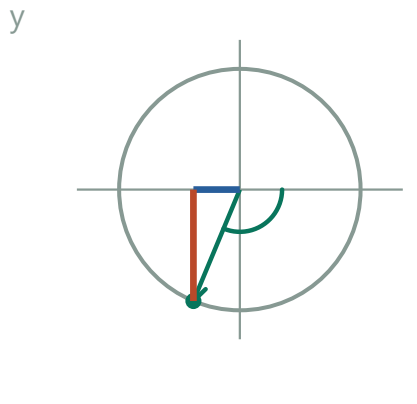
Move 5 to the left and 12 down.

1. 2.4=24/10=12/5 ⇒ (x,y)=(−5,−12)
Both coordinates are negative in quadrant III, preserving the positive slope.

2. r=√(25+144)=13
Use Pythagoras before forming the signed ratios.

3. Compute signed sine and cosine

Use reference legs 12 and 5 with hypotenuse 13. Both coordinates are negative: sine=−12/13, cosine=−5/13. Both are rational.



Negative cosine and sine

Divide the signed components by 13 to obtain −5/13 and −12/13.

Check. Their quotient is 12/5=2.4 and their squares sum to one.

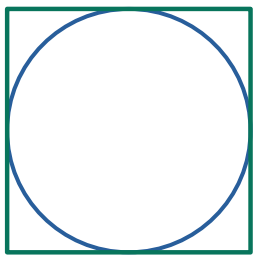
Exercise 363 · Sphere and cylinder areas

Printed page 36 · Surface area / Sphere geometry

Show that the surface area of a sphere is two thirds of the total surface area of any cylinder circumscribed around the sphere (the sphere therefore touches both bases of the cylinder as well as its lateral surface).

1. Read the touching conditions

Let the sphere radius be r. Contact at both cylinder bases puts the full diameter between those bases. Contact with the side places the cylinder wall exactly r away from its axis.

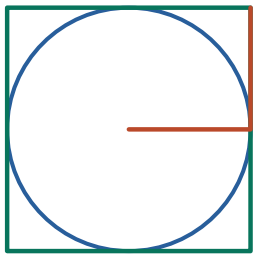


Identify the side and end contacts

The smallest enclosing cylinder has radius r and its two end planes touch opposite poles.

2. Find the cylinder dimensions and areas

The cylinder radius is r and height is 2r. Its two bases have combined area $2\pi r^2$ and its lateral area is $2\pi r \cdot 2r = 4\pi r^2$.

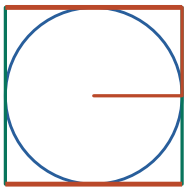


The sphere spans the cylinder height

Its height is 2r, so lateral area equals circumference $2\pi r$ times 2r.

3. Compare surface totals

The sphere's $4\pi r^2$ is two thirds of the $6\pi r^2$ total.



Include the two circular bases

The two bases add $2\pi r^2$. The sphere's $4\pi r^2$ is two-thirds of the total $6\pi r^2$.

1. $S_{cyl} = 2\pi r^2 + (2\pi r)(2r) = 6\pi r^2$
Add the two disks and the wrapped rectangular wall.

2. $S_{sphere}/S_{cyl} = 4\pi r^2/(6\pi r^2) = 2/3$
Cancel πr^2 because the common radius is positive.

Check. The sphere's area equals the cylinder's lateral area alone.

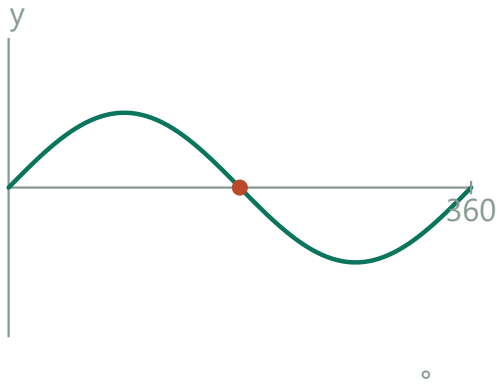
Exercise 364 · The first positive sine zero

Printed page 37 · Radian measure / The unit circle

With a graphing tool in degree mode, using the window $0 \le x \le 360$ and $-2 \le y \le 2$, examine the graph of $y = \sin x$. Explain why its first positive x-intercept has the value it does. Then graph the same equation *in radian mode*. The display will show many more oscillations than before, making it difficult to see the first positive x-intercept. To make that intercept clearer, reduce the window to $0 \le x \le 10$ and $-2 \le y \le 2$, then obtain an accurate reading of this x-value. Once you recognize it, explain how its value could have been predicted.

1. Translate a zero into a circle position

The sine of an angle is the height on a unit circle. Start at the positive horizontal axis, follow the rotation until it next reaches height zero, and count that fraction of a turn.

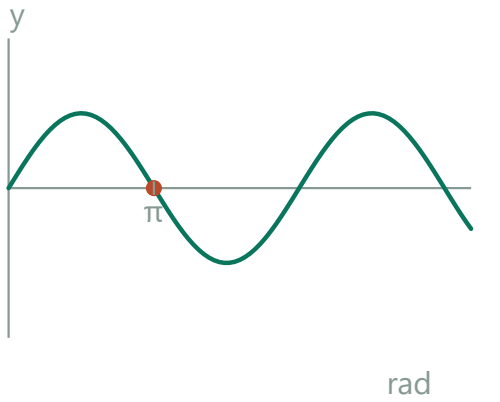


First positive zero in degrees

The sine height returns to zero after a half-turn.

2. Read the same position in either unit

A half revolution first returns the y-coordinate to zero, at 180° in degree mode or π≈3.14159 radians.

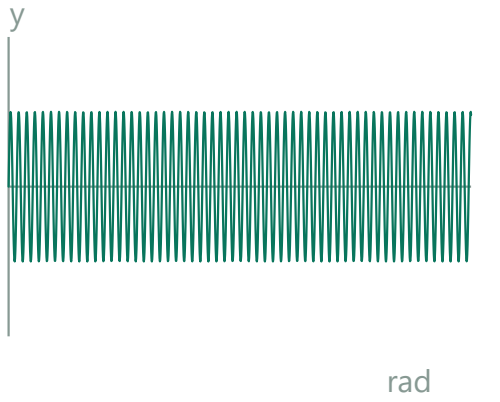


Switch the horizontal unit

In a 0–10 radian window the first zero after the origin is at π.

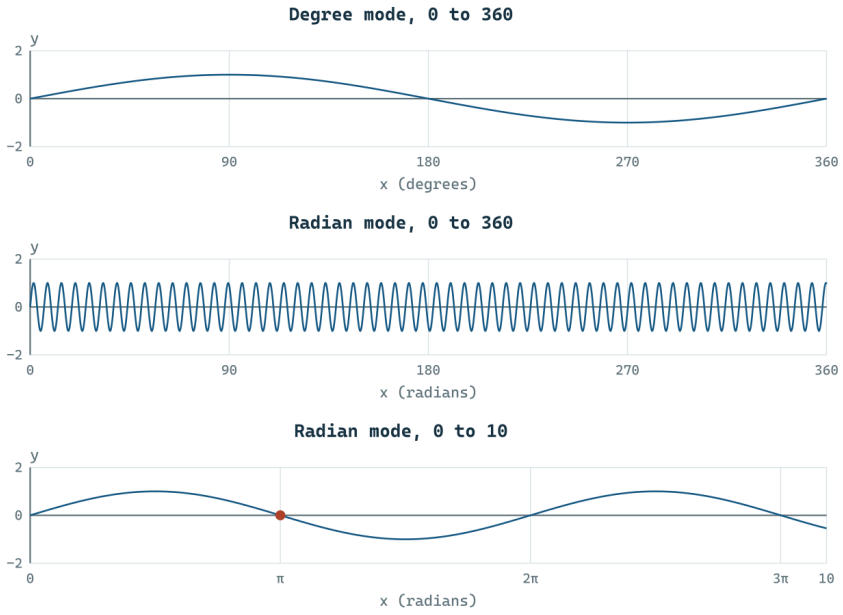
3. Explain the crowded display

A 0–360 radian window contains about 57 full turns; restricting it to 0–10 exposes the first zero.



Why the broad window looks crowded

The same 0–360 horizontal numbers now span about 57 full turns. Restrict the radian window to identify π.



Check. The conversion 180°·π/180°=π confirms the intercept.

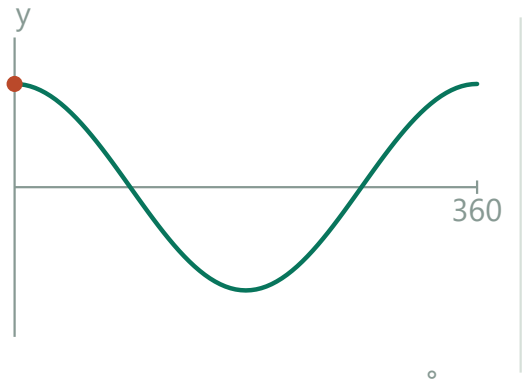
Exercise 365 · Compress cosine

Printed page 37 · Sinusoidal functions / Amplitude and period

With a graphing tool, graph each pair of equations on separate coordinate axes: **(a)** $y = \cos x$ and $y = \cos 2x$ **(b)** $y = \cos x$ and $y = \cos 3x$ **(c)** $y = \cos x$ and $y = \cos 0.5x$ What do the graphs of $y = \cos x$ and $y = \cos mx$ have in common, and how do they differ? Use what you have learned to graph $y = \cos 6x$ by hand.

1. Start cosine at a crest

Use heights 1,0,−1,0,1 as the phase inside cosine progresses through 0°,90°,180°,270°,360°. Changing m changes how fast that phase is reached.

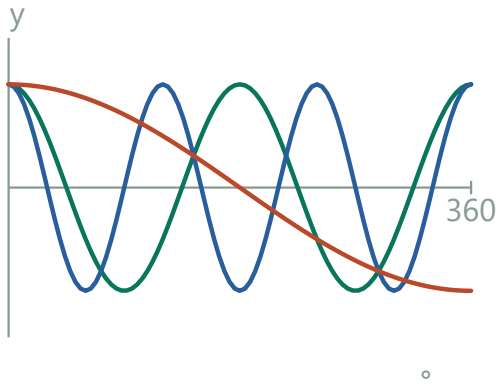


Measure a full baseline cycle

Follow the original curve until both the height and the direction of motion repeat.

2. Find the periods and sketch the curves

Each has amplitude 1 and midline 0. Divide horizontal distances by m: periods 180°,120°,720°,60°, compared with 360°. Use heights 1,0,−1,0,1 at successive quarter-periods; for m=6 these occur every 15°.

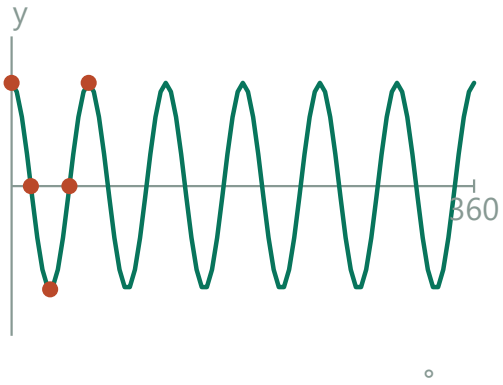


Change the input speed

The periods for multipliers 2, 3 and ½ are 180°, 120° and 720°. The vertical range stays fixed.

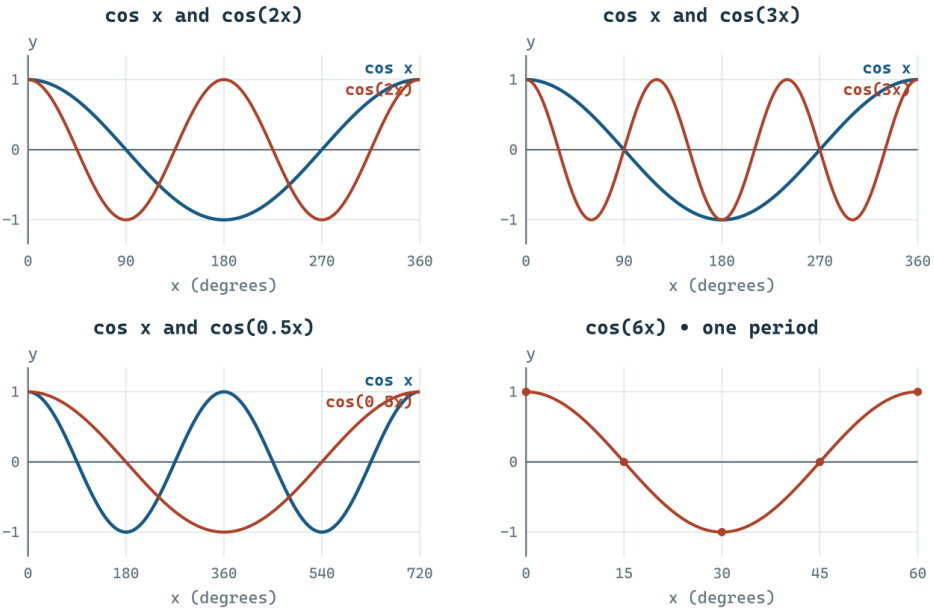
3. Extend the rule to other m

For m<0, cos(mx)=cos(|m|x), so the period is 360°/|m|. For m=0 the value is always 1, with no least positive period.



Construct one compressed cycle

The 6x curve completes a cycle in 60°. Mark its quarter-period landmarks.



Check. At 60° the m=6 graph has completed one full cycle.

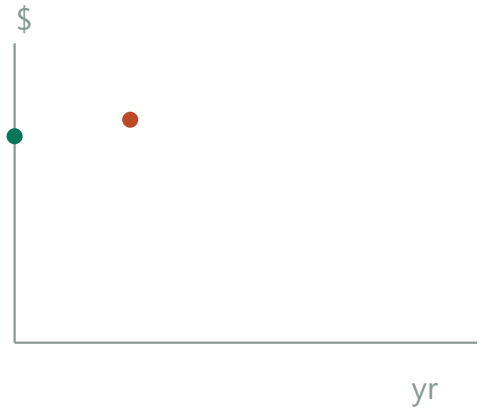
Exercise 366 · Compound annually

Printed page 37 · Compounding / Exponential functions

You have deposited \$1000 in a money-market account that earns 8 percent annual interest. Assuming no withdrawals or additional deposits are made, calculate how much money will be in the account one year later; two years later; three years later; t years later.

1. Translate 8% into a multiplier

If the opening balance is B dollars, the interest for one year is 0.08B dollars. The next balance is B+0.08B. On the next pass, start from the whole updated balance.

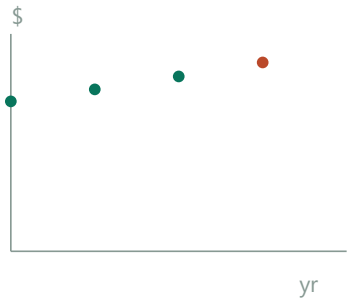


First credit

Start with \$1000 and multiply by 1.08 to reach \$1080.

2. Compute three successive balances

Multiply the current balance by 1.08 each year. After one, two, and three years it is \$1080, \$1166.40, and \$1259.71 (rounded).



Compound on each updated balance

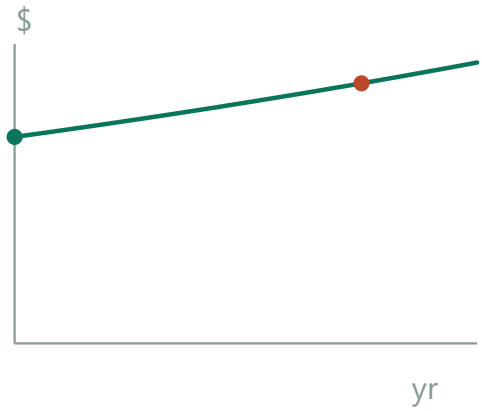
At years 2 and 3 the points are \$1166.40 and about \$1259.71.

1. $1000(1.08)^2 = 1080 + 0.08 \cdot 1080 = 1166.40$
Apply the second year's interest to the new balance.

2. $1166.40 \cdot 1.08 = 1259.712 \approx 1259.71$
Round the completed balance to cents.

3. Write a formula for t years

After t years the model gives 1000(1.08)^t dollars.



Repeat one factor per year

At integer year t, the model balance is $1000(1.08)^t$.

Check. The ratio of successive exact balances is always 1.08.

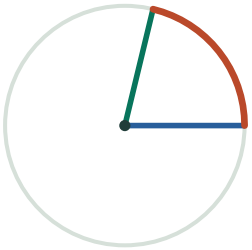
Exercise 367 · Arc to angle

Printed page 37 · Radian measure / Arc length

Find the size of central angle AOB, given that the length of arc AB is 16 cm and the length of radius OA is 12 cm.

1. Identify the curved length

AB is measured along the circumference. OA measures from center to rim. To express the opening at O in radians, count how many OA-lengths fit along the arc.

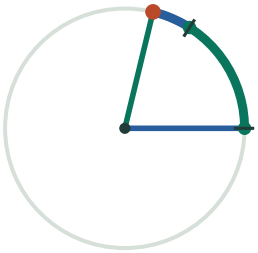


Separate the rim from the radius

The arc measures 16 cm and the straight radius measures 12 cm.

2. Divide with units

Both lengths are given in centimeters, so centimeters cancel in their quotient. Keeping the fraction exact makes a degree conversion easy.

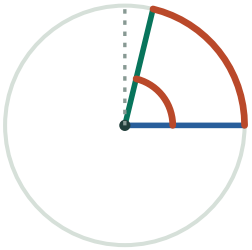


Count radii along the arc

The units cancel in 16 cm / 12 cm.

3. Compute the angle

Central angle equals length divided by radius, 16/12=4/3 radians, or about 76.39°.



Central angle 4/3 radians

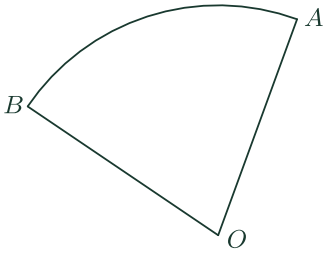
It is larger than 1 radian and smaller than a right angle.

Check. 12(4/3)=16 cm.

Exercise 368 · Half-size circle

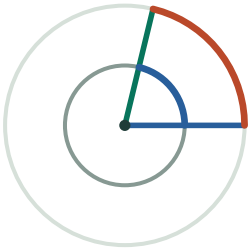
Printed page 37 · Radian measure / Arc length

(Continuation) Find the size of angle AOB, given that the length of arc AB is 8 cm and the length of radius OA is 6 cm.



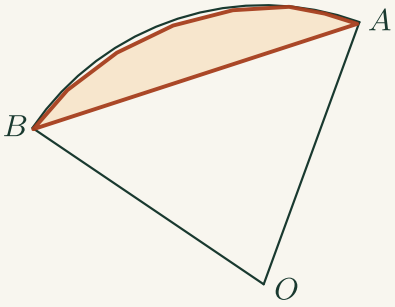
1. Compare the new drawing with the previous one

Both the arc and its radius have been halved. A dilation changes distances but preserves angles.



Shrink both measurements by ½

The smaller arc and radius have the same shape as before.

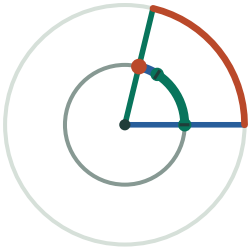


The arc from B to A

Measure this curved boundary along the rim. Arc length is the numerator in the radian ratio.

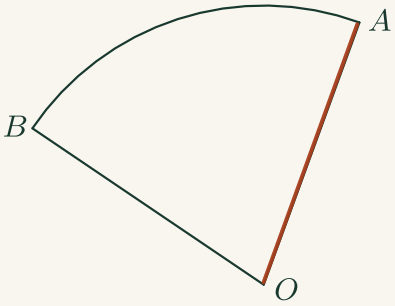
2. Use the definition in the new dimensions

Count 6-cm radius-lengths along an 8-cm arc. Write the result as a fraction and reduce it before comparing it to the preceding result.



Use the new radius as a unit

The arc now measures 8/6 radii, just as the original measured 16/12.

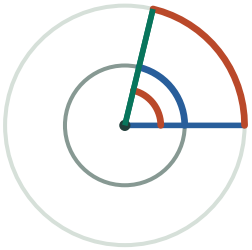


One reference radius

OA is a center-to-rim segment. Use its length as the measuring unit for the arc; OB has the same length.

3. Measure the unchanged angle

θ=8/6=4/3 radians, exactly the preceding angle. Halving both lengths preserves their ratio.



The angle stays at 4/3 radians

The two arcs subtend the same rays even though their lengths differ.

Check. Multiplying 4/3 by 180/π gives about 76.39° again.

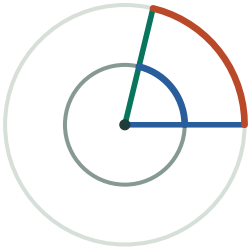
Exercise 369 · Determine an angle

Printed page 37 · Radian measure / Arc length

(Continuation) To find the size of the central angle AOB, it is enough to know the value of what ratio?

1. Consider what survives a change of scale

Doubling every length leaves the central angle unchanged. That suggests taking the ratio of the two lengths rather than their sum or difference.

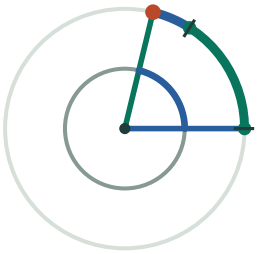


Change the scale, keeping the rays

Arc length and radius grow or shrink together.

2. Choose the order of division

An arc exactly as long as the radius defines one radian. A longer arc on the same circle should give a larger angle, so the arc length belongs in the numerator.

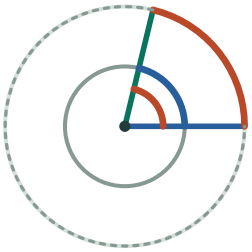


Divide the curved length by r

The scale factor cancels from s/r.

3. State the invariant ratio

Arc length divided by radius is exactly the central angle in radians. Use the arc corresponding to the intended sweep.



Use the intended arc

The ratio s/r measures the sweep along that arc in radians.

Check. Scaling the circle multiplies numerator and denominator by the same factor.

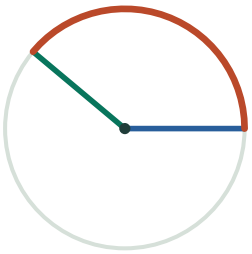
Exercise 370 · Compute a chord

Printed page 37 · Chords and inscribed angles / Radian measure

The radius of a circle is 9, and arc PQ has length 22. Find the length of chord PQ.

1. Translate arc into central angle

Its central angle is 22/9 radians. Divide the curved distance 22 by the radius 9. This gives the angle at the center in radians, the unit needed for the sine calculation next.

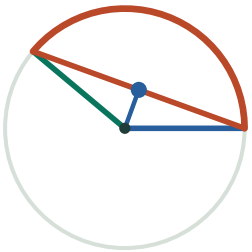


Arc PQ and its two radii

The known arc 22 and radius 9 give angle 22/9 radians.

2. Draw a perpendicular to the chord

The two radii to P and Q have the same length. The perpendicular from the center to the chord is also the angle bisector and splits the chord into equal halves.

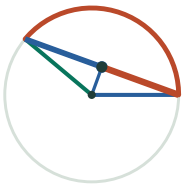


Bisect the isosceles triangle

The perpendicular from the center meets the chord at its midpoint.

3. Compute with one half-triangle

Bisect the isosceles triangle formed with its radii. Half the chord is 9 sin(11/9), hence the full length is 18 sin(11/9)≈16.9175.



Read the half-chord

Its length is 9 sin(11/9). Doubling gives the full chord, about 16.92.

1. sin(11/9)=(c/2)/9
The half-chord is opposite half of the central angle.

2. c/2=9sin(11/9) ⇒ c=18sin(11/9)
Multiply first by the radius, then by two; the angle is in radians.

Check. The cosine rule gives chord^2=162[1−cos(22/9)].

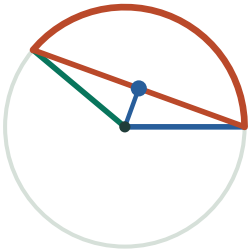
Exercise 371 · Why half an angle appears

Printed page 37 · Chords and inscribed angles / Right-triangle trigonometry

(Continuation) Working in *radian* mode, evaluate $2 \cdot 9 \cdot \sin(11/9)$. Notice that this expression provides the correct answer to the chord-length question. Why is this so? In particular, what angle does the number $11/9$ describe?

1. Use the chord midpoint

Join P and Q to the center to form an isosceles triangle. Mark its altitude onto PQ. The full arc subtends 22/9 radians, so each half-triangle has half of that angle at the center.

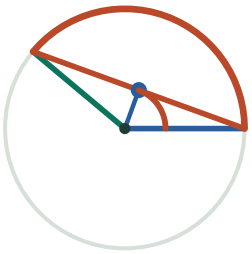


Find the chord midpoint

The perpendicular through the center splits the chord into equal parts.

2. Identify the angle and its opposite side

The perpendicular from the center bisects the chord and its central angle. Thus 11/9 radians is the angle between one endpoint radius and that perpendicular. The opposite side is half the chord, with hypotenuse 9.

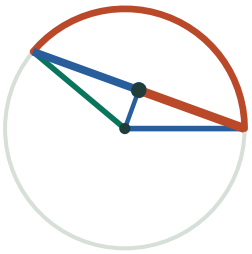


Identify the angle 11/9

It is half the angle swept by the 22-unit arc, measured at the center.

3. Recover both halves

In radian mode, 9 sin(11/9) gives one half-chord. Multiply by 2 to obtain about 16.92 length units for the whole straight segment.



Double the opposite leg

Each half-chord is 9 sin(11/9), so both halves total 2·9 sin(11/9).

Check. Doubling that opposite side yields about 16.92, less than diameter 18.

Exercise 372 · Degree to radian table

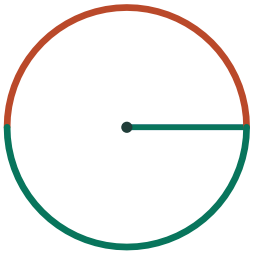
Printed page 37 · Angle measure / Radian measure

Consider a circle centered at O, and one of its arcs PQ. Determine, in exact form and then approximate to two decimal places, the *radian measure* of angle POQ, when its degree measure is

(a) 45 **(b)** 75 **(c)** 100 **(d)** 180 **(e)** 360

1. Connect the central angle to an arc

Join O to P and Q. A full turn cuts off an arc of length 2πr in a circle of radius r, so its radian measure, arc length divided by radius, is 2π. That same turn measures 360°. One degree therefore represents π/180 radians.

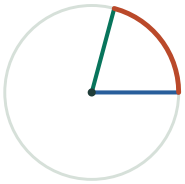


One turn spans 2π radians

Dividing the 360° revolution into degrees makes one degree equal to π/180 radians.

2. Convert 45° and 75°

For (a), 45(π/180)=π/4≈0.79 radians. The 45° angle occupies an eighth of a full turn. For (b), 75(π/180) reduces by dividing 75 and 180 by 15, giving 5π/12≈1.31 radians.



45° and 75°

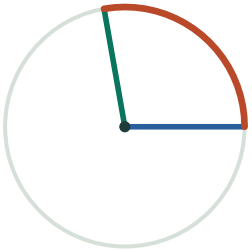
Multiply each degree measure by π/180 to obtain π/4 and 5π/12 radians. The arc shown is 75°.

1. 45°·π/180°=π/4
Cancel the degree units and divide 45 and 180 by 45.

2. 75°·π/180°=(75/180)π=5π/12
Reduce the fraction by 15.

3. Convert the obtuse angle

For (c), 100(π/180)=5π/9≈1.75 radians. A right angle is π/2≈1.57 radians, so the larger answer is consistent with a 100° angle.

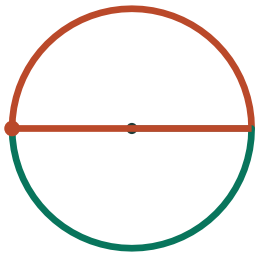


100° is beyond a quarter-turn

Its 5π/9 radians exceed π/2.

4. Use a half-turn and a full turn

For (d), 180(π/180)=π≈3.14 radians. For (e), 360(π/180)=2π≈6.28 radians. In each part the expression containing π is exact. The two-decimal number approximates the same angle.



Half-turn and full turn

Each semicircle contributes π radians; the whole circle contributes 2π.

Check. Dividing by 2π gives the same fraction of one turn as dividing the input by 360.

Exercise 373 · Map the unit square

Printed page 37 · Matrix transformations / Determinants and area

You have seen how rotations and reflections can be represented using matrices, but most 2×2 matrices do not represent isometries. Consider the matrix $\mathbf{K} = \begin{bmatrix} 2 & 1 \\ 1 & 3 \end{bmatrix}$, for example. Apply $\mathbf{K}$ to the unit square. What is the image polygon and what is its area?

1. Use the columns as images of the basis

The bottom and left edges are represented by (1,0) and (0,1). Multiplication by K picks out its first and second columns respectively.

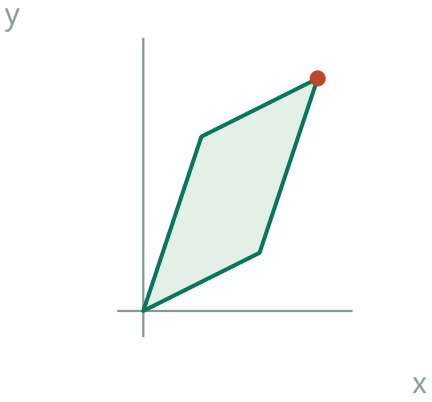


Read the two matrix columns

The images of (1,0) and (0,1) are the edges (2,1) and (1,3).

2. Find the image vertices

The edges from the origin become (2,1) and (1,3); the opposite corner becomes (3,4).

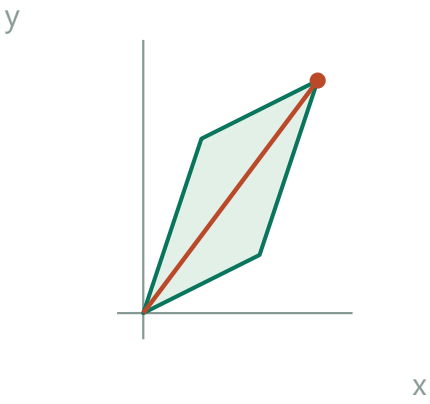


Add the two edges

Their sum (3,4) is the opposite vertex of the mapped square.

3. Measure area from two edge vectors

For vectors (a,b) and (c,d), the parallelogram they span has area equal to the magnitude of ad−bc. Here the four entries are the two column coordinates in the same order.

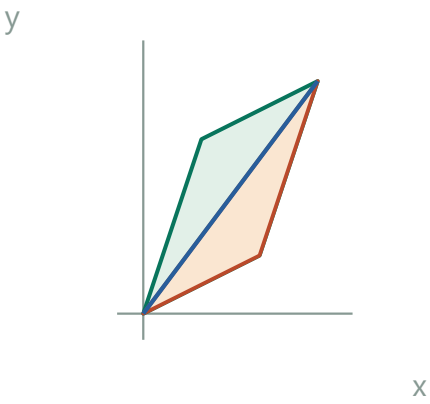


Calculate signed area

The crossed component products give 2·3−1·1.

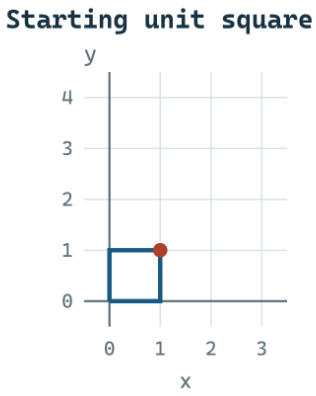
4. Evaluate the image area

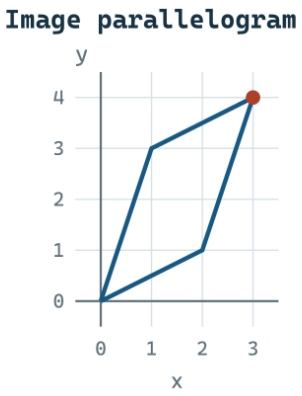
The image is a parallelogram of area 2·3−1·1=5.



Interpret the determinant

The filled parallelogram has area 5 square units.





Check. The shoelace formula on (0,0),(2,1),(3,4),(1,3) gives 5.

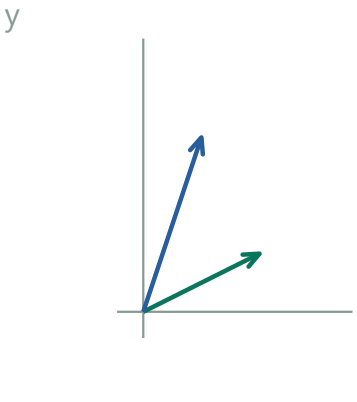
Exercise 374 · Area of transformed square

Printed page 37 · Matrix transformations / Determinants and area

(Continuation) Let $\mathcal{R}$ be the region that results when the matrix $\mathbf{M} = \begin{bmatrix} a & c \\ b & d \end{bmatrix}$ is applied to the unit square. Show that the area of $\mathcal{R}$ is $|ad - bc|$. The number $ad - bc$ is called the *determinant* of the matrix $\mathbf{M}$. Where have you seen this before?

1. Label the two transformed edges

The unit horizontal vector maps to (a,b) and the unit vertical vector to (c,d). By linearity their sum maps to the far corner, and parallel opposite edges survive.

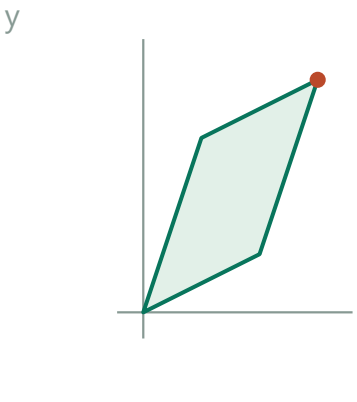


Label the two image edges

The green and blue vectors represent (a,b) and (c,d) for a nondegenerate example.

2. List the boundary points

The vertices in order are (0,0),(a,b),(a+c,b+d),(c,d).

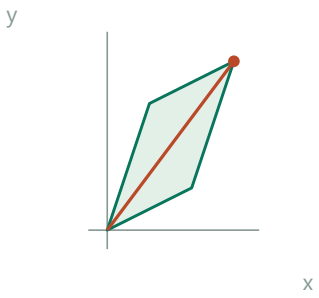


Complete the parallelogram

The opposite vertex must be (a+c,b+d).

3. Show the cancellation

For the ordered vertices, twice the signed area is a(b+d)+(a+c)d−b(a+c)−c(b+d). Expanding cancels ab and cd, leaving 2ad−2bc.



Split into oriented triangles

The shoelace contributions cancel in pairs except for the signed area ad−bc.

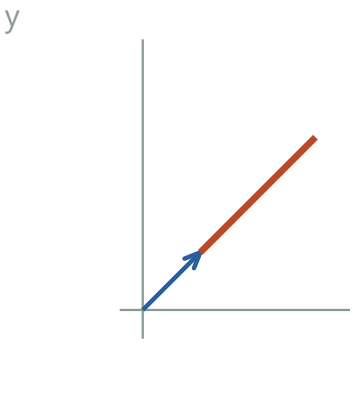
1. $2A_s = a(b+d)+(a+c)d-b(a+c)-(b+d)c$
Use successive vertex pairs in the signed shoelace sum.

2. $2A_s = ab+ad+ad+cd-ab-bc-bc-cd$
Distribute each product; ab and cd each cancel.

3. $A_s = ad-bc$
Divide by two. Take the absolute value for ordinary area.

4. Interpret sign and degeneracy

The shoelace formula gives signed area ad−bc, so ordinary area is its absolute value. Zero determinant means collapse onto a line or point.



A zero determinant collapses area

Parallel image edges lie on one line, producing ordinary area zero. In general area is |ad−bc|.

Check. The same determinant is the denominator in the 2×2 inverse and measures independence of the columns.

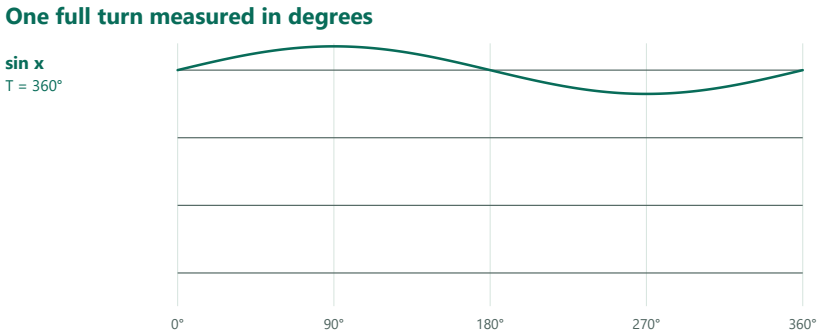
Exercise 375 · Periods in degrees

Printed page 38 · Periodic functions / Amplitude and period

When working in degree mode, we say that the *period* of the graph $y = \sin x$ is 360. What does this statement mean? What is the period of the graph of $y = \cos x$? What is the period of the graph $y = \tan x$? What is the period of the graph $y = \sin 2x$? What is the period of the graph $y = \sin mx$?

1. Define a repetition

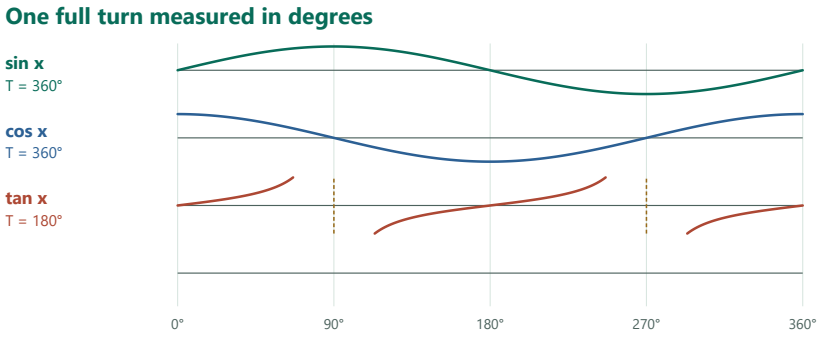
Translate a graph horizontally by T degrees. Its shape lines up with itself if f(x+T)=f(x) for every allowed x and the domain repeats too. The usual period is the smallest positive T that does this. For sine, a full revolution returns a point on the unit circle to its original height, so sin(x+360°)=sin x. Adjacent crests are 360° apart.



Each row has its own zero line. Dashes in the tangent row show undefined angles.

2. Recognize full and half turns

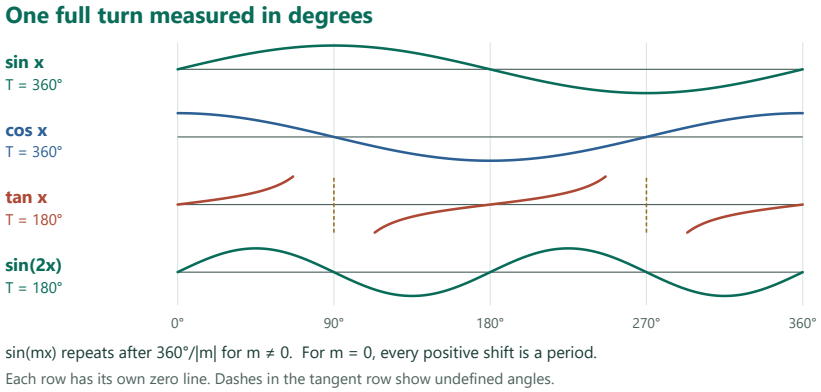
Cosine reads the horizontal coordinate, which also completes one cycle in 360°. Tangent reads the slope of the ray. After 180° both its vertical and horizontal components have opposite signs, leaving their ratio unchanged. Adjacent tangent branches repeat every 180°, including their breaks at vertical rays. The cosine row and tangent row thus have periods 360° and 180°, respectively.



Each row has its own zero line. Dashes in the tangent row show undefined angles.

3. Adjust for the internal multiplier

For sin(2x), advancing x by 180° advances the angle 2x by a full 360°, so its period is 180°. For sin(mx) with m≠0, the inside angle changes by mT. The smallest positive T that changes it by a full turn in either direction satisfies |m|T=360°, giving T=360°/|m|. When m=0 the graph is the constant 0. Every positive shift repeats it, so it has no smallest positive period.



Check. Adding one such period advances the underlying angle by one full turn (a half-turn for tangent).

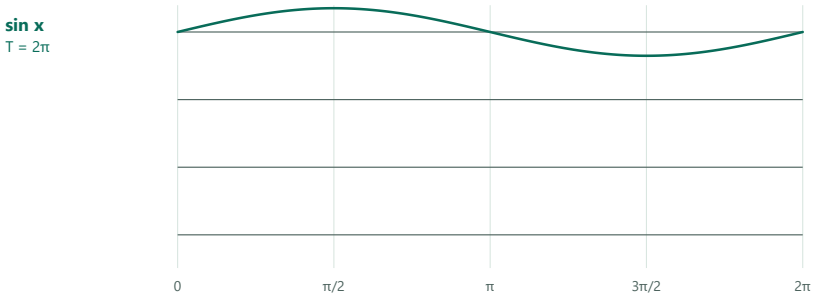
Exercise 376 · Periods in radians

Printed page 38 · Periodic functions / Radian measure

When working in radian mode, we say that the *period* of the graph $y = \sin x$ is 2π. What does this statement mean? What is the period of the graph of $y = \cos x$? What is the period of the graph $y = \tan x$? What is the period of the graph $y = \sin 2x$? What is the period of the graph $y = \sin mx$?

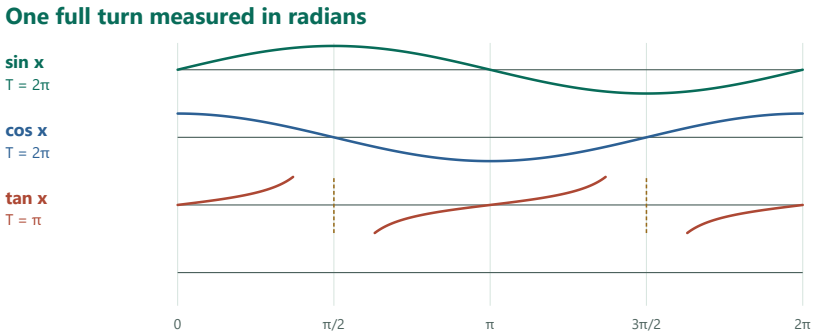
1. Change the unit, keeping the motion

One complete trip around the unit circle is 2π radians. Tangent compares rise to run, so reversing both coordinates after only π radians already gives the same quotient.



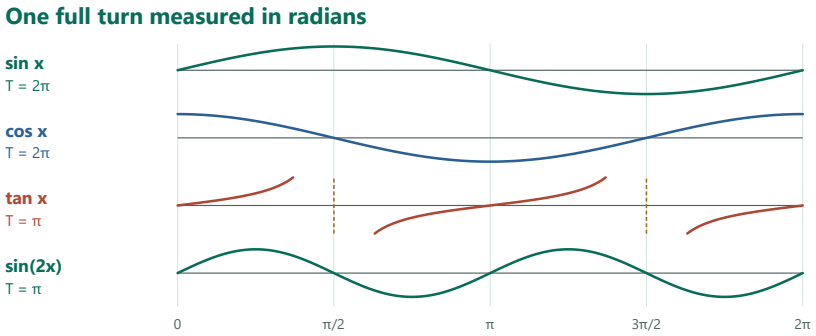
2. List the basic periods

Sin x and cos x each repeat first after 2π, tan x after π, and sin 2x after π.



3. Scale the period and handle m=0

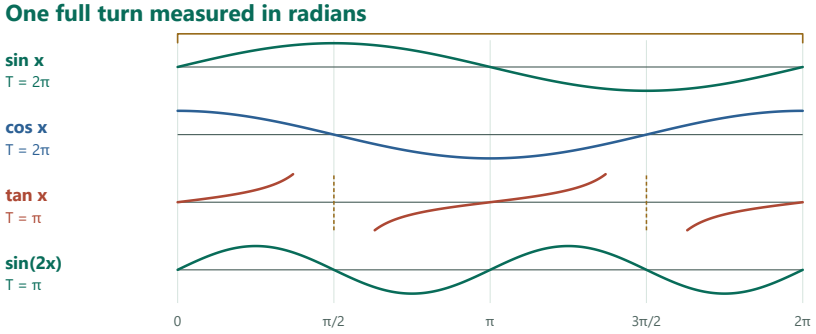
For m≠0 the fundamental period of sin(mx) is 2π/abs(m). At m=0 the function is constant.



sin(mx) repeats after 2π/|m| for m ≠ 0. For m = 0, every positive shift is a period.

4. Specify what smallest repeat means

Specifically, a period T satisfies f(x+T)=f(x) throughout the domain; the fundamental period is the least positive such T. A constant has every positive T as a period and hence no least one.



sin(mx) repeats after 2π/|m| for m ≠ 0. For m = 0, every positive shift is a period.

The bracket spans the shortest positive shift that reproduces every sine value.

Check. Converting the degree periods by multiplication by π/180 gives the same list.

Exercise 377 · Exponent mistakes with subtraction

Printed page 38 · Exponent rules / Algebraic manipulation

Are there errors in the following student responses? If so explain the error.

(a) Asked to find an expression equivalent to $x^8 - x^5$, a student responded x^3.

(b) Asked to find an expression equivalent to $\dfrac{x^8 - x^5}{x^2}$, a student responded $x^6 - x^3$.

(c) Another student said that $\dfrac{x^2}{x^8 - x^5}$ is equivalent to $\dfrac{1}{x^6} - \dfrac{1}{x^3}$.

1. (a) Test the proposed subtraction

The polynomial difference is defined for all real x. One unequal value disproves an identity claimed for every x.

1 **$x^8 - x^5 = x^5(x^3 - 1)$**

Factoring x^5 out of both terms leaves a difference inside parentheses.

2 **$2^8 - 2^5 = 256 - 32 = 224$**

Evaluate the original at the permitted input x = 2.

3 **$2^3 = 8 \neq 224$**

The proposed x^3 fails. Subtracting exponents describes a quotient.

2. (b) Divide each numerator term

Here x^2 is a denominator, so the original expression requires $x \neq 0$.

1 **$(x^8 - x^5)/x^2 = x^8/x^2 - x^5/x^2$**

A common nonzero denominator divides each term in the numerator.

2 **= x^(8 − 2) − x^(5 − 2)**

Apply the quotient law separately.

3 **$= x^6 - x^3,\ x \neq 0$**

Both exponents simplify as claimed on the original domain.

3. (c) Factor the entire denominator

The denominator is $x^5(x^3 - 1)$. For real x, its factors vanish at 0 and 1 respectively. Both values are excluded.

1 **$x^2/(x^8 - x^5) = x^2/[x^5(x^3 - 1)]$**
Keep the denominator together and factor out x^5.

2 **$= 1/[x^3(x^3 - 1)],\ x \neq 0, 1$**
Cancel the common nonzero factor x^2.

3 **At x = 2: original = 1/(8·7) = 1/56**
Substitute a value allowed by both expressions.

4 **$1/2^6 - 1/2^3 = 1/64 - 8/64 = -7/64$**
The proposed difference has the opposite sign, so it is not equivalent.

Check. At x=2 the third original quantity is positive, whereas $1/2^6 - 1/2^3$ is negative.

Exercise 378 · Regress on square-root height

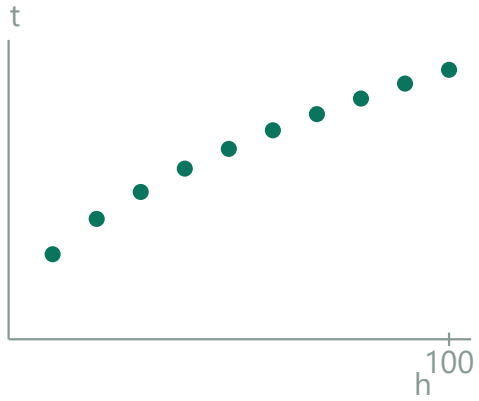
Printed page 38 · Linear regression / Data linearization

The table at right shows how many seconds are needed for a stone to fall to earth from various heights (measured in meters). You have already made a scatter plot of this nonlinear data. This time, calculate the *square roots* of the heights and enter them in a new column. A scatter plot of the relation between the second column and the new column should reveal a linear relationship. Find it, then use it to extrapolate how much time is needed for a stone to fall 300 meters.

height	*time*
10	1.42
20	2.01
30	2.46
40	2.85
50	3.18
60	3.49
70	3.76
80	4.02
90	4.27
100	4.50

1. Choose a transformed horizontal axis

Keep time as the vertical response. Replace each horizontal height by its square root, since the earlier free-fall pattern suggested time grows roughly as the square root of height.



Original fall-time measurements

Compare the curved trend in measured time versus height.

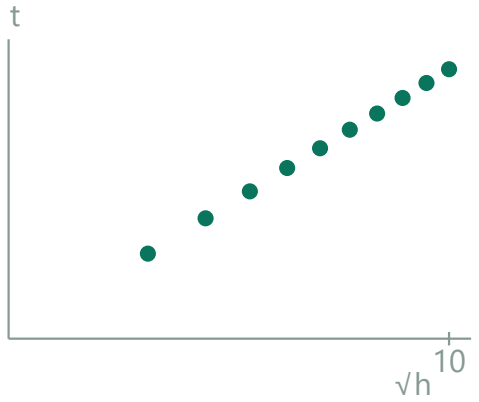
height	*time*
10	1.42
20	2.01
30	2.46
40	2.85
50	3.18
60	3.49
70	3.76
80	4.02
90	4.27
100	4.50

Original height column

Take the square root of each left-hand height to obtain the new horizontal coordinate. Carry its row along with it.

2. Construct the transformed table

The square-root height column, rounded to four decimals, is 3.1623, 4.4721, 5.4772, 6.3246, 7.0711, 7.7460, 8.3666, 8.9443, 9.4868, 10.0000. Pair these with the measured times in their original order.

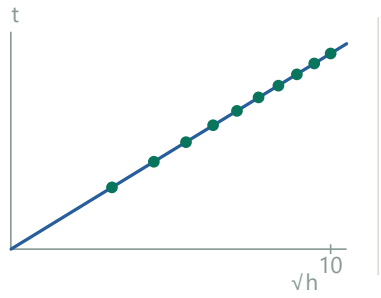


Change only the horizontal coordinates

The height 100 moves to horizontal coordinate 10. Times keep their original row partners.

3. Fit using full precision

Use the unrounded roots for regression. Transform heights to $\sqrt{10}, \sqrt{20}, \ldots, \sqrt{100}$. With $\sqrt{h}$ as predictor, regressing the measured time gives $t \approx 0.450283\sqrt{h} - 0.003301$.



Fit time against √height

The transformed points follow $t \approx 0.450283\sqrt{h} - 0.003301$.

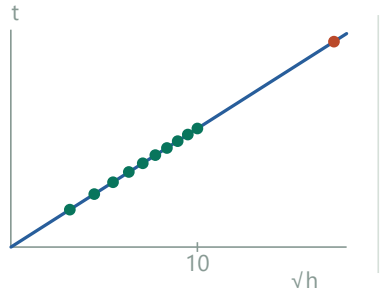
1. $x = \sqrt{h}$, $y = t$; mean $x \approx 7.1050934$, mean $y = 3.196$
First transform every height. Then average the transformed heights and the recorded times separately.

2. $\Sigma(x - \text{mean } x)^2 \approx 45.17648$; $\Sigma(x - \text{mean } x)(y - \text{mean } y) \approx 20.34219$
Collect squared horizontal deviations and paired products from all ten measurements.

3. $m \approx 20.34219 / 45.17648 \approx 0.450283$; $b = \text{mean } y - m \cdot \text{mean } x \approx -0.003301$
Use the least-squares slope and the fact that the fitted line passes through the means. Round only the displayed results.

4. Predict for 300 meters

At $h = 300$ the estimate is 7.80 s (approximately).



Predict at √300

Locate $\sqrt{300} \approx 17.32$ on the transformed axis; the line gives approximately 7.80 s.

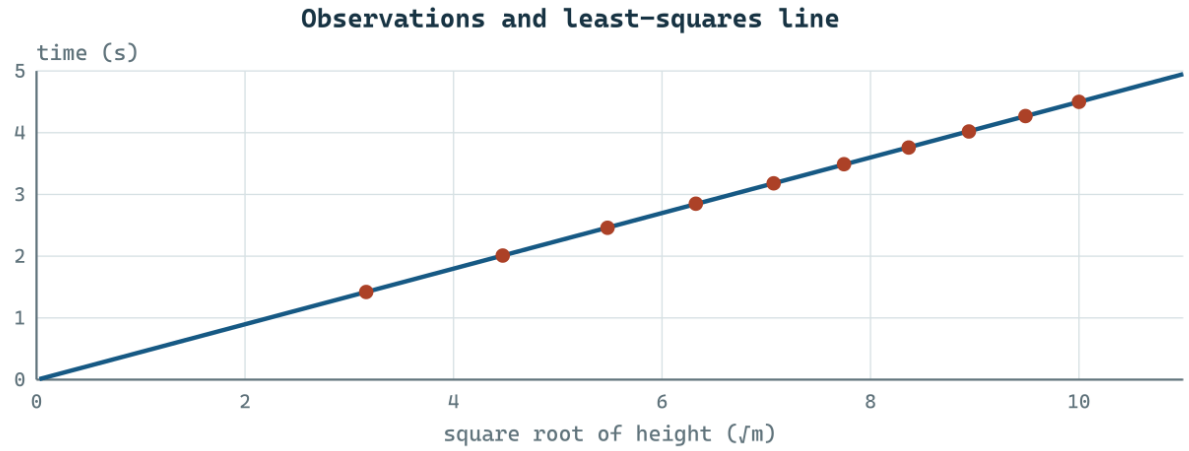
1. $u = \sqrt{300} \approx 17.3205$
The fitted predictor is square-root height, not height.

2. $t \approx 0.450283 \cdot 17.3205 - 0.003301 \approx 7.80$ s
Use the transformed predictor in the fitted line.

height	*time*
10	1.42
20	2.01
30	2.46
40	2.85
50	3.18
60	3.49
70	3.76
80	4.02
90	4.27
100	4.50

Largest observed drop

The measured heights stop at 100 meters. The requested 300-meter prediction extends the fitted relationship beyond the observed drops.



Check. The alternative regression of t^2 on h in exercise 336 predicts about 7.79 s.

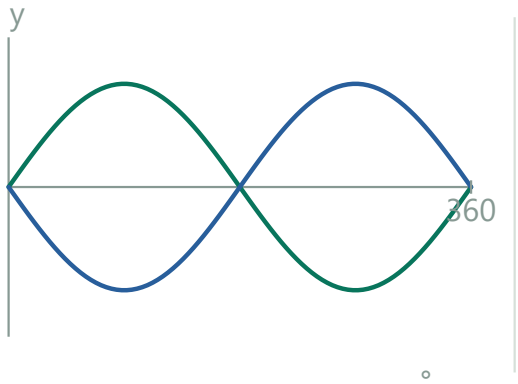
Exercise 379 · Parity of circular functions

Printed page 38 · The unit circle / Symmetry

Graph each of the following pairs of functions on a separate system of coordinate axes, and account for what you see:
(a) $y = \sin x$ and $y = \sin(-x)$ **(b)** $y = \cos x$ and $y = \cos(-x)$

1. Trace equal positive and negative turns

Start at (1,0). Rotate by x in one direction and by the same magnitude in the opposite direction. The two tips have the same horizontal position and opposite heights.

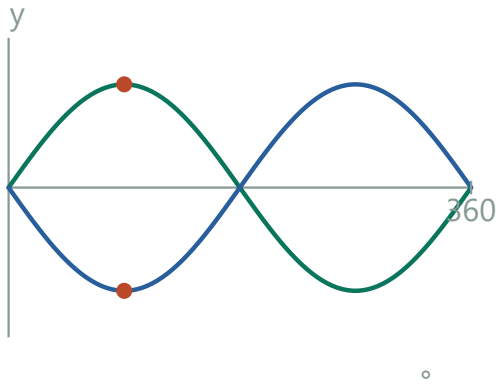


Reflect the angle in the x-axis

At opposite inputs on the circle, sine heights have opposite signs.

2. Interpret the sine pair

Reflecting a unit-circle point across the x-axis negates its y-coordinate only. Thus sin(−x)=−sin x; the two sine graphs are vertical reflections.

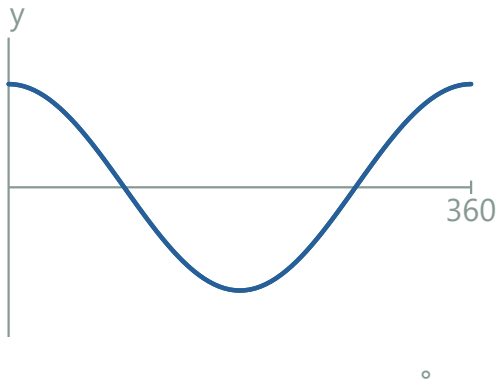


Compare the two sine graphs

The two traces are reflections across y=0; for example their outputs at 90° are +1 and −1.

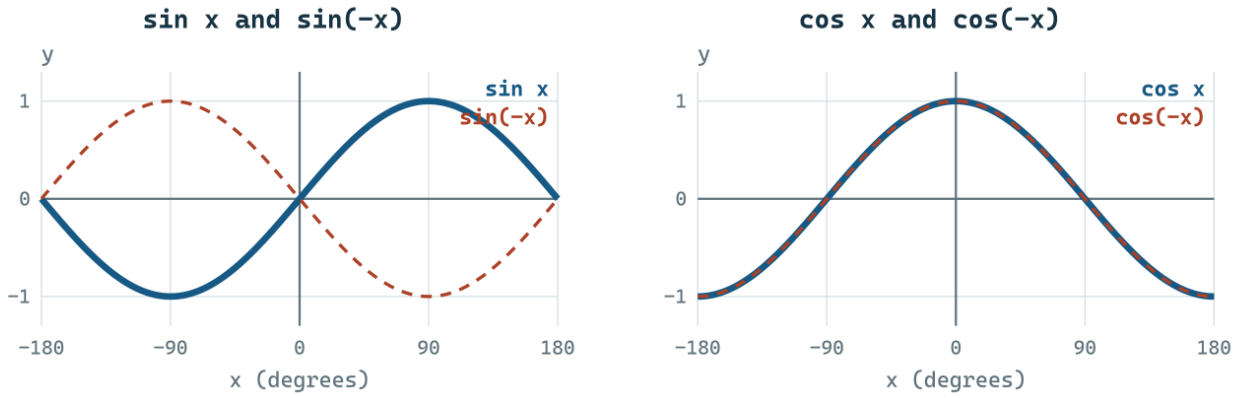
3. Interpret the cosine pair

Cos(−x)=cos x, so the cosine graphs coincide. Cosine uses only the horizontal coordinate. Reversing the direction angle leaves that coordinate unchanged, so at every input the two plotted heights agree.



Cosine traces coincide

The horizontal unit-circle coordinate stays fixed at opposite angles, giving cos(−x)=cos x.



Check. At 30° and −30° the sine values are ±½; the two cosines are both √3/2.

Exercise 380 · All points on a tangent level

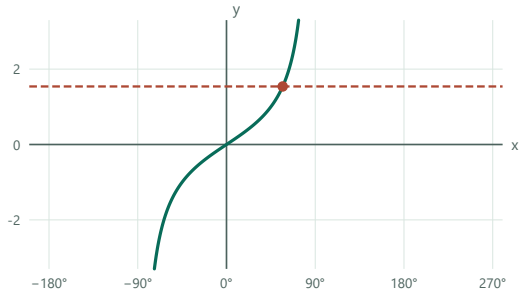
Printed page 38 · The tangent function / Trigonometric equations

Find a way of describing all the intersections of the line $y = 1.5399$ and the graph of $y = \tan x$. For what numbers m is it possible to solve the equation $m = \tan x$?

1. Intersect one tangent branch

On the branch from −90° to 90°, tangent increases through every finite slope. The horizontal level 1.5399 therefore has exactly one intersection on this branch.

One intersection on every tangent branch

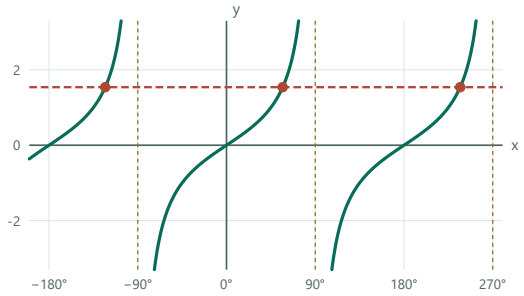


Set the height
The red line is y = 1.5399.
The central hit is x ≈ 57.00°.

2. Use periodicity to find the others

Moving the ray by half a turn preserves its slope. Keep y fixed at the chosen level and add integer multiples of half a turn to its angle.

One intersection on every tangent branch



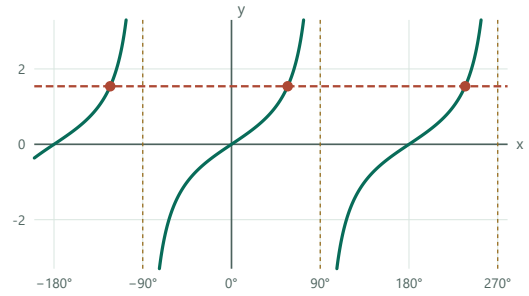
Set the height
The red line is y = 1.5399.
The central hit is x ≈ 57.00°.

Move one branch over
Each new intersection is a shift of 180° along the x-axis.

3. Give both angle conventions

In degree mode the intersections are (arctan(1.5399)+180°k,1.5399), where k ranges over the integers. In radian mode the x-coordinates are arctan(1.5399)+πk, with arctan also evaluated in radians.

One intersection on every tangent branch



Set the height
The red line is y = 1.5399.
The central hit is x ≈ 57.00°.

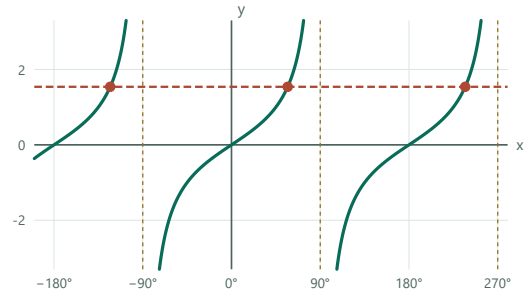
Move one branch over
Each new intersection is a shift of 180° along the x-axis.

Radian description
Add kπ to arctan(1.5399), for any integer k.

4. Determine all possible horizontal levels

Every real horizontal level m occurs once per branch since the slope of a nonvertical ray can be any real number.

One intersection on every tangent branch



Set the height
The red line is y = 1.5399.
The central hit is x ≈ 57.00°.

Move one branch over
Each new intersection is a shift of 180° along the x-axis.

Radian description
Add kπ to arctan(1.5399), for any integer k.

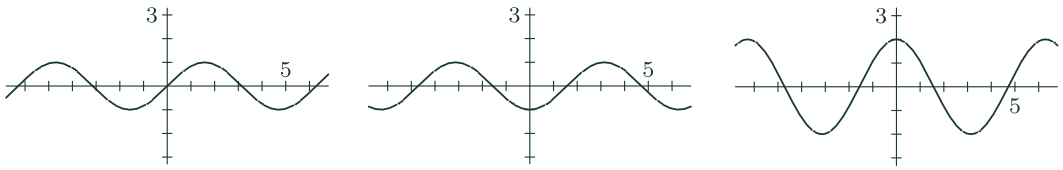
Each branch runs through every real height between its two vertical asymptotes.

Check. A ray through (1,m) has tangent m.

Exercise 381 · Read three radian waves

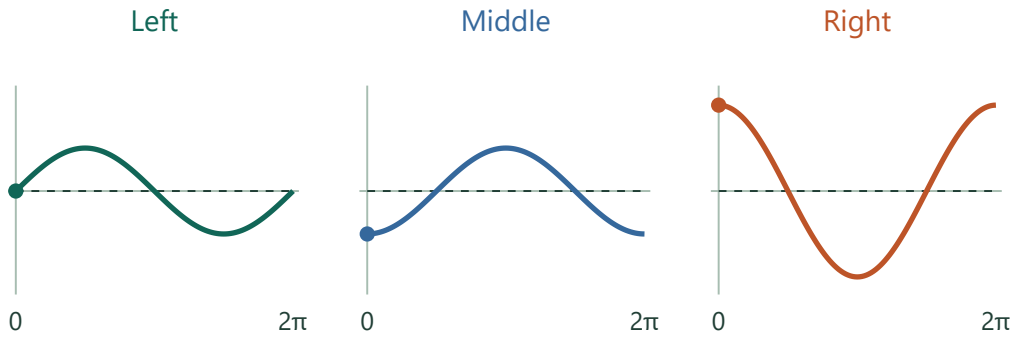
Printed page 38 · Sinusoidal functions / Graph interpretation

The *sinusoidal* graphs shown below appeared on a calculator in radian mode. Find equations that might have produced the graphs.



1. Read each graph's scale and phase

Find the halfway height between crest and trough. Then inspect x=0: an upward crossing of the midline suggests sine, a maximum suggests positive cosine, and a minimum suggests negative cosine. Check how far it is to the next matching point.



Compare the traces on the same horizontal and vertical scales. The dotted line is the midline. The colored points show where each trace meets the vertical axis.

2. Read the rising left curve

Left: crosses the origin rising, amplitude 1 and period about 2π, so y=sin x.

Position along the selected wave · 0



Left panel. Starting at the midline, the curve rises to height 1 at π/2, then returns to its original height and direction at 2π.



The left origin crossing

Observe the left curve pass through its centerline while rising. That motion determines its initial phase.

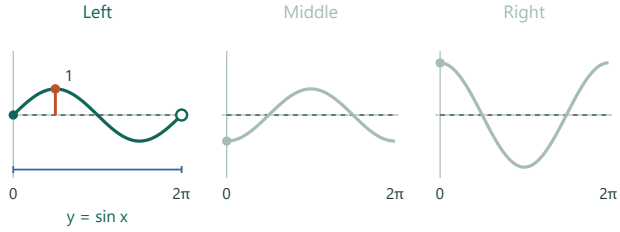
Compare settings · Position along the selected wave

π



At x = π, the height is 0.

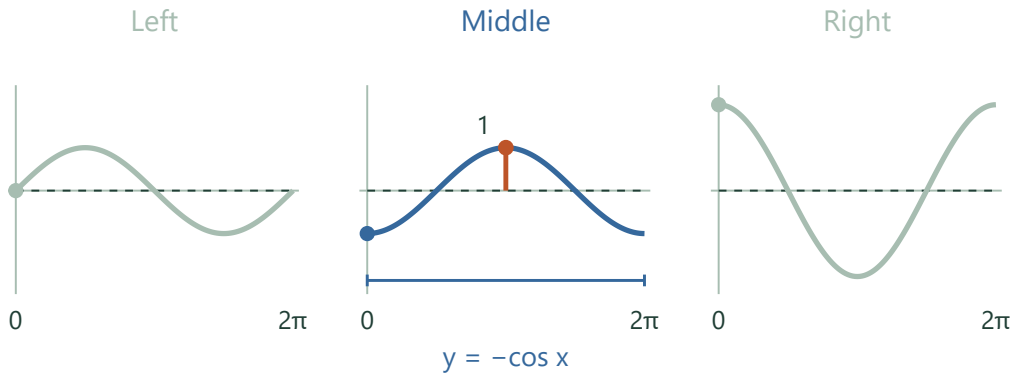
2π



At x = 2π, the height is 0.

3. Read the trough in the middle

Middle: starts at a trough -1 and has the same period, so $y=-\cos x$.



Middle panel. The curve starts one unit below the midline and reaches a crest at π. This start at a trough explains the negative cosine.



The middle trough

The middle curve is lowest at its own vertical axis. The sign of a cosine coefficient can encode this starting point.

4. Read the larger crest on the right

Right: starts at a crest $+2$, ranges down to -2, and repeats in 2π, so $y=2\cos x$.



Right panel. The curve starts at a crest, and the measured height from midline to crest is 2. The matching crest at 2π gives the period.

Check. At $x=\pi$ these predict $0, 1, -2$, consistent with the plotted midcycle positions.

Exercise 382 · Elbert population growth

Printed page 39 · Exponential growth and decay / Logarithms

The population of *Elbert* has been increasing at the rate of 2.4% per year. It has just reached 5280 (a milestone). What will the population be after ten years? after t years? After how many years will the population be 10 560?

1. Start with the reference population

The milestone is time zero. An increase of 2.4% in one year makes the next population the previous count multiplied by 1+0.024. After two years the same factor has been applied twice.

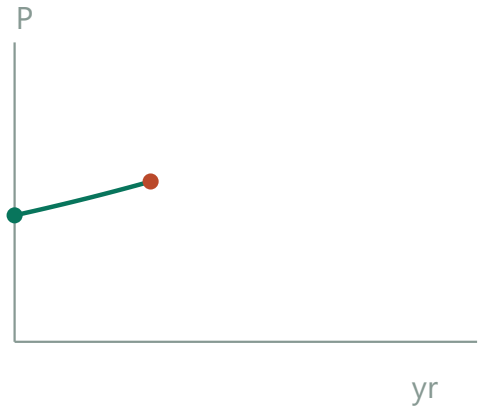


Use the current population as time zero

One year multiplies 5280 people by 1.024.

2. Write the model and evaluate ten years

P(t)=5280(1.024)^t. At 10 years this predicts about 6693 people.

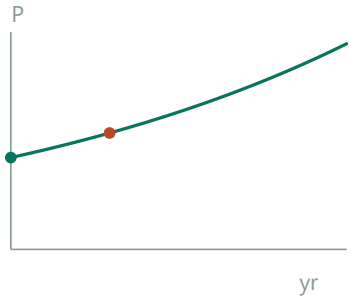


Ten annual increases

At year 10 the exponential model predicts about 6693 people.

3. Invert an exponential

Divide the desired future population by the starting population, then take logarithms. The exponent comes down as a factor because $\ln(a^t)=t\ln a$.



Let the time vary

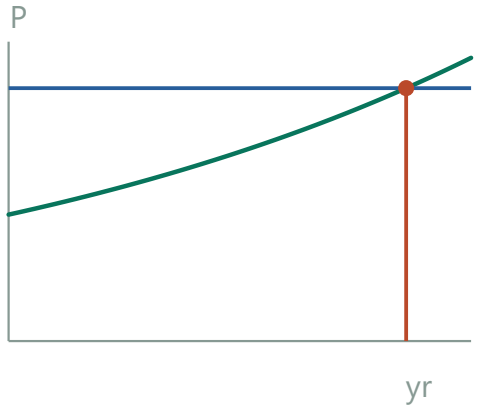
Each further year makes the same relative increase in a larger population.

1. $10560/5280=1.024^t \Rightarrow 2=1.024^t$
Divide out the initial population.

2. $\ln 2=t\ln(1.024) \Rightarrow t=\ln 2/\ln(1.024)$
Logarithms turn the exponent into a factor. The denominator is positive.

4. Find the doubling time

Doubling requires (1.024)^t=2, so t=ln 2/ln 1.024≈29.23 years in the continuous interpolation.



Read a doubled population

The horizontal level 10,560 meets the curve around year 29.23.

Check. At 29 whole years the model is below double; at 30 it is above.

Exercise 383 · Read a growth graph

Printed page 39 · Exponential growth and decay / Graph interpretation

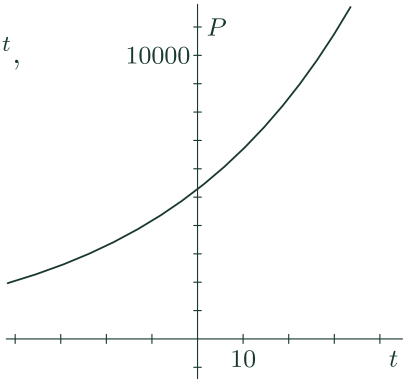
(Continuation) The graph of a function, $P(t) = 5280(1.024)^t$, that describes the population of *Elbert*, is shown on the right.

(a) What is $P(0)$, and what is its meaning?

(b) Use the graph to estimate the solution of $P(t) = 10\,560$.

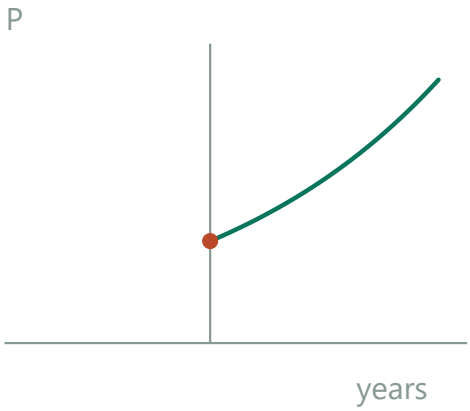
(c) Calculate $P(-30)$. What does this number mean?

(d) Comment on the part of the graph that lies outside the borders of the illustration. How would it look if you could see it, and what does it mean?



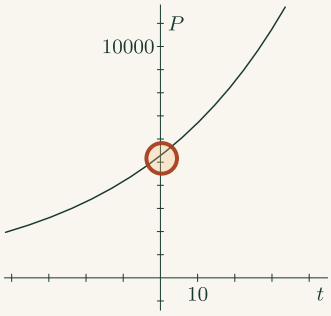
1. Read the origin and crossing

P(0)=5280, the starting population. The graph crosses 10,560 near year 29.



Locate the initial amount

The y-intercept gives 5,280 people at t=0.

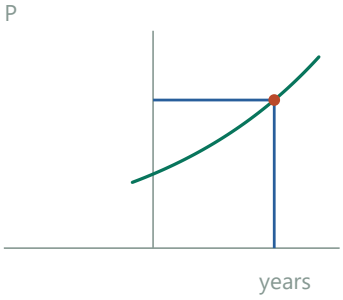


The reference-time intercept

At time zero the curve meets the vertical axis. Its height represents the reference population.

2. Interpret a negative time

P(−30)≈2592 models the population 30 years earlier.

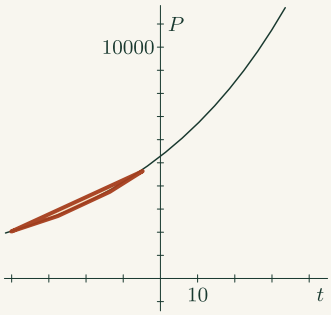


Trace the doubled height

From 10,560 on the vertical axis, move right to the curve and down to about 29 years.

1. $P(-30)=5280\cdot1.024^{-30}=5280/(1.024^{30})$
A negative time reverses thirty yearly growth factors.

2. P(−30)≈2592
Evaluate the denominator before rounding the population.

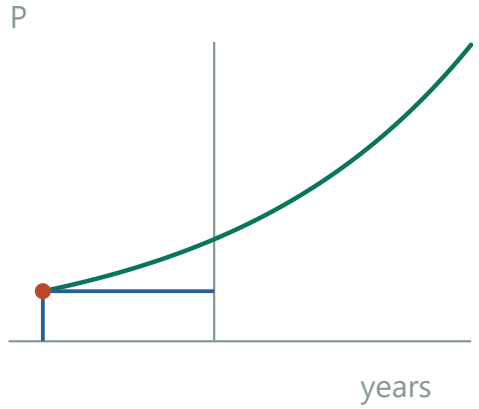


Backward extension

Following the curve left models earlier years. Its height decreases while remaining positive.

3. Extend the plot to either side

To the right the curve increases without bound. To the left it approaches zero from above. These are extensions of a constant-rate model; actual populations are whole people and rates can change.

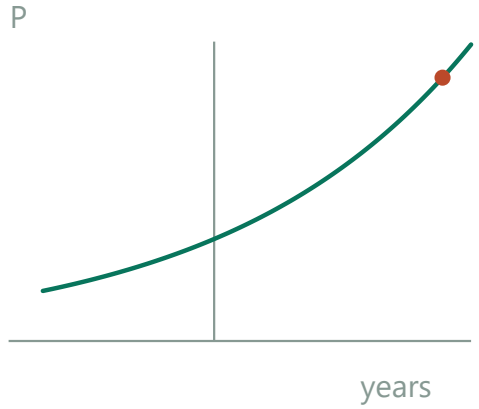


Extend backward in time

The model at t = −30 predicts about 2,592 people.

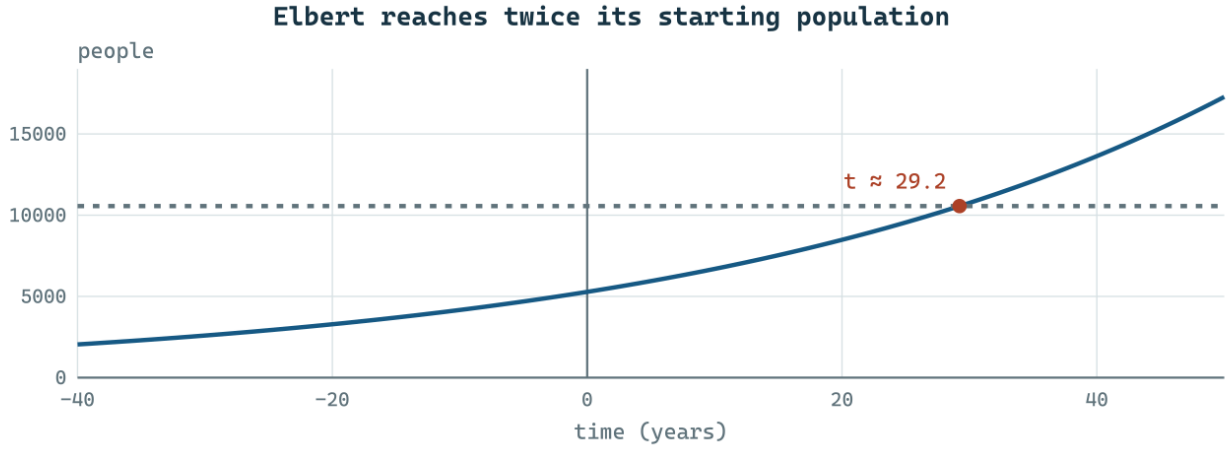
4. Decide what the extended model means

At each step backward divide by the growth factor 1.024. That stays positive and becomes smaller and smaller. Forward steps repeatedly multiply by a number greater than 1. Both statements assume the same rate continues.



Inspect long-term growth

The curve tends to zero far to the left and increases without a finite ceiling to the right.



Check. Multiplying 2592 by 1.024^{30} approximately restores 5280.

Exercise 384 · Invert real powers

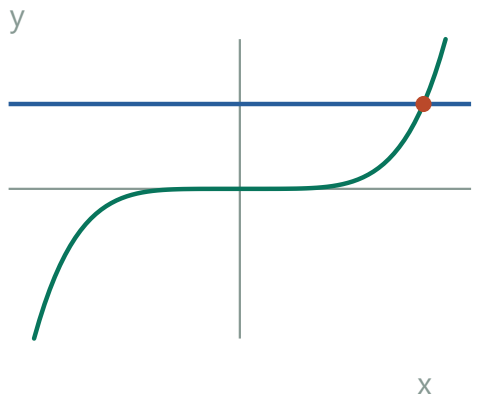
Printed page 39 · Radicals and rational powers / Exponent rules

Solve for x by hand.

(a) $x^5 = a^3$ **(b)** $x^{1/5} = a^3$ **(c)** $(1+x)^{15.6} = 2.0$ **(d)** $x^{-2} = a$

1. Invert an odd power

(a) x=the real fifth root of a^3. The real fifth-power function is increasing and accepts negative and positive inputs. Taking its real fifth root recovers exactly one x for any real a.

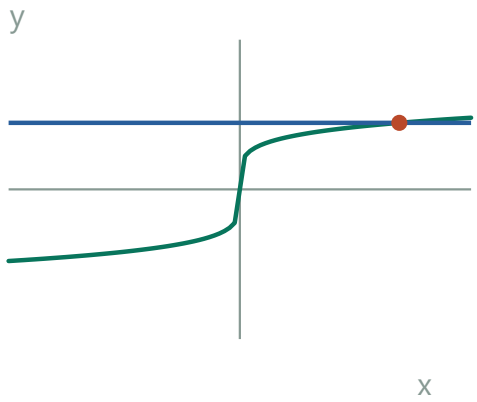


Use a one-to-one fifth power

Every horizontal line crosses $y=x^5$ once. Positive and negative targets each have one real fifth root.

2. Undo an odd root

(b) Raise both sides to the fifth power to get $x=a^{15}$.

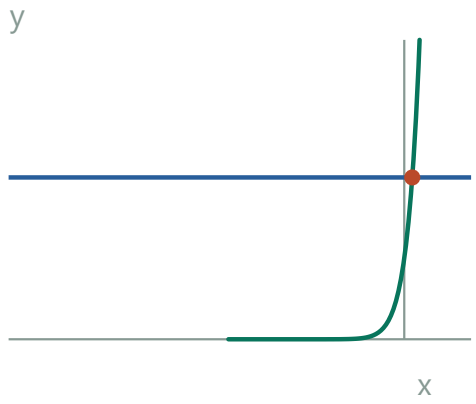


Undo the odd root

The fifth-root curve also has a single crossing at each output. Raising both sides to the fifth power recovers x.

3. Solve a fractional-power equation

(c) On the positive-base convention, x=2^(1/15.6)−1. If rational powers of negative bases are admitted, 15.6=78/5 also gives x=−2^(5/78)−1.

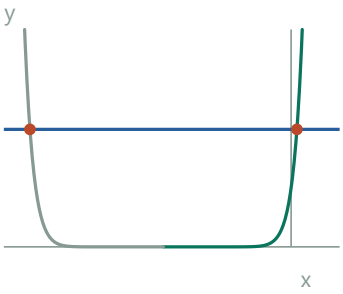


Use the positive-base branch

The plotted branch uses x+1>0 and meets y=2 once.

4. Explain the second convention

Since 78/5 is a rational exponent with odd denominator, the real fifth root can be taken first. Applying an even power removes its sign, which permits two base values on that convention.



Extend the rational power carefully

For exponent 78/5, taking the fifth root first then an even power permits the mirror branch for x+1<0.

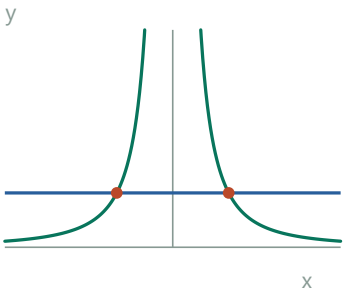
1. $u=x+1$; $(u^{(1/5)})^{78}=2$
For the rational-power convention the fifth root exists even when $u<0$.

2. $u^{(1/5)}=\pm 2^{(1/78)} \Rightarrow u=\pm 2^{(5/78)}$
Taking the even root gives both signs; the fifth power retains each sign.

3. $x=-1\pm 2^{(5/78)}$
Subtract one. The positive-base convention retains just the plus branch.

5. Undo a reciprocal square

(d) $x^{-2}=a$ requires $a>0$ and $x=\pm 1/\sqrt{a}$; otherwise no real solution.



Compare a reciprocal square

The sample level a=1 has two intersections, ±1. Every positive a gives two; zero and negative a give none.

1. $1/x^2=a \Rightarrow 1=ax^2$
Multiply by x^2 only on the original domain $x\neq 0$.

2. $x^2=1/a \Rightarrow x=\pm 1/\sqrt{a}$
A positive reciprocal square requires $a>0$; both nonzero roots work.

Check. In (c) the odd fifth root exists for negative inputs and the even 78th power removes its sign. In (d) neither x=0 nor a=0 is allowed.

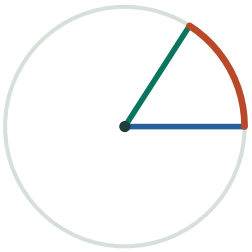
Exercise 385 · Two arcs, one circle

Printed page 39 · Radian measure / Arc length

In a circle of radius 5 cm, how long is a 1-radian arc? How long is a 2.2-radian arc?

1. Mark a one-radius arc

One radian is defined by a central angle cutting off an arc equal to one radius. Use the 5-cm radius as the measuring unit along the curve.

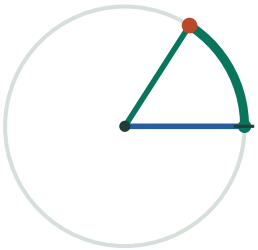


Mark one radian

A radius-length of arc on this circle measures 5 cm.

2. Scale to a noninteger angle

A 2.2-radian arc consists of two radius-lengths plus another one fifth of a radius-length. Convert the full count back to centimeters.

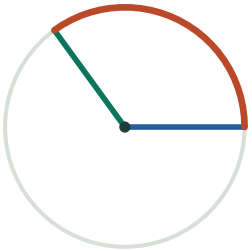


One radius on the rim

For $\theta = 1$ the arc is $r\theta = 5 \cdot 1 = 5$ cm.

3. Calculate the two lengths

Length $s = r\theta$ gives 5 cm and 11 cm. For $\theta = 1$ the arc is $5 \cdot 1$ cm. For $\theta = 2.2$ the arc is $5 \cdot 2.2$ cm. Both outputs are lengths because the angle counts units of one radius.



Extend the sweep to 2.2 radians

Its arc is 2.2 times 5 cm, or 11 cm.

Check. Dividing each length by 5 recovers its angle.

Exercise 386 · An equation in different dimensions

Printed page 39 · Planes in space / Coordinate geometry

On a number line (a one-dimensional context), the equation $x = 3$ describes a single point. In a two-dimensional context, what does the equation $x = 3$ describe? What does the equation $x = 3$ describe in a three-dimensional context?

1. List which coordinates are free

On a number line x is the only coordinate. Once x is fixed at 3 there is no choice left. Add a second coordinate y and its value can change independently.

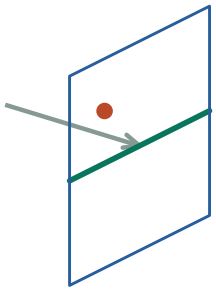


Keep x fixed in the xy-plane

The y-coordinate remains free, giving the line of points (3,y).

2. Build up the plane and space pictures

With x=3, moving vertically in the xy-plane never changes x. In space one can move independently in both the y and z directions while keeping that same x.

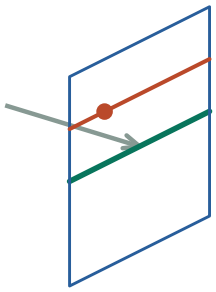


Allow z to vary as well

Each fixed height z adds another copy of the same line.

3. Name the resulting sets

In the plane it is the vertical line of all (3,y); in space it is the plane of all (3,y,z), parallel to the yz-plane.



Sweep out a plane

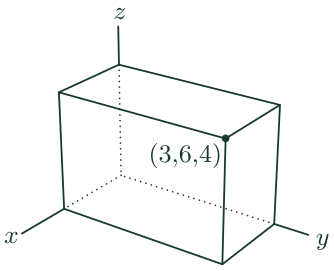
All (3,y,z) form the infinite plane parallel to the yz-plane.

Check. One fixed coordinate leaves one free variable in 2D and two in 3D.

Exercise 387 · Faces of a rectangular box

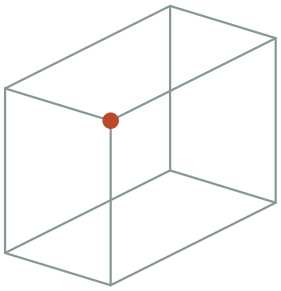
Printed page 39 · Planes in space / Coordinate geometry

The illustration at right shows a rectangular box, three of whose edges coincide with the coordinate axes, and one of whose vertices is $(3, 6, 4)$. Give an equation for the plane containing the front face of the box. Do the same for the top and right faces.



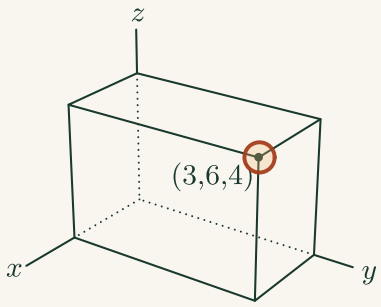
1. Trace the marked corner toward the axes

In the drawing, x measures frontward, y rightward and z upward. Project (3,6,4) in each direction to identify the coordinate reached on each far face.



Read the three containing planes

The corner (3,6,4) belongs to all three far faces.

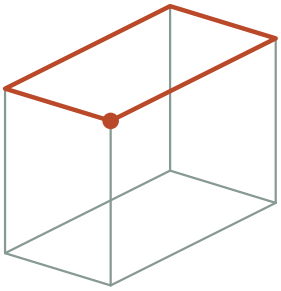


Their shared corner

The labeled opposite corner belongs to all three far faces and supplies one coordinate for each plane.

2. Describe a face by its constant coordinate

A motion along the top can change x and y but keeps z fixed. Use the same test on the front and on the right face.

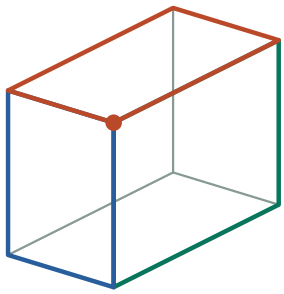


Hold one coordinate fixed

On the top face z=4 while x and y may vary within the box.

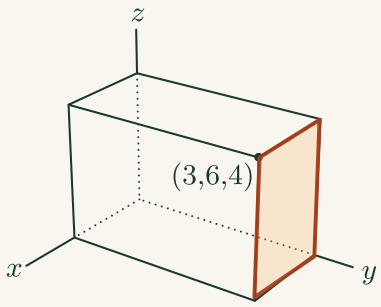
3. Write the three plane equations

Front face: x=3. Top: z=4. Right: y=6. These are the equations of the infinite containing planes; the faces occupy only the box's coordinate ranges.



Identify all far faces

Green is x=3, blue is y=6 and orange is z=4. Their containing planes extend beyond this box.



The right face

On the right-hand face, horizontal front-to-back motion and vertical motion both leave the y-coordinate fixed.

Check. The marked vertex satisfies all three simultaneously.

Exercise 388 · Two years of interest

Printed page 39 · Compounding / Algebraic manipulation

Show that $P + Pr + (P + Pr)r = P(1 + r)^2$. Based on your work with exponential growth, interpret the three individual terms on the left side of this equation, and explain why their sum should equal the expression on the right side.

1. Read the first year

Begin with principal P dollars. At rate r per year its first interest payment is Pr dollars. Put this back in the account so the second year begins with P+Pr.



Start with principal P

The initial blue-outlined bar represents P. For illustration, later bars use r=0.30 so the segment proportions are visible.

2. Interpret the second-year term

The new interest is r times this enlarged balance. Thus the two years have each multiplied the total by 1+r.

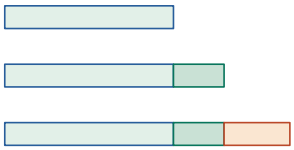


Return the interest to the balance

The second bar includes the original amount and Pr. The next interest acts on this whole bar.

3. Match each term and factor

The terms are original principal, first-year interest, and second-year interest on the increased balance. Factor the sum as $(P+Pr)(1+r)=P(1+r)^2$.



Calculate interest on the enlarged base

Orange adds r(P+Pr), larger than the first interest. The final balance is $P(1+r)^2$.

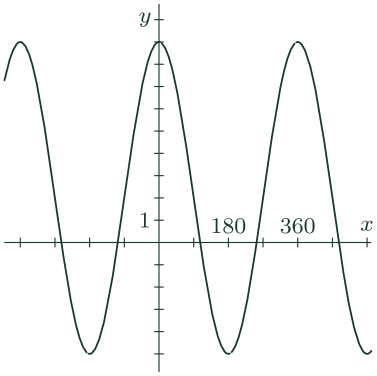
1. $P+Pr+(P+Pr)r=(P+Pr)\cdot 1+(P+Pr)r$
The first two terms are exactly last year's balance.

2. $=(P+Pr)(1+r)=P(1+r)(1+r)$
Factor that balance, then factor P out of its first occurrence.

Check. Expansion gives $P+2Pr+Pr^2$, with Pr^2 representing interest on interest.

Exercise 389 · Read a shifted cosine

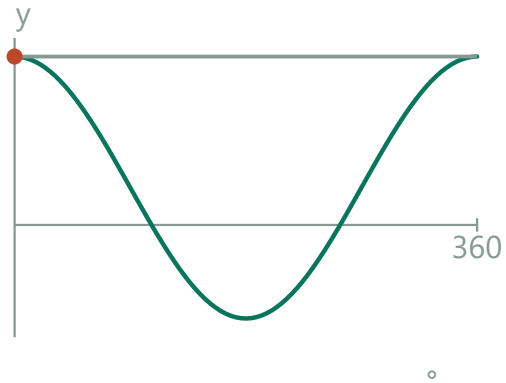
Printed page 39 · Sinusoidal functions / Amplitude and period / Graph interpretation



The equation whose graph is shown at right has the form $y = k + a\cos x$. Working in degree mode, find believable values for the coefficients a and k, and explain how these numbers affect the appearance of the graph.

1. Read off the extremes

The maximum is about 9 and the minimum about −5.

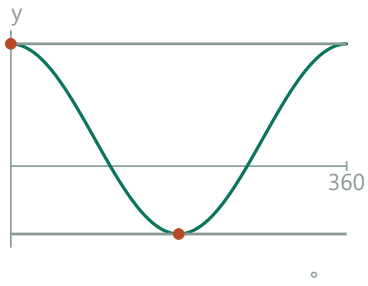


Locate the top at zero input

The pictured curve has its crest on the vertical axis.

2. Recover the center and amplitude

The midline is their average k=2, and the half-range is 7.

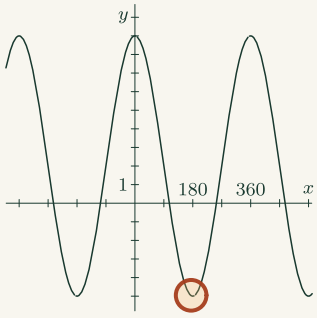


Read the lower turning point

The full vertical span from −5 to 9 is 14.

1. k+A=9; k−A=−5
Write the midline plus and minus the positive amplitude at the extremes.

2. 2k=4; 2A=14 ⇒ k=2, A=7
Add the equations to recover the center; subtract to recover the amplitude.

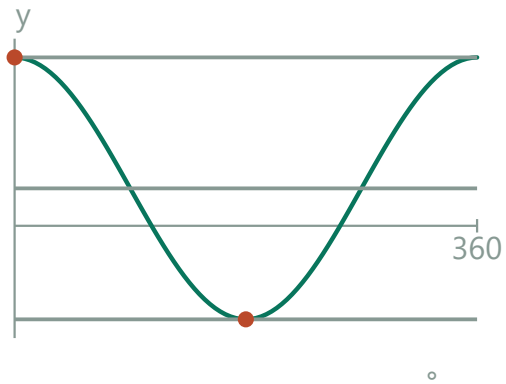


Neighboring trough

The lower turning point and the crest determine the full vertical span. Half that span is the amplitude.

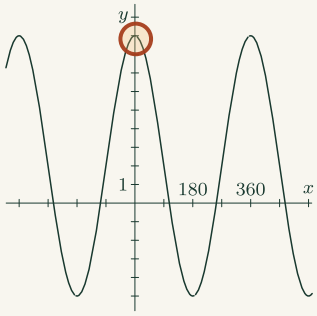
3. Choose the sign from the initial crest

A peak at x=0 requires positive a, giving y=2+7cos x.



Bisect the span

The midline is 2 and the amplitude is 7.

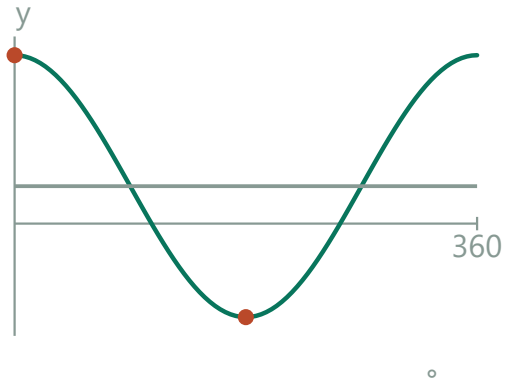


High point at the vertical axis

Read the y-coordinate of the crest at zero input. Its position distinguishes an upright cosine from an inverted one.

4. Explain the two coefficients

The constant k lifts every point equally. The coefficient a multiplies the cosine height measured from that new centerline. With a positive coefficient the high point stays at x=0. From the initial crest follow the graph to the next crest at 360°. This is one complete period; the shift and vertical stretch both leave this horizontal spacing unchanged.



Choose the positive cosine

Beginning at a crest gives y=2+7 cos x.

Check. At 180° this formula gives −5 and at 360° it gives 9.

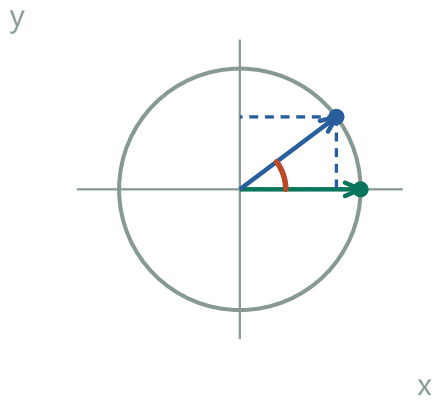
Exercise 390 · Circular motion from two samples

Printed page 39 · Circular motion / Parametric equations / Angular and linear speed

An object moves around $x^2 + y^2 = 25$ (which represents a circle whose radius is 5 meters) at a constant speed. At time $t = 0$ seconds, the object is at $(5, 0)$. When $t = 1$, it is at $(4, 3)$. Where is the object when $t = 2$? when $t = 3$? when $t = n$? What is the object's speed? At what times does the object return to $(5, 0)$? To arrive at your answers, what assumptions did you make? Did your classmates make the same assumptions?

1. Choose a direction and interpret the first second

The radius from the origin to (4,3) has horizontal component 4 and vertical component 3, so cos α=4/5 and sin α=3/5. Take α=arctan(3/4)≈0.64350 radians. First assume the object travels counterclockwise along the shorter arc, never reverses, and makes no extra circuit during that second.

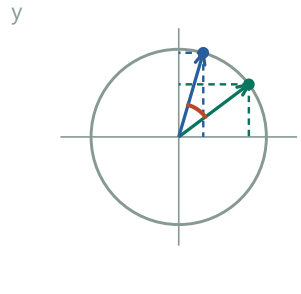


First one-second movement

The 3–4–5 triangle fixes sin α=3/5 and cos α=4/5.

2. Apply the same rotation twice

With constant directed motion, each second adds another angle α. A rotation through α sends any point (x,y) to ((4x−3y)/5,(3x+4y)/5). Apply it to (4,3) to get ((16−9)/5,(12+12)/5)=(7/5,24/5) at t=2.



Add the same angle again

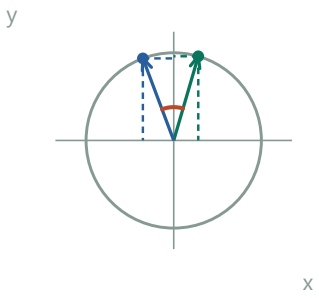
At two seconds the rotated coordinates are (7/5, 24/5).

1. $x_2 = 4\cdot(4/5) - 3\cdot(3/5) = 7/5$
Rotate the point (4,3), using cosine 4/5 and sine 3/5.

2. $y_2 = 4\cdot(3/5) + 3\cdot(4/5) = 24/5$
Use the second row of the same rotation.

3. Take a third step and describe any time

Rotate (7/5,24/5) again. The new coordinates are ((28−72)/25,(21+96)/25)=(−44/25,117/25) at t=3. After n seconds the total angle is nα, giving position (5 cos(nα),5 sin(nα)). The radius stays 5 because $\cos^2(n\alpha)+\sin^2(n\alpha)=1$.



Third one-second interval

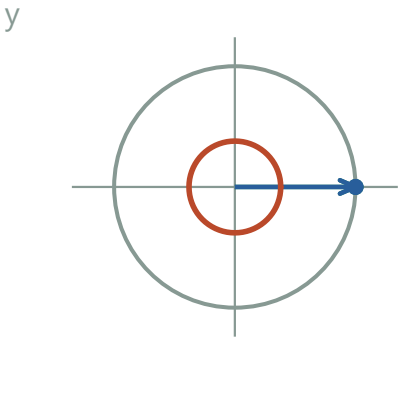
The next point is (−44/25,117/25). Its distance from the center stays 5.

1. $x_3=(7/5)(4/5)-(24/5)(3/5)=-44/25$
Apply one more rotation to the exact second point.

2. $y_3=(7/5)(3/5)+(24/5)(4/5)=117/25$
Adding the vertical projections preserves the radius.

4. Use arc length to find speed and returns

In one second the traveled arc has length radius times angle, 5α≈3.218 meters. That is the speed in meters per second. The point returns when its angle is a whole number of full turns, so t=2πj/α for j=0,1,2,… . The first positive return occurs at about 9.764 seconds.

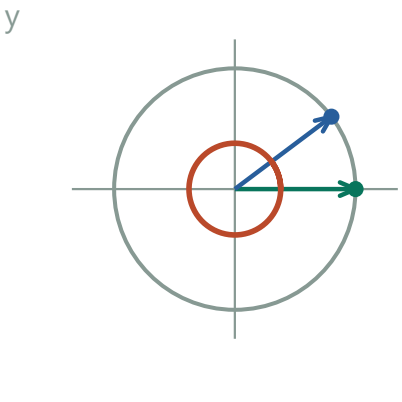


A return requires a full circuit

At angular speed α rad/s the return interval is 2π/α seconds.

5. Check which conclusions depend on the assumption

The two observations also allow any constant angular velocity ω=α+2πk radians per second, with integer k. Negative ω describes clockwise travel. All these choices give the same points at integer times n, since 2πkn is a whole number of turns. Their speeds 5|ω| and positive return periods 2π/|ω| differ. Comparing answers therefore requires saying what happened between samples, including whether direction was maintained.



Extra complete turns are invisible

At integer times adding 2πk to the one-second turn produces the same inspected point.

Check. The two explicit positions have norm 5; successively multiply their complex coordinates by (4+3i)/5.

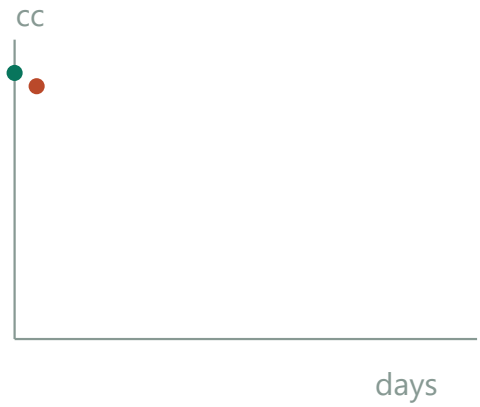
Exercise 391 · An exponentially shrinking balloon

Printed page 40 · Exponential growth and decay / Logarithms

A helium-filled balloon is slowly deflating. During any 24-hour period, it loses 5 percent of the helium it had at the beginning of that period. The balloon held 8000 cc of helium at noon on Monday. How much helium did it contain 3 days later? 4.5 days later? 20 days later? n days later? 12 hours later? k hours later? Approximately how much time is needed for the balloon to lose half its helium? This time is called the *half-life*. Be as accurate as you can.

1. Use the amount remaining

Losing 5% of the current amount leaves 95%, so each full day multiplies the volume by 0.95. Start the clock at Monday noon with 8000 cc.

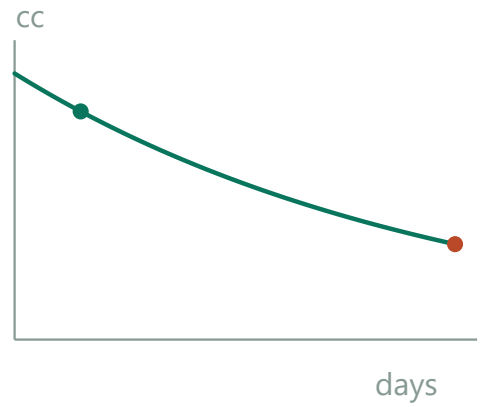


One day’s retention

The balloon keeps 95% of its previous volume each full day.

2. Evaluate complete and fractional days

After n days V=8000(0.95)^n cc. At 3, 4.5, and 20 days this is about 6859, 6351.06, and 2867.89 cc.



Compare successive days

Integer-day points follow $8000(0.95)^n$. The later volume is smaller.

3. Change the clock to hours

At 12 hours it is 7797.44 cc; at k hours use exponent k/24. Fractional times use the constant exponential interpolation.



Interpolate between noons

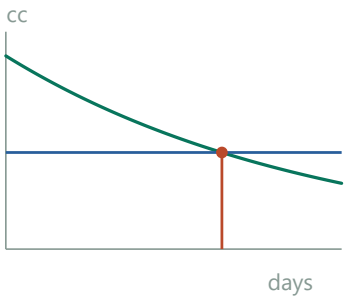
A fractional-day position on this smooth curve assumes exponential loss within each day.

1. t days$=k$ hours$\cdot(1$ day$/24$ hours$)=k/24$ days
Convert the new time unit before using the daily factor.

2. $V(12\text{ h})=8000\cdot0.95^{(12/24)}=8000\sqrt{0.95}$
Half a day uses the square root of the daily factor in the continuous model.

4. Solve the half-volume equation

For half the helium, solve $0.95^t=\frac{1}{2}$: $t=\ln(\frac{1}{2})/\ln(.95)\approx13.5134$ days.



Find half the initial helium

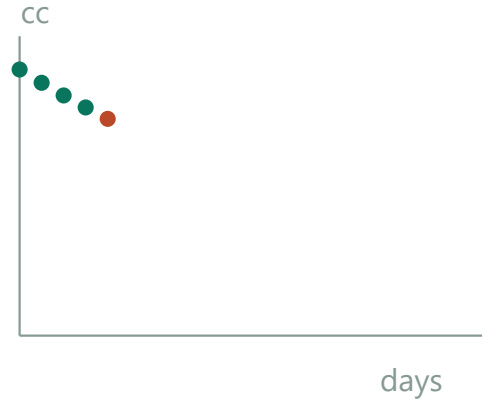
A horizontal line at 4000 cc reaches the model at about 13.51 days.

1. $4000=8000\cdot0.95^t\Rightarrow 1/2=0.95^t$
Divide by the initial volume.

2. $\ln(1/2)=t\ln(0.95)\Rightarrow t=\ln(1/2)/\ln(0.95)$
Both logarithms are negative, yielding a positive elapsed time.

5. Distinguish measured rule from interpolation

The 24-hour rule alone fixes integer-day volumes. It allows different patterns within each day, repeated after scaling by 0.95. The fractional-day values and half-life above additionally assume the same proportional decay rate throughout the day.



Keep the information the daily rule fixes

Daily endpoint data alone allow other patterns between successive noons.

Check. Advancing any starting time by 24 hours multiplies these volumes by 0.95.

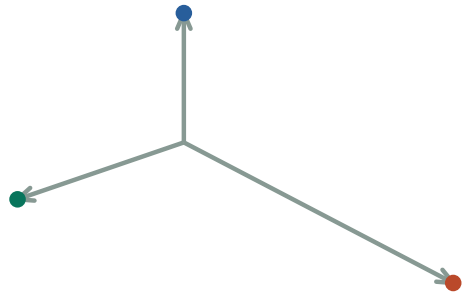
Exercise 392 · A plane from intercepts

Printed page 40 · Planes in space / Coordinate geometry

Plot the points where the graph of $x+2y+3z=27$ intersects the three coordinate axes. Also plot the points $(0, y, z)$ in the yz-plane that fit the equation. Then plot one point, all of whose coordinates are positive, that fits the equation. What geometric object is described by the equation $x + 2y + 3z = 27$?

1. Find intercepts one axis at a time

On the x-axis both y and z vanish. On each remaining axis, likewise set the two other coordinates to zero. This leaves one simple equation per intercept.

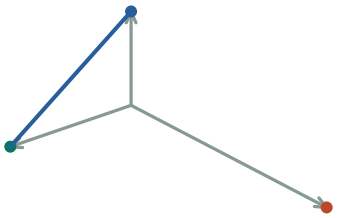


Mark three axis intercepts

The intercepts are 27 on x, 13.5 on y, and 9 on z.

2. Plot intercepts and trace a coordinate plane

Axis intercepts are (27,0,0),(0,13.5,0),(0,0,9). In the yz-plane all solutions form the line (0,(27−3s)/2,s).



Set x=0

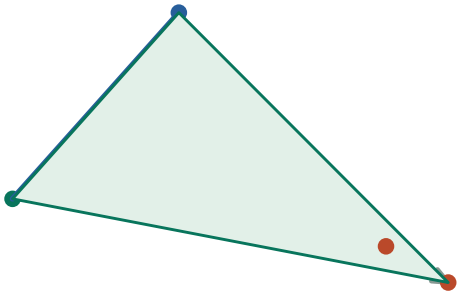
The marked segment in the yz-plane joins (0,13.5,0) and (0,0,9) and extends to a full line.

1. x=0 ⇒ 2y+3z=27
Points on the yz-plane have vanishing first coordinate.

2. z=s ⇒ y=(27−3s)/2
Choose any z, subtract 3s, and divide by 2.

3. Explain the free parameter

Within the yz-plane, picking z=s fixes y from 2y+3s=27. In the full three-dimensional equation two coordinates can be chosen freely and the third is then determined.

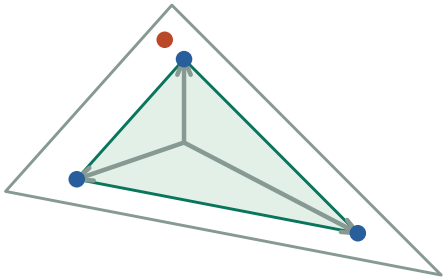


Include points off the axes

The shaded section belongs to the full infinite plane, for example the point (22,1,1).

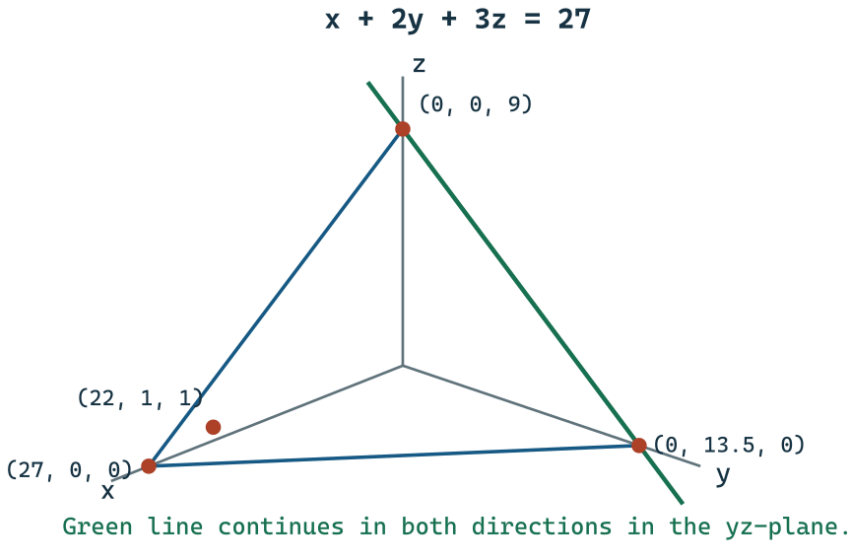
4. Give a point in the first octant and identify the surface

One strictly positive point is (22,1,1). The full graph is a plane.



Continue the plane

The intercept triangle is only the part in the first octant. The unrestricted plane continues past its edges.



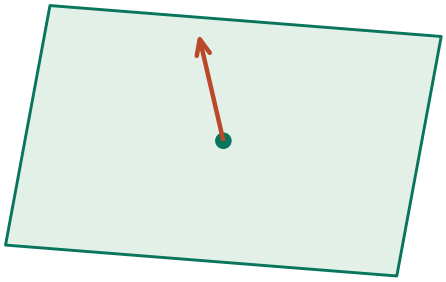
Check. All displayed points substitute to 27 and the three intercepts are noncollinear.

Exercise 393 · Perpendicular bisector in space

Describe the configuration of points equidistant from $A = (2, 4, 1)$ and $B = (4, 8, 7)$. Write and simplify an equation that says that $P = (x, y, z)$ is equidistant from A and B.

1. Compare squared distances

Equidistance means $(x-2)^2+(y-4)^2+(z-1)^2=(x-4)^2+(y-8)^2+(z-7)^2$. Squaring loses no distances here since both sides before squaring were nonnegative.

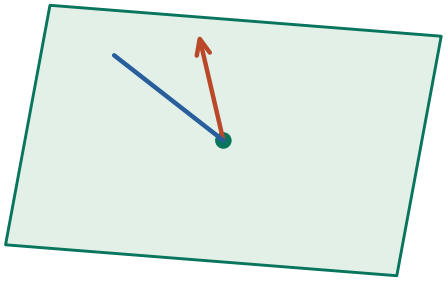


Bisect the segment AB

The midpoint is (3,6,4); the displacement from A to B points across the equidistant surface.

2. Use symmetry to identify the plane

The midpoint belongs to the locus, and moving from it perpendicular to AB keeps the right triangles to A and B congruent. AB itself is therefore a normal to the plane.

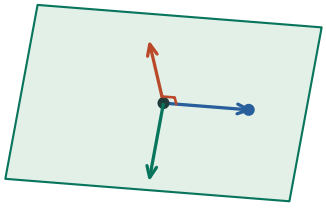


Slide sideways from the midpoint

Every in-plane displacement is perpendicular to (1,2,3), half the direction from A to B.

3. Write the normal form and simplify

The midpoint is (3,6,4) and a normal is the separation vector (2,4,6). Dot (1,2,3) with $(x-3,y-6,z-4)$ and set to zero, giving $x+2y+3z=27$.



The whole bisector is a plane

Dotting (1,2,3) with the displacement from (3,6,4) gives $x+2y+3z=27$.

1. $(x-3)+2(y-6)+3(z-4)=0$
An in-plane displacement has zero dot product with (1,2,3).

2. $x+2y+3z-3-12-12=0 \Rightarrow x+2y+3z=27$
Expand and move the three constants to the other side.

Check. Expanding the two squared-distance expressions cancels all quadratic terms and gives the same equation.

Exercise 394 · Read a decay graph

Printed page 40 · Exponential growth and decay / Graph interpretation

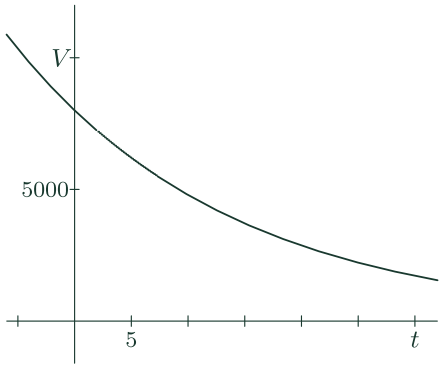
The picture at right shows part of the graph of $V(t) = 8000(0.95)^t$. This function tells the story of a shrinking balloon that loses 5 percent of its helium each day.

(a) What is $V(0)$, and what is its significance?

(b) Use the graph to estimate the t-value that solves the equation $V(t) = 4000$.

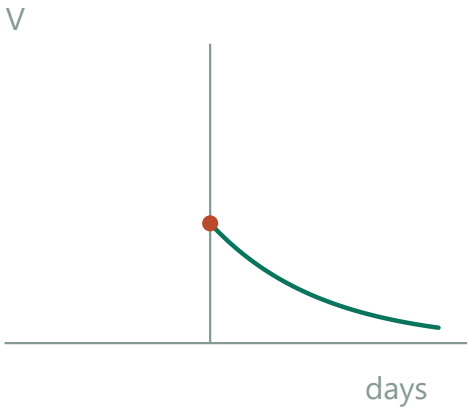
(c) Calculate $V(-3)$. What does this value mean?

(d) Comment on the part of the graph that lies outside the borders of the illustration. How would it look if you could see it, and what does it mean?



1. Read the initial gas volume

At t=0 the graph meets the vertical V-axis. The formula gives $V(0)=8000(0.95)^0=8000$ cc, the amount of helium at the reference time. The daily retention factor is 0.95 because losing 5% leaves 95% of the preceding day's volume.

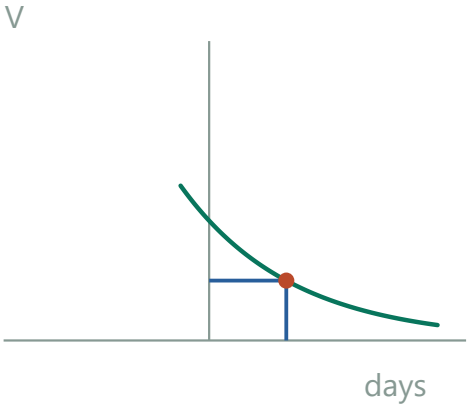


Locate the initial balloon volume

The y-intercept is 8000 cc, before applying any daily loss.

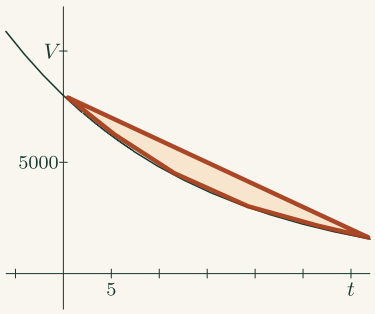
2. Trace the half-height crossing

Locate 4000 cc halfway between the starting height 8000 and zero. Go horizontally to the descending curve, then vertically to the time axis. The reading is about 13½ days. Indeed V(13)≈4107 cc and V(14)≈3901 cc, bracketing 4000. Solving $0.95^t=½$ gives t=log(½)/log(0.95)≈13.51 days.



Find the half-height crossing

Around day 13.5 the height has fallen from 8000 to 4000 cc.

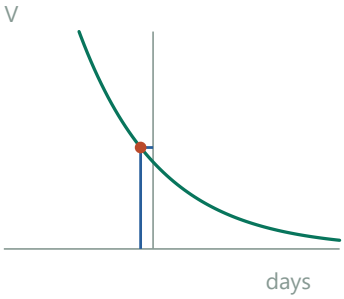


The future branch

Reading right follows the decreasing volume after the reference time. Use a horizontal level to locate a target amount.

3. Move backward three days

At t=−3 the exponent undoes three days of shrinkage. Calculate V(−3)=8000/(0.95³)≈9330.81 cc. This is the modeled volume three days before the reference time. Multiply it by 0.95 three times to check that the present 8000 cc is recovered.

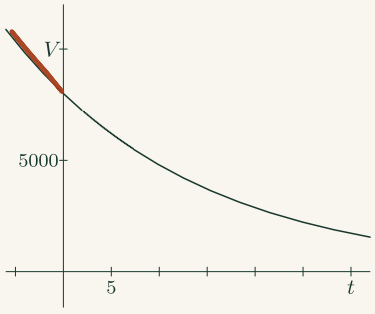


Undo three days of shrinkage

A backward step divides by 0.95. The day −3 reading is above the initial level.

1. $V(-3)=8000\cdot0.95^{-3}=8000/0.95^3$
Undo rather than apply three daily losses.

2. =8000/0.857375≈9330.81 cc
The earlier volume is larger than the initial reference value.

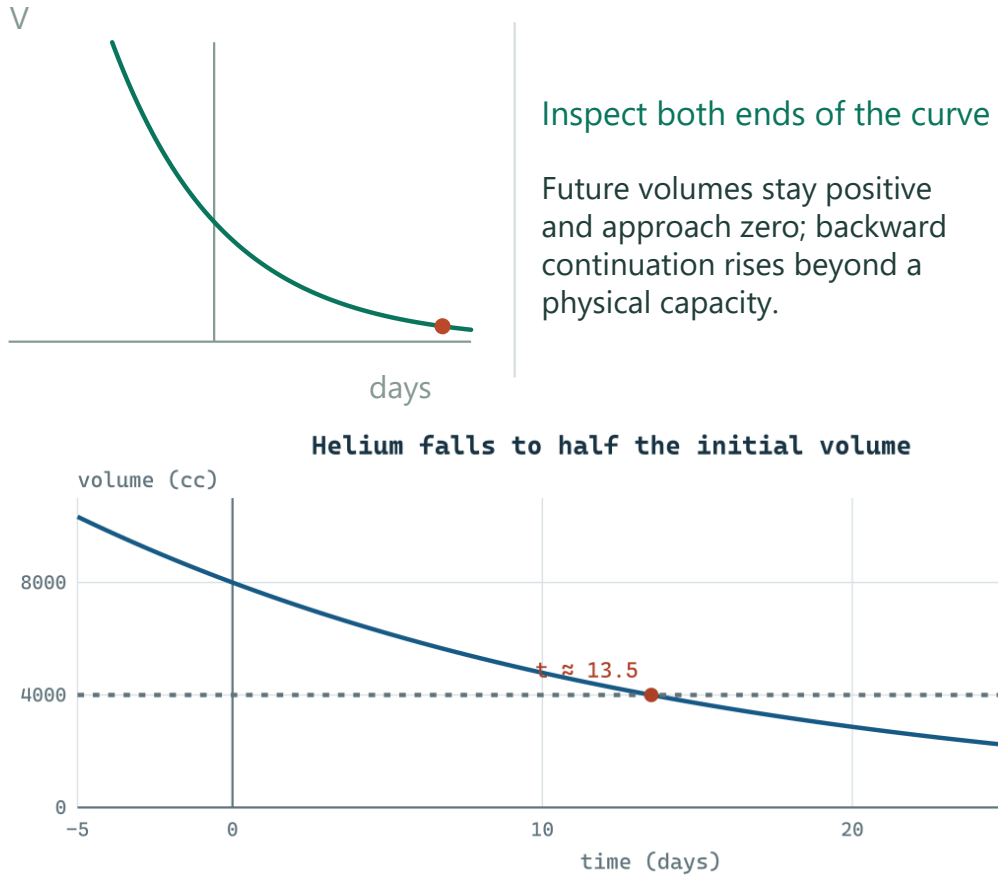


The earlier branch

Follow the curve left of the vertical axis to interpret a time before the first observation.

4. Follow both ends of the curve

To the right, each equal time step keeps multiplying by the same fraction. The curve stays positive, flattens, and approaches the horizontal axis as time increases. To the left, successive backward steps divide by 0.95 and the model rises without bound. Extending it far enough into the past exceeds a real balloon's capacity, so that part describes the mathematical formula beyond a plausible physical interval.



Check. 0.95^3 times the calculated past volume recovers 8000.

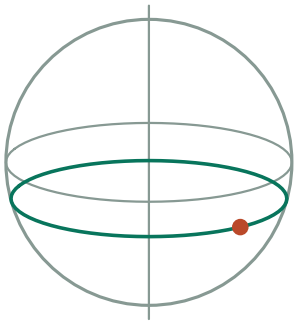
Exercise 395 · Lusaka coordinates

Printed page 40 · Spherical coordinates / Coordinate geometry

An xyz-coordinate system is placed with its origin at the center of the earth, so that the equator is in the xy-plane, the North Pole has coordinates $(0, 0, 3960)$, and the xz-plane contains the prime meridian. Find the coordinates (x, y, z) of Lusaka, Zambia, whose latitude and longitude are 15.3 degrees south and 28.5 degrees east.

1. Orient the point on the globe

Latitude south is negative, so Lusaka lies below the equatorial plane. Longitude 28.5° east lies between the positive x and y axes.

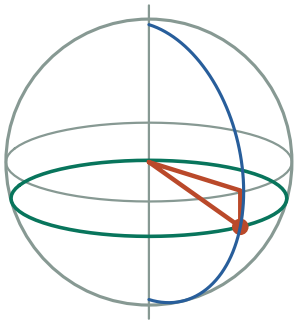


Locate Lusaka below the equator

South latitude gives a negative vertical component.

2. Resolve the sphere radius

The projection onto the equator has length 3960 cos 15.3°. Multiply it by cos 28.5° for x and sin 28.5° for y. Take the signed vertical component of the original radius for z.

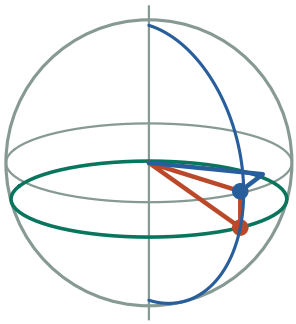


Project at 28.5° east

Use R cos 15.3° for the equatorial radius; longitude splits it into positive x and y.

3. Evaluate the Cartesian position

Coordinates are $3960(\cos 15.3°\cos 28.5°, \cos 15.3°\sin 28.5°, -\sin 15.3°)$. Numerically this is about (3357, 1823, −1045) miles.



Read the result

The Cartesian location is approximately $(3357, 1823, -1045)$ miles.

Check. Southern latitude gives negative z; east longitude below 90° gives positive x and y.

Exercise 396 · Remove negative exponents

Printed page 40 · Exponent rules / Algebraic manipulation

Convert the following to equivalent forms in which no negative exponents appear:

(a) $\left(\frac{2}{5}\right)^{-1}$ (b) $\frac{6}{x^{-2}}$ (c) $\left(\frac{-3}{2}\right)^{-3}$ (d) $\frac{6xy}{3x^{-1}y^{-2}}$ (e) $\left(\frac{2x^2}{3x^{-1}}\right)^{-2}$

1. Use the reciprocal rule

Any nonzero factor with a negative exponent can be moved across the fraction bar to change the sign of the exponent. Keep the original restrictions on denominators.

$$\frac{1}{x^{-2}} \longrightarrow \frac{x^2}{1} \qquad x \text{ must be nonzero}$$

A reciprocal cancels the negative exponent when the factor crosses the fraction bar.

2. Invert the first fraction and the negative power below the bar

(a) 5/2. (b) $6/(x^{-2})=6x^2$. A power of −1 takes the reciprocal, switching 2/5 to 5/2. Dividing 6 by $1/x^2$ instead multiplies 6 by x^2.

$$\frac{2}{5} \longrightarrow \frac{5}{2} \qquad \text{reciprocal}$$

$$\frac{6}{x^{-2}} \longrightarrow \frac{6x^2}{1} \qquad \text{divide by } 1/x^2$$

Read each row independently. In the lower row, dividing by a reciprocal multiplies by its reciprocal.

3. Use odd-power signs for a negative base

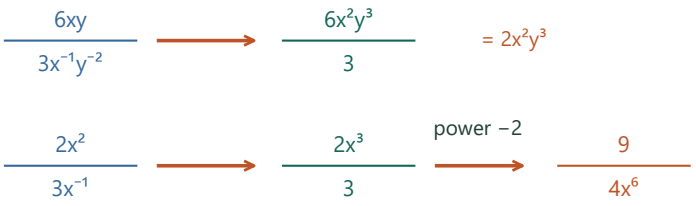
(c) $(-3/2)^{-3} = -8/27$. Invert $-3/2$ to get $-2/3$, then cube numerator and denominator. An odd number of negative factors keeps the result negative.



Invert first, then cube. Keeping the minus sign with the numerator shows why the result stays negative.

4. Collect like variables before the outer power

(d) $6xy/(3x^{-1}y^{-2}) = 2x^2y^3$. (e) $(2x^2/(3x^{-1}))^{-2} = 9/(4x^6)$.



1. $6xy/(3x^{-1}y^{-2}) = 2x^{(1-(-1))}y^{(1-(-2))}$
Subtract the denominator exponents; x and y were both nonzero.

2. $= 2x^2y^3$
Simplify 1+1 and 1+2.

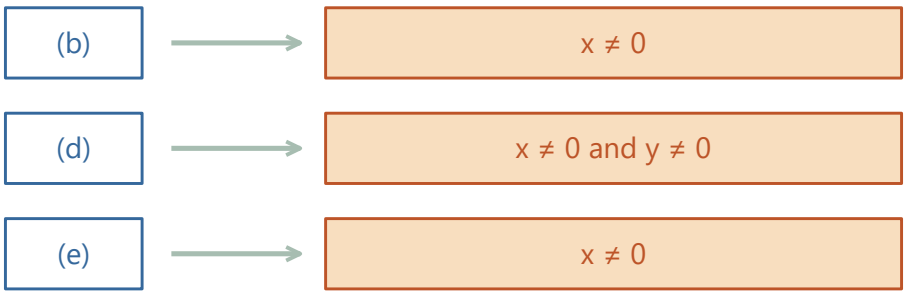
3. $(2x^2/(3x^{-1}))^{-2} = (2x^3/3)^{-2} = (3/(2x^3))^2$
First combine x powers, then invert the whole nonzero fraction.

4. $= 9/(4x^6)$, $x \neq 0$
Apply the square to numerator and denominator.

In the lower row, simplify inside the parentheses first. The last arrow takes the reciprocal and squares it.

5. Keep all domain restrictions

Retain $x \neq 0$ in b,d,e and $y \neq 0$ in d. A simplified expression may accept zero even when the original contains a reciprocal at zero. Retain the exclusions coming from the original forms.



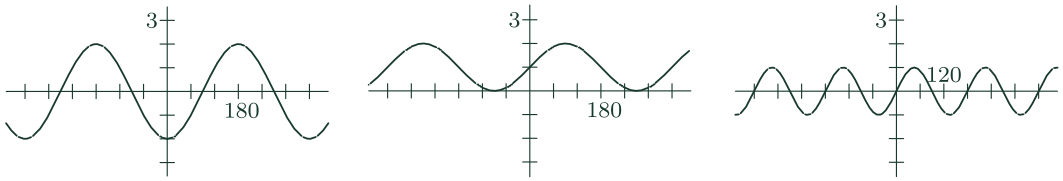
Carry the restrictions from the starting denominators into the simplified answers.

Check. For (e), simplify inside first to $2x^3/3$, then take its reciprocal squared.

Exercise 397 · Read three degree waves

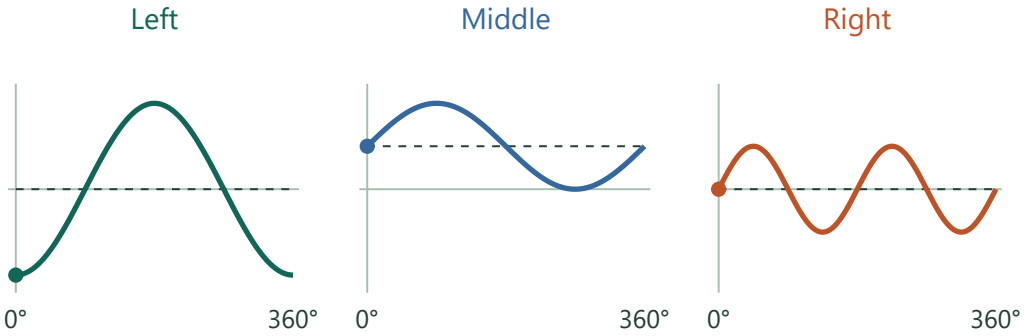
Printed page 40 · Sinusoidal functions / Amplitude and period / Graph interpretation

Working in degree mode, find plausible equations for each of the sinusoidal graphs below:



1. Find four features of each curve

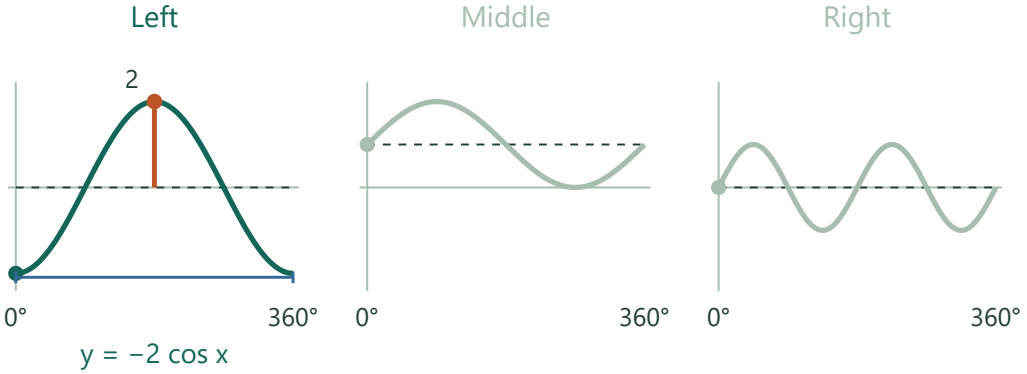
Read the midline from the extremes, half their separation for amplitude, and the horizontal repeat distance for the period. The height and direction at x=0 determine which simple sine or cosine fits.



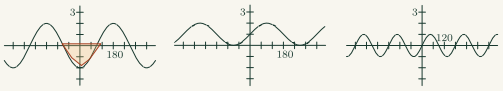
Read the three panels in degrees. For each trace, compare its starting height with the dotted midline before measuring a repeat.

2. Identify the inverted left cosine

Left starts at −2 and has a crest at 180°, so y=−2cos x.



Left panel. The low point is two units below the midline at 0°. The curve is back at that low point at 360°, one full period later.

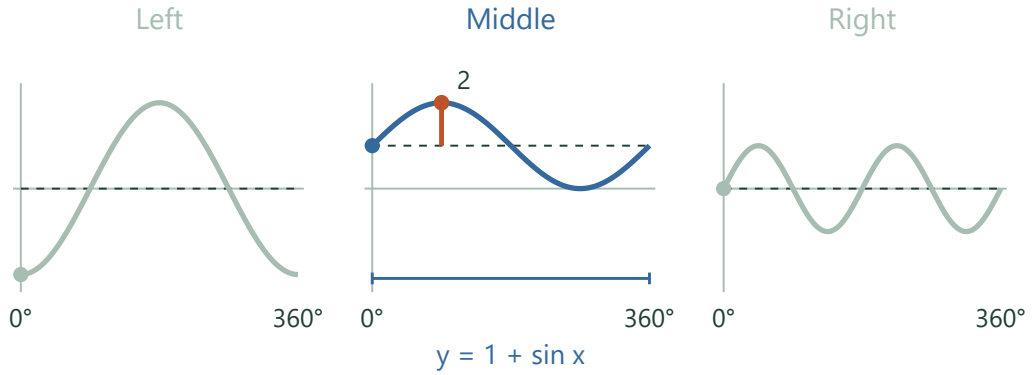


Left graph's starting trough

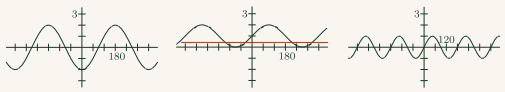
The first curve begins at a low point on its vertical axis and reaches a crest after half a cycle.

3. Fit the vertically shifted middle curve

Middle has range [0,2], rises through 1 at x=0 and has period 360°, so y=1+sin x.



Middle panel. The dotted midline rises to 1. The wave climbs one more unit to the crest at 90° and drops one unit from the midline to the trough at 270°.



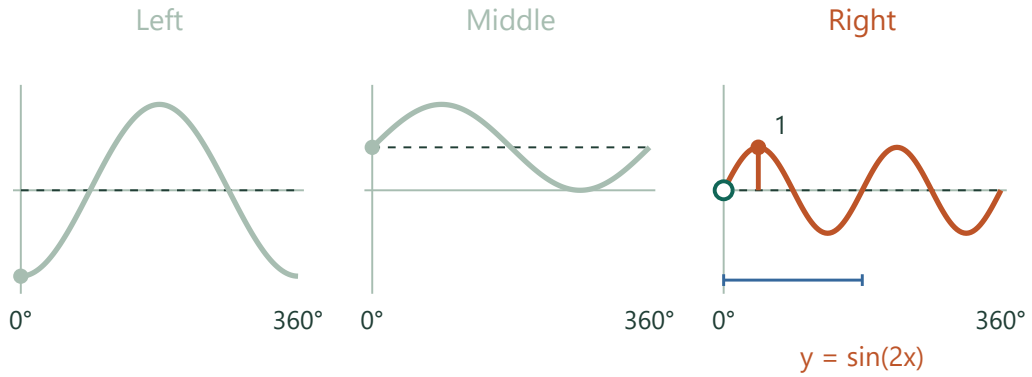
Middle graph's elevated centerline

Compare the highest and lowest points in the middle panel. Their halfway line sits above the horizontal coordinate axis.

4. Find the faster right-hand oscillation

Right has amplitude 1, rises through zero at the origin, and repeats every 180°, giving y=sin(2x).

Position along the selected wave · 0°

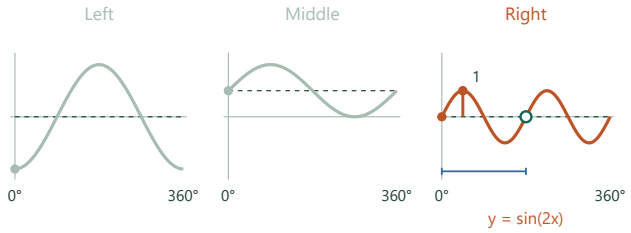


Right panel. The curve passes through the midline rising at both 0° and 180°. The second complete wave in the same window explains the factor 2.

1. m·180°=360° ⇒ m=2	The right curve completes a full internal turn in 180 input degrees.
2. y=sin(2x)	Its upward origin crossing selects sine with positive amplitude one.

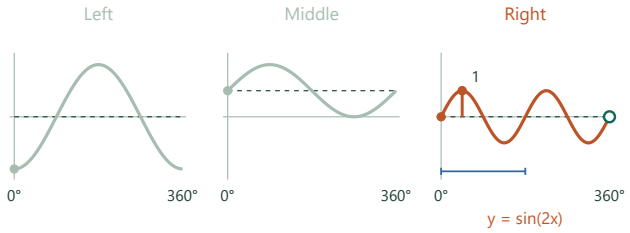
Compare settings · Position along the selected wave

180°



At x = 180°, the height is 0.

360°



At x = 360°, the height is 0.

Check. At a quarter period the three outputs are 0,2,1, matching the midline or peak shown.

Exercise 398 · Double in seven years

Printed page 40 · Compounding / Radicals and rational powers

In order that a \$10000 investment grow to \$20000 in seven years, what must be the annual rate of interest? Seven years could be called the *doubling time* for this investment. Notice that it is being assumed that the interest is *compounded*.

1. Model one year of compound interest

If r is the annual rate as a decimal, one dollar becomes 1+r dollars after one year. The same balance multiplier is applied seven times.

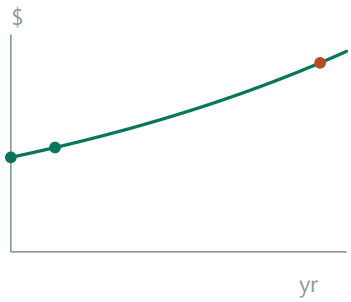


Mark the given endpoints

The balance starts at \$10,000 and must reach \$20,000 after seven annual credits.

2. Separate rate from growth factor

Dividing the desired balance by the principal leaves the total growth factor 2. Take its seventh root to get the one-year factor. Subtract 1 before changing the decimal to a percentage.



Use seven identical growth factors

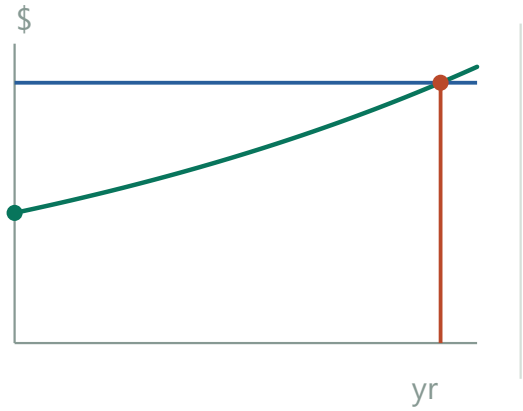
Multiplying (1+r) seven times has to double the starting balance.

1. $(1+r)^7 = 20000/10000 = 2$
Divide both balances by the initial deposit.

2. 1+r=2^(1/7) ⇒ r=2^(1/7)−1
Undo the seventh power, then subtract one to isolate the rate.

3. Calculate the required rate

Solve $10{,}000(1+r)^7=20{,}000$ to get r=2^(1/7)−1≈10.41%.



Take a seventh root

Each year multiplies the balance by 2^(1/7), corresponding to about 10.41%.

Check. Raising the annual factor to the seventh power gives 2.

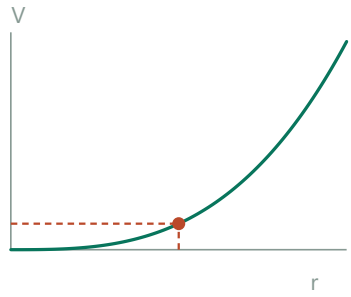
Exercise 399 · Sphere radius from volume

Printed page 41 · Volume / Radicals and rational powers

Express the radius r of a sphere as a function of the volume V it encloses.

1. Start from the sphere formula

The radius occurs cubed in $V=(4\pi/3)r^3$. To solve for it, first multiply both sides by the reciprocal $3/(4\pi)$. This isolates the cube of the radius.



Trace volume from a radius

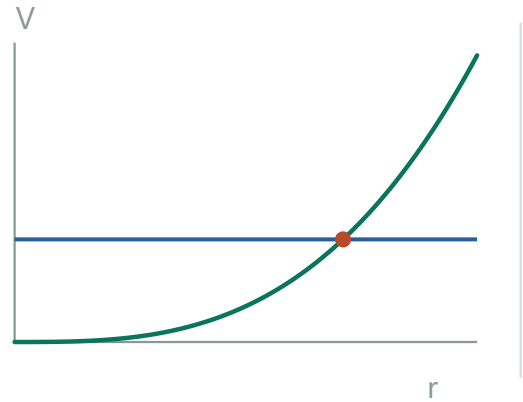
Radius one gives volume $4\pi/3$. Multiplication by $3/(4\pi)$ removes this factor and leaves r^3.

1. $3V=4\pi r^3 \Rightarrow r^3=3V/(4\pi)$
Multiply by 3 and divide by the nonzero factor 4π.

2. $r=\sqrt[3]{(3V/(4\pi))}$
Volume is nonnegative, so this real cube root is a valid radius.

2. Choose the meaningful root

For a sphere the volume and radius are nonnegative. Taking the real cube root now gives one allowed radius for each volume; cubic units become ordinary length units.

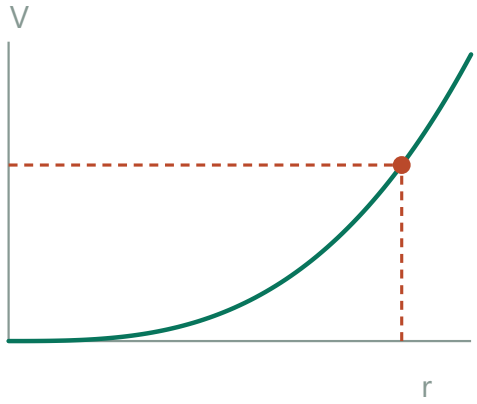


Trace back to a radius

A positive volume level crosses the radius-volume curve once. For example V=12 fixes r=(9/π)^(1/3).

3. Write radius as a function of volume

From $V=4\pi r^3/3$ obtain $r=(3V/(4\pi))^{(1/3)}$ for $V\geq 0$.



Invert the cubic dependence

The orange guides locate a single radius for the chosen volume: $r=(3V/(4\pi))^{(1/3)}$. At zero, both coordinates vanish.

Check. Cubing this expression and multiplying by $4\pi/3$ restores V.

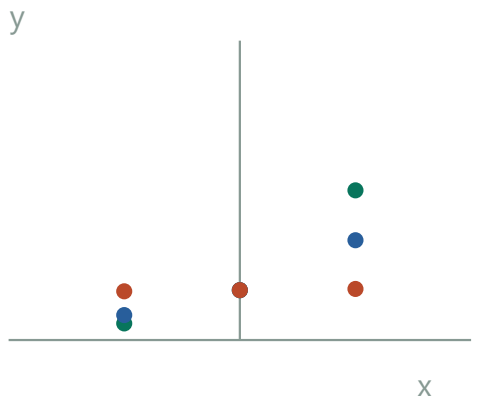
Exercise 400 · Four exponential curves

Printed page 41 · Exponential functions / Graph interpretation

On one system of coordinate axes, graph the equations $y = 3^x$, $y = 2^x$, $y = 1.024^x$, and $y = \left(\frac{1}{2}\right)^x$. What do graphs of the form $y = b^x$ have in common? How do they differ?

1. Anchor all four graphs

At x=0 any positive base gives 1. At x=1 you read the base itself; at x=−1 you read its reciprocal. Plotting those three points makes the contrasting slopes visible.

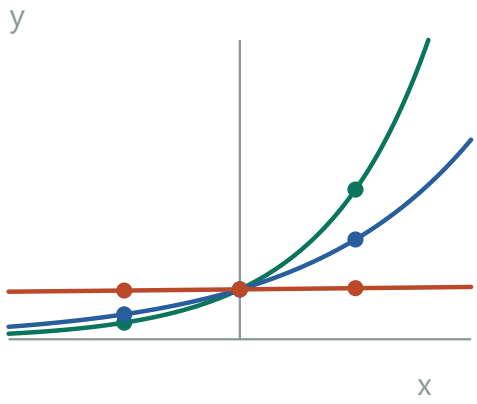


Compare common inputs

All bases give y=1 at x=0.

2. Compare growth, decay and reflection

All pass through (0,1) and remain positive. For bases above 1 the curves increase, with larger bases growing faster for positive x. The half-base curve decreases and reflects 2^x across the y-axis.

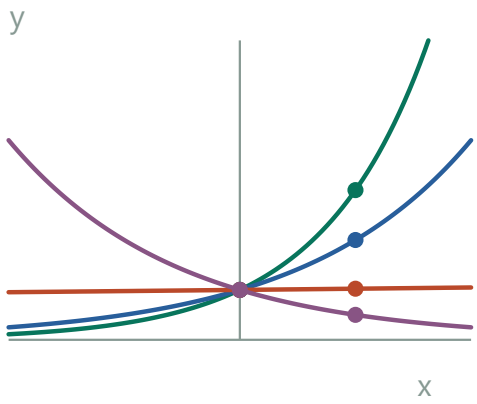


Trace the one-unit change

For positive x, the larger growth base rises faster. The base 1.024 changes little over this window.

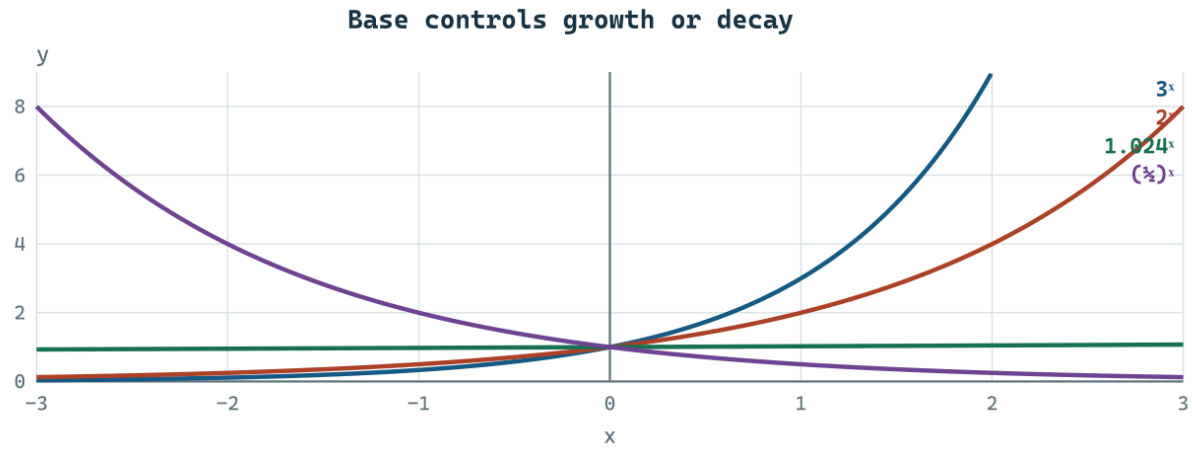
3. Extend the curves from their anchor points

Plot at x=−1,0,1 using 1/b,1,b, then extend toward the horizontal asymptote y=0 on the appropriate side.



Distinguish the remaining branch

The half-base curve decreases and reflects 2^x across the vertical axis. None reaches y=0.



Check. Each one-unit increase in x multiplies y by its base; the exceptional base b=1 gives a constant.

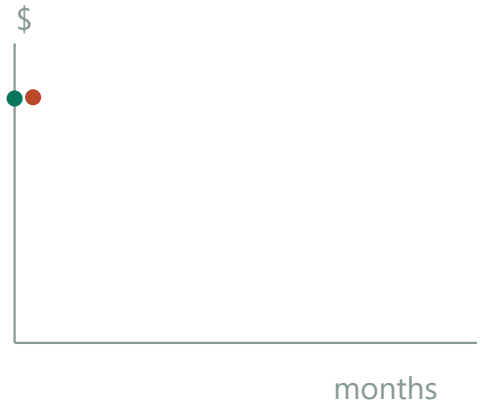
Exercise 401 · From monthly to yearly growth

Printed page 41 · Compounding / Percentage change

Make up a context for the expression $4000(1.005)^{12}$, in which the "12" counts months. In this context, what do the expressions $4000\left((1.005)^{12}\right)^{n}$ and $4000(1.0617)^{n}$ mean?

1. Attach units to the factors

The expression starts at 4000 in some unit, for instance dollars. Each application of 1.005 keeps the whole balance and adds 0.005 of it, and the exponent counts twelve applications.

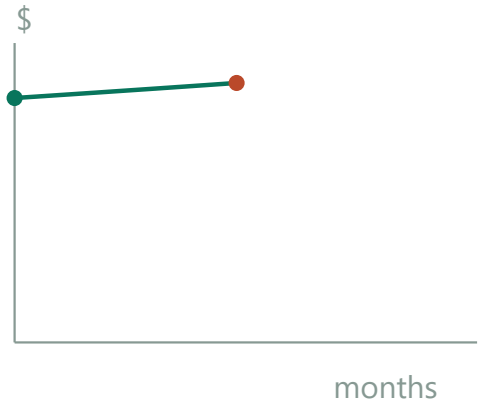


Model one account

Each month's interest increases the current $4000 starting balance by a factor 1.005.

2. Interpret the first year's model

An account opens with $4000 and gains 0.5% per month. The expression is its 12-month balance.

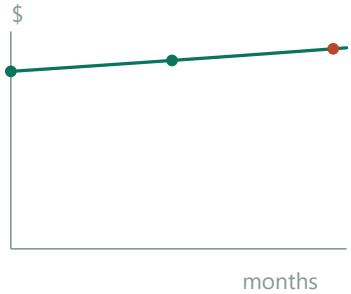


Group twelve credits

One full year uses multiplier 1.005^{12}.

3. Construct the annual multiplier

Raising the whole annual factor to n gives the n-year balance.



Group credits by year

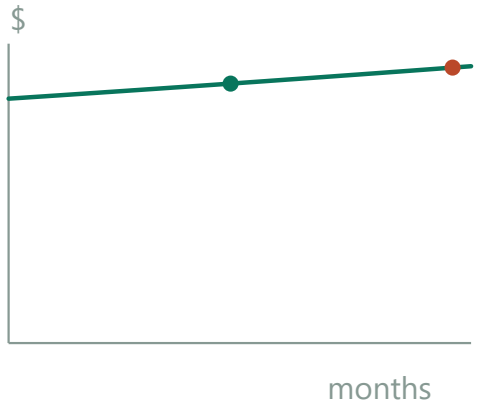
The span from month 12 to 24 repeats the same annual factor.

1. $4000 \cdot 1.005^{(12n)} = 4000 \cdot (1.005^{12})^{n}$
Twelve months per year make 12n monthly periods.

2. $1.005^{12} \approx 1.0616778$
Round only after forming the entire annual growth factor.

4. Judge the rounded substitute

Using 1.0617 gives the same model to rounded annual-factor precision.



Use a rounded annual factor

1.0617 approximates the exact one-year multiplier; compounding preserves that small rounding difference.

Check. $1.005^{12} \approx 1.061678$, an effective annual gain of about 6.17%.

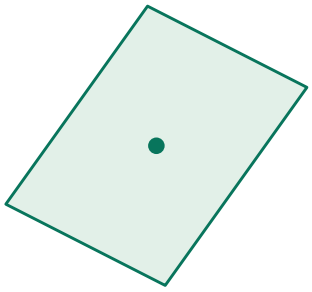
Exercise 402 · A line in a plane

Printed page 41 · Parametric lines / Planes in space

Find coordinates for two points that belong to the plane $2x + 3y + 5z = 15$. Write a parametric description of the line through your two points.

1. Pick two easy points

Set two coordinates to zero when seeking an intercept. On the y-axis the equation becomes 3y=15, while on the z-axis it becomes 5z=15.

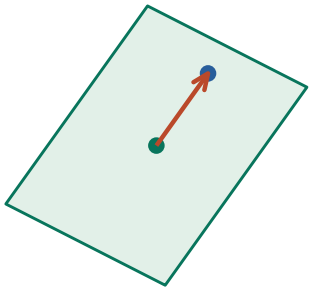


Start with a point on the plane

Setting x=z=0 gives (0,5,0).

2. Give a direction to the line

Subtract the first point from the second to get a displacement. Start at the first point and take any real multiple of that displacement to reach every point on the line.

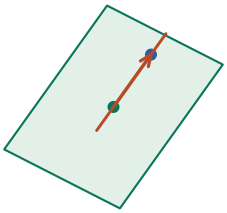


Choose another intercept

Setting x=y=0 gives (0,0,3); their difference is (0,−5,3).

3. Write the parametric description

Choose (0,5,0) and (0,0,3). Their line is (0,5,0)+t(0,−5,3) for all real t.



Extend the connecting line

The red line is (0,5,0)+t(0,−5,3), which stays on the plane for every real t.

1. (x,y,z)=(0,5−5t,3t)
Add t times the endpoint difference to the first point.

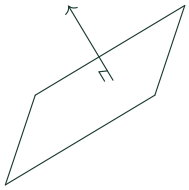
2. 2x+3y+5z=3(5−5t)+15t=15
The t-terms cancel, verifying every point stays in the plane.

Check. Substitution gives 3(5−5t)+5(3t)=15 identically.

Exercise 403 · Why the coefficients are a normal

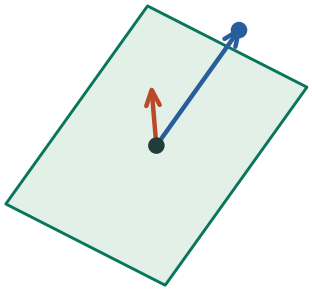
Printed page 41 · Planes in space / The dot product / Orthogonality

(Continuation) Show that the vector $[2, 3, 5]$ is perpendicular to the line that joins your two points. Explain why $[2, 3, 5]$ is perpendicular to the plane. Vectors that are perpendicular to planes are frequently called *normal vectors*.



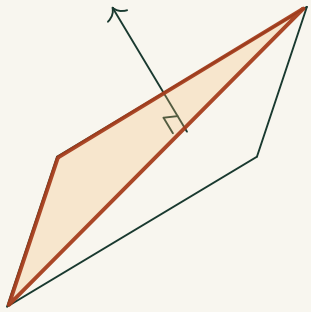
1. Use the line from the preceding construction

From (0,5,0) to (0,0,3) the displacement has components (0,−5,3). A dot product of zero with (2,3,5) will show the directions meet at right angles.



Compare a line and a normal

An in-plane example goes from (0,5,0) to (0,0,3).

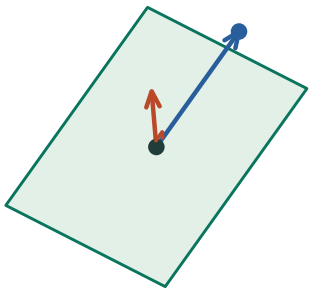


Two directions within the plane

The shaded-sheet outline represents a plane. Displacements between any two of its points lie along the plane.

2. Verify the particular direction

For the chosen line, $(2,3,5)\cdot(0,-5,3)=0$. Multiplying matching components gives $2\cdot0+3\cdot(-5)+5\cdot3=-15+15$. Their cancellation is the numerical perpendicularity test.

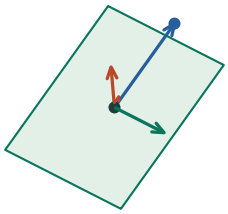


The in-plane change has zero dot product

$(2,3,5)\cdot(0,-5,3)=-15+15=0$.

3. Prove the general normal property

More generally subtract the two plane equations for any pair of points to see their displacement has zero dot product with (2,3,5).

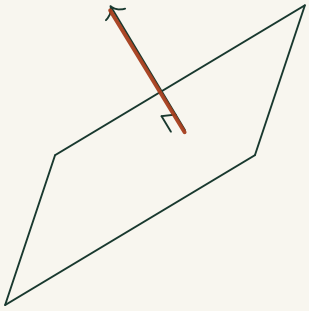


Every in-plane direction works

Subtracting the plane equations at any two points cancels their common constant.

1. $2x_2+3y_2+5z_2-(2x_1+3y_1+5z_1)=15-15$
Subtract two equations valid throughout the plane.

2. $2\Delta x+3\Delta y+5\Delta z=0$
Factor the coefficients. This is the normal dotted with the displacement.



The normal arrow

The arrow represents the direction normal to the entire plane. Its angle to any displacement lying in the plane is a right angle.

Check. The same works for independent plane directions, for example (3,−2,0) as well as (0,−5,3).

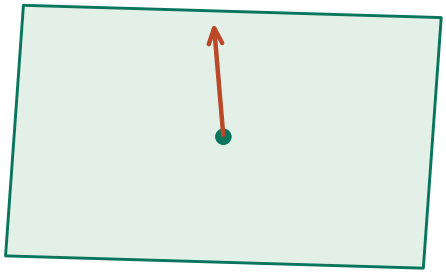
Exercise 404 · Subtract two plane equations

Printed page 41 · Planes in space / The dot product

Suppose that (x_1, y_1, z_1) and (x_2, y_2, z_2) are points that fit the equation $ax+by+cz = d$. In other words, suppose that $ax_1 + by_1 + cz_1 = d$ and $ax_2 + by_2 + cz_2 = d$ are both true. Subtract one of these equations from the other, and interpret the result.

1. Match each coefficient by coordinate

Both points obey an equation with the same right-hand number d. Subtract in the same order for x, y and z so the result records one displacement between the points.

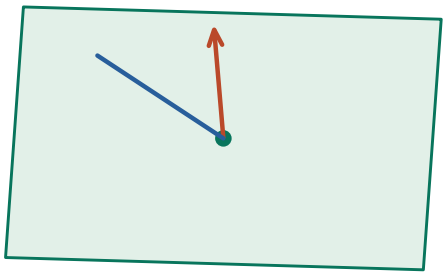


Pick two points on a common plane

The diagram represents a typical plane with a nonzero normal (a,b,c).

2. Read the new left side as a dot product

The coefficients form one vector, and the coordinate differences form another. The right side d−d is zero.

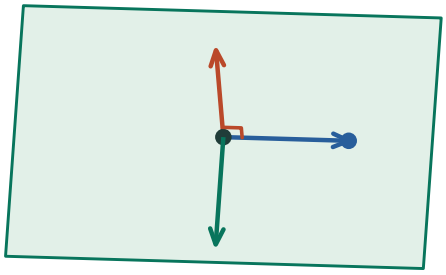


Subtract the two point equations

The constant d disappears, leaving the dot product of the normal with the displacement.

3. Interpret the identity

Subtraction yields $a(x_2-x_1)+b(y_2-y_1)+c(z_2-z_1)=0$. Every in-plane displacement is perpendicular to the nonzero vector (a,b,c).



A perpendicular to all directions

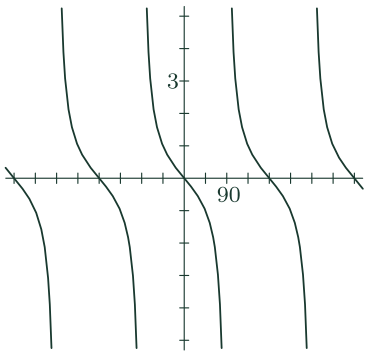
A vector with zero dot product against every in-plane displacement is normal to the plane.

Check. Adding a displacement with this zero dot product to any one solution produces another solution.

Exercise 405 · A descending tangent curve

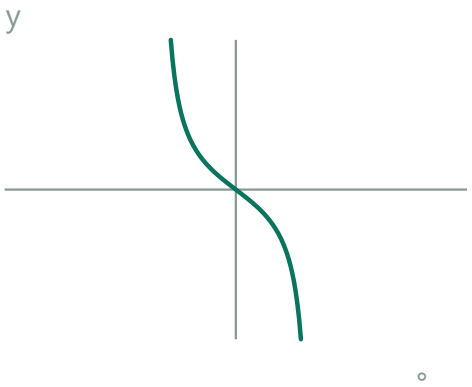
Printed page 41 · The tangent function / Asymptotes

The graph at right appeared on a calculator in degree mode. Find an equation that might have produced this graph.



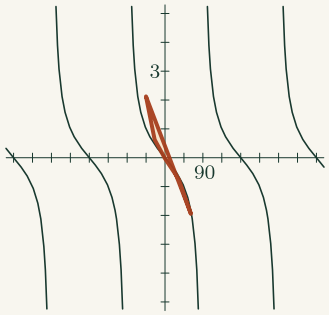
1. Read the shape before guessing a formula

The repeated steep branches approach vertical lines. The central branch crosses at zero and runs downhill as x increases. This suggests a tangent with its vertical direction reversed.



A decreasing central branch

Follow the curve downward through the origin.

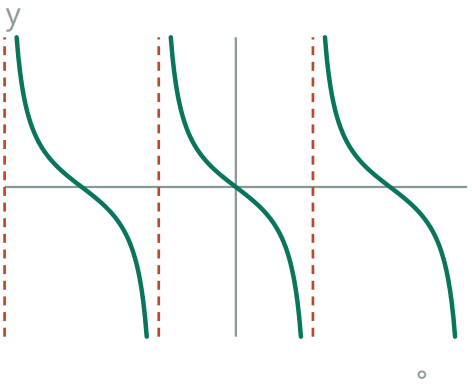


The descending central branch

As the input increases through zero the middle branch slopes downward. This is the feature that fixes the sign of the tangent term.

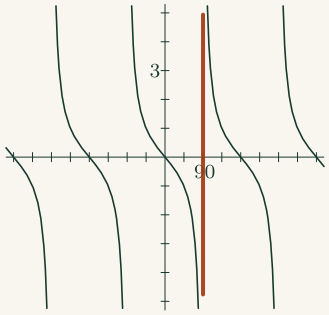
2. Check the degree spacing

Ordinary tangent is undefined at odd multiples of 90° and repeats after 180°. A negative multiplier changes the direction but preserves both of those features.



Restore its neighbors

The nearest vertical breaks are at −90° and 90°. Each branch repeats after 180°.

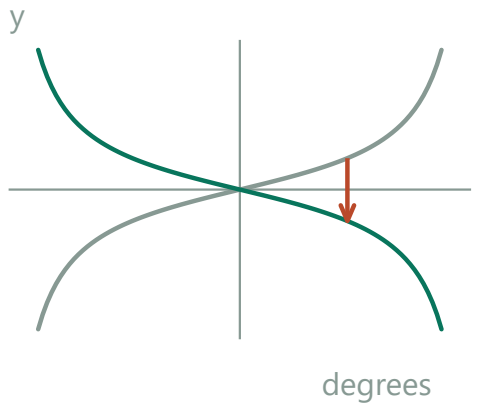


A gap near a vertical direction

At an odd quarter-turn the slope function is undefined. Separate branches approach this vertical direction.

3. Give a matching equation

A possible equation is y=−tan x. Successive branches repeat after 180°, the graph decreases through the origin, and vertical asymptotes occur at 90°+180°k.



Match the negative tangent

Reflection of tan x across the horizontal axis gives the observed descending branches.

Check. At x=45° the graph should give −1 and at −45° it should give +1.

Exercise 406 · Antipode on a translated sphere

Printed page 41 · Sphere geometry / Symmetry

The point $P = (11, 2, 5)$ is on the sphere $x^2 + y^2 + z^2 - 10x + 2y - 14z = -26$. What are the coordinates of the point on this sphere that is farthest from P?

1. Locate the sphere center

Rewrite x^2-10x as $(x-5)^2-25$, y^2+2y as $(y+1)^2-1$ and z^2-14z as $(z-7)^2-49$. Move the constants to the other side to get 49.



Locate the sphere center

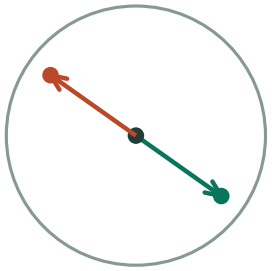
Completing squares puts the center at (5,−1,7) and makes the radius 7.

1. $x^2-10x=(x-5)^2-25$; $y^2+2y=(y+1)^2-1$
Add and subtract the square of half each linear coefficient.

2. $z^2-14z=(z-7)^2-49$
The three square centers are therefore (5,−1,7).

2. Reflect the given radius

Completing squares gives center $(5,-1,7)$ and radius 7. Reflect P across the center: $2(5,-1,7)-(11,2,5)=(-1,-4,9)$.

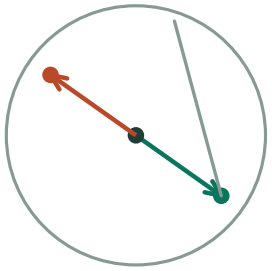


Reverse the radius through P

Move the same displacement on the other side of the center to find $(-1,-4,9)$.

3. Prove no point is farther away

For every point Q on this sphere, the triangle inequality gives $PQ \le PC+CQ=14$. Equality requires the two radius vectors to be opposite, as at the point found.



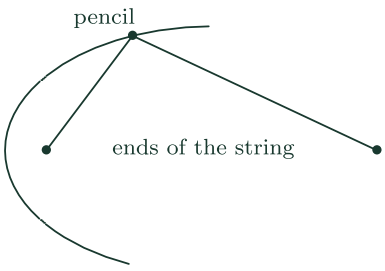
Compare every possible chord

A diameter reaches length 14. Every other chord is at most the sum of its two radius lengths.

Check. The two radius vectors are opposites and the separation is 14, the diameter.

Exercise 407 · Trace a taut string

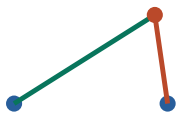
Printed page 42 · Ellipses / Foci and directrices



Kepler's First Law: Planets follow *elliptical* orbits around the sun. As the following exercise shows, these curves are typically not circles, but are very much like circles. You need to find a piece of string that is approximately ten inches long, a large piece of paper, a sharp pencil, and a partner. Mark two points about 6 inches apart near the center of the paper. While your partner anchors the ends of the string on the marks, you use the pencil point to pull the string taut. Keep the string taut as you drag the pencil on a tour that surrounds the ends of the string. The resulting curve is called an *ellipse*. (This and the next exercise are well-suited to the classroom, where board space is also available.)

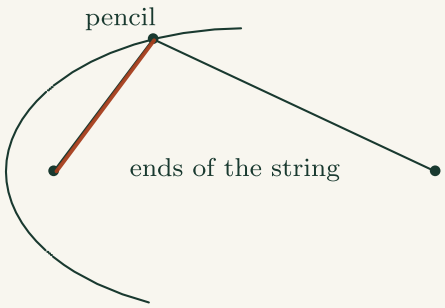
1. Connect the experiment to two distances

Pulling the string taut holds the sum of distances to the two anchors at 10 inches. The pencil therefore traces an ellipse.



Anchor a single 10-inch string

The pencil joins the two anchors with two straight pieces, whose total length stays 10.

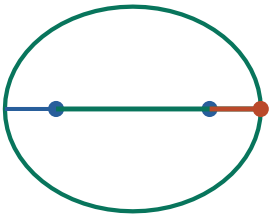


Left string segment

One part of the fixed-length string joins the pencil to the left eyelet.

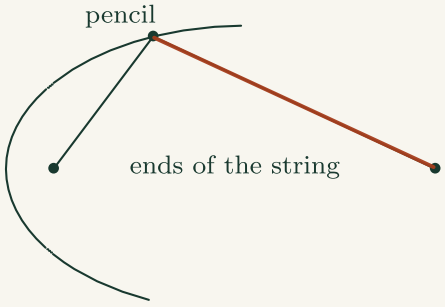
2. See why the semimajor axis is 5

At a horizontal tip beyond both anchors, the two string pieces have lengths $a-3$ and $a+3$. Their sum is $2a=10$. At the top the pieces have equal length 5, so the height is $\sqrt{(5^2-3^2)}$.



Move the pencil to the right vertex

Its two distances now add along one straight axis. The complete major diameter equals the string length.

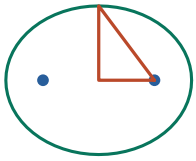


Right string segment

The other part joins the pencil to the right eyelet. When the pencil moves, the length lost by one segment is gained by the other.

3. Specify the reproducible dimensions

Taking the midpoint as origin and anchors on the x-axis gives semiaxes 5 and 4.



Separate anchors by 6 inches

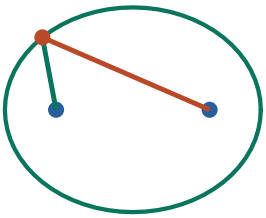
The foci are 3 inches on each side of the center. At the top, a 3–4–5 triangle gives height 4.

1. $2\sqrt{(3^2+b^2)}=10 \Rightarrow 9+b^2=25$
At the top point both anchor distances are equal. Divide by two, then square.

2. $b^2=16 \Rightarrow b=4$ in
The vertical semiaxis is positive.

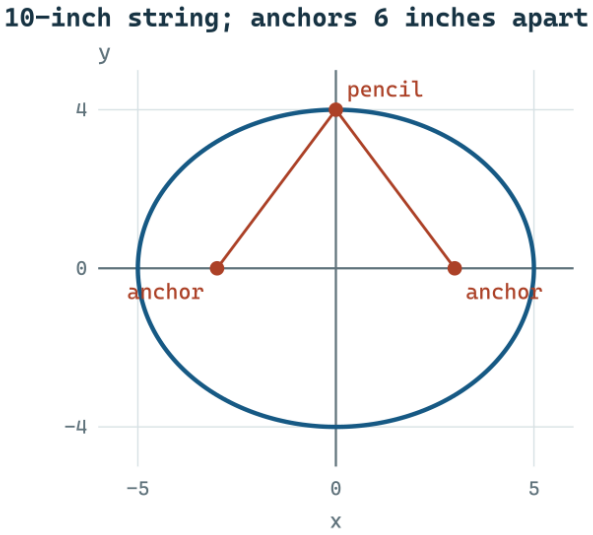
4. Trace the complete curve

For a reproducible drawing, anchor the two ends at (−3,0) and (3,0) inches and sweep the pencil once around both marks, keeping the full 10-inch string taut. With approximate supplies, the dimensions scale with the actual string length and separation.



Keep the string taut during the sweep

The pencil at every position has focal distances summing to 10, tracing the ellipse.



Check. At the highest point the two string pieces each measure 5 inches and form 3–4–5 right triangles.

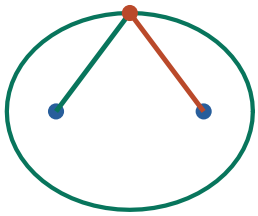
Exercise 408 · Move the anchors

Printed page 42 · Ellipses / Eccentricity

(Continuation) Let your partner do the drawing now. Try anchoring the ends of the string closer together or farther apart, and notice the effect that this has on the shape of the finished curve. To obtain an actual circle, how should the ends of the string be placed?

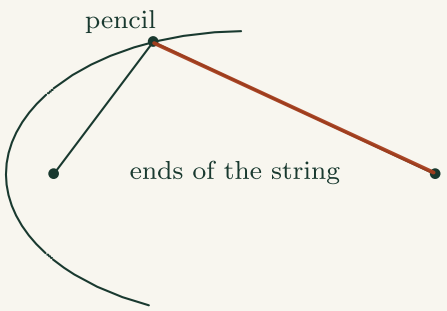
1. Hold the total string length fixed

At either far tip, the sum of the two distances equals twice the midpoint-to-tip distance a. Thus a is set by the string, regardless of how the anchors move.



Fix total length 10

The major semiaxis remains 5 while the anchors move.

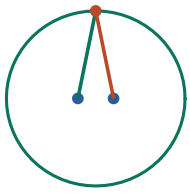


Right string segment · Figure from exercise 407

The other part joins the pencil to the right eyelet. When the pencil moves, the length lost by one segment is gained by the other.

2. Watch the top point

At the top the pencil is equally distant from the anchors, with both distances a. If their half-separation is c, the vertical height is $\sqrt{(a^2-c^2)}$. Larger c leaves less vertical height.

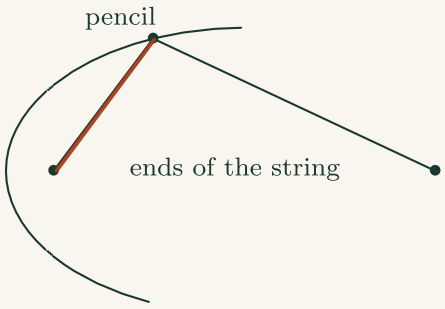


Move the anchors together

With c smaller, $b=\sqrt{(25-c^2)}$ grows. The upper tip is farther from the major axis.

1. $2\sqrt{(c^2+b^2)}=2a \Rightarrow c^2+b^2=a^2$
At the top, each half of the string has length a.

2. $b=\sqrt{(a^2-c^2)}$
Increasing half-separation c at fixed a decreases the height, reaching zero at c=a.



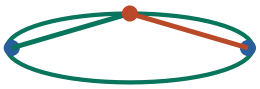
Left string segment · Figure from exercise 407

One part of the fixed-length string joins the pencil to the left eyelet.

3. Describe the shape and limiting cases

The long axis remains equal to the string length. Closer anchors make the ellipse rounder; farther anchors make it thinner. Coincident anchors give a circle of radius half the string length. At separation equal to the length the construction degenerates to a segment.

Distance between the pins · 9.6 in

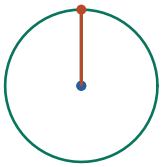


Move the anchors outward

The ellipse narrows as c approaches 5; at c=5 the limiting shape is the segment between anchors.

Compare settings · Distance between the pins

0 in



Change the pin separation

The pins are 0 inches apart. The upper tip has height 5 inches, from $b^2=25-c^2$, while the taut length stays 10 inches.

10 in

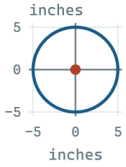
Change the pin separation

The pins are 10 inches apart. The upper tip has height 0 inches, from $b^2=25-c^2$, while the taut length stays 10 inches.

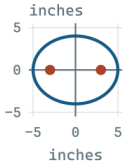
The pins are 0 inches apart. The upper tip has height 5 inches, from $b^2=25-c^2$, while the taut length stays 10 inches.

The pins are 10 inches apart. The upper tip has height 0 inches, from $b^2=25-c^2$, while the taut length stays 10 inches.

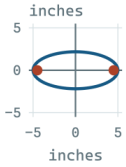
Anchor separation 0 inches; fixed 10-inch string



Anchor separation 6 inches; fixed 10-inch string



Anchor separation 9 inches; fixed 10-inch string



Check. With semimajor length a fixed and half-separation c, the minor radius is $\sqrt{(a^2-c^2)}$.

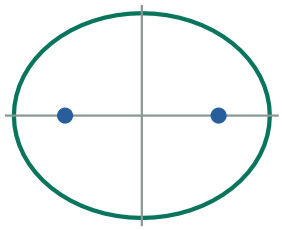
Exercise 409 · Axis intercepts of a string ellipse

Printed page 42 · Ellipses / Foci and directrices

(Continuation) Imagine using graph paper for the preceding construction. Suppose that the string is 10 units long, and that its ends are held at $F_1 = (-3, 0)$ and $F_2 = (3, 0)$. Calculate the four axis intercepts of the ellipse. You should notice that there is a simple relationship between the length of the string and the distance between the x-intercepts.

1. Use symmetry to reduce the work

The anchors lie equal distances from the origin on the x-axis. The four intercepts therefore occur in opposite pairs, so one positive x and one positive y intercept suffice.

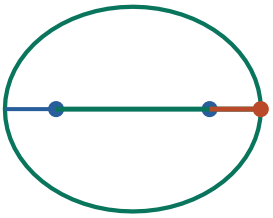


Reflect the construction in either axis

Both anchors lie on the horizontal centerline, equally far from the origin.

2. Find the two horizontal tips

On the far right the distances add as $(x+3)+(x-3)=2x=10$, so vertices are $(\pm 5,0)$.

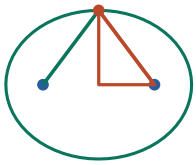


Find the two horizontal intercepts

At (5,0), the distances are 8 and 2. Moving to $(-5,0)$ exchanges them.

3. Find the vertical tips

On the y-axis, $2\sqrt{(9+y^2)}=10$ gives $(0,\pm 4)$. On $x=0$ the horizontal distance from P to either focus is 3. Both string segments must be 5 units long, so $y^2=25-9=16$. There are two choices of vertical sign.



Find the two vertical intercepts

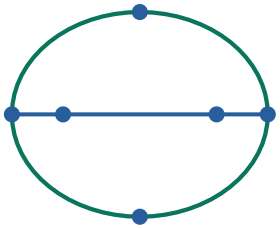
At the upper tip each distance is 5 and the horizontal leg is 3, giving height 4.

1. $2\sqrt{(9+y^2)}=10 \Rightarrow \sqrt{(9+y^2)}=5$
Divide by two because the two distances match.

2. $9+y^2=25 \Rightarrow y^2=16 \Rightarrow y=\pm 4$
Square the nonnegative lengths and take both vertical positions.

4. Compare the string length to the span

The vertex separation equals the full 10-unit string.



Match diameter and taut length

The complete span from -5 to 5 equals the 10-unit string.

Check. At either vertex the distances are 2 and 8; at either other intercept they are 5 and 5.

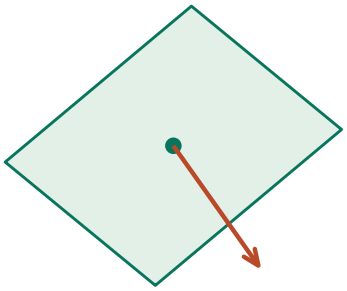
Exercise 410 · A plane halfway between two points

Printed page 42 · Planes in space / Midpoints / The dot product

Create an equation that describes the configuration of points equidistant from the points $P = (1, 5, 6)$ and $Q = (5, -3, -8)$.

1. Find the equal-distance surface

The midpoint of PQ has equal distances to the endpoints. Moving from that midpoint at right angles to PQ preserves the equality by Pythagoras. These allowed directions make a plane.

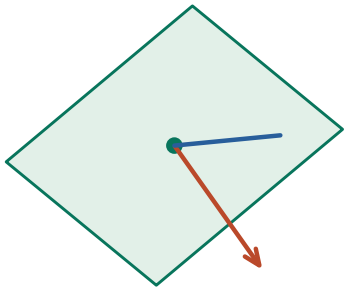


Locate the midpoint

Halfway between the two sites is (3,1,−1).

2. Use the line of the two sites as a normal

Compute Q−P and halve its entries to obtain the simpler normal (2,−4,−7). The displacement from the midpoint to a typical point must have zero dot product with it.

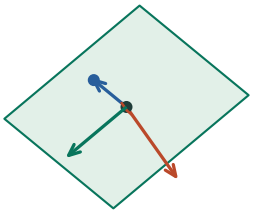


Use the segment direction as normal

The entire equidistant locus is perpendicular to (4,−8,−14) at that midpoint.

3. Write and simplify the equation

The midpoint is (3,1,−1); Q−P=(4,−8,−14). Thus (2,−4,−7)·(x−3,y−1,z+1)=0, or 2x−4y−7z=9.



Keep all sideways displacements

The dot-product condition gives 2x−4y−7z=9.

1. $2(x-3)-4(y-1)-7(z+1)=0$
Use the point halfway between the sites and the reduced normal.

2. $2x-4y-7z-6+4-7=0 \Rightarrow 2x-4y-7z=9$
Expand, combine constants, and add 9 to both sides.

Check. Expanding equal squared distances cancels x^2, y^2 and z^2 and leaves that linear equation.

Exercise 411 · Transform planetary data

Printed page 42 · Kepler's third law / Data linearization

The table at right contains astronomical data for the eight planets of our solar system. Each entry in the d-column stands for the *mean distance* from the planet to the sun, measured using the earth-to-sun mean distance (93 million miles) as the *astronomical unit*. Each entry in the t-column stands for the amount of time needed for the planet to make one revolution of its orbit, measured using earth years. Verify that this is non-linear data. Johannes Kepler, after working with such data for many years, empirically discovered in 1626 the relationship you are about to confirm. Create two new columns of transformed data, putting the square roots of the d-entries in one column and the cube roots of the t-entries in the other. Use the simple linear relationship between these variables to write an equation that expresses *Kepler's Third Law*.

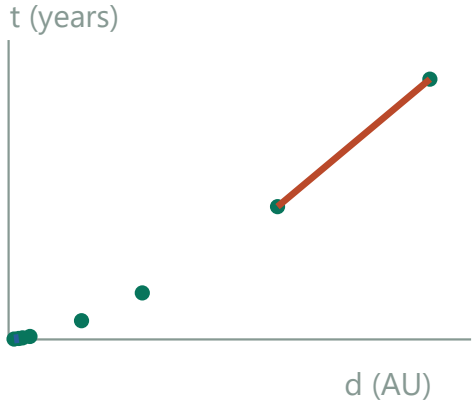
d	t
0.387	0.241
0.723	0.615
1.000	1.000
1.524	1.881
5.203	11.86
9.539	29.46
19.19	84.07
30.06	164.8

(a) Calculate the length of time needed for the asteroid Ceres to orbit the sun, given that Ceres is 251 million miles from the sun. This is an *interpolation* question, by the way.

(b) If there were a planet that took 50 years to complete its orbit, how far from the sun would this planet have to be?

1. Test whether the original columns are linear

On a linear relation the slope between points would be constant. Between Mercury and Venus it is $(0.615-0.241)/(0.723-0.387)\approx 1.11$. Between Uranus and Neptune it is $(164.8-84.07)/(30.06-19.19)\approx 7.43$.

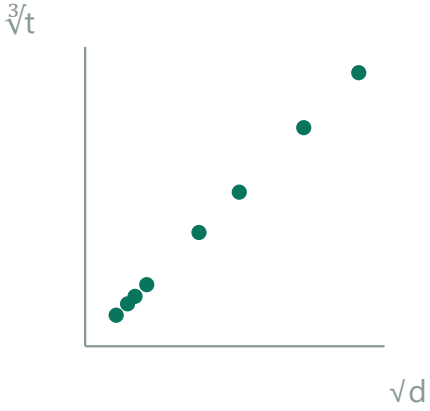


Compare the end pairs

Blue joins Mercury to Venus, orange Uranus to Neptune. The outer pair gains more years for each additional AU.

2. Transform the distances and times

In planetary order, √d gives 0.622, 0.850, 1.000, 1.235, 2.281, 3.089, 4.381, 5.483. The corresponding cube roots of t are 0.622, 0.850, 1.000, 1.234, 2.280, 3.088, 4.381, 5.483.



Transform both coordinates

Place each planet at (√d, ∛t). The two transformed coordinates nearly agree throughout the table.

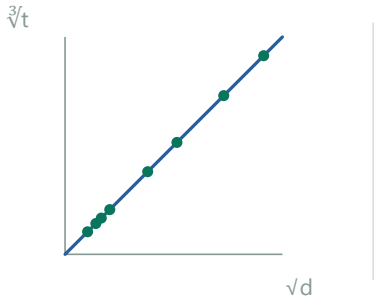
d	t
0.387	0.241
0.723	0.615
1.000	1.000
1.524	1.881
5.203	11.86
9.539	29.46
19.19	84.07
30.06	164.8

The distance column d

Read the original mean orbital distances in astronomical units. Convert every distance by the same transformation.

3. Turn the near-equality into a law

Their near equality gives ∛t≈√d, or $t^2=d^3$ in the ideal model with years and AU.



Test equality of the roots

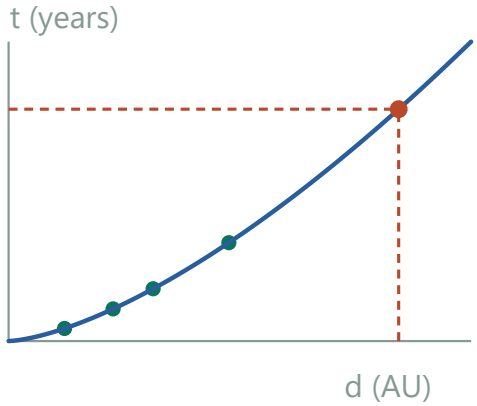
The blue diagonal has height equal to horizontal position. The agreement describes ∛t ≈ √d.

1. $\sqrt[3]{t}=\sqrt{d} \Rightarrow (\sqrt[3]{t})^6=(\sqrt{d})^6$
Use the sixth power to clear both root indices in the idealized law.

2. $t^2=d^3$
Orbital periods and distances are positive, so the transformation can be reversed.

4. Interpolate the orbit of Ceres

Ceres has d=251/93 AU, so t=(251/93)^(3/2)≈4.43 years.

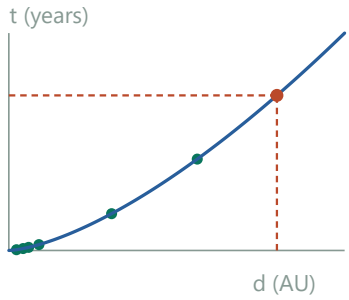


Read an orbit forward

At d = 251/93 AU the power curve reaches about 4.43 years.

5. Invert the law for a 50-year orbit

A 50-year orbit gives d=50^(2/3)≈13.57 AU, about 1.26 billion miles.



Read the power curve backward

From an orbital time of 50 years, follow the guides to d ≈ 13.57 AU.

1. $d^3=50^2=2500 \Rightarrow d=\sqrt[3]{2500}\approx 13.57$ AU
The physical semimajor axis is a positive length.

2. 13.57·93 million miles≈1.26 billion miles
Convert the AU prediction with the exercise's approximate mile scale.

d	t
0.387	0.241
0.723	0.615
1.000	1.000
1.524	1.881
5.203	11.86
9.539	29.46
19.19	84.07
30.06	164.8

Earth's reference row

Earth sets the reference units: the two entries on its row both equal one, so it is a useful check on the transformed columns.

Check. Earth gives (1,1) exactly in these units. The transformed Neptune row gives about 5.48 in both columns.

Exercise 412 · Equivalent orbital laws

Printed page 42 · Kepler's third law / Exponent rules

(Continuation) Kepler's Third Law can be written in a variety of equivalent forms. To see this, solve the equation **(a)** for d in terms of t, and **(b)** for t in terms of d. **(c)** Also raise both sides of $d^{1/2} = t^{1/3}$ to the sixth power.

1. (a) Isolate orbital distance

Let d be the orbital distance and t the period in the specified units. Both physical quantities are positive, so the roots and powers below are defined.

1 **$d^{1/2} = t^{1/3}$**
Begin with the given relation.

2 **$(d^{1/2})^2 = (t^{1/3})^2$**
Square both positive sides. Squaring is one-to-one on positive inputs.

3 **$d = t^{2/3}$**
Multiply the exponents. One half times two is one.

2. (b) Isolate orbital period

Return to the original relation so that cubing immediately undoes the cube root of t.

1 **$(d^{1/2})^3 = (t^{1/3})^3$**
Cube both sides. Cubing is one-to-one on the real numbers.

2 **$d^{3/2} = t$**
Multiply the exponents on both sides.

3 **$t = d^{3/2}$**
Read the equality with the requested quantity first.

d	t
0.387	0.241
0.723	0.615
1.000	1.000
1.524	1.881
5.203	11.86
9.539	29.46
19.19	84.07
30.06	164.8

The period column t · Figure from exercise 411

Each orbital time, in Earth years, belongs to the planet represented by the distance on the same row.

3. (c) Remove both root indices at once

Six is divisible by both 2 and 3. Because d and t are positive, taking the positive sixth root reverses this step.

1 **(d^(1/2))⁶ = (t^(1/3))⁶**
Raise the two equal quantities to the same sixth power.

2 **d^(6/2) = t^(6/3)**
Multiply each original fractional exponent by six.

3 **d³ = t²**
Reduce the two integer exponents.

Check. Substituting either explicit formula into the other returns the original variable.

Exercise 413 · Simplify signed powers

Printed page 43 · Exponent rules / Radicals and rational powers

Convert the following to simpler equivalent forms:

(a) x^6x^{-6} (b) $(8a^{-3}b^6)^{1/3}$ (c) $\left(\frac{x^{1/2}}{y^{2/3}}\right)^6\left(\frac{x^{1/2}}{y^{2/3}}\right)^{-6}$

1. (a) Cancel a power and its reciprocal

The negative exponent creates a denominator x^6. Therefore $x = 0$ is excluded before any cancellation.

1 **x⁶x⁻⁶ = x⁶ · (1/x⁶)**
A power with negative exponent denotes a reciprocal.

2 **= x⁶/x⁶ = 1, x ≠ 0**
A nonzero power divided by itself is one.

2. (b) Cube-root the individual factors

The factor a^{-3} requires $a \neq 0$. A real cube root permits both signs and zero for b.

1 **$(8a^{-3}b^6)^{(1/3)} = \sqrt[3]{(8b^6/a^3)}$**
Move a^{-3} into the denominator.

2 **$= (\sqrt[3]{8})(\sqrt[3]{(b^6)})/\sqrt[3]{(a^3)}$**
Real cube roots distribute over products and nonzero quotients.

3 **$= 2b^2/a, \ a \neq 0$**
Use $2^3 = 8$, $(b^2)^3 = b^6$ and $\sqrt[3]{(a^3)} = a$.

3. (c) Identify the domain of the common base

Give the repeated fraction a name. Its denominator $y^{(2/3)} = (\sqrt[3]{y})^2$ allows negative y, but it vanishes at zero.

1 **B = x^(1/2)/y^(2/3)**
The square root requires $x \geq 0$; the fraction requires $y \neq 0$.

2 **$B^{-6} = 1/B^6$**
The second factor additionally requires $B \neq 0$.

3 **$B \neq 0 \Leftrightarrow \sqrt{x} \neq 0 \Leftrightarrow x > 0$**
A defined fraction is zero exactly when its numerator is zero.

4. (c) Multiply the two powers

Retain both conditions from the original expression even though the simplified result is a constant.

1 **$B^6B^{-6} = B^6/B^6$**
Replace the second power by its reciprocal.

2 **$= 1$ for $x > 0$ and $y \neq 0$**
The denominator is nonzero throughout exactly this domain.

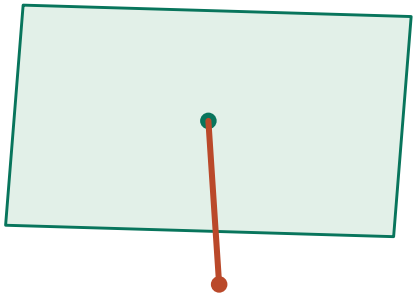
Check. For (b), cubing the simplified expression gives $8a^{-3}b^6$.

Exercise 414 · Nearest point to the origin

Of all the points that lie on the plane $2x + 3y + 6z = 98$, which one is closest to the origin? Remember that the vector $[2, 3, 6]$ is a normal vector for the plane.

1. Follow a perpendicular from the origin

The plane's coefficient vector is n=(2,3,6), so the line through the origin in direction n consists of all points (2t,3t,6t). Find the one parameter at which it reaches the plane.

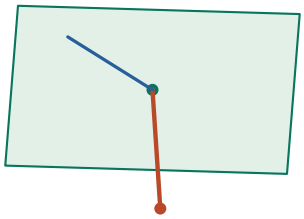


Drop the perpendicular from the origin

The shortest candidate lies along the plane's normal (2,3,6).

2. Locate the candidate point

The nearest point lies on the normal line t(2,3,6). Substitution gives 49t=98, so the point is (4,6,12).



Locate the foot

The line t(2,3,6) meets the plane at t=2, giving (4,6,12).

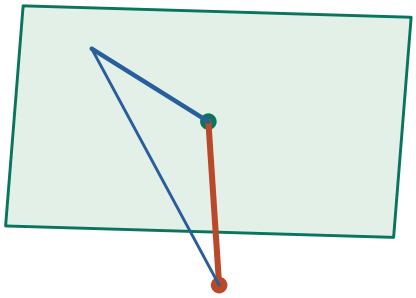
1. 2(2t)+3(3t)+6(6t)=98
Place the normal-line point into the given plane equation.

2. (4+9+36)t=98 ⇒ t=2
Combine the coefficients before dividing by 49.

3. t(2,3,6)=(4,6,12)
Return from the line parameter to a position.

3. Prove it wins against every other point

For any point Q on the plane, Q−(4,6,12) is perpendicular to the normal (2,3,6). By Pythagoras, $|Q|^2 = 14^2 + |Q-(4,6,12)|^2$.

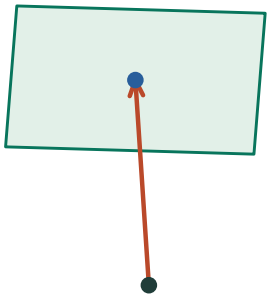


Try a different point

The segment from the origin to any other choice is a hypotenuse with the perpendicular as one leg.

4. Explain uniqueness

The minimum is unique. The added squared distance in the Pythagorean identity vanishes only when Q equals (4,6,12). At every distinct plane point the squared distance from the origin is strictly greater.



Equality only at the foot

The other leg must vanish to match the minimum distance of 14.

Check. Its norm is 14, equal to the plane distance 98/√49.

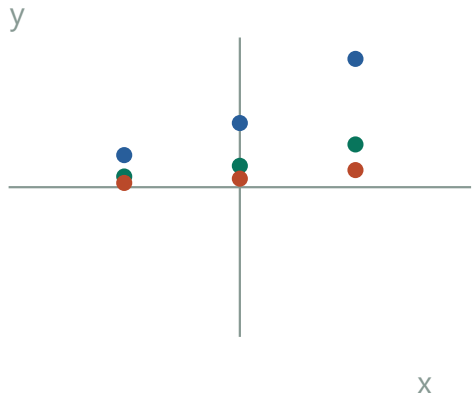
Exercise 415 · Vertical scaling of exponentials

Printed page 43 · Exponential functions / Graph transformations

On one system of coordinate axes, graph the equations $y = 2^x$, $y = 3 \cdot 2^x$, $y = (0.4)2^x$, and $y = (-3)2^x$. What do all graphs $y = k \cdot 2^x$ have in common? How do they differ?

1. Sample at three common inputs

At x=−1,0,1 each curve has y-values k/2,k,2k. Draw through these points, approaching 0 as x tends left.

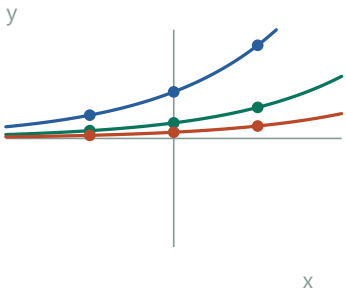


Compare common inputs

At x=0 each vertical intercept is the multiplier k.

2. Explain one-step growth

Each one-unit step doubles the signed value. The ratio comes from $2^{(x+1)}=2\cdot2^{x}$. It applies to the output including its sign, so a negative initial value grows in magnitude downward.



Trace the one-unit change

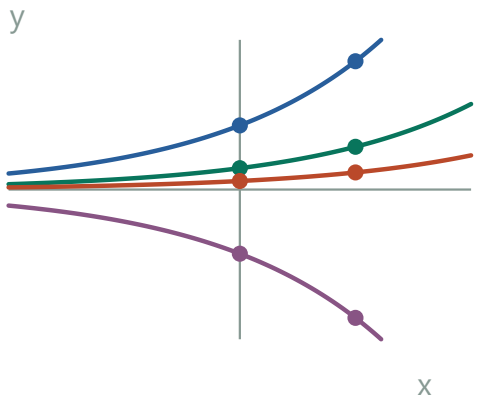
Increasing x by 1 doubles the current signed output on every nonzero curve.

1. $f(x+1)=k\cdot2^{(x+1)}=k\cdot2^{x}\cdot2$
Split off one factor of the exponential base.

2. $f(x+1)=2f(x)$
The identity remains valid even when k is zero or negative.

3. Compare positive and negative multipliers

Positive k gives positive increasing curves; negative k reflects a positive curve across the x-axis.

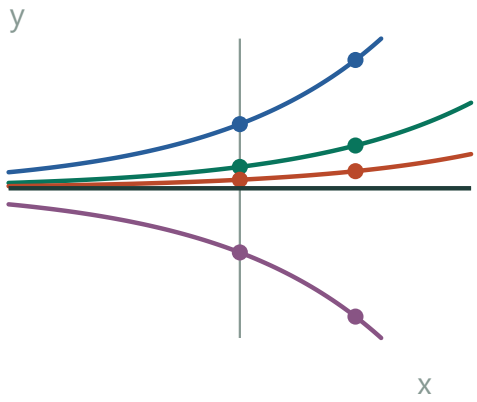


Distinguish the remaining branch

For $k=-3$, the reflection lies below the horizontal axis.

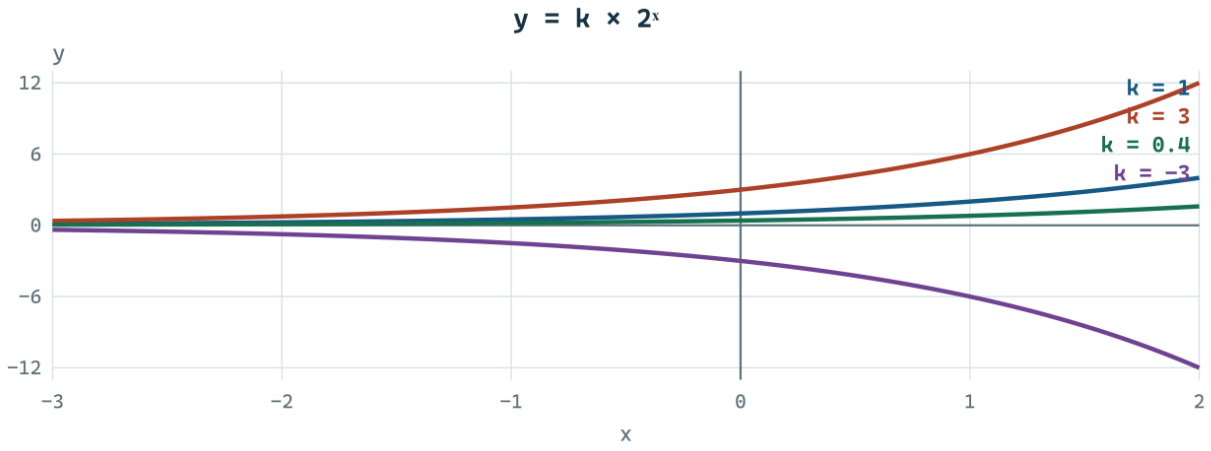
4. Describe the zero multiplier and asymptotes

For $k=0$ the graph is the x-axis. For $k\neq0$ the x-axis is a horizontal asymptote and the magnitude doubles every unit.



Consider the zero multiplier

Setting $k=0$ gives the horizontal axis; each nonzero member approaches that axis in the far-left direction.

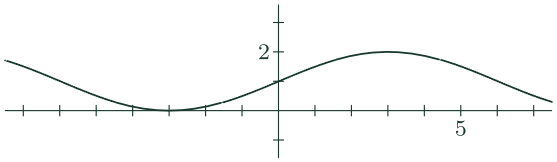


Check. Each graph crosses the y-axis at k, and the absolute values have the same doubling time.

Exercise 416 · A wave with a specified trough

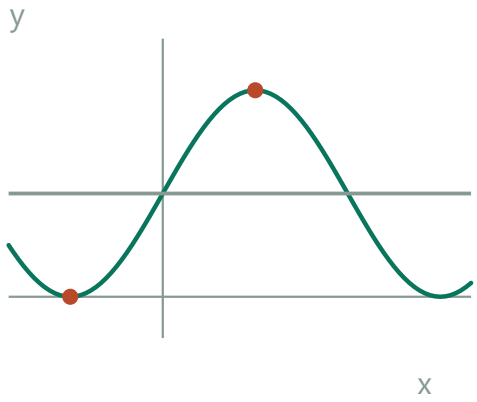
Printed page 43 · Sinusoidal functions / Amplitude and period / Graph interpretation

The sinusoidal graph at right touches the x-axis at -3. Write an equation that could have produced it. Does it affect your answer whether you choose to work in degree mode or radian mode?



1. Reconstruct the midline and amplitude

The trough is 0, crest 2 at x≈3, midline 1. Read the bottom and top heights on the graph. Their midpoint gives the vertical offset and half their difference gives amplitude 1.



Find the central height

The trough 0 and crest 2 place the midline at 1.

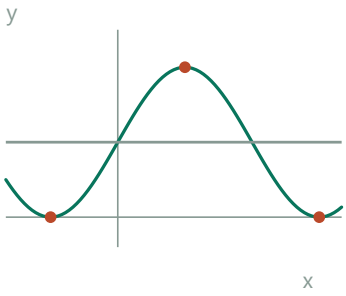


The trough against the axis

Here the graph just touches the horizontal axis. Read the height of that turning point.

2. Convert half a wave to angular speed

Going from the trough at −3 to the crest at 3 takes six x-units and half a turn. The full period is twelve x-units, so one x-unit advances the phase by 2π/12 radians or 360°/12.



Measure half a period

The horizontal distance from trough −3 to crest 3 is 6, so a whole wave spans 12.

1. T/2=3−(−3)=6 ⇒ T=12
The adjacent trough and crest are half a period apart.

2. ω=2π/12=π/6 radians per x-unit
One whole period must advance the argument by 2π.

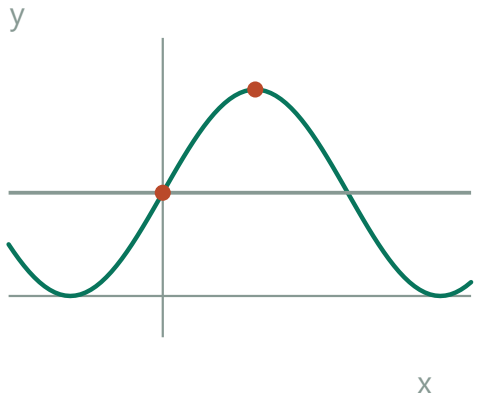


The neighboring crest

The neighboring high point is half a wave beyond the trough. Its horizontal separation from the trough determines half the period.

3. Write the two equivalent conventions

Six units is half a period, so use y=1+sin(πx/6) in radian mode, or y=1+sin(30x°) in degree mode.



Use the appropriate unit

In radians the curve is 1+sin(πx/6); in degrees use 1+sin(30x°).

Check. At x=−3,0,3 the heights are 0,1,2 in both conventions.

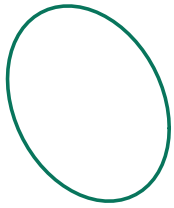
Exercise 417 · A circle equation in space

Printed page 43 · Three-dimensional geometry / Cross sections / Circle geometry

In the xz-plane, $x^2 + z^2 = 9$ describes a circle. What does the same equation describe in xyz-space?

1. Identify the unconstrained coordinate

The equation has only x and z, so any pair on the original radius-3 circle can be given an arbitrary y-coordinate. Moving a valid point straight along the y-axis never changes its squared distance x^2+z^2 from that axis.

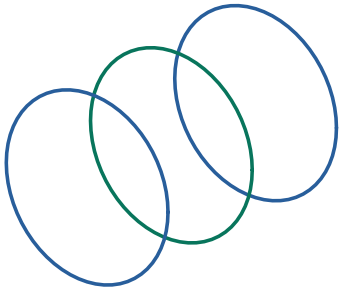


Begin with $x^2+z^2=9$

At any one y-value this is a radius-3 circle in the xz directions.

2. Stack the cross sections

Draw the same xz-circle in the planes $y=0$, $y=1$ and $y=-1$. Continue through all real values of y to form the entire surface.

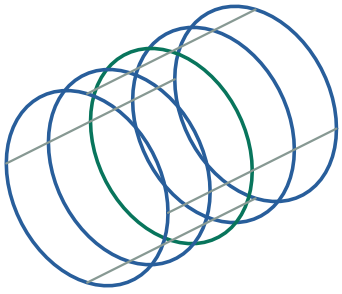


Translate the cross section along y

Changing the free y-coordinate leaves x^2+z^2 unchanged.

3. Name the three-dimensional figure

Every horizontal-in-y slice is the same circle in the xz directions. The union is the surface of an infinite circular cylinder of radius 3 along the y-axis.



Extend the stack both ways

The union is the surface of a circular cylinder, without end caps or a bound on y.

Check. The parametrization $(3\cos t, s, 3\sin t)$, with s arbitrary, satisfies the equation everywhere.

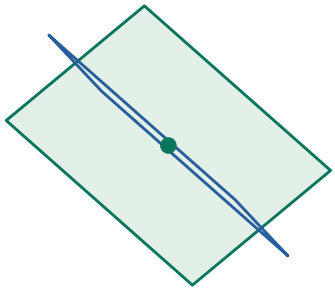
Exercise 418 · Two intersecting planes

Printed page 43 · Planes in space / Parametric lines / Systems of equations

Consider the equations $2x + 3y + 5z = 15$ and $x - 2y + 2z = 3$. There are many points (x, y, z) whose coordinates fit *both* equations. Two of them are $P = (1, 1, 2)$ and $Q = (17, 2, -5)$. Find another point whose coordinates fit both equations. What does the configuration of *all* the common solutions look like?

1. Expect a line of intersection

Each linear equation gives a plane. Their normals (2,3,5) and (1,−2,2) are not multiples, so these are distinct nonparallel planes. Their common points form a line.

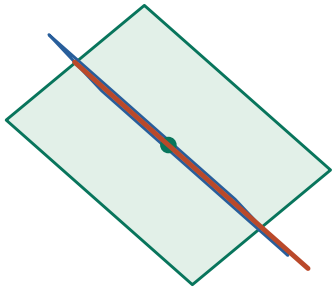


Two different planes

Their independent normals rule out coincident or parallel planes.

2. Use the known points to choose its direction

The direction Q−P=(16,1,−7) is perpendicular to both normals.

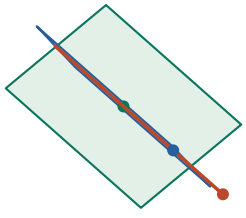


Connect P to Q

The shared displacement Q−P=(16,1,−7) runs along both surfaces.

3. Parametrize and choose a third point

All solutions are P+t(16,1,−7), since the normals are independent. At t=2 one more point is (33,3,−12).



Take a second equal step

At t=2 the parameter line reaches (33,3,−12); all its points satisfy both equations.

1. P+t(Q−P)=(1+16t,1+t,2−7t)
Multiply each displacement coordinate by the same parameter.

2. t=2 ⇒ (1+32,1+2,2−14)=(33,3,−12)
Two steps along the line give a third common point.

Check. At this point the plane left sides are 66+9−60=15 and 33−6−24=3.

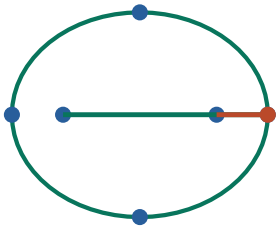
Exercise 419 · Five distance-sum checks

Printed page 43 · Ellipses / Foci and directrices

Let $F_1 = (-3, 0)$ and $F_2 = (3, 0)$, and let $P = (x, y)$ be a point on the graph of the equation $16x^2 + 25y^2 = 400$. Recall that PF_1 and PF_2 are the distances from P to F_1 and F_2, respectively. After you have verified that $PF_1 + PF_2 = 10$ holds whenever P is one of the four axis intercepts, calculate $PF_1 + PF_2$ for a fifth point P of your choosing. Solve $16x^2 + 25y^2 = 400$ for y, finding two functions that together generate this curve.

1. Locate and test the four intercepts

The four intercepts are (±5,0),(0,±4). Their focal distances sum to 8+2 or 5+5.

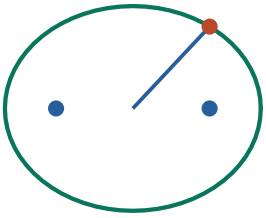


Check the four axial tips

The two distances total 8+2 at a horizontal vertex and 5+5 at a vertical vertex.

2. Choose a fifth point from the equation

For example set x=3. Then $25y^2$=400−144=256, giving y=16/5 if the upper point is chosen. One focus has the same x-coordinate; its distance is simply the vertical difference.

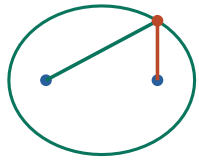


Choose x=3

The positive upper branch gives y=16/5 and the point (3,16/5).

3. Check that point's two distances

At (3,16/5) they are 34/5 and 16/5, again totaling 10.



Measure from both foci

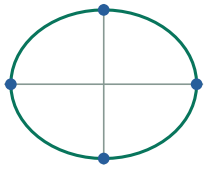
At the chosen point the two segments measure 34/5 and 16/5, totaling 10.

1. $L^2=(3+3)^2+(16/5)^2=(900+256)/25$
Measure to the left focus using squared displacement.

2. $L=34/5$; $R=16/5 \Rightarrow L+R=50/5=10$
The vector to the right focus is vertical.

4. Find all upper and lower points

Solving for y gives the two branches $y=\pm(4/5)\sqrt{(25-x^2)}$, for $-5\le x\le 5$.



Use both y signs

Solving for y gives an upper and a lower value at each interior x between −5 and 5.

1. $25y^2=400-16x^2=16(25-x^2)$
Move the x-term and factor 16.

2. $y=\pm(4/5)\sqrt{(25-x^2)}$, $-5\le x\le 5$
Divide by 25 and take both roots where the radicand is nonnegative.

Check. Substituting either branch eliminates the sign and recovers the displayed ellipse.

Exercise 420 · Eliminate ellipse radicals

Printed page 43 · Ellipses / Foci and directrices / Algebraic manipulation

Let $F_1 = (-3, 0)$, $F_2 = (3, 0)$, and $P = (x, y)$. Use the distance formula to convert the equation $PF_1 + PF_2 = 10$ into *Cartesian form*. Then simplify the result by removing the radicals from the equation. To do this, start by arranging the terms so that there is one radical on each side of the equal sign. Then square both sides, as in $(PF_1)^2 = (10 - PF_2)^2$. Remember to make use of the identity $(m - n)^2 = m^2 - 2mn + n^2$. If all goes well, you will eventually discover that the equation $PF_1 + PF_2 = 10$ is equivalent to $16x^2 + 25y^2 = 400$.

1. Label the two focal distances

Write $L=\sqrt{((x+3)^2+y^2)}$, $R=\sqrt{((x-3)^2+y^2)}$. A horizontal displacement to the left focus uses $x-(-3)=x+3$. To the right focus it uses $x-3$ instead. Both vertical displacements are y.

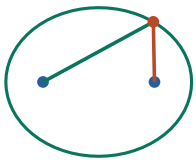


Name the two segment lengths L and R

L starts at $(-3,0)$, R at $(3,0)$. Their taut sum is 10.

2. Move one square root across and square

From $L=10-R$ get $L^2-R^2=100-20R=12x$, so $R=5-3x/5$.



Compare the squares of the distances

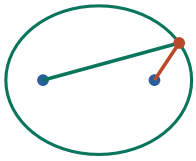
The y^2 terms cancel when the squared lengths from the two horizontal anchors are subtracted.

1. $L^2-R^2=(x+3)^2-(x-3)^2=12x$
Cancel y^2, the shared x^2, and the constant terms.

2. $L^2=(10-R)^2=100-20R+R^2$
Square the isolated distance and keep the middle product.

3. Isolate the second radical

The subtraction of the squared distances cancels both y^2 terms, leaving $(x+3)^2-(x-3)^2=12x$. This explains where the linear term in the previous equation came from.



Separate the right-hand distance

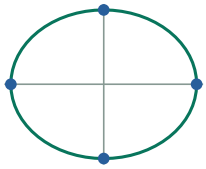
The remaining equation $R=5-3x/5$ depends only on horizontal position.

1. $12x=100-20R \Rightarrow 20R=100-12x$
Use the two expressions for the difference of squares.

2. $R=5-3x/5$
Divide every term by 20.

4. Eliminate the final square root

Square this against $R^2=(x-3)^2+y^2$. The linear terms cancel, leaving $16x^2+25y^2=400$.



Eliminate the last square root

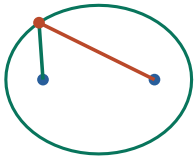
Squaring yields $16x^2+25y^2=400$, an ellipse with horizontal and vertical intercepts 5 and 4.

1. $x^2-6x+9+y^2=25-6x+9x^2/25$
Expand both the distance formula and the squared expression for R.

2. $16x^2/25+y^2=16 \Rightarrow 16x^2+25y^2=400$
Cancel $-6x$, collect x^2 terms, and multiply by 25.

5. Check the reverse implication

The resulting quadratic bounds x between -5 and 5. On this interval both candidate distances $5-3x/5$ and $5+3x/5$ are positive. Substituting the quadratic gives their squares equal to the actual squared distances, so the positive roots are the distances and their sum is 10.



Check the original distance sum

For points on this ellipse the isolated distances stay nonnegative and recombine to 10.

1. $16x^2 \le 400 \Rightarrow -5 \le x \le 5$
The omitted term $25y^2$ is nonnegative.

2. $5 \pm 3x/5 \ge 2 \Rightarrow L+R=(5+3x/5)+(5-3x/5)=10$
Both reconstructed square roots have the required positive sign.

Check. On the resulting curve $-5 \le x \le 5$, both $5 \pm 3x/5$ are positive; their sum 10 confirms the squaring introduced no extra branch.

Exercise 421 · Decimal exponents

Printed page 43 · Exponent rules / Radicals and rational powers

Explain why calculating $z^{2.5}$ is a square-root problem. What does $z^{0.3}$ mean?

1. Turn each decimal into a reduced fraction

A terminating decimal gives an exact rational number. Here 2.5 is $25/10=5/2$, whereas 0.3 is $3/10$. The denominator identifies a root and the numerator identifies a power.

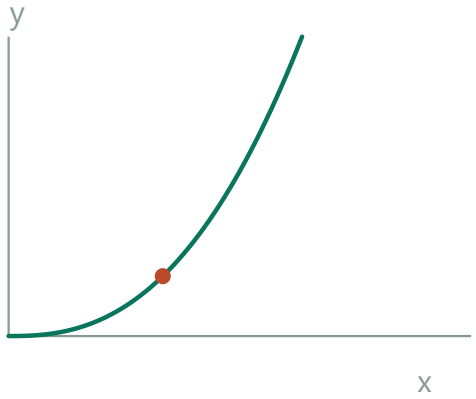


Read terminating decimals exactly

The two decimal exponents lie at 3/10 and 5/2; the denominator specifies a root index.

2. Interpret the denominator 2

Because 2.5=5/2, take √z and raise it to the fifth (or $z^2\sqrt{z}$).

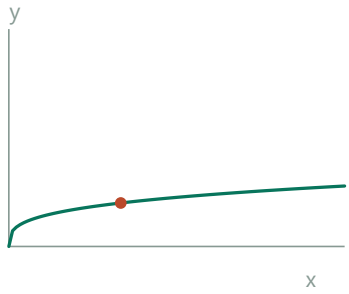


Square-root then raise to the fifth

The positive branch of z^(5/2) rises faster than linearly. Its values are $(\sqrt{z})^5$.

3. Interpret the denominator 10

Since 0.3=3/10, take the tenth root of z and cube it.



Take the tenth root and cube

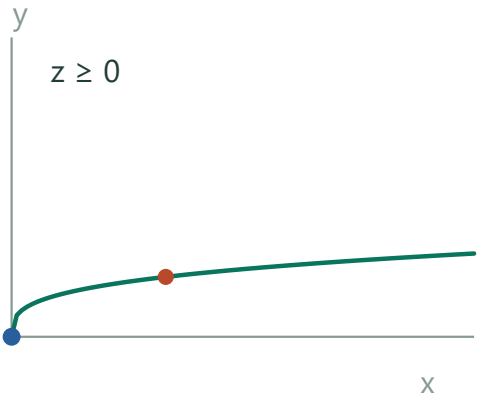
The exponent 3/10 gives a positive increasing curve through (1,1), with a different shape from the 5/2 power.

1. 0.3=3/10 ⇒ z^0.3=(z^(1/10))³
The denominator specifies which root; the numerator is the power.

2. z≥0
A real tenth root is defined only for nonnegative radicands.

4. Track the real domain

Real formulas here require z≥0. The square root and the tenth root both have even index. A negative input cannot be an even power of a real number.



Include zero and check negative inputs

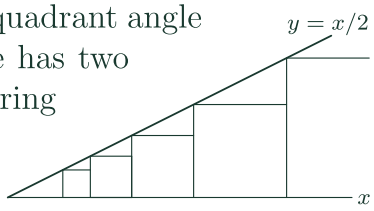
Both even-index roots accept z=0. Neither supplies a real value at a negative z.

Check. Squaring the first expression gives z^5; raising the second to the tenth gives z^3.

Exercise 422 · A growing train of squares

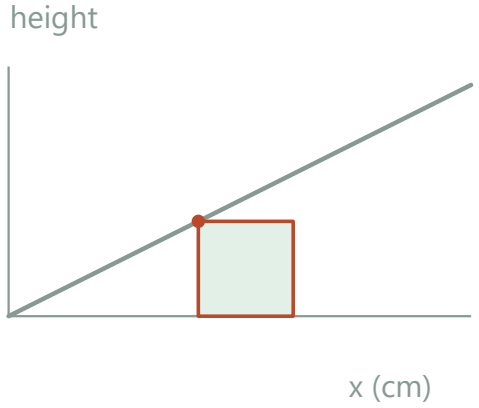
Printed page 43 · Geometric sequences and series / Similarity and scaling

The figure shows a sequence of squares inscribed in the first-quadrant angle formed by the line $y = \frac{1}{2}x$ and the positive x-axis. Each square has two vertices on the x-axis and one on the line $y = \frac{1}{2}x$, and neighboring squares share a vertex. The first (smallest) square is 8 cm tall. How tall are the next four squares in the sequence? How tall is the nth square in the sequence?



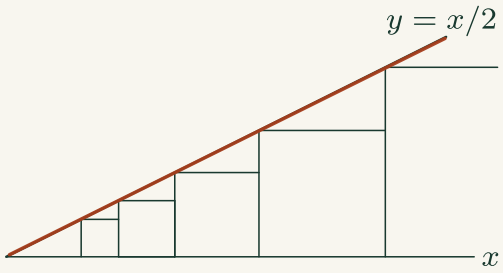
1. Read the contact on the sloping line

The top-left corner of the first square sits at (x,s) on y=x/2, so x=2s. Its bottom-left corner has the same horizontal coordinate. The square extends to the right by exactly its own height.



Meet the line at the upper left

The first square touches y=x/2 at (16,8). Its bottom edge starts at x=16.

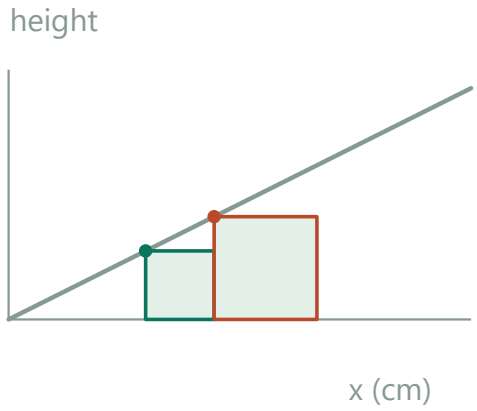


The slanted boundary

Every square touches the same line at its upper-left corner. Read its height from the line at that corner's x-coordinate.

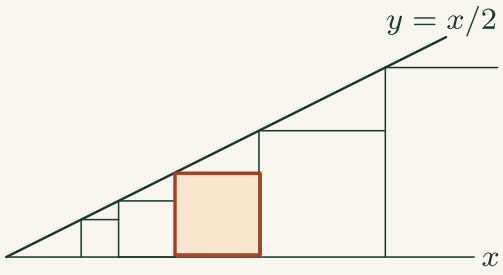
2. Map one square to the next

A square of height s starts at horizontal coordinate 2s, where its top left meets the line. Its right edge is at 3s. The next height is therefore 3s/2.



Use the right-hand edge as the next start

The first right edge is x=24. The line there has height 12, setting the next side.

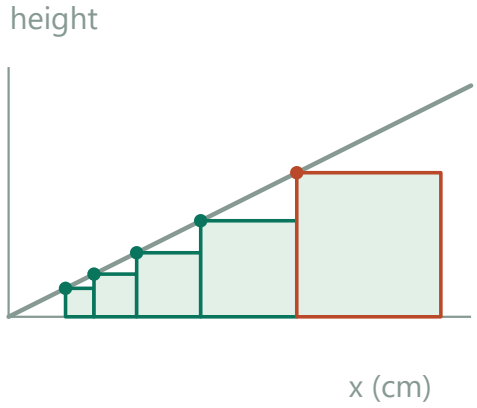


One square and the next starting edge

The current square's width equals its height. Its right edge determines where the next square begins.

3. Generate successive heights

Next four: 12,18,27,40.5 cm; nth height 8(3/2)^(n−1) cm.

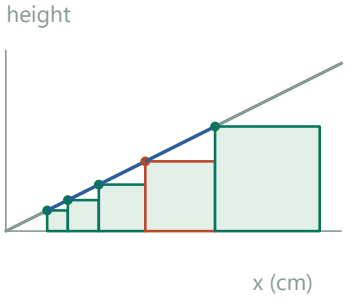


Grow four more touching squares

The successive heights are 12, 18, 27, and 40.5 cm. The same line fixes every top-left corner.

4. Explain the exponent

Start the index at n=1 for the 8-cm square. To reach the nth square from the first takes n−1 multiplications by the same scale factor.



Count the transitions between contacts

The colored sloping segments join successive top-left corners. There are n−1 jumps from square one to square n.

1. $s_2/s_1=3/2 \Rightarrow s_n=s_1(3/2)^{(n-1)}$
Going from the first square to square n crosses n−1 boundaries.

2. $s_1=8 \Rightarrow s_n=8(3/2)^{(n-1)}$ cm
Check n=1 gives a multiplier of one.

Check. The first square runs from x=16 to x=24, placing the next top left at (24,12).

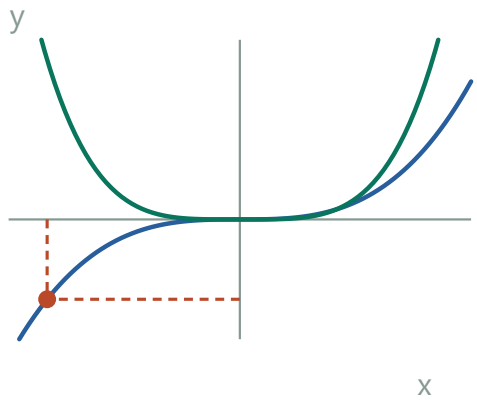
Exercise 423 · Negative bases and fractional powers

Printed page 44 · Radicals and rational powers / Domain and range

Verify that $(-8)^{1/3}$ can be evaluated, but that $(-8)^{1/4}$ cannot, and explain why $(-8)^{2/6}$ is ambiguous. To avoid difficulties like these, it is customary to restrict the base, b, of an exponential function, $f(x) = b^x$, to be a positive number.

1. Test odd and even powers

The sign of a real number survives an odd power: multiplying three copies of −2 gives −8. An even power pairs off minus signs, producing a nonnegative result from either sign.

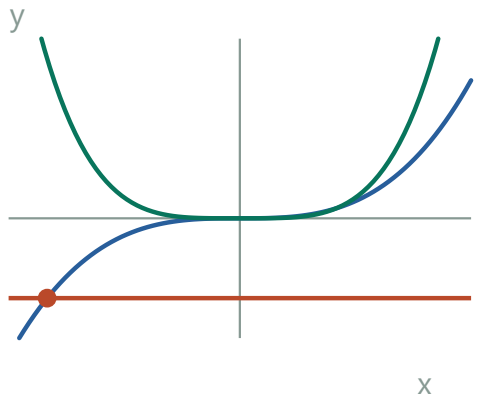


Compare odd and even powers

Blue cubes preserve the sign of −2. The green fourth power is nonnegative on both sides.

2. Compare the two literal root operations

The real cube root is −2; a real fourth root is impossible because fourth powers are nonnegative.

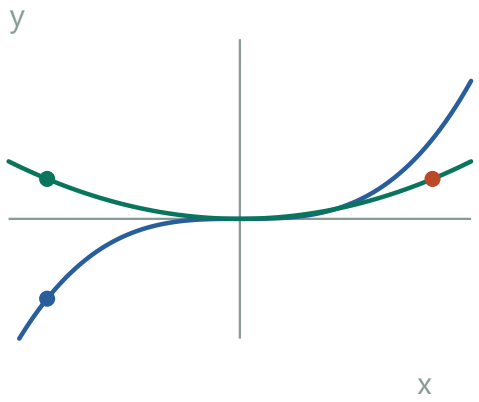


Test a negative target

The level −8 has a cube-root input −2 and no real fourth-root input.

3. See what happens with an unreduced exponent

Reducing 2/6 to 1/3 gives −2, but squaring first and taking the positive sixth root of 64 gives +2.

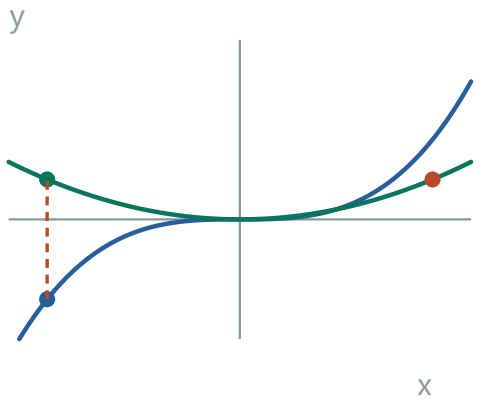


Watch the sign lost by squaring

The parabola gives the same square at ±2. Thus the positive sixth root of 64 cannot recover the sign of the original −8.

4. State the convention

The usual reduced-fraction definition chooses −2; unrestricted rearrangements fail here.



Use the reduced fraction

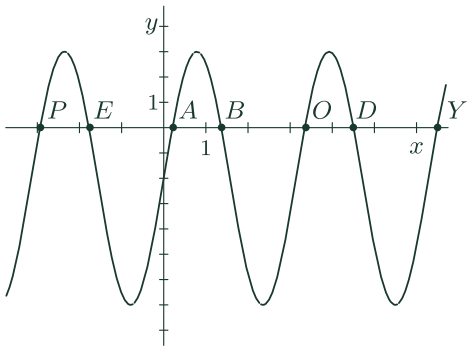
For (−8)^(2/6) reduce to 1/3 and follow the blue cube backward from −8 to −2. Squaring early discarded that sign.

Check. Positive bases avoid the sign conflict, which is why real-valued exponential functions use them.

Exercise 424 · Read the remaining intercepts

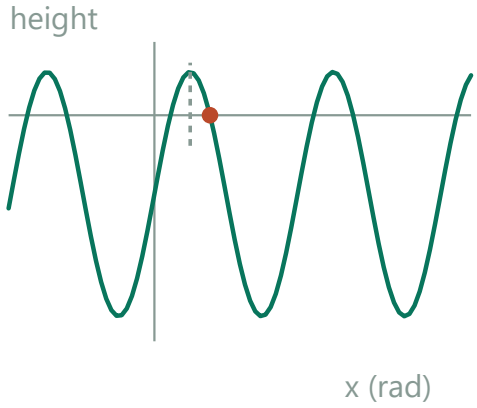
Printed page 44 · Sinusoidal functions / Symmetry / Periodic functions

The equation whose graph is shown at right has the form $y = k + a\sin 2x$. As shown in the diagram, one of the x-intercepts is $B = (b, 0)$. Working in radian mode, find the values of the other x-intercepts shown in the figure in terms of b.



1. Locate the peak between the crossings

The argument of sine is 2x, so its first positive peak for a positive coefficient occurs when 2x=π/2. Reading the paired roots about that center gives two equally spaced horizontal offsets.

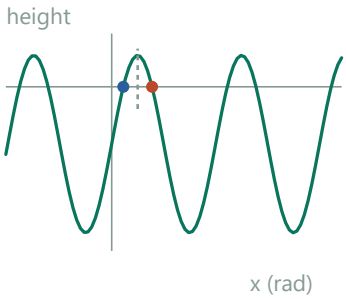


Locate the descending root B

An illustrative b = 1.22 rad places B to the right of the crest at π/4. The symbolic reasoning works for the given b.

2. Reflect B across the crest

The period is π and the first peak is centered at π/4. Its rising zero A therefore has x=π/2−b.

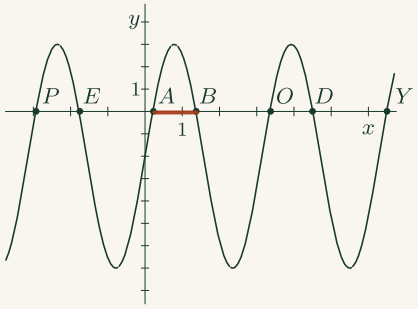


Reflect B across the crest

The rising root A has the opposite offset from π/4, hence its coordinate is π/2−b.

1. (xA+b)/2=π/4
The left and right roots are symmetric around the peak.

2. xA=2(π/4)−b=π/2−b
Double the midpoint and subtract the known root.

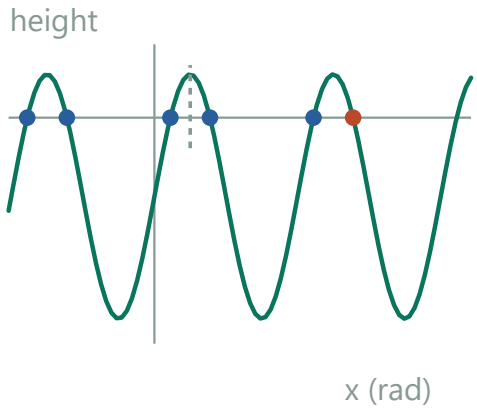


The two crossings A and B

Both marked roots flank the same crest. Their mean horizontal coordinate lies directly underneath that crest.

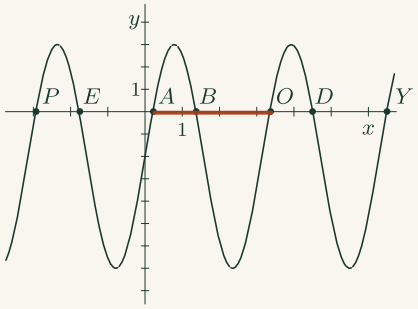
3. Translate a whole period to get the nearby pairs

Shift the pair (A,B) one period left for P=−π/2−b and E=b−π, and one period right for O=3π/2−b and D=b+π.



Translate both roots by one period

Horizontal steps of π carry A and B to the adjacent pairs (P,E) and (O,D).

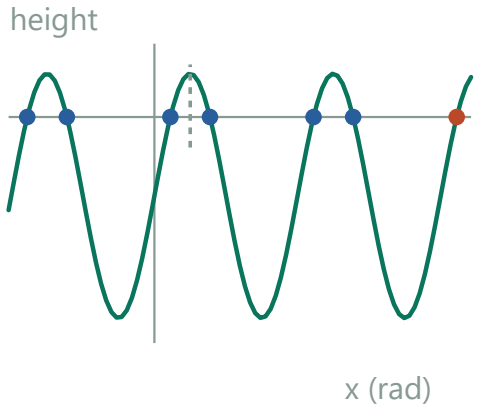


One complete cycle

A and the next rising zero O play the same role one cycle apart. The other corresponding roots use that same horizontal displacement.

4. Place the remaining root

Shift A two periods right for Y=5π/2−b. All ordinates are zero.



Move the rising root two periods

Y is the orange point two periods to the right of A. Its coordinate is 5π/2−b.

Check. Within each pair the mean x is a peak; corresponding roots are π apart.

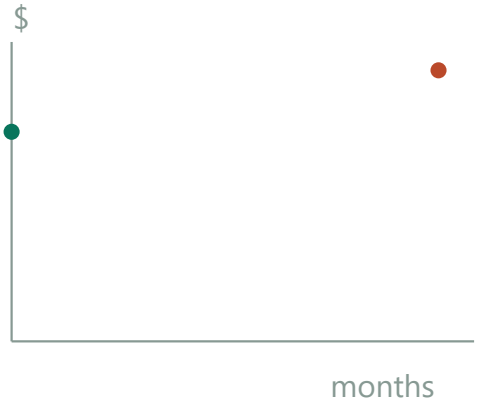
Exercise 425 · Compound inflation

Printed page 44 · Compounding / Percentage change

Suppose that a basket of groceries that costs \$68.80 today cost only \$53.20 a year ago. What is the annual rate of inflation? What is the *monthly* rate of inflation, assuming that it is constant?

1. Measure the entire price factor

Compare the same basket at the two times. Divide the later price by the earlier price to get dollars now per dollar then. Subtract 1 to isolate the fractional increase.

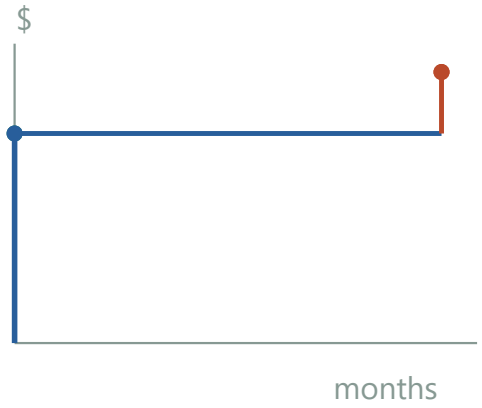


Compare the same basket

The old price \$53.20 and current price \$68.80 are twelve months apart.

2. Compute annual inflation

Divide prices to get annual factor 68.8/53.2, so the annual rate is about 29.32%.

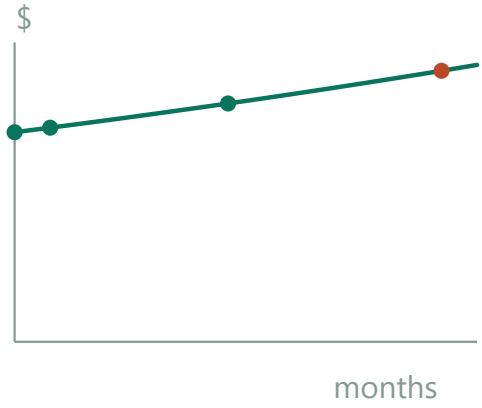


Divide final price by initial price

Their ratio is the annual factor; subtract 1 to get the annual inflation rate.

3. Relate one month to one year

If the monthly rate is m, after twelve equal compounded increases the annual factor is $(1+m)^{12}$. Take a twelfth root before subtracting 1.

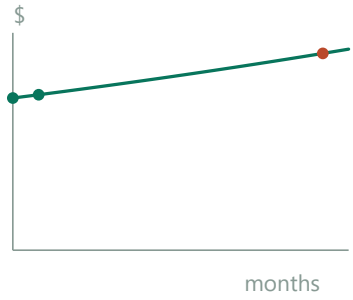


Use a common monthly factor

Twelve identical factors must jointly produce 68.8/53.2.

4. Compute the monthly rate

If every month has equal compounded inflation, the monthly rate is (68.8/53.2)^(1/12)−1≈2.166%.



Extract the monthly multiplier

The twelfth root, less 1, is approximately 2.166% growth per month.

1. $(1+r)^{12}=68.80/53.20$
Twelve equal monthly factors reproduce the observed annual price ratio.

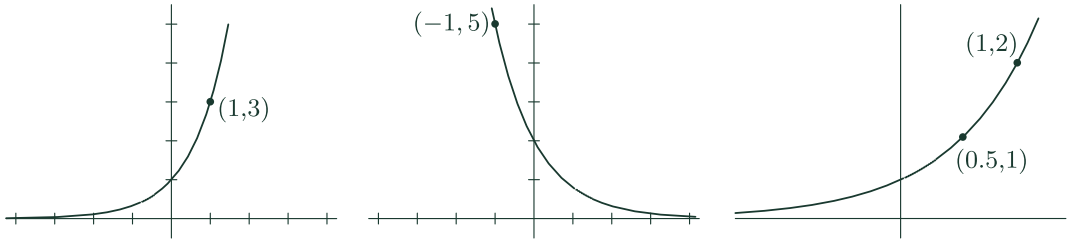
2. r=(68.80/53.20)^(1/12)−1
Take the positive monthly factor, then remove the original one.

Check. The monthly factor raised to the twelfth power reproduces the observed price ratio.

Exercise 426 · Read exponential curves

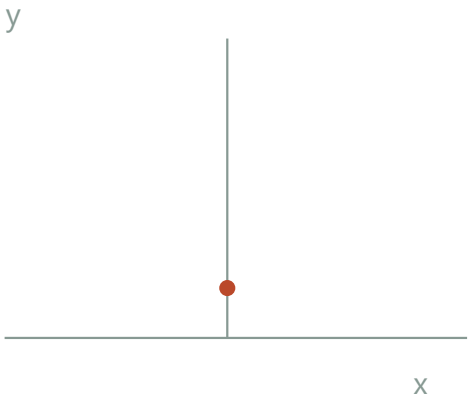
Printed page 44 · Exponential functions / Graph interpretation

Find a plausible equation $y = a \cdot b^x$ for each of the exponential graphs shown below:



1. Read the initial value and growth factor

For y=ab^x the intercept at x=0 is a. A move one unit to the right multiplies the height by b. A half-unit move multiplies by √b.

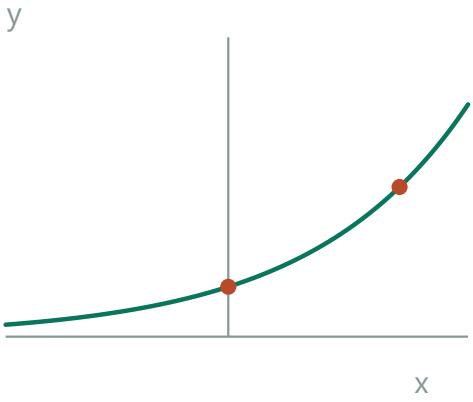


Identify what an intercept reveals

In y=a·b^x, the vertical intercept immediately gives a.

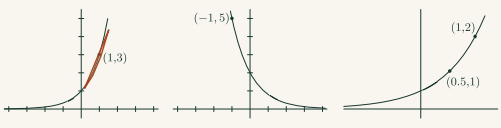
2. Fit the left graph

Left passes through (0,1),(1,3), so y=3^x. The intercept fixes a=1. Moving from x=0 to x=1 takes the height from 1 to 3, giving the one-step factor.



Fit the left graph

A rise from 1 to 3 in one x unit fixes b=3.

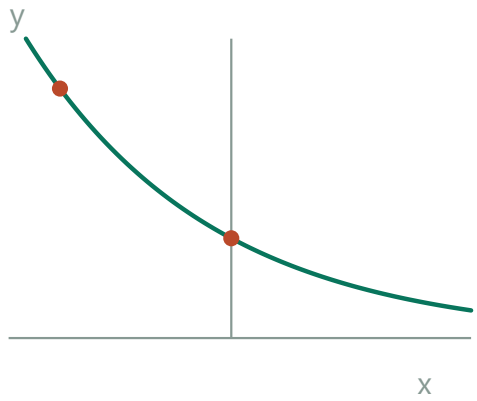


Left curve and its marked value

In the left panel compare where the curve crosses its vertical axis with the labeled point one input unit later.

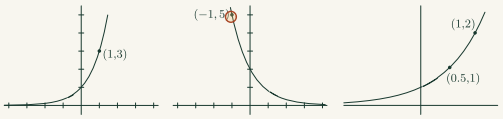
3. Fit the middle graph

Middle crosses at (0,2) and goes through (−1,5); 2/b=5 gives y=2(0.4)^x.



Fit the middle decay graph

The left-to-right ratio from 5 to 2 is 0.4. Its equation is $2(0.4)^x$.

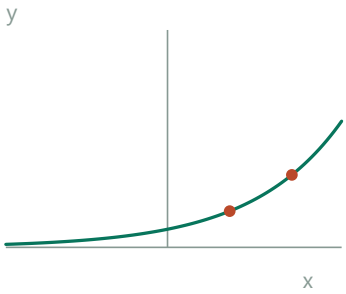


Middle negative-input observation

The middle curve has a labeled value one step before its intercept. Moving backward a unit divides by the growth factor.

4. Fit the right graph

Right doubles from y=1 to y=2 over half a unit, so b=4 and a=½; y=½·4^x.



Fit the right graph

Doubling in a half-step means multiplying by 4 per full step; the intercept is ½.

1. y(x+½)/y(x)=b^(1/2)=2
The marked half-unit step doubles the nonzero height.

2. $b=2^2=4$; a=y(0)=½
Square the factor to obtain the base per full unit and read the intercept.

Check. Substitution recovers both labeled points on the last graph: ½·2=1 at x=.5 and ½·4=2 at x=1.

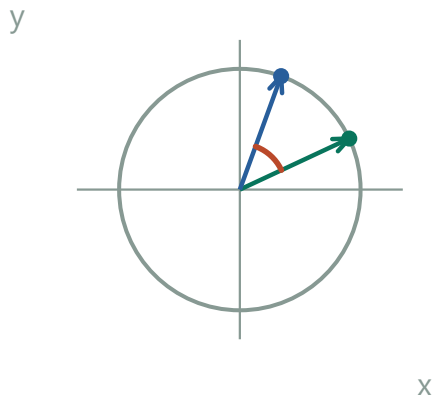
Exercise 427 · Dot product of two moving directions

Printed page 44 · The dot product / The unit circle

The value of $[\cos\theta, \sin\theta]\bullet[\cos(\beta+\theta), \sin(\beta+\theta)]$ does depend on the value of β, but does not depend on the value of θ. Explain why.

1. Read the vectors as two radius arrows

The first endpoint is at direction θ and the second at θ+β. Imagine rotating the whole picture through −θ: the first arrow then lies on the positive horizontal axis and the second at direction β.

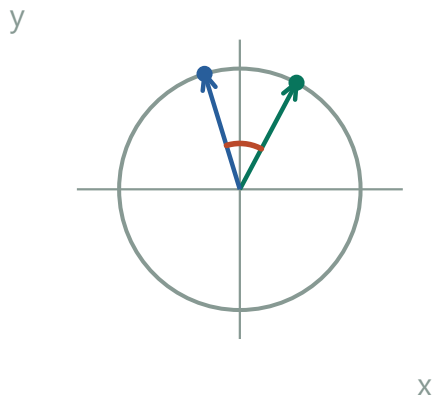


Example fixed separation β

Two unit radii enclose the difference between θ+β and θ.

2. Use an angle-based dot product

The lengths of both arrows stay 1 during that rotation. The dot product uses the product of those lengths and the cosine of the included angle.

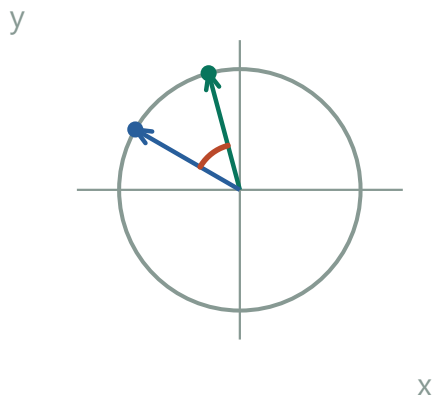


Rotate both directions together

Their tip positions move but the angle between them stays β.

3. Identify the surviving parameter

Their angle separation is β. Their dot product is 1·1·cos β, independent of their common rotation θ.



Projection depends only on β

The dot product is cos β because both vectors have unit length.

Check. The coordinate expansion is cos θ cos(θ+β)+sin θ sin(θ+β)=cos β.

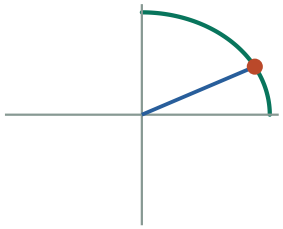
Exercise 428 · Parametrize an ellipse

Printed page 44 · Ellipses / Parametric equations

Graph the curve that is described parametrically by $(x, y) = (5\cos t, 4\sin t)$. If you are working in degree mode, the *parameter* interval should of course be $0 \le t \le 360$. This presentation should remind you of the parametric description of a circle. The curve is actually an ellipse. Confirm this by substituting the parametric equations into the ellipse equation $16x^2 + 25y^2 = 400$.

1. Connect to the unit circle

The pair (cos t,sin t) describes one trip around a circle. Multiplying its x-coordinate by 5 and y-coordinate by 4 stretches that circle into an oval. The angular parameter is the angle on the original circle.

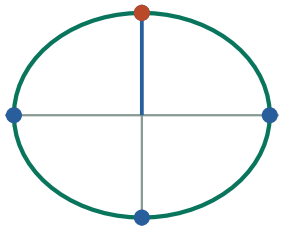


Start with the parameter point

The horizontal coordinate is 5 cos t and the vertical coordinate is 4 sin t.

2. Mark the quarter turns

The four quarter turns give (5,0),(0,4),(−5,0),(0,−4), in counterclockwise order.

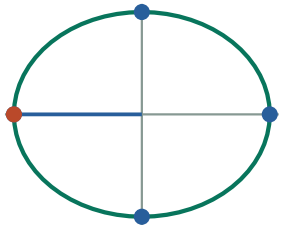


Advance by quarter turns

The curve visits (5,0), (0,4), (−5,0) and (0,−4) in this order.

3. Verify the Cartesian equation

Substitution gives $16\cdot25\cos^2t+25\cdot16\sin^2t=400$, so all points lie on the ellipse.

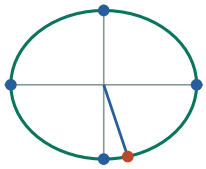


Check a moving point

Both squared and scaled coordinates contribute to $400(\cos^2t+\sin^2t)$.

4. Show the parameter covers the whole ellipse

Conversely (x/5,y/4) is a unit vector for every ellipse point, hence has an angle t.

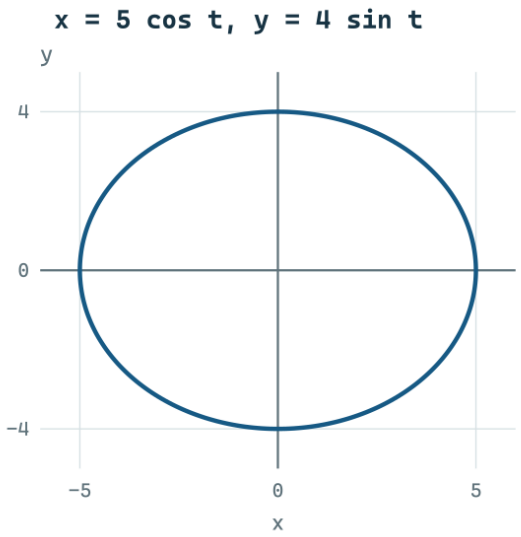


Complete one turn

Any normalized ellipse point (x/5,y/4) lies on the unit circle and has a parameter angle.

1. $16x^2+25y^2=400 \Rightarrow (x/5)^2+(y/4)^2=1$
Divide through by 400 to normalize both coordinates.

2. $(x/5,y/4)=(\cos t,\sin t) \Rightarrow (x,y)=(5\cos t,4\sin t)$
Every unit-circle point has a direction t; rescaling recovers the original point.



Check. The two semiaxes read off directly from the maximum component sizes, 5 and 4.

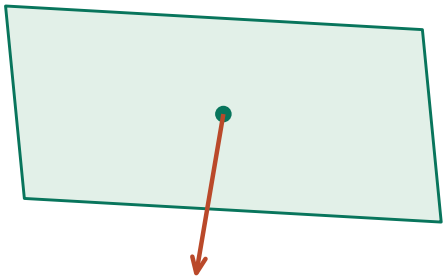
Exercise 429 · Point–normal plane

Printed page 44 · Planes in space / The dot product

Write an equation for the plane that is perpendicular to the vector $[4, 7, -4]$ and that goes through the point $(2, 3, 5)$.

1. Identify the displacement in the plane

For a test point (x,y,z), subtract the given anchor (2,3,5). The resulting displacement lies along the plane precisely when it is perpendicular to the given normal.

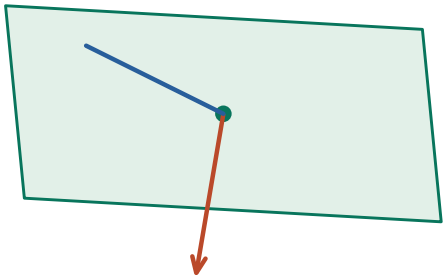


Anchor a displacement

Start the plane at (2,3,5); points on the plane differ by sideways vectors.

2. Use the dot product as a perpendicularity test

Pair the normal's three components with the corresponding three coordinate differences. A zero total means the displacement has no component along the normal.

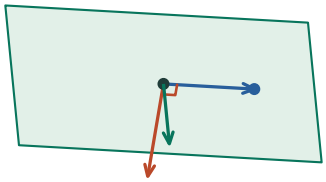


Require a right angle

Each sideways displacement must have zero dot product with (4,7,−4).

3. Expand the plane equation

Set $4(x-2)+7(y-3)-4(z-5)=0$, or $4x+7y-4z=9$. Expanding the dot product gives constant terms $-8-21+20=-9$ on the left; moving their sum to the right gives 9.



Expand the point-normal test

The constant is the normal dotted with the anchor, giving $4x+7y-4z=9$.

1. $4x-8+7y-21-4z+20=0$
Distribute the normal components, including the negative z coefficient.

2. $4x+7y-4z=8+21-20=9$
Move the constants to the other side.

Check. The supplied point gives $8+21-20=9$ and each in-plane difference has dot product zero with the normal.

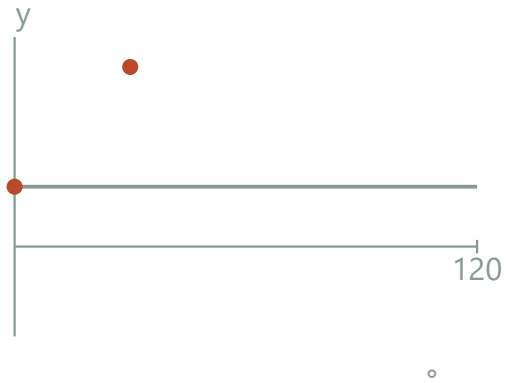
Exercise 430 · Fit a wave with a known maximum

Printed page 44 · Sinusoidal functions / Amplitude and period

Write an equation of the form $y = k + a\sin mx$ for a curve that has a maximum point at $(30, 6)$ and that has 2 as its y-intercept. Use a graphing tool to check that the graph of your equation fits the given description. Because you have the freedom to work in either degree mode or radian mode, and because there are many curves that fit the given description, there are many correct answers to this question. Find another one.

1. Fix the vertical shift

At x=0 the sine factor is zero regardless of m. The remaining height is the y-intercept. The maximum is four units higher, fixing the amplitude magnitude.

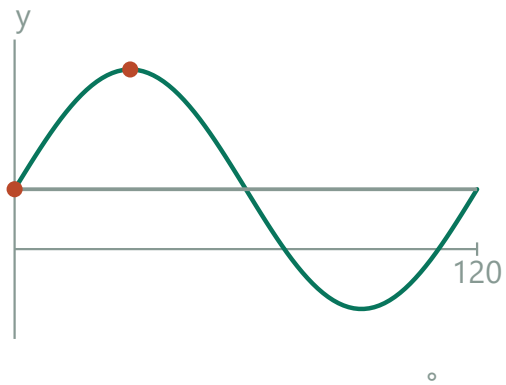


Fix the intercept

At x=0 a sine term vanishes. The intercept therefore sets the centerline at 2.

2. Choose a frequency for the first crest

The intercept fixes k=2. Choose amplitude 4. In degrees, y=2+4sin(3x°) peaks at x=30 because 3·30=90.

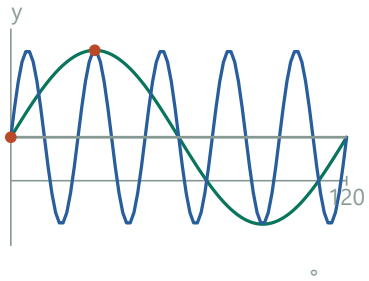


Place the first crest

With amplitude 4 the height reaches 6 at x=30 when the inside angle is 90°.

3. Choose a different crest at the same position

Another model is y=2+4sin(15x°), since 450° is also a sine peak.



Choose a later crest

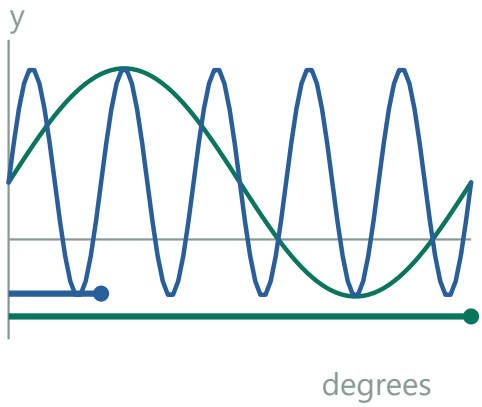
With m=15 the same input makes a 450° inside angle. This is also a crest.

1. 30m=90+360k ⇒ m=3+12k
A peak occurs whenever the sine argument is a quarter-turn plus whole turns.

2. k=0,1 ⇒ m=3,15
These different rates both give the specified maximum at x=30.

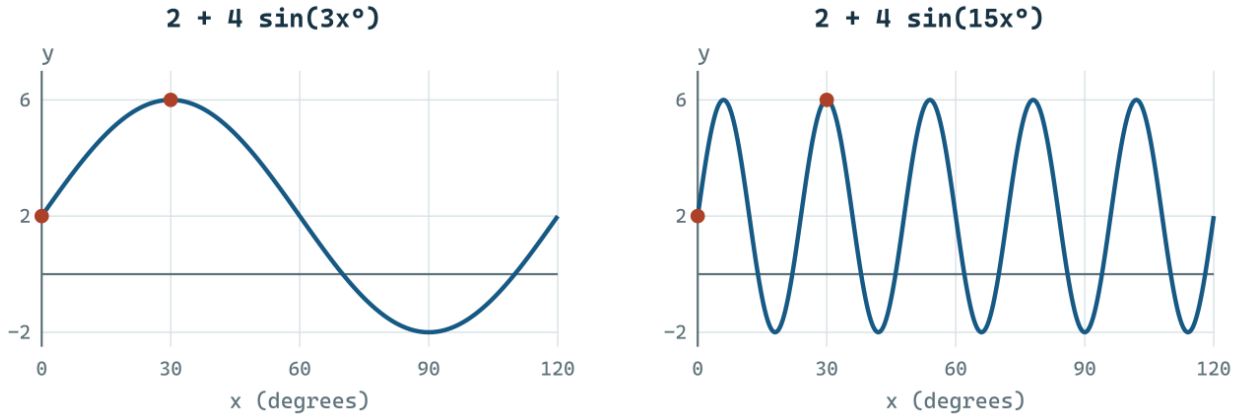
4. Check that the models differ

The first model makes one full wave every 120 horizontal units. The second completes one every 24, even though both hit the specified intercept and maximum.



Compare the two periods

Both graphs pass through the required points. One cycles every 120° and the other every 24°.

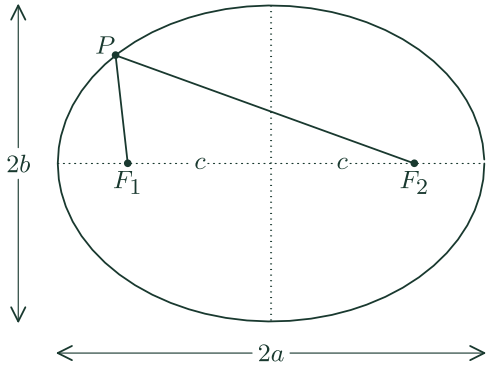


Check. Both have range [−2,6] and value 2 at x=0, but their periods are 120 and 24.

Exercise 431 · Why the focal sum is 2a

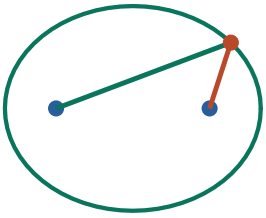
Printed page 45 · Ellipses / Foci and directrices

Some ellipse terminology. For reasons that will eventually become clear, the anchor points F_1 and F_2 for the string are called *focal points* (or *foci*). The focal points are located on the *major* symmetry axis, with the ellipse *center* midway between them. The *vertices* are the points where the ellipse intersects the major axis. As shown, it is customary to let $2a$ be the distance between the vertices, $2c$ be the distance between the foci, and $2b$ be the distance between the intersections of the ellipse with the *minor* symmetry axis. As you know, $PF_1 + PF_2$ is constant for any point P on the ellipse. Explain why this constant equals $2a$. (Hint: Try a special position for P.) Each of PF_1 and PF_2 is called a *focal radius*.



1. Put the pencil at a special position

Choose the right vertex on the major axis. Each focus and that vertex now lie on the same line, so focal distances are simple differences or sums of their distances from the center.

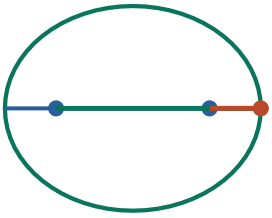


Keep the same total string length

Both marked focal segments change as P moves, but their sum remains fixed.

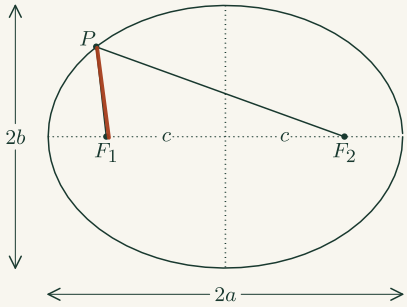
2. Read the two marked distances

At a major-axis vertex, the distances to the two foci are a−c and a+c. Their sum is 2a.



Put P at a horizontal vertex

One segment measures a−c and the other a+c, totaling 2a.

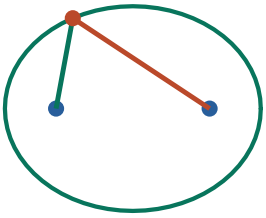


Focal radius to F_1

One leg of the taut-string path joins P to the left focus. Its length changes as P moves along the ellipse.

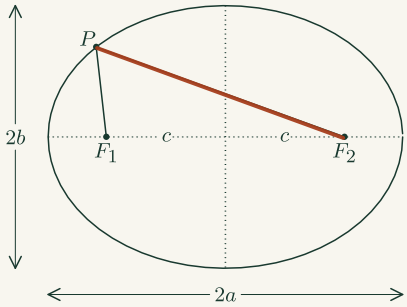
3. Transfer the sum to every ellipse point

Because the sum is constant everywhere, this determines it.



Transfer the sum to a new P

The string construction keeps that same 2a at every point around the ellipse.



Focal radius to F_2

The second leg reaches the right focus. The ellipse definition constrains the sum of these two leg lengths.

Check. At the opposite vertex the two distances simply exchange places.

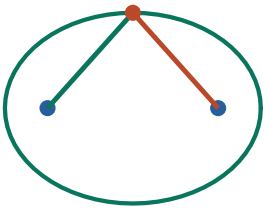
Exercise 432 · Find the minor semiaxis

Printed page 45 · Ellipses / The Pythagorean theorem

(Continuation) Given $a = 12$ and $c = 8$, find b. (Hint: Try a special position for P.)

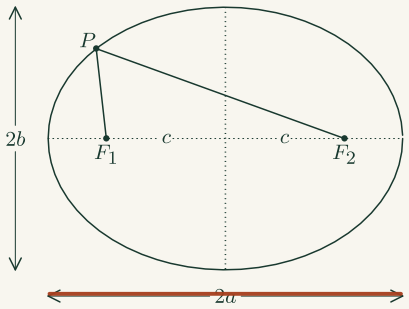
1. Move the pencil to the minor-axis tip

This tip lies on the perpendicular bisector of the segment joining the foci. Both string pieces therefore have the same length and must each account for half the full length 2a.



Use the top point

Symmetry gives both focal segments the same length a=12.

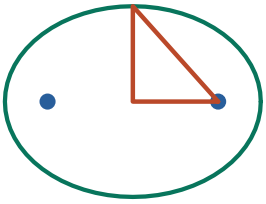


The major diameter · Figure from exercise 431

The dimension arrow underneath the ellipse spans the two vertices and measures the full major-axis length.

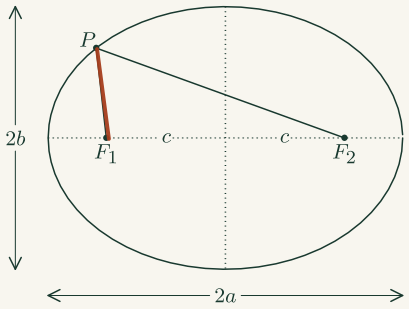
2. Build the right triangle

Use the center, one focus and the tip. The horizontal leg is c and the vertical leg b. The hypotenuse is one of the equal focal distances.



Drop to the center

The hypotenuse is 12 and the horizontal leg c is 8. The remaining leg is b.

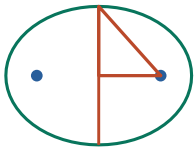


Focal radius to F_1 · Figure from exercise 431

One leg of the taut-string path joins P to the left focus. Its length changes as P moves along the ellipse.

3. Calculate the minor semiaxis

At a minor-axis endpoint each focal distance is a by symmetry. A right triangle gives $b=\sqrt{(a^2-c^2)}=\sqrt{80}=4\sqrt{5}$.



Solve for b

The vertical semiaxis measures $\sqrt{(12^2-8^2)}=4\sqrt{5}$.

1. $b^2=12^2-8^2=144-64=80$
Subtract the squared focal offset from the squared semimajor axis.

2. $b=\sqrt{(16\cdot5)}=4\sqrt{5}$
Use the nonnegative root for a length.

Check. Squaring gives $b^2+c^2=80+64=144=a^2$.

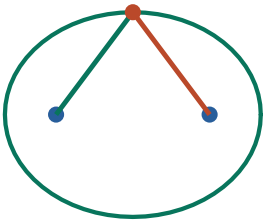
Exercise 433 · The ellipse right triangle

Printed page 45 · Ellipses / The Pythagorean theorem

(Continuation) For any ellipse, what can be said about the quantity $b^2 + c^2$?

1. Choose the minor endpoint again

At the tip of the minor axis, symmetry makes the two focal radii equal. Their sum is the length 2a from the construction, so each focal radius is a.

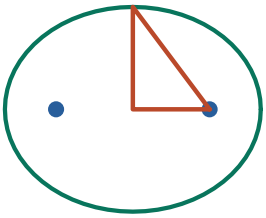


Look along the perpendicular bisector

At the upper minor vertex both focal segments have length a.

2. Relate the three symbols in one triangle

The path from the center to the focus uses distance c along the major axis; from the center to the tip it uses distance b along the perpendicular minor axis.

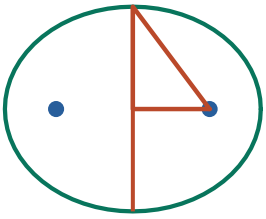


Complete one right triangle

The lateral leg is c, the vertical leg b and the hypotenuse a.

3. Apply Pythagoras for every ellipse

A minor-axis endpoint is equally distant from both foci; each distance must be a. The leg lengths to one focus via the center are b and c, so $b^2+c^2=a^2$.



Apply Pythagoras

The same construction gives $b^2+c^2=a^2$ for every ellipse.

Check. A circle is the special case c=0, reducing this to b=a.

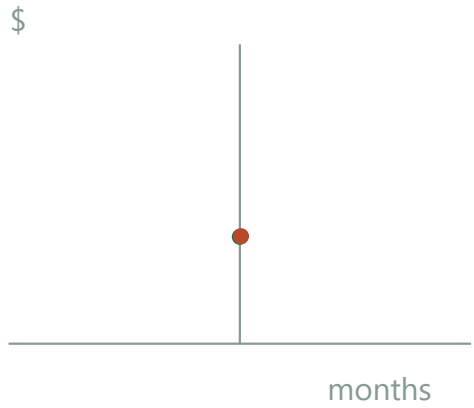
Exercise 434 · Interpret exponential time

Printed page 45 · Exponential functions / Logarithms

Make up a context for the equation $y = 5000(1.005)^x$. Find the values of x that make $y = 12\,500$ and $y = 2000$. Interpret these values in the context you chose.

1. Attach a time unit to the growth factor

For an ideal account worth $5000 today with monthly growth factor 1.005, x measures elapsed months.

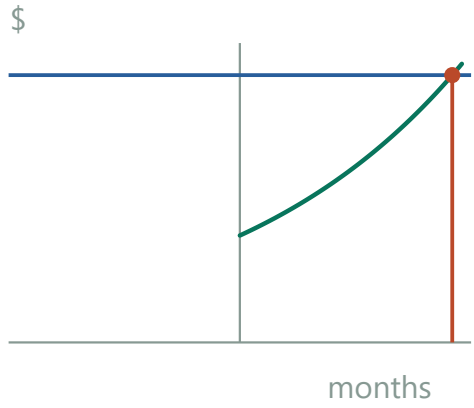


Set the current month to zero

At month zero the modeled balance is $5000.

2. Undo the compound growth

Divide the target balance by 5000. Taking logarithms turns the unknown exponent into a multiplier of ln(1.005). The sign of the time tells whether the target is before or after today.

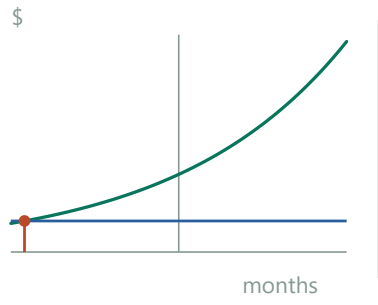


Read a larger target

A balance $12,500 occurs after the reference date.

3. Solve once for a general target

x=ln(y/5000)/ln(1.005). Keep both log arguments dimensionless: y/5000 is a ratio of two balances and 1.005 is one monthly growth factor.



Extend to an earlier balance

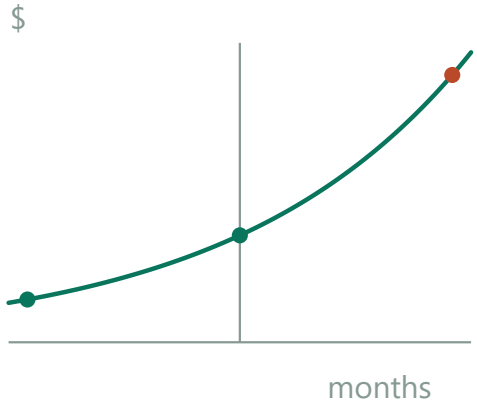
$2000 is smaller, so its matching time lies left of month zero.

1. $y/5000 = 1.005^x \Rightarrow \ln(y/5000) = x \ln(1.005)$
Divide by the initial balance, then take logs of positive numbers.

2. $x = \ln(y/5000)/\ln(1.005)$, $y > 0$
Because the growth factor exceeds one, its logarithm is nonzero.

4. Interpret the two balances

For \$12,500 this gives +183.716 months; for \$2000 it gives −183.716 months, before today, assuming growth extended backward.



Compare opposite elapsed times

The factors 2.5 and 0.4 are reciprocals, placing the two target balances at opposite signed times.

Check. The two factors 2.5 and 0.4 are reciprocals, so the times must have opposite signs.

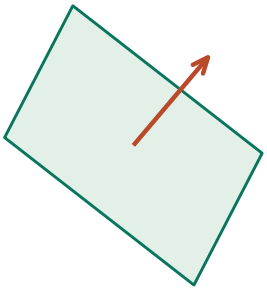
Exercise 435 · Perpendicular planes

Printed page 45 · Planes in space / Orthogonality

Write an equation for a plane that is perpendicular to the plane $2x - y + 3z = 6$ and that passes through the origin.

1. Use the angle between normals

Two planes meet at a right angle when their normal vectors are perpendicular. The original plane's coefficients supply its normal, leaving a choice for the other normal.

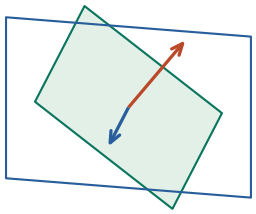


Compare surface orientations

Only the two orientations matter here; the supplied plane can be translated parallel to itself for this angle comparison.

2. Choose an easy zero dot product

For an unknown normal (a,b,c), require 2a−b+3c=0. Take c=0 and a=1 to make b easy to find. A plane through the origin also has constant term zero.



Choose both normals

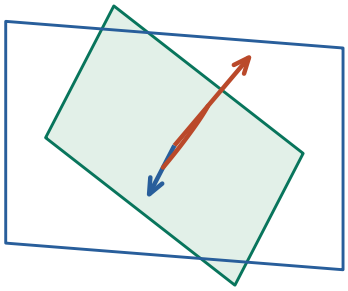
The two chosen normals (2,−1,3) and (1,2,0) have dot product zero.

1. 2a−b+3c=0
A candidate normal (a,b,c) must be perpendicular to the given normal.

2. a=1, c=0 ⇒ b=2
Choose two convenient coordinates and solve for the remaining one.

3. Give one valid plane

Pick a normal perpendicular to (2,−1,3), for example (1,2,0). The desired plane can be x+2y=0.



Read the angle from the normals

A plane through the origin with normal (1,2,0) has equation x+2y=0.

Check. The normals have dot product 2−2=0, giving a right dihedral angle.

Exercise 436 · Effective annual rate

Printed page 45 · Compounding / Percentage change

A constant monthly interest rate of 1.4% is equivalent to what annual interest rate?

1. Track a single dollar

One month's interest adds 0.014 of the month's opening balance. A dollar becomes 1.014 dollars, then 1.014^2 dollars after two months. Reinvestment makes the original monthly factor repeat.

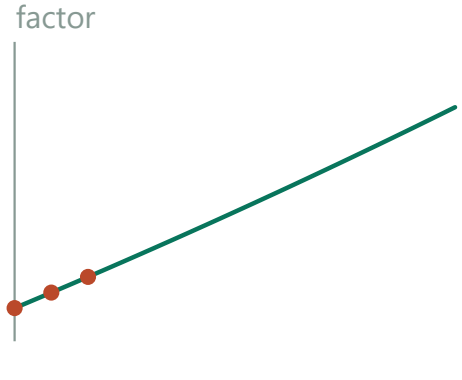


Begin with one monetary unit

One month takes $1 to $1.014.

2. Convert balance to a rate

After twelve months the resulting number is the full ending balance per original dollar. Subtract the original dollar and multiply the increase by 100 to express a percentage.

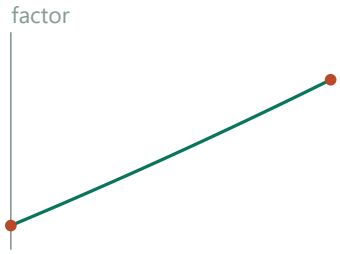


Accumulate equal intervals

Successive monthly factors multiply the current balance.

3. Compute the annual equivalent

The growth factor is 1.014^{12}. Subtract 1 to get approximately 18.16% annual growth.



Compare the equivalent rates

After 12 steps the multiplier is 1.014^{12}, giving about 18.16% annual growth.

1. $A_{12}/A_0 = 1.014^{12} \approx 1.18156$
Twelve monthly multiplications give the final-to-initial balance ratio.

2. $100(A_{12}/A_0 - 1) \approx 18.16\%$
Remove the original principal before expressing growth as a percentage.

Check. The answer exceeds the simple sum 16.8% because later months earn on earlier gains.

Exercise 437 · A vertical focal ellipse

Printed page 45 · Ellipses / Foci and directrices

Let $F_1 = (0, 2)$, $F_2 = (0, -2)$, and $P = (x, y)$. Use the distance formula to convert the equation $PF_1 + PF_2 = 6$ into Cartesian form. Simplify your answer until it reaches the form $hx^2 + ky^2 = m$.

1. Write the two distances

The lower focus gives $L=\sqrt{(x^2+(y+2)^2)}$; the upper gives $R=\sqrt{(x^2+(y-2)^2)}$. Subtracting their squares cancels x^2 and y^2. Since $L+R=6$, the difference of squares also equals $6(L-R)$.

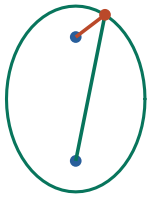


Mark the two vertical foci

The distances from P to (0,−2) and (0,2) sum to 6.

2. Solve for one distance

Let L and R be distances to the lower and upper focus. Then $L^2-R^2=8y$ and $L+R=6$, so $R=3-2y/3$.



Isolate one segment

Subtract squared distances; the common horizontal contribution cancels.

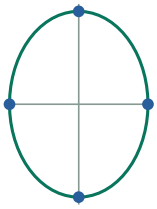
1. $L^2-R^2=(y+2)^2-(y-2)^2=8y$
Cancel the equal x^2 terms in the two squared distances.

2. $(L-R)(L+R)=8y \Rightarrow L-R=4y/3$
Divide by the given focal sum 6.

3. $2R=(L+R)-(L-R)=6-4y/3$
Subtract the difference from the sum to isolate the upper-focus distance.

3. Remove the remaining radical

Squaring $(x^2+(y-2)^2)=(3-2y/3)^2$ yields $9x^2+5y^2=45$.



Obtain the vertical ellipse

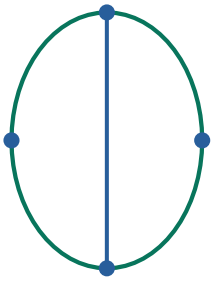
The resulting equation $9x^2+5y^2=45$ has vertical semiaxis 3.

1. $x^2+y^2-4y+4=9-4y+4y^2/9$
Square $R=3-2y/3$ and use the given coordinates of the upper focus.

2. $x^2+5y^2/9=5 \Rightarrow 9x^2+5y^2=45$
Cancel $-4y$, collect terms, and clear denominators.

4. Check for extra points after squaring

Conversely the quadratic forces $|y| \le 3$. Substitution gives $L^2=(3+2y/3)^2$ and $R^2=(3-2y/3)^2$.

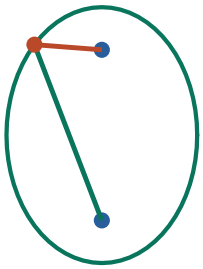


Bound the possible heights

Every point on the ellipse has $-3 \le y \le 3$, keeping $3 \pm 2y/3$ positive.

5. Recover the original sum

Both right-hand bases are positive on this interval, so their square roots add to 6. Thus squaring has preserved exactly the original locus.



Recover both original lengths

Taking the positive square roots yields distances $3+2y/3$ and $3-2y/3$. Their sum returns to 6.

Check. The vertical semimajor axis is 3; $b^2=3^2-2^2=5$ gives the same equation.

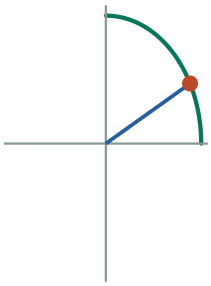
Exercise 438 · Trace the vertical ellipse

Printed page 45 · Ellipses / Parametric equations

(Continuation) Graph the curve described parametrically by $(x, y) = \left(\sqrt{5}\cos t, 3\sin t\right)$. Find a way of showing that this is the same ellipse that appeared in the preceding example.

1. Construct the vertical ellipse

Stretch a unit-circle point horizontally by $\sqrt{5}$ and vertically by 3. One whole parameter revolution then traces one complete stretched curve.

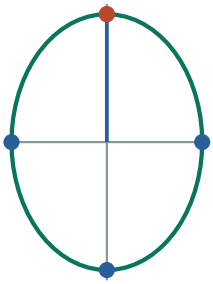


Trace from the positive horizontal tip

One quarter-turn carries $(\sqrt{5},0)$ toward the upper vertical vertex.

2. Mark its four intercepts

The curve traces from $(\sqrt{5},0)$ through $(0,3),(-\sqrt{5},0),(0,-3)$ and back.

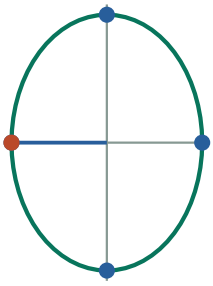


Mark the axis crossings

The vertical extremes are $(0,\pm3)$; the horizontal extremes are $(\pm\sqrt{5},0)$.

3. Check inclusion by substitution

Substitution into $9x^2+5y^2$ gives $45(\cos^2t+\sin^2t)=45$.

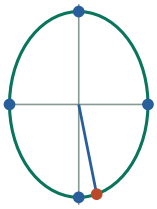


Use the moving point

Substituting into $9x^2+5y^2$ yields $45(\cos^2t+\sin^2t)=45$.

4. Show the reverse inclusion

If $9x^2+5y^2=45$, then $(x/\sqrt{5})^2+(y/3)^2=1$. That normalized point has a unit-circle angle t. Rescaling its coordinates recovers the original point, so none of the ellipse is missing.

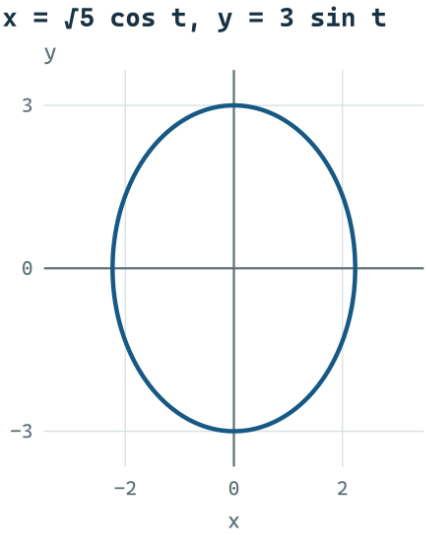


Recover an angle for every point

Divide the two coordinates by their own semiaxes to get a point on the unit circle.

1. $9x^2+5y^2=45 \Rightarrow (x/\sqrt{5})^2+(y/3)^2=1$
Divide the Cartesian equation by 45.

2. $x/\sqrt{5}=\cos t$, $y/3=\sin t$
A normalized point on the unit circle supplies a real parameter for every ellipse point.



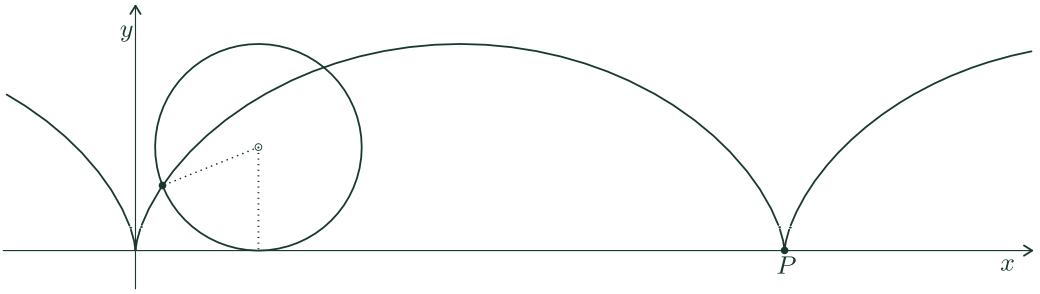
Check. Conversely the normalized coordinates $(x/\sqrt{5}, y/3)$ form a unit-circle point for every solution.

Exercise 439 · A mark on a rolling wheel

Printed page 46 · The cycloid / Circular motion / Parametric equations

The cycloid. A wheel of radius 1 rolls along the x-axis without slipping. A mark on the rim follows a path that starts at $(0, 0)$, as shown in the figure below.

(a) Find the x-coordinate of the point P where the mark first returns to the x-axis.
(b) Find both coordinates of the center after the wheel makes a quarter-turn.
(c) Find both coordinates of the mark after the wheel makes a quarter-turn.
(d) Find both coordinates of the mark after the wheel rolls a distance t, where $t < \frac{1}{2}\pi$.
(e) Check your formulas to see whether they are also correct for $\frac{1}{2}\pi \leq t$.



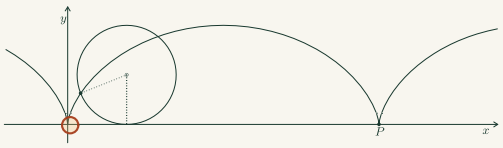
1. Relate rolling distance to angle

No slipping makes the distance traveled by the center equal to the length of rim rolled onto the floor. With radius one, that distance t is also the clockwise angle in radians.



Start with a floor contact

The radius-one wheel starts with its rim mark at (0,0).



The starting contact at the origin

The cusp on the floor is where the mark starts in contact with the ground. It provides a reference for one complete turn.

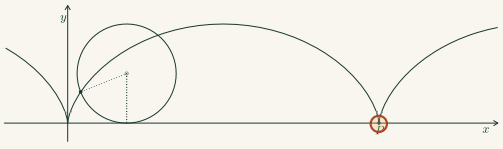
2. Find the first return and the quarter turn

The mark returns to the floor after one circumference, at $x=2\pi$. After a quarter-turn the center is $(\pi/2,1)$ and the mark lies one unit left of it at $(\pi/2-1,1)$.



Roll one quarter-turn

The center moves $\pi/2$ to the right. The original floor mark now sits one radius left of the center.



The next ground contact P

P is the next point at floor level on the traced path. It records the horizontal distance rolled during a whole turn.

3. Separate the moving center from the mark

After rolling distance t the center is (t,1). Its horizontal coordinate increases by the distance rolled, and its vertical coordinate stays one radius above the floor. The rim mark still changes position relative to that center.

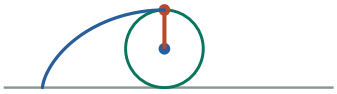


Separate translation from rotation

The center is (t,1); the highlighted segment is the changing offset from center to rim.

4. Rotate the displacement and add it back

Clockwise rotation of the starting offset (0,−1) gives (−sin t,−cos t), so the mark is (t−sin t,1−cos t).



Add the rotated offset

At the half-turn the mark is directly above the center. The offsets are (−sin t,−cos t).

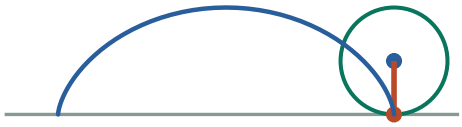
1. $x_{offset} = 0 \cdot \cos t + (-1)\sin t = -\sin t$
Use the first row of the clockwise rotation on the starting offset (0,−1).

2. $y_{offset} = -0 \cdot \sin t + (-1)\cos t = -\cos t$
Apply the second row and then add the moving center (t,1).

5. Extend beyond the first quadrant

The rotation argument works for every $t \geq 0$. Sine and cosine automatically describe every quadrant and later complete turns. The center continues translating by t even after the rim direction repeats.

Wheel rotation · 2π rad



Finish the first revolution

The mark reaches the floor again at $x = 2\pi$. The next turn starts a translated copy of the arch.

Compare settings · Wheel rotation

0 rad



Roll without slipping

The center has moved 0 radii. The rim mark is at (0, 0) in radius units.

The center has moved 0 radii. The rim mark is at (0, 0) in radius units.

π rad



Roll without slipping

The center has moved 3.14 radii. The rim mark is at (3.14, 2) in radius units.

The center has moved 3.14 radii. The rim mark is at (3.14, 2) in radius units.

Check. At t=π the mark is above the hub at (π,2); at t=2π the formula returns (2π,0).

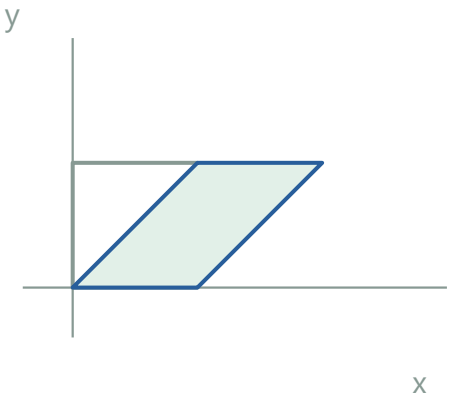
Exercise 440 · Inverse matrix notation

Printed page 46 · Matrix inverses

Explain the notation $\mathbf{M}^{-1}$ used for the inverse of a square matrix $\mathbf{M}$.

1. Start with the operation to be undone

Applying M moves vector v to Mv. Applying its inverse afterward must return v for every v. The matrix that leaves every vector fixed is the identity I.

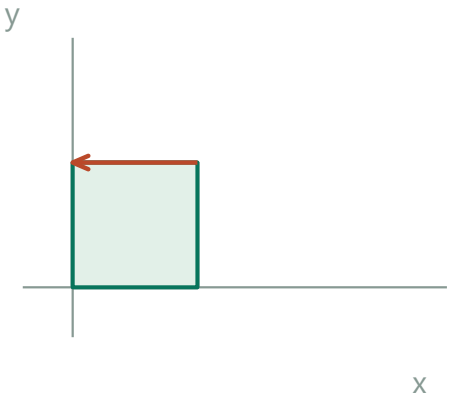


Apply an invertible transformation

A shear changes the square while keeping distinct input points distinguishable.

2. Connect inverse notation with reciprocal notation

It undoes M in both orders, so $MM^{-1}=M^{-1}M=I$. This mirrors nonzero number times its reciprocal equaling 1, with I taking the role of 1.

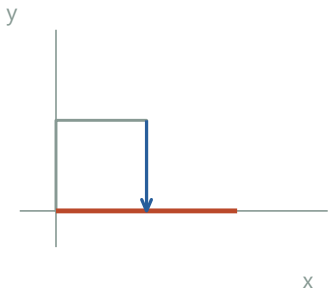


Undo the displacement

The inverse shear restores the original figure; applying both maps gives the identity.

3. Explain when undoing is possible

A square matrix has such an inverse exactly when it is nonsingular. A singular matrix sends some nonzero vector to zero, so its action cannot be undone.



Test a collapsed dimension

A map that sends the square to a line loses information about height and has no inverse.

1. Mv=0 for v≠0 ⇒ Mv=M0
Singularity makes distinct starting vectors have the same image.

2. N(Mv)=N(M0) would give v=0 if NM=I
An inverse would force a contradiction, so no such undoing matrix exists.

Check. Applying M^{-1} to Mv restores every vector v.

Exercise 441 · Change growth periods

Printed page 46 · Compounding / Percentage change

Robin has some money invested in an account that pays 6 percent interest per year. At what rate is the investment increasing each decade? What is the monthly rate of growth?

1. Work with a growth factor first

Each year one dollar of balance becomes 1.06 dollars. More than one year calls for repeated multiplication. A fraction of a year calls for a factor that reproduces 1.06 when repeated enough times.

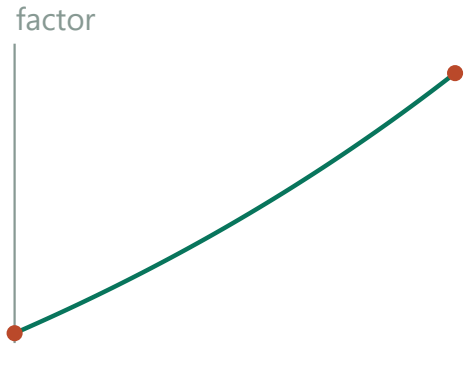


Begin with one monetary unit

A one-year interval multiplies the balance by 1.06.

2. Find the decade rate

Ten years have factor 1.06^{10}, so the decade gain is about 79.08%.

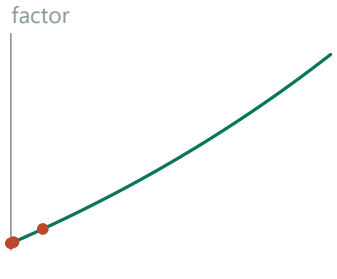


Accumulate equal intervals

Ten such factors give a decade gain of about 79.08%.

3. Find the equivalent monthly rate

An equivalent monthly factor is 1.06^(1/12), corresponding to about 0.4868% per month.



Compare the equivalent rates

A twelfth of one annual interval has factor 1.06^(1/12), a monthly gain about 0.4868%.

1. $(1+r)^{12}=1.06 \Rightarrow 1+r=1.06^{(1/12)}$
Use twelve equal compounded monthly factors.

2. $100r=100(1.06^{(1/12)}-1)\approx 0.4868$
Subtract one before converting the rate to percent.

Check. Twelve copies of the monthly factor multiply back to 1.06.

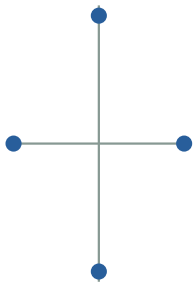
Exercise 442 · Ellipse with vertical major axis

Printed page 46 · Ellipses / Foci and directrices / Parametric equations

The equation $\frac{x^2}{4}+\frac{y^2}{9}=1$ describes an ellipse. Find the x- and y-intercepts of this curve, and use them to make a quick sketch. Once you have the values of a and b, use them to find c and add the focal points to your sketch. Finally, find a parametric description of this ellipse.

1. Read the axis intercepts

Intercepts are $(\pm 2,0),(0,\pm 3)$, giving $b=2, a=3$. Set $y=0$ for the horizontal tips and $x=0$ for the vertical tips. The larger intercept distance belongs to the vertical semimajor axis.

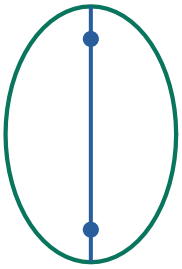


Mark the four intercepts

The x-axis meets the curve at ± 2; the y-axis at ± 3.

2. Locate the foci

Foci are $(0,\pm\sqrt{5})$ since $c^2=9-4$. Use $c^2=a^2-b^2$. Put this offset in the y-coordinate because the long diameter is vertical; the center has coordinates $(0,0)$.

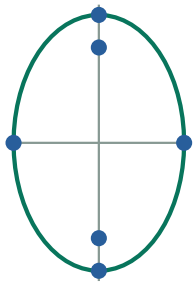


Place the foci on the longer direction

Their vertical displacement from the center is $\sqrt{(3^2-2^2)}=\sqrt{5}$.

3. Trace the graph through the four landmarks

Sketch around these four intercepts using $(2\cos t, 3\sin t)$ for a full turn.

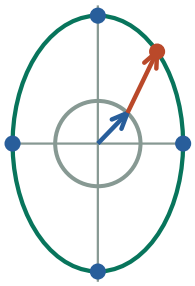


Draw the complete curve

The parameterization $(2\cos t, 3\sin t)$ passes through the four supplied landmarks.

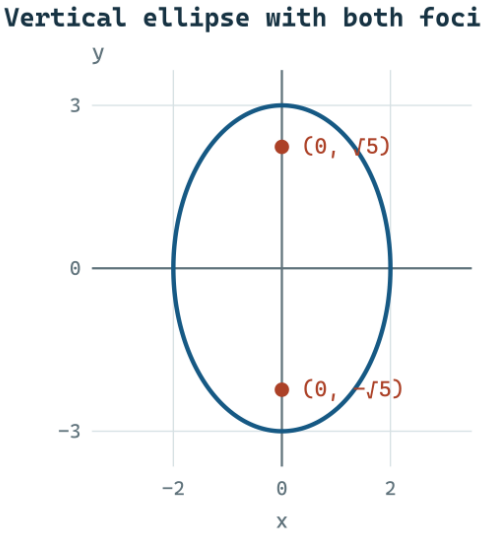
4. Explain the parametrization

Dividing the x-coordinate by 2 and y by 3 produces a unit-circle point. Running that point through one turn and rescaling each coordinate in its own direction reaches every point on the ellipse.



Sweep the unit circle and stretch

The blue unit point (cos t,sin t) moves to (2 cos t,3 sin t). Each ellipse point returns to that circle when its coordinates are divided by 2 and 3.



Check. The parametrization reduces the defining equation to $\cos^2 t+\sin^2 t=1$.

Exercise 443 · Powers of ten by inspection

Printed page 46 · Exponent rules / Radicals and rational powers

Write each of the following numbers as a power of 10. You should not need a calculator.

(a) 1000 **(b)** 1000000 **(c)** 0.01 **(d)** $\sqrt{10}$ **(e)** $100\sqrt{10}$ **(f)** $\dfrac{1}{\sqrt[3]{100}}$

1. Recognize whole and reciprocal powers

Count three and six factors of ten in 1000 and a million. A hundredth is 1/100, so its exponent has the opposite sign to the exponent of 100.



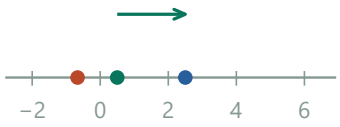
exponent

Mark the integer powers

1000 and a million land at 3 and 6. A hundredth lies at −2.

2. Use the root and product rules

The square root of 10 uses exponent 1/2. Multiplication by 100 adds 2 to it. For the last entry, $100=10^2$, the cube root divides the exponent by 3 and the reciprocal changes its sign.



exponent

Move by a factor of 100

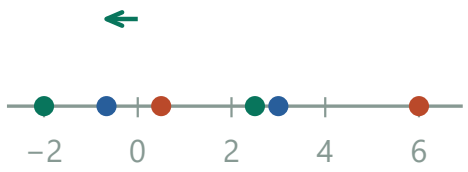
The square root of ten begins halfway from 0 to 1. Multiplication by 100 moves it two exponent units right.

1. $100\sqrt{10}=10^2\cdot10^{(1/2)}=10^{(2+1/2)}$
Multiply like bases by adding exponents.

2. $1/\sqrt[3]{100}=(10^2)^{(-1/3)}=10^{(-2/3)}$
A cube root divides the exponent by 3; a reciprocal negates it.

3. List the six exponents

The exponents for 1000, 1,000,000, 0.01, $\sqrt{10}$, $100\sqrt{10}$, and $1/\sqrt[3]{100}$ are $3, 6, -2, \tfrac{1}{2}, 5/2, -2/3$ respectively. Roots divide exponents; reciprocals negate them.



exponent

Trace the reciprocal cube root

The cube root of 100 has exponent 2/3. Its reciprocal is the same distance left of zero.

Check. Cubing the final number gives $1/100=10^{-2}$.

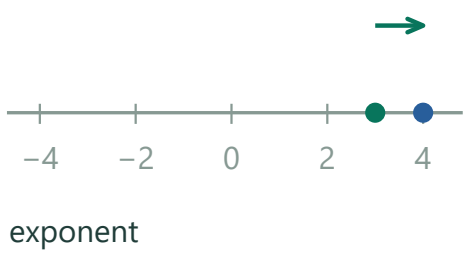
Exercise 444 · Exponent for 2016

Printed page 46 · Logarithms / Exponent rules

If 2016 is written as a power of 10, its exponent is irrational. Use your calculator to find an approximation of this exponent. Then use that approximation (but *no* calculator) to write $\frac{1}{2016}$ as a power of 10.

1. Name the unknown exponent

Write $10^k = 2016$. Because $1000 < 2016 < 10000$, the exponent lies between 3 and 4. The base-10 logarithm is the inverse operation that extracts it.

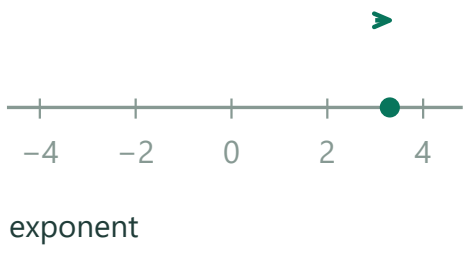


Bracket the exponent

2016 lies between 1000 and 10000, so its exponent lies between 3 and 4.

2. Approximate the exponent

Compute $\log_{10}(2016) \approx 3.304490528$. The calculator's common logarithm gives a rounded result inside the expected 3-to-4 interval. Store the number before changing its sign for the reciprocal.



Locate the calculator result

log 2016 ≈ 3.30449, a little over three tenths of the way through this decade.

3. Take the reciprocal without a calculator

The reciprocal has exponent −3.304490528. These finite decimals are approximations.



Reflect for the reciprocal

A number and its reciprocal correspond to opposite exponents.

Check. Adding the exact exponents gives zero, so the two numbers multiply to 1.

Exercise 445 · What the calculator log returns

Printed page 46 · Logarithms / Exponential functions

(Continuation) Use a calculator to find $\log 2016$. Compare the displayed value with the first of your two previous answers.

1. Translate the earlier exponential equation

The first answer in the preceding exercise was the exponent k needed to produce 2016 using base ten. The common logarithm asks for precisely that exponent.

1 **$10^k = 2016$**

Start with the earlier equation.

2 **$k = \log_{10}(2016)$**

Invert the base-ten exponential. Its range is positive, and 2016 is positive.

2. Evaluate and compare

Keep extra digits while checking. The calculator display and the previous rounded exponent represent the same exact number.

1 **$\log_{10}(2016) \approx 3.304490528$**

Evaluate the common logarithm and round the result.

2 **$10^3 < 2016 < 10^4 \Rightarrow 3 < k < 4$**

Increasing powers of ten bracket the result independently.

3 **10^(3.304490528) ≈ 2016**

Raise ten to the displayed approximation to check the earlier exponent.

Check. The output is between 3 and 4 since 1000<2016<10000.

Exercise 446 · Interpret six logarithms

Printed page 46 · Logarithms / Radicals and rational powers

Use a calculator to find the log of each of the following. Interpret the results. By the way, "log" is short for *logarithm*, to be discussed soon.

(a) 1000 **(b)** 1000000 **(c)** 0.01 **(d)** $\sqrt{10}$ **(e)** $100\sqrt{10}$ **(f)** $\frac{1}{\sqrt[3]{100}}$

1. Read a common logarithm as an exponent

The calculator's log key asks what power of 10 equals its input. For example, in log(1000)=u the statement to test is $10^u=1000$. All six inputs are positive, so their real logarithms exist.



Start with the defining exponent

The position 3 records $10^3 = 1000$.

2. Count powers of ten in the first three inputs

For (a), $1000=10\cdot10\cdot10=10^3$, so $\log(1000)=3$. For (b), $1000000=10^6$, giving 6. For (c), $0.01=1/100=10^{-2}$, giving -2. A negative logarithm fits a positive input smaller than 1.

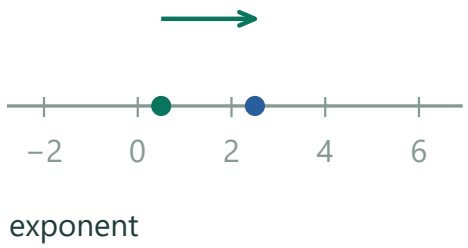


Locate the integer logarithms

10^6 is a million; 10^{-2} is 0.01. Their exponents occupy opposite sides of zero.

3. Use square roots and products

For (d), $\sqrt{10}=10^{(1/2)}$, so the logarithm is 1/2. For (e), $100\sqrt{10}=10^2\cdot10^{(1/2)}=10^{(5/2)}$, so the logarithm is 5/2. The factor 100 increases the preceding logarithm by exactly 2.

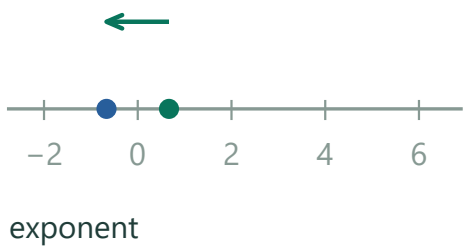


Add two exponent units

$\sqrt{10}$ sits at 1/2. Multiplying it by 100 gives the position 5/2.

4. Undo a cube root in the denominator

For (f), $\sqrt[3]{100}=(10^2)^{(1/3)}=10^{(2/3)}$. Taking its reciprocal gives $10^{(-2/3)}$, whose logarithm is $-2/3$. A calculator returns approximately -0.666667. Raising 10 to each reported exponent reconstructs the corresponding original input.



Reverse the cube-root exponent

100 has exponent 2. Its cube root has exponent 2/3, and the reciprocal sits at $-2/3$.

Check. For example $10^{(2.5)}=10^2\sqrt{10}=100\sqrt{10}$.

Exercise 447 · Factor large powers

Printed page 46 · Exponent rules / Algebraic manipulation

Solve for x: $\quad 4^{2016} - 4^{2015} - 4^{2014} + 4^{2013} = 90\,(2^{x})$

1. Factor out the smallest common power

Each term on the left contains 4^{2013}. The signs stay attached to their terms when this factor is removed.

1 **$4^{2016} - 4^{2015} - 4^{2014} + 4^{2013}$**
Read the four powers in their original order.

2 **$= 4^{2013}(4^{3} - 4^{2} - 4 + 1)$**
Subtract 2013 from each exponent to find its remaining factor.

3 **$= 4^{2013}(64 - 16 - 4 + 1) = 45 \cdot 4^{2013}$**
Compute the small powers and combine the signed coefficients.

2. Isolate the power of two

The unknown occurs only in the exponent on the right. Division by the constant 90 preserves equality.

1 **$45 \cdot 4^{2013} = 90 \cdot 2^{x}$**
Replace the original left side by the factored value.

2 **$2^{x} = 4^{2013}/2$**
Divide both sides by 90 and reduce 45/90.

3 **$2^{x} = (2^{2})^{2013}/2 = 2^{4026}/2$**
Rewrite 4 as 2^{2} and multiply the exponents.

4 **$2^{x} = 2^{4025} \;\Rightarrow\; x = 4025$**
Dividing by 2 removes one factor; powers of base 2 are one-to-one for all real x.

Check. Multiplying the answer's power by 90 gives $45 \cdot 2^{4026}$, the factored left side.

Exercise 448 · Powers do not distribute over sums

Printed page 47 · Exponent rules / Algebraic manipulation

Are there errors in the following student responses? If so, explain the error(s).

(a) Asked for an expression equivalent to $x^3 + x^{-3}$, a student responded x^0.

(b) Asked for an expression equivalent to $(x^{-1} + y^{-1})^{-2}$, a student responded $x^2 + y^2$.

1. (a) Add a power and its reciprocal

The negative exponent requires x ≠ 0. The product rule for exponents does not simplify a sum; a common denominator does.

1 **$x^3 + x^{-3} = x^3 + 1/x^3$**

Replace the negative exponent by a reciprocal.

2 **$= x^6/x^3 + 1/x^3 = (x^6 + 1)/x^3$**

Write x^3 with denominator x^3, then add numerators.

3 **At x = 1: $1^3 + 1^{-3} = 2$, while $1^0 = 1$**

One allowed input disproves the proposed identity.

2. (b) Find the quantity that must be nonzero

The inner reciprocals require x ≠ 0 and y ≠ 0. The negative outer power also requires their sum to be nonzero.

1 **$x^{-1} + y^{-1} = 1/x + 1/y$**

Write each reciprocal separately.

2 **$= y/(xy) + x/(xy) = (x + y)/(xy)$**

Use the common denominator xy, which the inner restrictions make nonzero.

3 **$(x + y)/(xy) \neq 0 \Leftrightarrow x + y \neq 0$**

The numerator determines whether this defined fraction is zero.

3. (b) Invert the whole sum and square

For $x \neq 0$, $y \neq 0$ and $x + y \neq 0$, the reciprocal is defined. Keep these restrictions with the final expression.

1 **$(x^{-1} + y^{-1})^{-2} = ((x + y)/(xy))^{-2}$**
Substitute the combined inner fraction.

2 **$= (xy/(x + y))^{2}$**
The exponent −2 inverts the entire fraction and then squares it.

3 **At x = y = 1: $(1/2)^2 = 1/4$, but $1^2 + 1^2 = 2$**
This permitted pair disproves the proposed $x^2 + y^2$.

Check. At x=y=1 the first claimed replacement would give 1 rather than 2, and the second 2 rather than ¼.

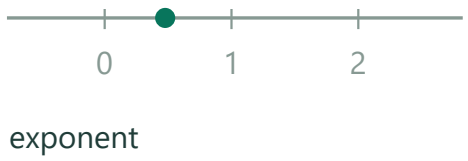
Exercise 449 · Logarithms as exponents

Printed page 47 · Logarithm rules / Exponent rules

Using the log function, solve each of the following for x:
(a) $10^x = 3$ **(b)** $10^x = 300$ **(c)** $10^x = 9$ **(d)** $10^x = 3^{-1}$ **(e)** $10^x = \sqrt{3}$
You should see a few patterns in your answers — try to explain them.

1. Use the first equation to set a reference

Taking the base-10 logarithm reverses raising 10 to a power. For (a), $10^x=3$ gives $x=L=\log_{10}3\approx0.477121$. Use $10^L=3$ in the other equations so the same quantity stays exact until the final decimal.

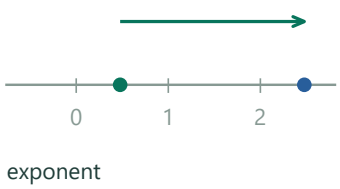


Place the exponent of 3

L = log 3 is about 0.4771.

2. Multiply by one hundred

For (b), $300=3\cdot100=10^{L}\cdot10^{2}=10^{(L+2)}$. Thus $x=L+2\approx2.477121$. Moving the decimal point two places to the right multiplies the original number by 100 and adds 2 to its logarithm.



Shift two units for 300

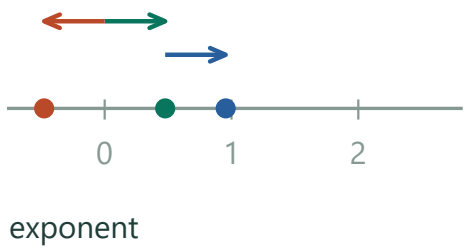
Multiplying the input by 100 moves L to L+2.

1. $10^{L}=3 \Rightarrow 300=10^{2}\cdot10^{L}=10^{(2+L)}$
A hundredfold increase contributes two more factors of ten.

2. $\log_{10}(300)=2+L$
The logarithm reads the exponent.

3. Square or invert the common input

For (c), $9=3^{2}=(10^{L})^{2}=10^{(2L)}$, giving $x=2L\approx0.954243$. For (d), $3^{-1}=1/10^{L}=10^{(-L)}$, giving $x=-L\approx-0.477121$. A reciprocal changes the sign of the exponent.

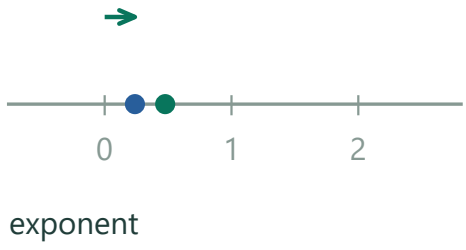


Double or reverse the displacement

Squaring 3 gives exponent 2L, and taking its reciprocal gives exponent −L.

4. Take the square root

For (e), $\sqrt{3}=(10^{L})^{(1/2)}=10^{(L/2)}$, giving $x=L/2\approx0.238561$. Squaring $10^{(L/2)}$ returns 3. Together the examples show why products add logarithms and why raising an input to a power multiplies its logarithm by that power.



Halve the distance for a square root

A square root uses halfway to L from zero. Doubling this exponent recovers L.

Check. Exponentiating each answer with base 10 recovers the listed input.

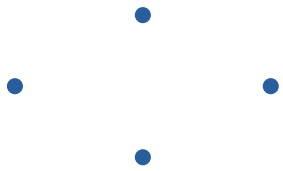
Exercise 450 · Horizontal ellipse from endpoints

Printed page 47 · Ellipses / Coordinate geometry

The vertices of an ellipse are $(9, 0)$ and $(-9, 0)$, and the y-intercepts of the ellipse are 5 and -5. Write an equation for the ellipse.

1. Find the ellipse center

The midpoint of the two vertices is (0,0). Their common y-coordinate shows that the major axis is horizontal, so its semilength goes with x.

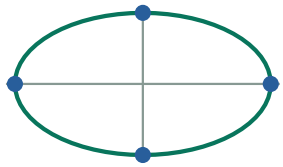


Place the endpoints

The farthest horizontal points are nine units from the origin. The vertical endpoints are five units away.

2. Normalize the two distances

Divide x by the horizontal semiaxis 9 and y by the vertical semiaxis 5. On the ellipse these normalized coordinates act like coordinates on a unit circle.

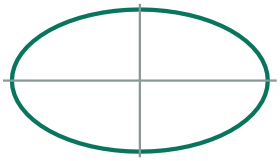


Read the semiaxes

The length read from one center-to-tip segment is used in each denominator.

3. Write the equation

The ellipse is centered at the origin with horizontal semiaxis 9 and vertical semiaxis 5. Its equation is $x^2/81+y^2/25=1$.



Normalize by each semiaxis

$x^2/81+y^2/25=1$ traces the ellipse through those four endpoints.

Check. Substituting each of the four points yields 1.

Exercise 451 · Generic ellipse

Printed page 47 · Ellipses / Parametric equations

Find the x- and y-intercepts of the generic ellipse described by $\dfrac{x^2}{m^2}+\dfrac{y^2}{n^2}=1$. Find a parametric description of this ellipse.

1. Solve separately on each axis

Set y=0 to find where the graph crosses the x-axis. Then set x=0 to find the y-intercepts. The denominators require m and n to be nonzero.



Keep the two scale factors separate

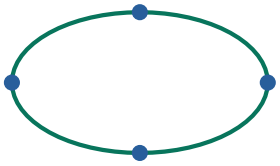
The schematic uses |m| horizontally and |n| vertically.

1. $y=0 \Rightarrow x^2/m^2=1 \Rightarrow x=\pm m$
Multiply by m^2 and take both roots; $m\neq 0$ is required in the equation.

2. $x=0 \Rightarrow y^2=n^2 \Rightarrow y=\pm n$
The vertical coordinate has the analogous two intercepts for $n\neq 0$.

2. Use the unit-circle identity

Call the two normalized coordinates x/m and y/n. Their squares sum to 1, so they can be written cos t and sin t through a complete turn.

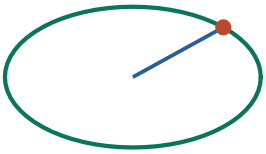


Read all intercepts

One coordinate vanishes at a time, giving (±m,0) and (0,±n).

3. Give intercepts and parameter interval

For nonzero m,n the intercepts are (±m,0),(0,±n). A parametrization is (m cos t,n sin t), for 0≤t≤2π in radians.



Parametrize a full turn

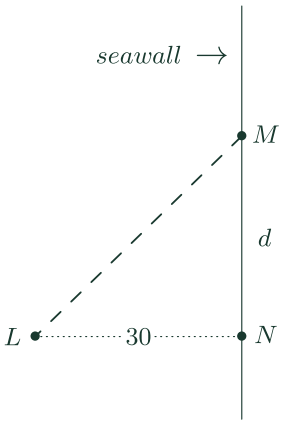
(m cos t,n sin t) stretches or reflects each unit-circle coordinate according to its own factor.

Check. Its normalized coordinates run over the entire unit circle.

Exercise 452 · A rotating beacon

Printed page 47 · The tangent function / Circular motion / Domain and range

A lighthouse is 30 feet from a seawall. The diagram shows this arrangement from above, as if the viewer were in a helicopter. The light L on top of the lighthouse rotates counterclockwise, once every six seconds, casting a single moving beam of light onto the seawall. Let N be the point on the seawall that is nearest the light. Let M be the moving spot of light. Let d be the distance from N to M, and let t be the time, in seconds, since M last passed N. Find d when $t = 0.00$, $t = 0.30$, $t = 0.75$, and $t = 1.49$. Are d and t related linearly? What does the graph of this relationship look like? Why is "distance" not an appropriate word to use when describing the situation at the instant $t = 5$?



1. Build the triangle LNM

LN is the perpendicular from the rotating light L to the wall, so it is the fixed adjacent leg of length 30. NM runs along the wall and is the signed opposite leg. One turn in six seconds gives angular speed 2π/6 radians per second.

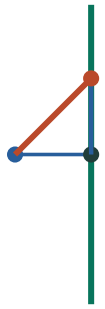


Build the top-view ray

The wall is green, light is at the blue dot, and the fixed perpendicular segment is 30 ft.

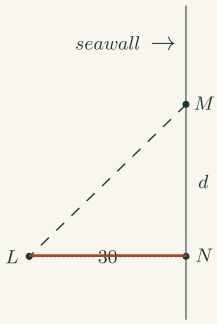
2. Construct and evaluate the tangent model

For signed wall displacement, d=30tan(πt/3) ft while the beam points toward the wall. At t=0,.30,.75,1.49, values are 0,9.75,30,2864.68 ft approximately.



Turn through 45°

At t=0.75 s the perpendicular and wall legs are equal. In general NM=30 tan(πt/3).



The perpendicular LN

The lighthouse's fixed perpendicular distance to the wall gives the adjacent leg of the right triangle.

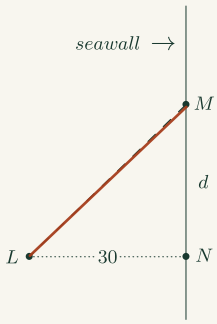
3. Follow the beam near parallel

The tangent graph is nonlinear and tends to infinity as the beam approaches parallel. When the beam faces away there is no spot.



Aim almost parallel to the wall

The ray still hits, but the spot is far outside this window. At exactly parallel there is no intersection.



The single directed beam LM

This segment follows the outgoing ray, which has a direction. Extending the underlying infinite line backward would create a fictitious hit while the light faces away.

4. Interpret the negative displacement

At 5 s the beam points below N: signed displacement is −30√3 ft, while ordinary distance is +30√3 ft.



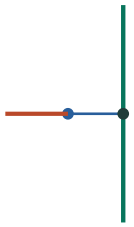
Aim below the perpendicular

The spot occurs below N, so its signed coordinate is negative; the length remains positive.

5. Restrict the formula to a ray that hits the wall

In the first six seconds the signed-position graph exists for 0≤t<1.5 and 4.5<t<6, with the spot back at N at t=6. From 1.5 through 4.5 seconds the single ray misses the seawall. The physical graph repeats every six seconds.

Elapsed time in the six-second rotation · 3 s



Turn away from the wall

A backward extension of the line would cross the wall. The emitted ray has no such spot.

1. λ cos(πt/3)=30, λ≥0
λ measures distance forward along the ray to the wall.

2. λ=30/cos(πt/3)>0 only if cos(πt/3)>0
A negative parameter would lie behind the beacon; zero cosine makes the ray parallel.

3. 0≤t<1.5 or 4.5<t≤6
These are exactly the forward-facing times within the first six seconds.

Compare settings · Elapsed time in the six-second rotation

0 s



Turn the beam

The beam reaches the wall at a signed distance of 0 feet from its foot.

The beam reaches the wall at a signed distance of 0 feet from its foot.

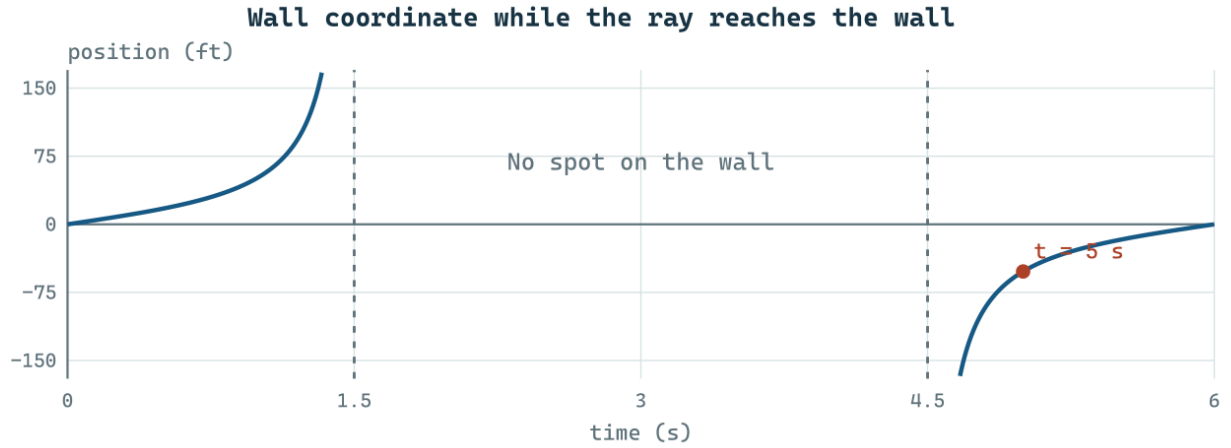
6 s



Turn the beam

The beam reaches the wall at a signed distance of -0 feet from its foot.

The beam reaches the wall at a signed distance of -0 feet from its foot.



Check. The right triangle gives d/30=tan θ; at .75 s, θ=45°, so the legs must agree.

Exercise 453 · Switch logarithmic and exponential notation

Printed page 47 · Logarithms / Exponential functions

Rewrite

(a) the *logarithmic equation* $4 = \log 10000$ as an exponential equation;

(b) the exponential equation $10^{4.49713\ldots} = 31415$ as a logarithmic equation.

1. Identify the three roles

In $\log_{10}N=k$, the fixed base is 10, the exponent is k, and the number produced by the power is N. These roles stay the same when the equation is rewritten.

base	exponent	value
10	k	N

$$\log_{10} N = k \quad \Leftrightarrow \quad 10^k = N$$

The base, exponent and value have fixed jobs in both forms of the statement.

2. Change from logarithm to power

$4=\log_{10}10000$ means $10^4=10000$. The original logarithm says which power of the base 10 gives 10,000, so 4 goes in the exponent position.

base	exponent	value
10	4	10,000

$$\log_{10}(10{,}000) = 4 \quad \Leftrightarrow \quad 10^4 = 10{,}000$$

Place the exponent 4 above base 10. The base and result keep their values when the equation is rewritten.

3. Change from power to logarithm

Writing 10^(4.49713…)=31415 logarithmically gives $\log_{10}31415=4.49713\ldots$.

base	exponent	value
10	4.49713…	31,415

$$10\text{^}(4.49713\ldots) = 31{,}415 \quad \Leftrightarrow \quad \log_{10}(31{,}415) = 4.49713\ldots$$

In reverse, put the power's result into the logarithm. Its exponent becomes the logarithm's output.

Check. A logarithm always occupies the exponent position in the corresponding power equation.

Exercise 454 · Population parameters

Printed page 47 · Exponential growth and decay / Percentage change

The function p defined by $p(t) = 3960(1.02)^t$ describes the population of the fictional town *Mirzakhani* at t years after it was founded.

(a) Find the founding population.

(b) At what annual rate has the population of Mirzakhani been growing?

(c) Calculate $p(65)/p(64)$.

1. Locate the founding instant

The variable t counts years since founding, so the founding population is the output at t=0. A nonzero base to the zero power equals 1.

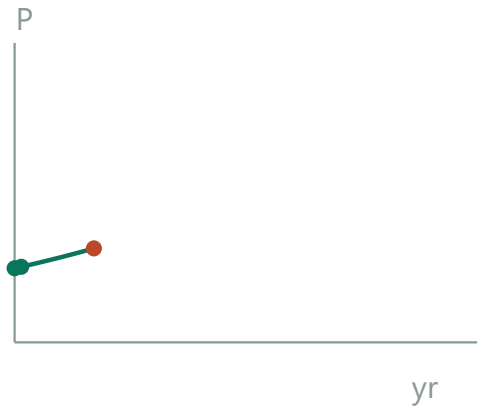


Find the founding population

Setting t=0 leaves P=3960.

2. Read the starting value and rate

At t=0 the founding population is 3960. The annual factor 1.02 represents 2% growth.

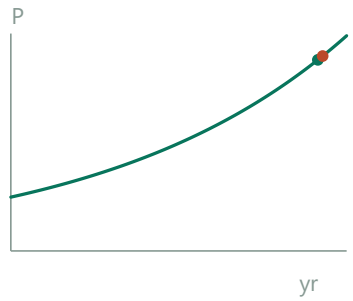


Apply a factor 1.02 per year

An annual step adds 2% of that year's starting population.

3. Compare successive years

The ratio p(65)/p(64)=1.02 by canceling the common factor.



Compare consecutive later years

The ratio of the year-65 population to the year-64 population is still 1.02.

1. $p(65)/p(64)=[3960\cdot1.02^{64}\cdot1.02]/[3960\cdot1.02^{64}]$
Separate the sixty-fifth growth factor.

2. =1.02
The common balance is positive, so it can be cancelled.

Check. The ratio is independent of which adjacent years are chosen.

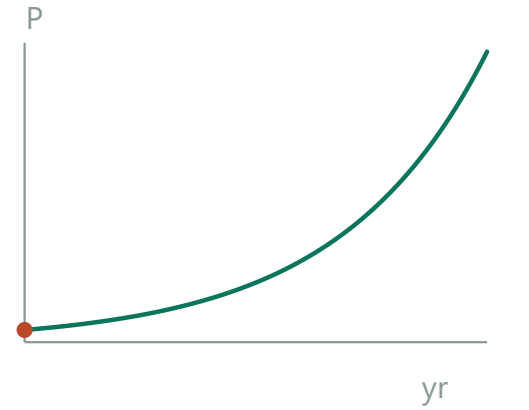
Exercise 455 · When does the town reach 77,218?

Printed page 47 · Exponential growth and decay / Logarithms

(Continuation) Using a graphing device, find an estimate for the t-value that makes $p(t) = 77218$. What is the meaning of your answer? By the way, notice that this question asks you to find an exponent. This is a typical logarithm question, for which you will soon learn a special technique.

1. Set up the graph comparison

Draw the increasing population model and the horizontal line y=77,218 on a common year axis. At their intersection both give the same number of residents. Project that point down to read the year.

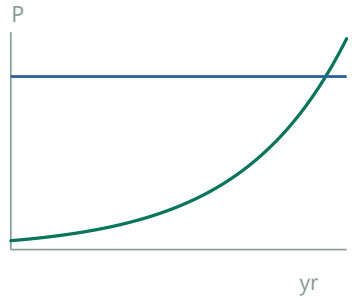


Plot the exponential population

The curve rises from 3960 as time advances.

2. Explain the inverse calculation

Divide by 3960 first so the unknown t is isolated in 1.02^t. Taking logarithms produces a linear equation in t.



Place the target level

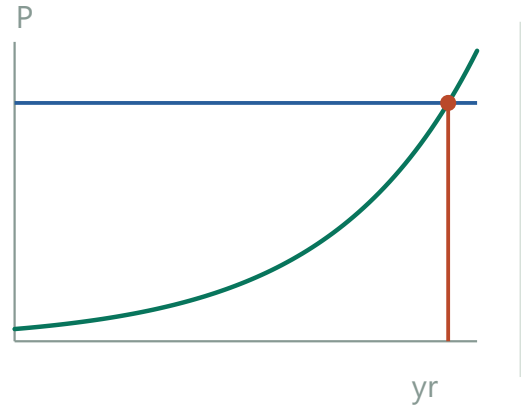
Trace horizontally from population 77,218 to the growth curve.

1. $1.02^t=77218/3960 \Rightarrow t \ln 1.02=\ln(77218/3960)$
Divide by the founding population and log both positive sides.

2. $t=\ln(77218/3960)/\ln 1.02$
The logarithm of a base greater than one is positive.

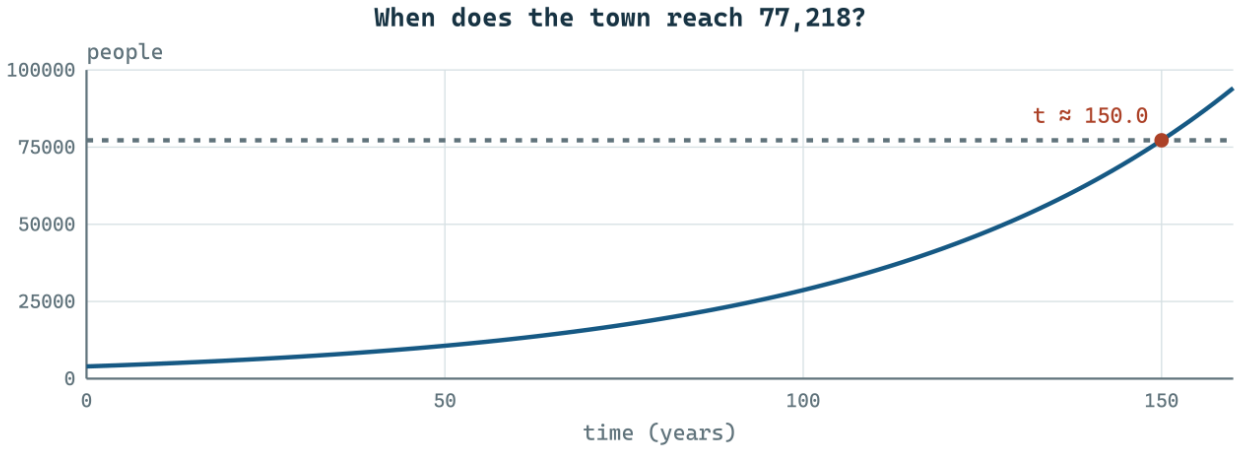
3. Estimate and interpret the time

Divide by 3960 and take logs: t=ln(77218/3960)/ln(1.02)≈150 years after founding. On a graph this is the abscissa of the horizontal level 77218.



Read the time coordinate

The projection to the time axis is approximately 150 years after founding.



Check. 3960(1.02)^150 rounds to 77,218 residents.

Exercise 456 · Similarity and eccentricity

Printed page 48 · Ellipses / Eccentricity / Similarity and scaling

Show that there is a dilation that transforms $9x^2 + 4y^2 = 36$ onto $9x^2 + 4y^2 = 225$. These ellipses are therefore *similar*. For each one, calculate c, a, and the ratio c/a, which is called the *eccentricity*. Notice that these ellipses have the same eccentricity. Explain why similar ellipses must have the same eccentricity. Is the converse true?

1. Find the scale factor

A dilation by k multiplies both coordinates by k and every quadratic term by k^2. The target right side is 225 in place of 36, so k^2 must equal their ratio.

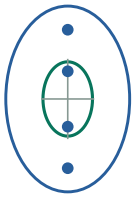


Start with the smaller ellipse

Its semiaxes are 3 and 2 and its focal offset is √5.

2. Compute both sets of lengths

Dilate about the origin by √(225/36)=5/2. First ellipse: a=3, b=2, c=√5. Second: a=15/2, b=5, c=5√5/2.



Dilate every length by 5/2

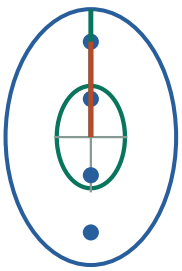
The semiaxes become 7.5 and 5. The foci also move outward by the same factor.

1. $9x^2+4y^2=225 \Rightarrow x^2/25+y^2/(225/4)=1$
Divide every term by 225 to read the squared semiaxes.

2. $c^2=225/4-25=125/4 \Rightarrow c=5\sqrt{5}/2$
The longer semiaxis is vertical; take the positive focal offset.

3. Explain why the ratio survives

Both have eccentricity √5/3. Scaling multiplies a,c by the same factor.

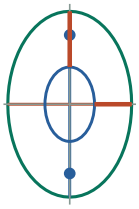


Take the ratio of scaled lengths

Both focal offset and semimajor axis scale by 5/2. Their ratio c/a stays √5/3.

4. Prove the converse

Conversely equal e gives the same $b/a=\sqrt{(1-e^2)}$; align centers and axes and dilate to match both semiaxes.



Use equal eccentricity in reverse

Equal e fixes $b/a=\sqrt{(1-e^2)}$. Aligning centers and axes then scaling the major semiaxis matches the minor one as well.

1. $e^2=c^2/a^2=(a^2-b^2)/a^2=1-(b/a)^2$
Divide the ellipse right-triangle relation by $a^2>0$.

2. $b/a=\sqrt{(1-e^2)}$
The semiaxes are positive, so matching e fixes their ratio.

Check. Test the dilation at (0,3): the image (0,7.5) lies on the second ellipse.

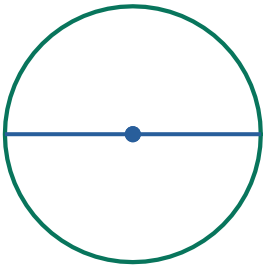
Exercise 457 · Range of eccentricity

Printed page 48 · Ellipses / Eccentricity

(Continuation) What are the possible values for the eccentricity of an ellipse? What is the eccentricity of a circle? What happens as the ellipse becomes less like a circle?

1. Start from $b^2+c^2=a^2$

Divide by a^2. The minor-to-major ratio b/a is positive and at most one, giving $(c/a)^2=1-(b/a)^2$. A real nondegenerate ellipse requires $b>0$.

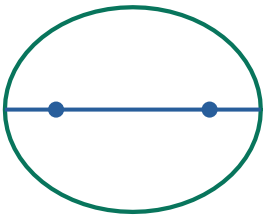


Start with a circle

With coincident foci c=0, the ratio c/a vanishes.

2. Compare circle and elongated shapes

For a nondegenerate ellipse $0 \le c/a < 1$. A circle has $c=0$. Increasing elongation decreases b/a and moves $e=\sqrt{1-(b/a)^2}$ toward 1.



Separate the foci

Increasing c/a narrows the ellipse; here $e=0.6$ and $b/a=0.8$.

3. Construct an ellipse for any allowed value

Every value in this range occurs: for any $0 \le e < 1$ choose semiaxes $a=1$ and $b=\sqrt{1-e^2}>0$. Then $c=e$, giving the specified eccentricity.



Approach the upper eccentricity bound

For any $0 \le e < 1$ the semiminor length $a\sqrt{1-e^2}$ stays positive. At $e=1$ the ellipse collapses.

Check. Equality $e=1$ would force $b=0$ and collapse the ellipse.

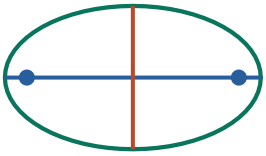
Exercise 458 · Which axis is major?

Printed page 48 · Ellipses / Symmetry

An ellipse has two symmetry axes. Why is one called *major* and the other called *minor*? Is it always true that the major axis is the x-axis?

1. Compare diameters through the center

The major diameter is the longest one joining opposite points. The shorter diameter meets it at a right angle. Their names describe these lengths and have no dependence on how graph paper is oriented.

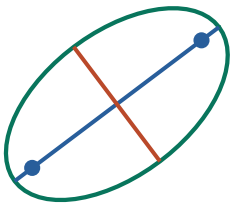


Compare the two diameters

The longer diameter is major; it passes through both foci.

2. Use the foci to orient the figure

The major axis contains the longer diameter and foci. The minor axis is perpendicular to it through the center.

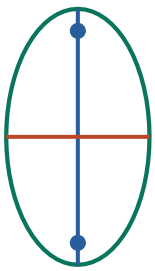


Turn the ellipse on the page

Rotating the figure changes the coordinate directions of its axes while preserving their lengths.

3. Rotate the axes and consider a circle

Rotation changes their orientation, so either can be vertical, horizontal or tilted. In a circle no direction is uniquely major.



Put the major axis vertically

The major axis can be vertical or tilted. For a circle all diameters have the same length.

Check. The ellipse $x^2/4+y^2/9=1$ has major axis along y.

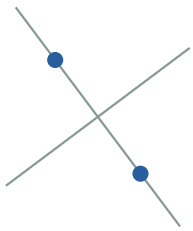
Exercise 459 · A tilted string ellipse

Printed page 48 · Ellipses / Foci and directrices / Vectors and components

The length of a piece of string is 18, and its ends are anchored at $(-3, 4)$ and $(3, -4)$. An ellipse is traced with a pencil, while keeping the string taut. Sketch a graph of this curve, and find coordinates of its two vertices.

1. Find the midpoint and axis lengths

The two foci are opposite vectors, so the midpoint is at the origin. Half the 18-unit string is the semimajor length. The center-to-focus distance is $\sqrt{(3^2+4^2)}$.

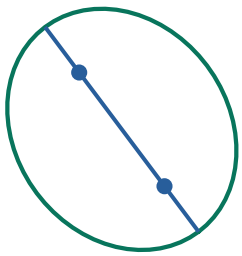


Locate the midpoint of the anchors

The foci (−3,4) and (3,−4) are opposite about the origin and five units from it.

2. Follow the sloping major axis

Center is (0,0), a=9, c=5. The major-axis unit direction is (3/5,−4/5), so vertices are ±(27/5,−36/5).

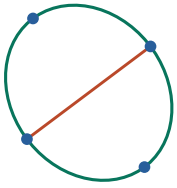


Extend along the focal direction

The long-axis unit vector is (3/5,−4/5); each vertex is 9 units from the center.

3. Construct the other two landmarks

For a sketch, the minor-axis endpoints are $\pm\sqrt{56}(4/5,3/5)$.



Place the short-axis tips

Along the perpendicular direction (4/5,3/5), measure the semilength $\sqrt{56}$.

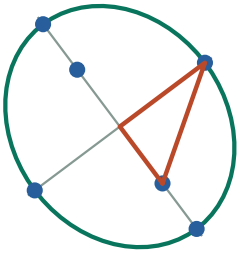
1. $b^2=a^2-c^2=81-25=56$
Use half the string and half the focal separation.

2. $(3,-4)\cdot(4,3)=12-12=0$
The second vector is perpendicular to the major axis.

3. $B\pm=\pm\sqrt{56}\cdot(4/5,3/5)$
Normalize the perpendicular vector and scale it by b.

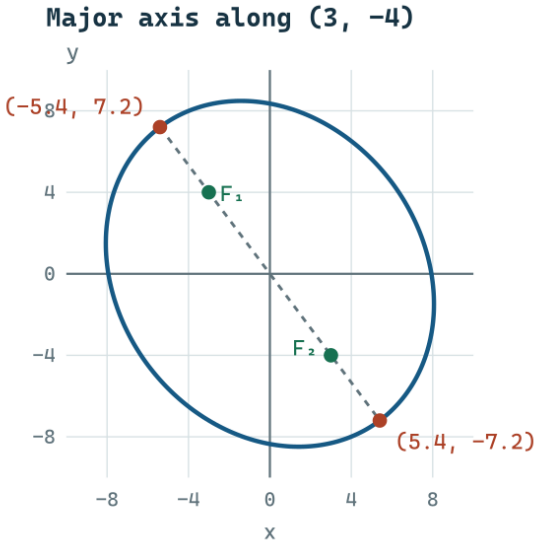
4. Explain the minor-axis direction

Turning (3/5,−4/5) by 90° gives (4/5,3/5). The minor semilength is $\sqrt{(9^2-5^2)}$, so going that far in both perpendicular directions fixes the two remaining tips.



Use both perpendicular axes

The four endpoint landmarks and the fixed 18-unit focal sum orient the entire sketch.



Check. At either vertex the focal distances are 9−5=4 and 9+5=14.

Exercise 460 · Rotate a parametrization

Printed page 48 · Ellipses / Rotations / Parametric equations

(Continuation) After learning about transforming points in a plane using rotation matrices, a curious Math 3 student wondered if the same could be done to find parametric equations for this ellipse.

(a) Determine the acute angle, θ, between the major axis of the ellipse and the x-axis.

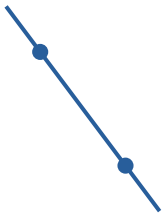
(b) Write parametric equations for a congruent ellipse, centered at the origin, whose major axis is the x-axis.

(c) Orient the congruent ellipse appropriately by rotating it by θ. Do you want to use a matrix to rotate clockwise or counterclockwise?

(d) Use a graphing tool to sketch the ellipse.

1. Measure the tilt of the major axis

The 18-unit string from the preceding exercise gives semimajor length $a=9$. From the origin to the focus $(3,-4)$ the horizontal change is 3 and the downward change is 4. The acute angle with the horizontal is $\theta=\arctan(4/3)\approx 53.13°$. The focal distance is $c=5$, so $b^2=a^2-c^2=81-25=56$.

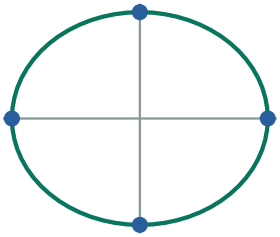


Read the major-axis inclination

The direction from the origin to $(3,-4)$ goes 3 right and 4 down, making an acute angle about $53.13°$.

2. Start with a horizontal ellipse

Stretch a unit circle by 9 horizontally and by $\sqrt{56}$ vertically. Its point at parameter t becomes $(u,v)=(9\cos t,\sqrt{56}\sin t)$. Substitution gives $u^2/81+v^2/56=1$. At $t=0$ the point is the right-hand vertex $(9,0)$.

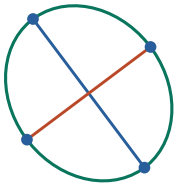


Build the unrotated ellipse

Begin with $(9\cos t,\sqrt{56}\sin t)$, oriented horizontally.

3. Choose and apply a clockwise turn

The desired axis goes down as it goes right. Rotate clockwise by θ to send the unit direction $(1,0)$ to $(3/5,-4/5)$. Its perpendicular $(0,1)$ turns to $(4/5,3/5)$. A point u units along the first direction and v along the second therefore becomes $x=(3u+4v)/5$, $y=(-4u+3v)/5$.



Rotate the two basis directions

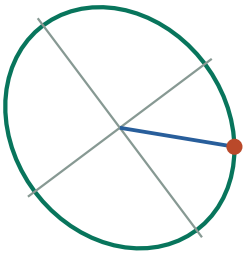
Turning clockwise sends $(1,0)$ to $(3/5,-4/5)$ and $(0,1)$ to $(4/5,3/5)$.

1. $x=(3/5)(9\cos t)+(4/5)(\sqrt{56}\sin t)$
Project the major- and minor-axis vectors horizontally.

2. $y=(-4/5)(9\cos t)+(3/5)(\sqrt{56}\sin t)$
Project those vectors vertically; the major axis slopes down to the right.

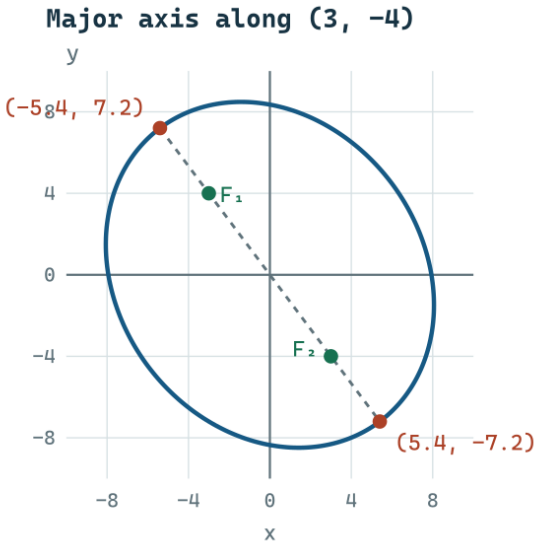
4. Trace the transformed coordinates

Enter x(t)=(27 cos t+4√56 sin t)/5 and y(t)=(−36 cos t+3√56 sin t)/5 in radian parametric mode. Trace 0≤t≤2π with equal axis scales. At 0 and π the curve reaches opposite major vertices ±(27/5,−36/5). At π/2 and 3π/2 it reaches the opposite minor endpoints ±(4√56/5,3√56/5), which fixes the orientation of the sketch.



Trace the rotated curve

The major and minor vertices are the rotated horizontal and vertical tips, anchoring the sketch.



Check. At t=0,π the two vertices match exercise 459.

Exercise 461 · Cartesian tilted ellipse

Printed page 48 · Ellipses / Rotations / Coordinate transformations

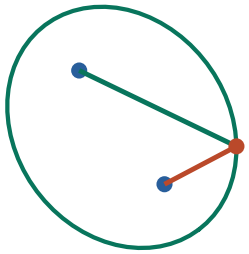
(Continuation)

(a) If $P = (x, y)$, $F_1 = (-3, 4)$, and $F_2 = (3, -4)$, write (though do not attempt to simplify) an equation that says $PF_1 + PF_2 = 18$. Sketch the graph using a graphing tool.

(b) As an algebra challenge, simplify the equation by hand so that it is in the form $Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0$, where the coefficients are *relatively prime*.

1. Convert each string segment to a distance

The locus is $\sqrt{((x+3)^2+(y-4)^2)}+\sqrt{((x-3)^2+(y+4)^2)}=18$. The sketch has major direction $(3,-4)$.

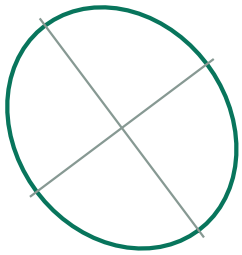


Use the two sloping focal anchors

Distances from the two points $(-3,4)$ and $(3,-4)$ must sum to 18.

2. Align new axes with the foci

Rotate into axial coordinates $u=(3x-4y)/5$, $v=(4x+3y)/5$.

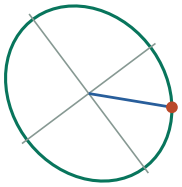


Resolve along the ellipse's own axes

The major direction is $(3/5,-4/5)$ and its perpendicular is $(4/5,3/5)$.

3. Clear denominators before expanding

The axial equation is $56u^2+81v^2=4536$. Since u and v each divide their linear numerator by 5, substituting gives $56(3x-4y)^2+81(4x+3y)^2=113400$. Expand and collect x^2, xy and y^2.



Translate the axial equation into x and y

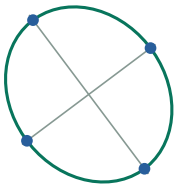
The sloped coordinate axes produce the cross term 24xy in the Cartesian equation.

1. $56(3x-4y)^2+81(4x+3y)^2=25\cdot81\cdot56$
Clear the two semiaxis denominators and the rotation's denominator 5^2.

2. $56(9x^2-24xy+16y^2)+81(16x^2+24xy+9y^2)=113400$
Expand both binomial squares with opposite signs in their mixed terms.

4. Obtain the relatively prime coefficients

Substitute into $u^2/81+v^2/56=1$ and expand: $72x^2+24xy+65y^2-4536=0$. The coefficients have no common integer factor.

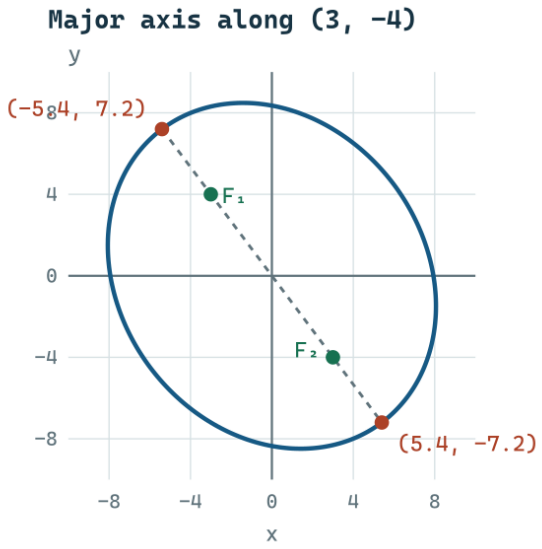


Recover the complete conic

$72x^2+24xy+65y^2=4536$ is the same rotated ellipse.

1. $1800x^2+600xy+1625y^2=113400$
Collect the coefficients for each power separately.

2. $72x^2+24xy+65y^2=4536$
Divide by 25 to obtain relatively prime coefficients.



Check. At $(27/5,-36/5)$, axial coordinates are $(9,0)$, satisfying the equation.

Exercise 462 · Ellipses scaled to their foci

Printed page 48 · Ellipses / Eccentricity / Similarity and scaling

Many ellipses are similar to $9x^2 + 25y^2 = 225$. Write an equation for the one whose focal points are:

(a) $(8, 0)$ and $(-8, 0)$; **(b)** $(0, 12)$ and $(0, -12)$.

1. Read the reference ellipse

Divide its equation by 225 to get $x^2/25+y^2/9=1$. This makes the semiaxes 5 and 3 and puts the major axis horizontally. The distance c comes from $c^2=a^2-b^2$.



Find the original focal offset

For semiaxes 5 and 3 the foci are four units from the center.

2. Scale to the first focus pair

The model has a=5,b=3,c=4. For horizontal foci ±8 scale by 2: $x^2/100+y^2/36=1$.



Double both semiaxes

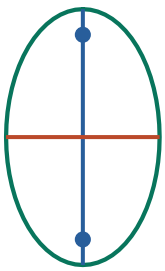
To move horizontal foci to ±8, scale by 2. The new semiaxes are 10 and 6.

1. $8/4=2 \Rightarrow (a,b)=(2\cdot5,2\cdot3)=(10,6)$
Use the ratio of focal offsets to scale both axes.

2. $(x/10)^2+(y/6)^2=1$
The new focal pair is horizontal, so the major denominator belongs to x.

3. Scale and rotate to the second pair

For vertical foci ±12 scale by 3 and quarter-turn: $x^2/81+y^2/225=1$.



Triple and quarter-turn

To put the foci at $y=\pm12$, rotate the tripled ellipse vertically.

Check. $100-36=64$ and $225-81=144$ give the specified squared focal offsets.

Exercise 463 · Equal exponents across three bases

Printed page 48 · Exponent rules / Radicals and rational powers

Solve each of the following equations by hand.

(a) $8^x = 32$ (b) $27^x = 243$ (c) $1000^x = 100000$

Explain why all three equations have the same solution.

1. (a) Write both sides as powers of two

A positive base greater than one gives a strictly increasing exponential. Equal outputs with that base require equal exponents.

1 **$8^x = 32 \Leftrightarrow (2^3)^x = 2^5$**

Recognize 8 and 32 as powers of 2.

2 **$2^{(3x)} = 2^5 \Leftrightarrow 3x = 5$**

Multiply nested exponents and use the one-to-one property.

3 **$x = 5/3$**

Divide the exponent equation by the nonzero constant 3.

2. (b) Repeat with base three

Every real x is allowed. The original positive base makes its real powers defined.

1 **$27^x = 243 \Leftrightarrow (3^3)^x = 3^5$**

Write 27 and 243 as powers of 3.

2 **$3^{(3x)} = 3^5 \Leftrightarrow 3x = 5$**

Match exponents because powers of 3 are strictly increasing.

3 **$x = 5/3$**

The same linear equation gives the same solution.

3. (c) Explain the common result

In each part a cube is raised to x and equated to the fifth power of the same positive base.

1 **$1000^x = 100000 \Leftrightarrow (10^3)^x = 10^5$**
Use powers of ten for the final pair.

2 **$10^{(3x)} = 10^5 \Leftrightarrow 3x = 5$**
Use the one-to-one property for base ten.

3 **$x = 5/3$: $(b^3)^{(5/3)} = b^5$**
Substitution works with b = 2, 3 and 10. Each equation has one real solution.

Check. In each case cube-rooting the base and raising it to the fifth produces the target.

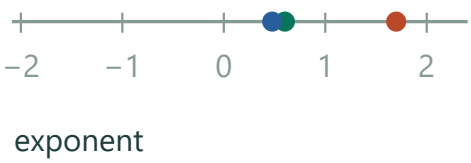
Exercise 464 · Three predictable logarithms

Printed page 49 · Logarithm rules

Given a positive number p, the solution to $10^x = p$ is called the *base-10 logarithm of* p, expressed as $x = \log_{10} p$, or simply $x = \log p$. For example, $10^4 = 10000$ means that 4 is the base-10 logarithm of 10000, or $4 = \log 10000$. The log function provides immediate access to such numerical information. Using a calculator for confirmation, and remembering that *logarithms are exponents*, explain why it is predictable that
(a) $\log 64$ is three times $\log 4$;
(b) $\log 12$ is the sum of $\log 3$ and $\log 4$;
(c) $\log 0.02$ and $\log 50$ differ only in sign.

1. Translate the three numeric relationships

The numbers 64 and 4 are related by cubing, the numbers 12,3,4 by multiplication, and 0.02 and 50 by reciprocation. A logarithm tracks what these operations do to exponents of 10.

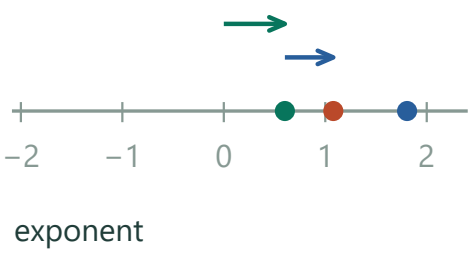


Locate the original inputs

The three marked exponents correspond to 4, 3 and 50.

2. Prove the cube and product relations

$64=4^3$ gives $\log 64=3\log 4$. The product $12=3\cdot4$ gives $\log 12=\log 3+\log 4$.

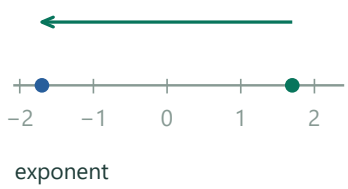


Compare powering and multiplying

Tripling $\log 4$ gives $\log 64$. Adding $\log 3$ to $\log 4$ gives $\log 12$.

3. Prove the reciprocal relation

Since $.02=1/50$, $\log .02=-\log 50$. Writing $50=10^k$ gives $1/50=10^{-k}$ directly. This explains the sign reversal even before entering numbers on the calculator.



Reflect the reciprocal

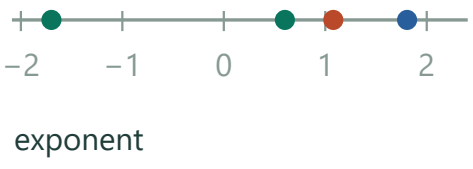
The exponents for 50 and 0.02 lie at equal distances on opposite sides of zero.

1. $50=10^L \Rightarrow 1/50=1/10^L=10^{-L}$
Define L as the logarithm of 50 and use the reciprocal rule.

2. $\log_{10}(0.02)=-L$
The reciprocal $1/50$ equals 0.02.

4. Verify the three results numerically

A calculator gives $\log 64\approx1.80618$ and $\log 4\approx0.60206$. Also $\log 12\approx1.07918$ while $\log 3\approx0.47712$. Finally $\log 0.02\approx-1.69897$ and $\log 50\approx1.69897$.



Compare the numerical positions

The logs of 4, 64, 12, and 0.02 are about 0.602, 1.806, 1.079, and −1.699.

Check. Raising 10 to both sides of each equality recovers the corresponding arithmetic identity.

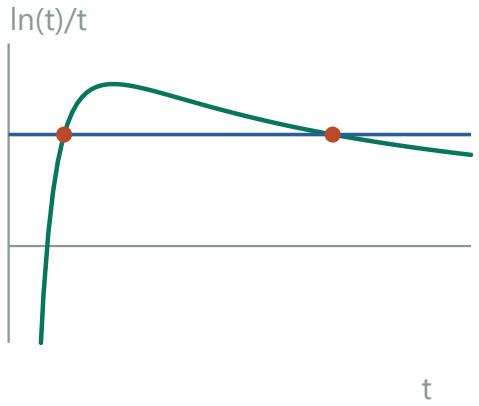
Exercise 465 · When two growth formulas agree

Printed page 49 · Power functions / Exponential functions / Systems of equations

In summarizing the growth of a certain population, Bailey writes $G(t) = 747t^{1.44}$ by mistake, instead of $747(1.44)^t$. Are there t-values for which the expressions agree in value?

1. Separate the positive-time case

For positive elapsed time cancel 747 and take logs: (ln t)/t=(ln 1.44)/1.44. One solution is t=1.44, and numerical solving gives t≈8.40854.

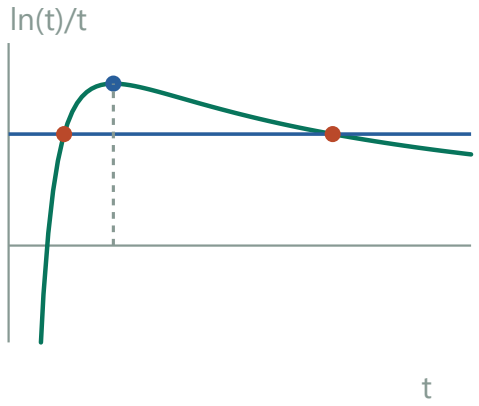


Compare a curve with one fixed level

For t>0 the reformulated equation compares (ln t)/t with (ln 1.44)/1.44. Their crossings are about 1.44 and 8.40854.

2. Prove how many positive intersections can occur

Because (ln t)/t rises until t=e then falls, these are the only positive solutions.

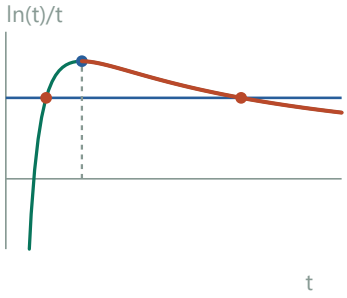


Locate the one crest

The high blue point is at t=e. The level crosses the rising and falling sides once each.

3. Justify the turning point

For $t>0$ the derivative of $\ln(t)/t$ is $(1-\ln t)/t^2$. Its sign switches once at e. Together with the two solutions on opposite sides of e, this proves the list is complete. At $t=0$, the two original formulas give different values.



Explain the two branches

The derivative has sign $1-\ln t$. It turns negative after e, so the highlighted descending branch never returns.

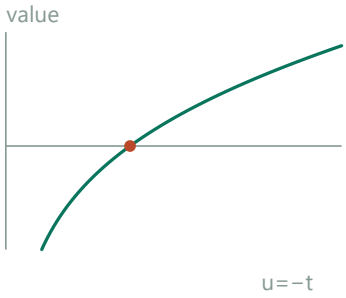
1. $g(t)=\ln(t)/t \Rightarrow g'(t)=[(1/t)t-\ln t]/t^2$
Use the quotient rule only on $t>0$.

2. $g'(t)=(1-\ln t)/t^2$
The positive denominator leaves the numerator to determine the sign.

3. $g'=0$ at $t=e$; $g'>0$ before e and $g'<0$ afterward
The two monotone branches can each meet a horizontal level at most once.

4. Explore the negative-time convention

If negative times and real rational powers $1.44=36/25$ are admitted, there is also $t\approx-0.81378$.



Admit a negative-time convention

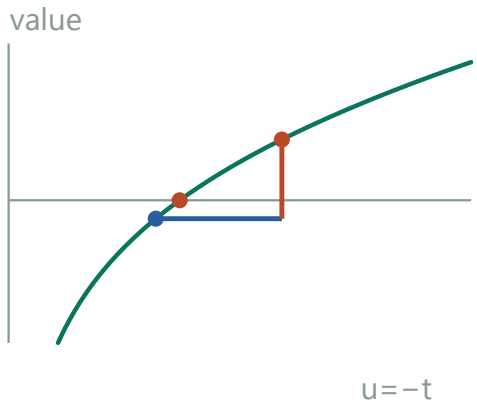
For real rational powers with odd denominator, let $u=-t>0$. The crossing gives $u\approx0.81378$.

1. $t=-u$, $u>0 \Rightarrow (-u)^{(36/25)}=u^{(36/25)}$
The odd 25th root can be negative; its even 36th power is positive.

2. $u^{1.44}=1.44^{(-u)} \Rightarrow 1.44\ln u+u\ln 1.44=0$
Logarithms apply to positive sides; transfer the right-hand term to the left.

5. Prove uniqueness there too

For the negative-time convention put $u=-t>0$. The equation becomes $1.44\ln u+u\ln(1.44)=0$. Its left side increases strictly from $-\infty$ to $+\infty$ as u runs from 0 to ∞, so the negative solution is unique.



Count its roots

Both $1.44\ln u$ and $u\ln 1.44$ increase. Their sum runs from negative to positive infinity and has exactly one zero.

Check. At either positive solution the log difference $1.44\ln t-t\ln 1.44$ vanishes; $t=0$ gives different values.

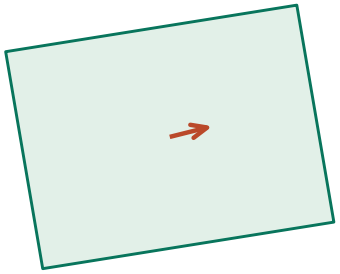
Exercise 466 · Parallel plane

Printed page 49 · Planes in space / Vectors and components

Find an equation for the plane that goes through the point $(3, 2, 1)$ and that is parallel to the plane $7x + 4y + 4z = 65$.

1. Read the normal vector

Moving parallel to the target plane must be the same as moving parallel to the given plane, so the vectors perpendicular to them point along the same direction. Keep the three coefficients and allow the constant to change.

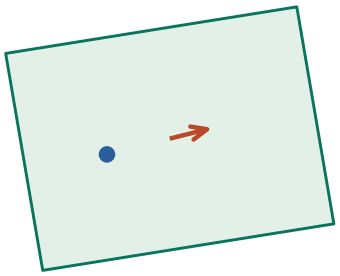


Keep the given normal

The coefficients (7,4,4) determine the orientation of the supplied plane.

2. Determine the new constant

Substitute coordinates (3,2,1) for x,y,z. Whatever value the left side gives is the required right side of the new plane.

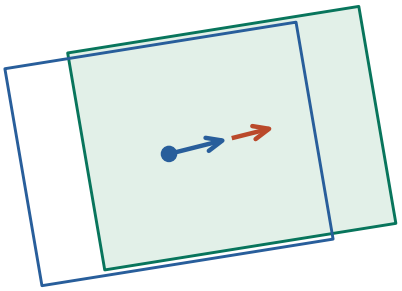


Mark the prescribed point

The required parallel surface passes through (3,2,1), where the same left side gives 33.

3. Write the parallel plane

Keep its normal (7,4,4); the required constant is $7\cdot3+4\cdot2+4\cdot1=33$. Thus $7x+4y+4z=33$.



Translate the entire plane

The two planes have the same normal and different constants, 65 and 33.

Check. Subtracting the two equations rules out any common point while preserving parallel normals.

Exercise 467 · Determine exponential parameters

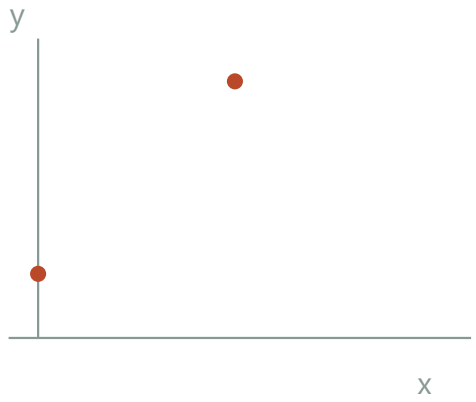
Printed page 49 · Exponential functions / Systems of equations

An *exponential function* f is defined by $f(x) = k \cdot b^x$. Find k and b, given that

(a) $f(0) = 3$ and $f(1) = 12$ **(b)** $f(0) = 4$ and $f(2) = 1$

1. Use the zero input first

At x=0, b^0=1, so the output of either exponential equals k. Once this factor is known, divide the second output by k to isolate a power of b.

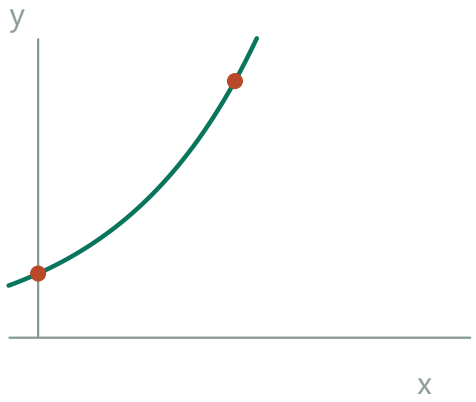


Mark the supplied intercept

An output at x=0 directly supplies k. In (a) this is 3.

2. Find the model from the one-step pair

(a) f(0)=3 gives k=3; f(1)=12 implies b=4. The output changes from 3 to 12 in exactly one input unit, so dividing 12 by 3 directly gives b.

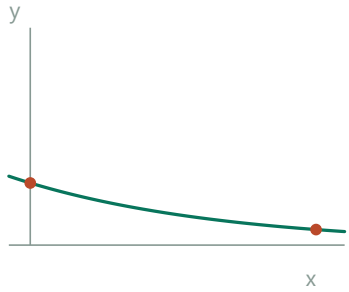


Read one full multiplier

The point (1,12) requires 12/3=4, so $f(x)=3\cdot4^x$.

3. Find the positive base from a two-step pair

(b) k=4; f(2)=1 gives b^2=1/4, so the positive exponential base is b=½.



Use two steps for the second data set

Moving from 4 to 1 over two x units multiplies by ¼, giving b=½ per unit.

1. k=4 ⇒ $4b^2$=1 ⇒ b^2=1/4
Substitute the given value at x=2 and divide by 4.

2. b=1/2
A real exponential defined at every real input uses a positive base.

Check. The models 3·4^x and 4·(½)^x reproduce both given values.

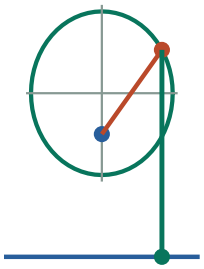
Exercise 468 · A focus and horizontal directrix

Printed page 49 · Ellipses / Foci and directrices

You have seen that an ellipse is in some ways like a circle. The following exercise, which illustrates an alternative definition, shows that an ellipse is in some ways also like a parabola: Let $F = (0, 0)$ be the *focus* and the line $y = -6$ be the *directrix*. Plot several points P that are half as far from the focus as they are from the directrix. The configuration of all such points is an ellipse. Identify the four points where the ellipse crosses the coordinate axes (two on each axis). Use the distance formula to write an equation for the ellipse.

1. Express both distances in coordinates

The distance to the focus at the origin is $\sqrt{(x^2+y^2)}$. Distance to a horizontal line is the absolute vertical coordinate difference. Here that difference is $y-(-6)$.

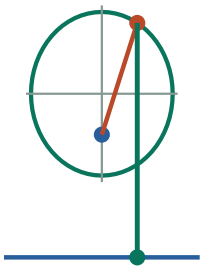


Draw two distances from P

Focus distance reaches the origin; perpendicular line distance reaches the horizontal directrix $y=-6$.

2. Write and square the ratio equation

Set $\sqrt{(x^2+y^2)}=\text{abs}(y+6)/2$. Squaring and completing the square gives $x^2/12+(y-2)^2/16=1$.

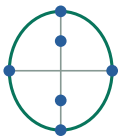


Keep a one-to-two ratio

For an ellipse point the red focal length is half its perpendicular distance to the line.

3. Complete the square explicitly

Multiplying the squared equation by 4 gives $4x^2+4y^2=y^2+12y+36$. Bring terms together, then add the missing square term: $4x^2+3(y-2)^2=48$.



Identify the center and tips

Completing squares gives an ellipse centered at (0,2), with vertical extremes −2 and 6.

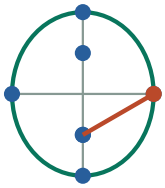
1. $4x^2+4y^2=y^2+12y+36 \Rightarrow 4x^2+3(y^2-4y)=36$
Expand $(y+6)^2$, then group the y-terms.

2. $y^2-4y=(y-2)^2-4 \Rightarrow 4x^2+3(y-2)^2=48$
Add back the 12 removed by completing the square.

3. $x^2/12+(y-2)^2/16=1$
Divide by 48. The original distance equation had nonnegative sides, so squaring is reversible.

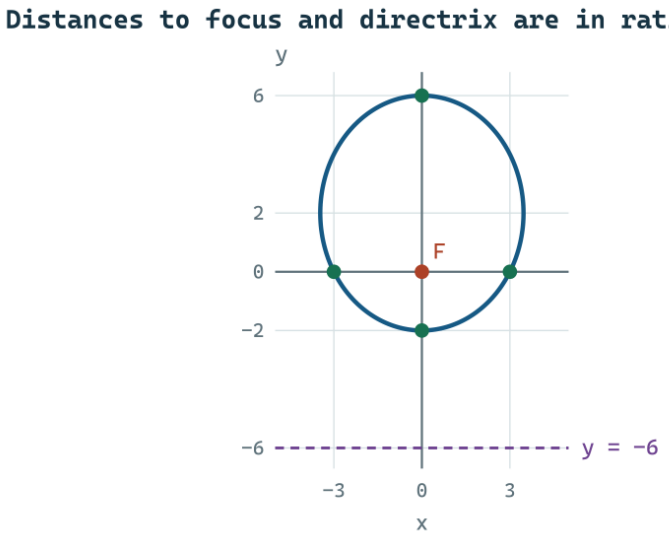
4. Locate the four intercepts and sketch

The four axis crossings are $(\pm3,0),(0,6),(0,-2)$. Plot the resulting vertical ellipse around (0,2).



Sketch the locus

The intersections with the x-axis are ±3, while the furthest horizontal extent occurs at y=2.



Check. At (3,0) the two distances are 3 and 6; at (0,−2) they are 2 and 4.

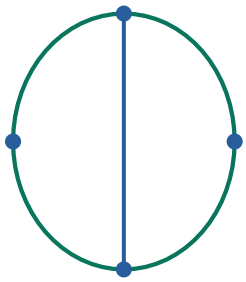
Exercise 469 · Dimensions of the translated ellipse

Printed page 49 · Ellipses / Foci and directrices

(Continuation) Notice that this ellipse does not have the origin as its center, unlike the previous examples you have seen, so the x-axis is not a line of symmetry. The value of a for this ellipse should be clear, and the value of b can be obtained by finding where the ellipse meets its minor axis. Now calculate the value of c and notice that, had this ellipse been drawn by the string method, one end of the string would be held at the origin F.

1. Locate the new center and long semiaxis

Its vertical extremes 6 and −2 give center y=2 and a=4.

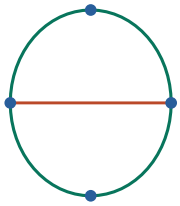


Bisect the vertical extremes

Halfway between y=−2 and y=6 is the center y=2.

2. Use the horizontal symmetry line

At y=2 the equation gives x=±√12, so b=√12 and c=√(16−12)=2.



Read the width at center height

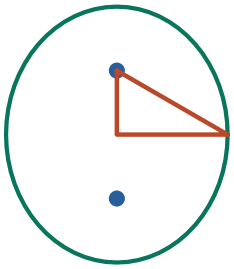
At y=2 the horizontal endpoints are x=±√12.

1. $y=2 \Rightarrow x^2/12=1 \Rightarrow x=\pm\sqrt{12}$
The shifted y-term vanishes along the horizontal symmetry line.

2. $c^2=a^2-b^2=16-12=4 \Rightarrow c=2$
Compute the center-to-focus distance using the semiaxes.

3. Explain why the foci move

The ellipse is vertical. Move c above and below the center (0,2) rather than left and right of it. This shows explicitly why one pin of the string construction can be at the origin.

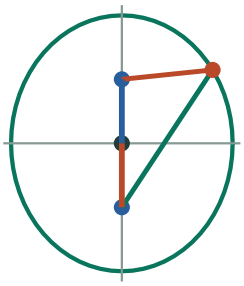


Calculate the focal offset

The foci are two units above and below the center, at (0,4) and (0,0).

4. Mark both pin positions

Foci are (0,0),(0,4). Start at the center (0,2) and move 2 units down and 2 units up along the vertical major axis. The lower pin coincides with F from the construction.



Pin both ends of the string

Starting at center (0,2), move two units up and down to (0,4) and (0,0). One pin coincides with the original focus F.

Check. The focal separation 4 is twice c.

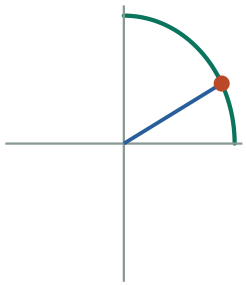
Exercise 470 · Trace a translated ellipse

Printed page 49 · Ellipses / Parametric equations / Translations

(Continuation) Graph the curve that is described parametrically by the equation $(x, y) = (\sqrt{12}\cos t, 2 + 4\sin t)$, for $0 \le t \le 360$.

1. Read the center from the parametrization

Before the +2 is applied, sine and cosine oscillate about zero. Adding 2 to the vertical coordinate moves the entire curve up by two units. The coefficients give the two semiaxis lengths.

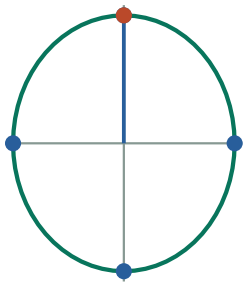


Start at the displaced center

The parameterization is built from (0,2) with horizontal scale $\sqrt{12}$ and vertical scale 4.

2. Evaluate the quarter turns

The horizontal semiaxis is $\sqrt{12}$ and the vertical semiaxis 4, with center (0,2). At quarter turns plot $(\sqrt{12},2),(0,6),(-\sqrt{12},2),(0,-2)$ and return to the start.

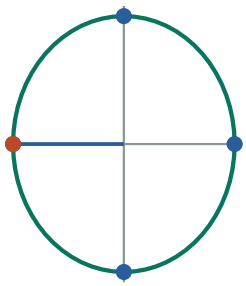


Pass through the upper vertex

At 90° the horizontal offset vanishes and the height is $2+4=6$.

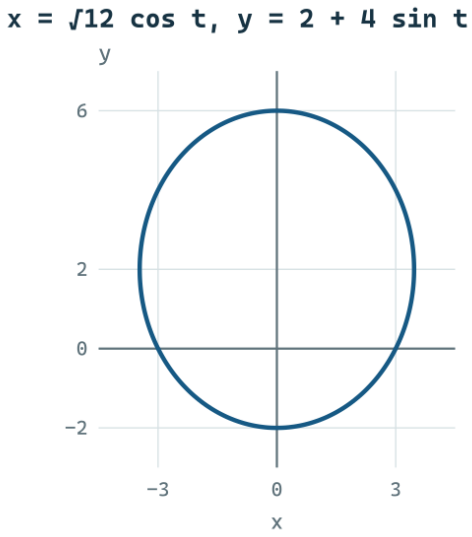
3. Connect them in time order

At small positive t, y rises from 2 as x begins moving left. Draw a smooth counterclockwise arc through the top, left and bottom extremities, ending back at the rightmost tip.



Follow the remaining quarter turns

The four tips are $(\sqrt{12},2)$, $(0,6)$, $(-\sqrt{12},2)$, $(0,-2)$.



Check. Eliminating t gives $x^2/12+(y-2)^2/16=1$.

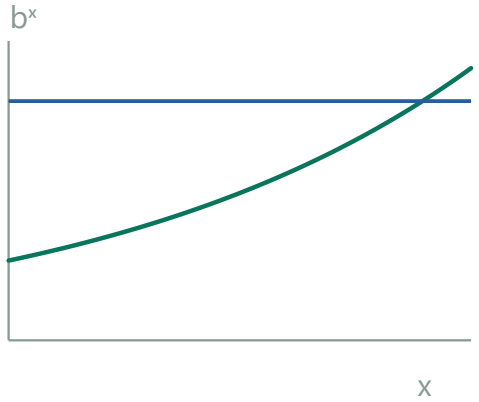
Exercise 471 · Different base, same exponent

Printed page 49 · Logarithms / Exponential functions

What if the base of an exponential equation isn't 10? One way of solving an equation like $1.02^x = 3$ is to use the log function to rewrite the equation as $(10^{0.0086})^x = 10^{0.4771}$. First justify this conversion, then solve $10^{0.0086x} = 10^{0.4771}$.

1. Replace a number with a base-ten power

For any positive number z, the relationship $z=10^{(\log_{10} z)}$ is exact. Apply it separately to 1.02 and to 3, then use the given decimals as approximations.

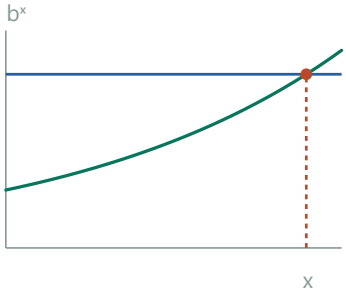


Set the target on the output axis

Base 1.02 reaches the height 3 when their base-ten exponents agree.

2. Use the displayed exponents

Since log 1.02≈.0086 and log 3≈.4771, the supplied equation with rounded exponents gives x≈.4771/.0086≈55.48.



Identify the needed horizontal input

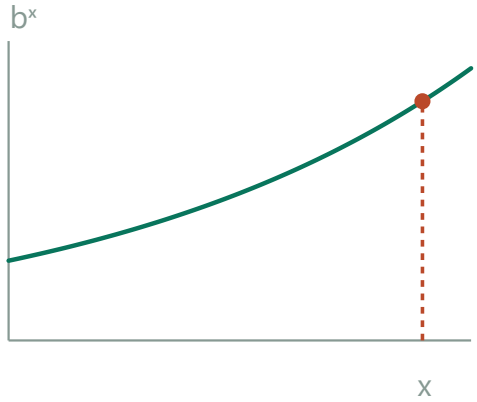
Write x log 1.02 = log 3; both sides represent the same output exponent.

1. (10^(log 1.02))ˣ=10^(log 3)
Rewrite both positive numbers as powers of ten.

2. 10^(x log 1.02)=10^(log 3) ⇒ x log 1.02=log 3
The two base-ten powers are equal only if their exponents agree.

3. Reduce rounding error

More accurately x=ln 3/ln 1.02≈55.4781. Using the unrounded logarithms keeps the small denominator from affecting the quotient unnecessarily. The base of the logarithms can be changed as long as both use the same one.



Read its logarithmic coordinate

x≈55.478 periods; rounding the supplied log values first gives a close estimate.

Check. Exponentiating 1.02 to this power gives 3; early rounding explains small discrepancies.

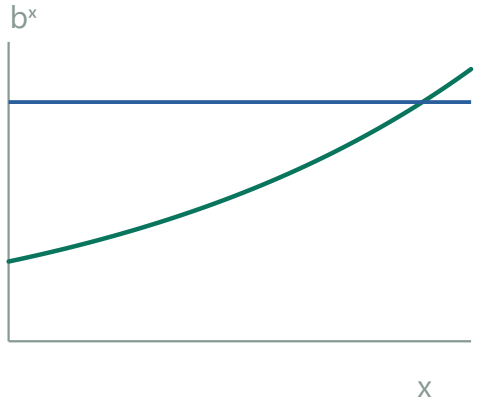
Exercise 472 · Why change of base works

Printed page 49 · Change of base / Exponential functions

(Continuation) You have now calculated the logarithm of 3 using the base 1.02, for which $\log_{1.02} 3$ is the usual notation. The usual ways of reading $\log_{1.02} 3$ are "log base 1.02 of 3" or "log 3, base 1.02", or "the base-1.02 logarithm of 3", or "log to the base 1.02 of 3." The desired value was obtained as a quotient of two base-10 logarithms. Explain. By the way, do you recall a context for the equation $1.02^x = 3$?

1. Identify what the unknown logarithm asks for

The subscript 1.02 is the growth factor and 3 the target. The logarithm is the exponent x needed so that the power reaches 3.

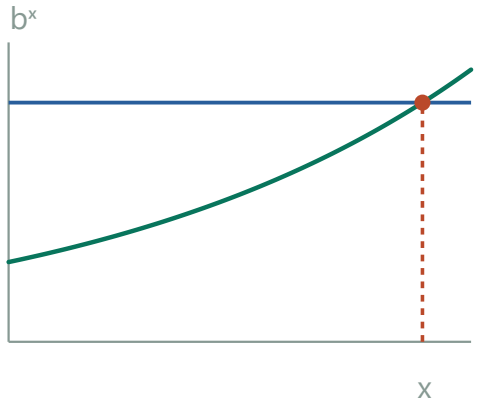


Set the target on the output axis

The target 3 asks how many 1.02-fold steps triple the starting unit.

2. Pass to a shared logarithm base

Taking base-ten logs of 1.02^x=3 gives xlog 1.02=log 3. Hence x=(log 3)/(log 1.02).

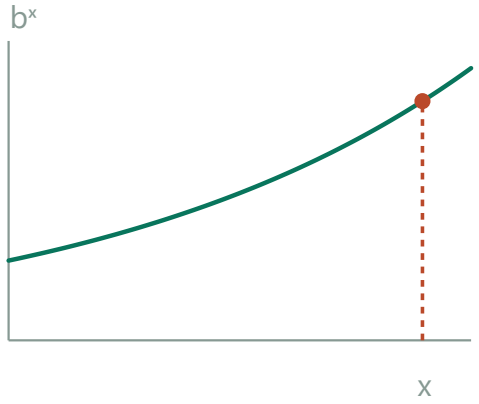


Identify the needed horizontal input

Taking common logs moves the unknown exponent into a coefficient.

3. Interpret x in a growth story

It is the elapsed time needed to triple under 2% growth per period.



Read its logarithmic coordinate

$\log_{1.02} 3 = \log 3 / \log 1.02 \approx 55.478$.

Check. The quotient is unchanged using natural logs because both log values receive the same conversion factor.

Exercise 473 · Annualize a partial-year return

Printed page 50 · Compounding / Percentage change

On 24 August 1997 (which is the 236th day of the year), the Fidelity Select Electronics Fund reported a 44.3 percent return on investments for the year to date. Calculate the annual growth rate for this fund in 1997. Explain your method and the assumptions you made.

1. Turn year-to-date gain into a balance multiplier

A 44.3% gain leaves 1.443 times the starting value. That is the cumulative factor for 236 days, so using it once per year would give the wrong time interval.

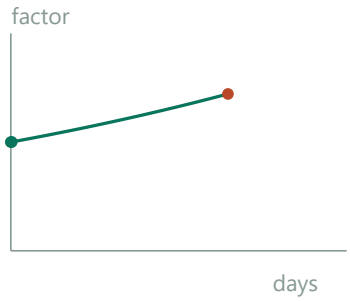


Mark the reported interval

A 44.3% to-date return means one invested unit became 1.443 units in 236 days.

2. Find the equivalent daily and annual factors

Assuming a constant compounded growth factor, the daily factor is 1.443^(1/236). The projected full-year gain is 1.443^(365/236)−1≈76.33%.



Infer a constant daily factor

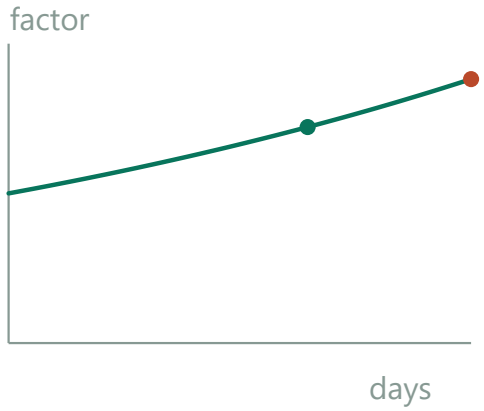
The modeled factor is the 236th root of 1.443.

1. $g^{236}=1.443 \Rightarrow g=1.443^{(1/236)}$
g is the positive constant daily balance factor.

2. $g^{365}-1=1.443^{(365/236)}-1$
Use 365 daily factors and remove the initial principal to obtain the annualized gain.

3. Distinguish an annualized rate from a known return

It is an annualized projection; the remaining days determine the realized annual result.



Extend to day 365

Maintaining that factor projects about 76.33% annual growth. The remaining days were unobserved.

Check. Raising the annual factor to 236/365 returns 1.443.

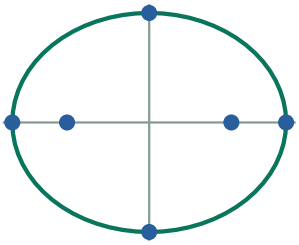
Exercise 474 · Test a focus–directrix ratio

Printed page 50 · Ellipses / Foci and directrices / Eccentricity

Let $F = (9, 0)$ and choose a point P that fits the equation $16x^2 + 25y^2 = 3600$. Confirm that the distance from P to F is exactly three fifths the distance from P to the vertical line $x = 25$. Repeat this verification for two more points P that fit the equation. Calculate a and c for this ellipse, to show that its eccentricity is $3/5$. Hmm . . .

1. Draw a horizontal ruler to the vertical directrix

From (x,y) to x=25 the shortest path is horizontal. Its length is the absolute difference between x and 25. To F=(9,0) instead use the ordinary point distance.

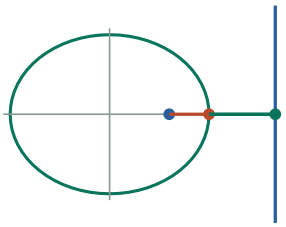


Read the focal offset

The semiaxes 15 and 12 give c=9 and e=3/5.

2. Get the ellipse parameters

The semiaxes are a=15 and b=12. Thus c=9 and e=3/5.



Test the right vertex

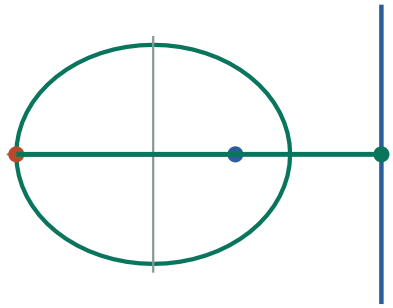
The two distances are 6 to the right focus and 10 to the vertical line x=25.

1. $16x^2+25y^2=3600 \Rightarrow x^2/225+y^2/144=1$
Divide by 3600 to read a=15 and b=12.

2. $c=\sqrt{(225-144)}=9$; $e=c/a=9/15=3/5$
Use the focal right triangle, then divide by the semimajor axis.

3. Test three boundary points

Choose (15,0),(−15,0),(0,12). Focal distances are 6,24,15 and directrix distances are 10,40,25 respectively.

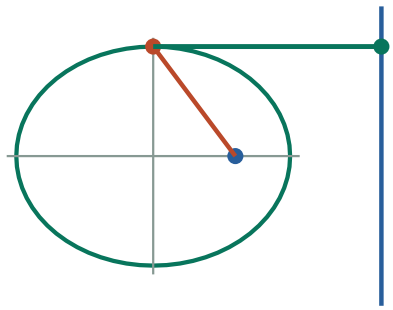


Test the opposite vertex

The distances 24 and 40 keep the same 3/5 ratio.

4. Compare each pair

The three ratios are 6/10, 24/40 and 15/25. Reducing them gives the same value as c/a, including for the point above the center.



Test the upper minor vertex

Its red distance is 15 and perpendicular line distance 25, again in the ratio 3/5.

Check. Every quotient equals 3/5; substituting the points verifies each lies on the ellipse.

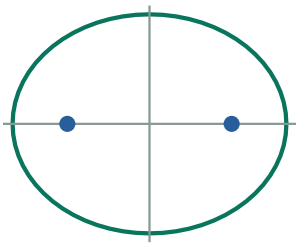
Exercise 475 · The other directrix

Printed page 50 · Ellipses / Foci and directrices

(Continuation) The line $x = 25$ is a directrix for the ellipse $16x^2 + 25y^2 = 3600$, and F is its focus. This ellipse has another focus G and another directrix. Find coordinates for G and an equation for its directrix. For any point P on the ellipse, what is the sum $PF + PG$?

1. Reflect the known focus and directrix

The ellipse is symmetric about the y-axis. Reflecting the positive-x focus across that axis gives its partner. The same symmetry changes a right-side vertical line into one on the left.

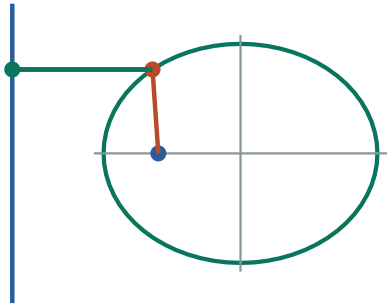


Reflect the right focus

Horizontal reflection places the second focus at x=−9.

2. Locate the reflected geometric objects

The second focus is (−9,0) and its directrix is x=−25.

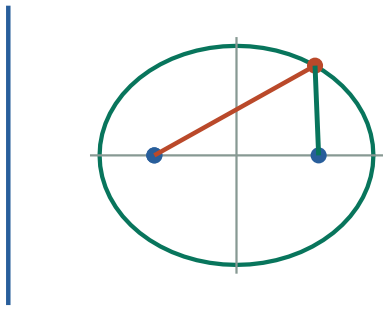


Reflect the directrix

The partner directrix is the vertical line x=−25.

3. Recover the constant string length

By the focal definition the distance sum for every ellipse point is 2a=30.



Return to the string description

The sum of the two focal distances stays equal to the major diameter 30.

Check. At the right-hand vertex the distances are 6 and 24.

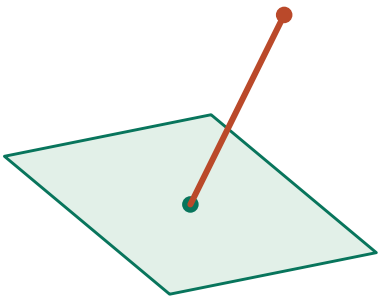
Exercise 476 · Distance to an oblique plane

Printed page 50 · Planes in space / Distance and loci / Vector projections

Find the distance from the point $(1, 3, 5)$ to the plane $2x - y + 5z = 9$.

1. Measure perpendicular distance

The coefficient vector n=(2,−1,5) points perpendicular to the plane. Moving a distance s along the unit vector n/√30 changes the plane expression n·X by s√30.

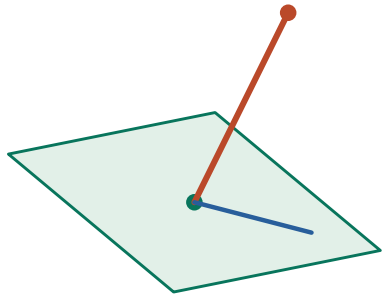


Measure along the normal

The perpendicular red segment links (1,3,5) to the plane.

2. Compute the plane-expression difference

The point's left side is 24, exceeding 9 by 15.

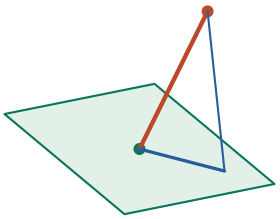


Locate the foot of the perpendicular

The point's plane-expression value is 24 and the target is 9, a difference of 15.

3. Convert that difference to length

Divide by normal magnitude √30: distance=15/√30=√30/2.



Turn that difference into length

Dividing 15 by the normal length √30 gives √30/2; any slanted segment is longer.

1. $\|n\|=\sqrt{(2^2+(-1)^2+5^2)}=\sqrt{30}$
The equation difference measures separation in units of normal length.

2. d=abs(24−9)/√30=15√30/30=√30/2
Take the absolute offset, divide by the normal magnitude, and rationalize.

Check. Subtract half the normal (2,−1,5) from the point to obtain (0,3.5,2.5) on the plane.

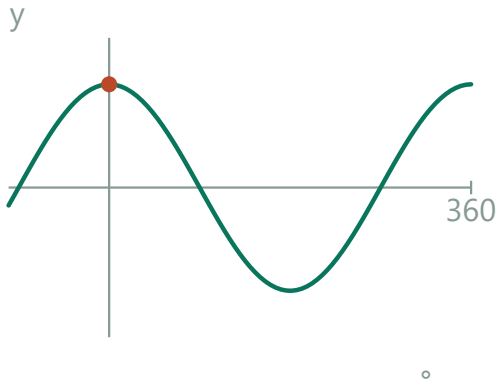
Exercise 477 · Shift a cosine

Printed page 50 · Sinusoidal functions / Phase shifts

How are the graphs of $y = \cos x$ and $y = \cos(x - 90)$ related? Work in degree mode.

1. Track the original crest

In cos(x−90°), what used to happen when the input was zero now happens when x−90°=0. Every other feature similarly appears 90° later.

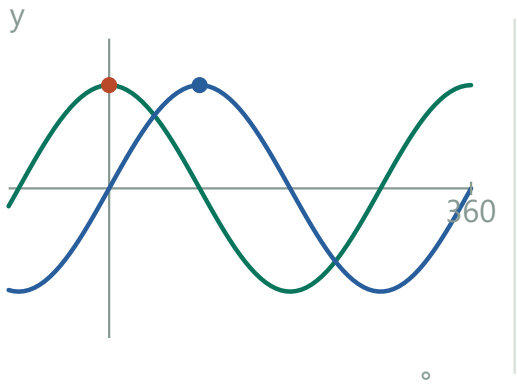


Track the crest at input zero

The original cosine reaches height 1 at x=0.

2. Identify the graph's new landmarks

At x=0°,90°,180°,270° the shifted cosine gives 0,1,0,−1. Those are the familiar quarter-turn heights of sine.

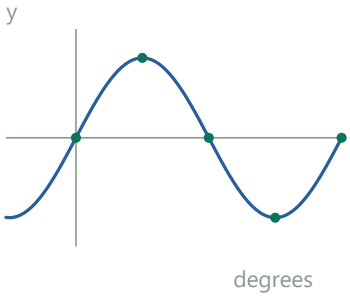


Move the crest right by 90°

Replacing x by x−90° moves all old landmarks to the right.

3. Give the transformation and equivalent function

Replacing x by x−90° shifts the entire graph 90° right. Via the cosine difference identity the shifted graph is sin x.



Compare the shifted graph with sine

At x=0 the shifted curve rises through zero, matching sine.

1. cos(x−90°)=cos x cos 90°+sin x sin 90°
Use the cosine difference identity on the shifted argument.

2. =0·cos x+1·sin x=sin x
Insert the quarter-turn values.

Check. An original crest at x=0 reappears at 90°.

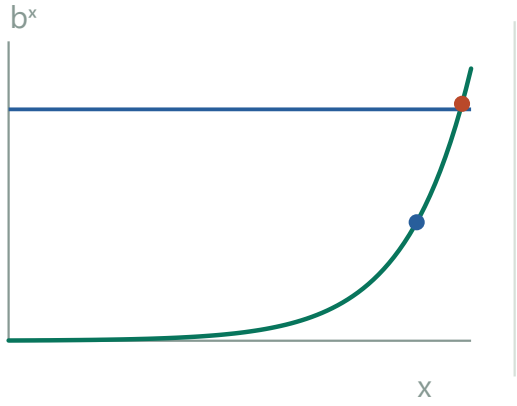
Exercise 478 · Base-two log of 1000

Printed page 50 · Logarithms / Change of base

Solve $2^x = 1000$. In other words, find $\log_2 1000$, the base-2 logarithm of 1000.

1. Bracket the exponent

Because 2^9=512 and 2^{10}=1024, the exponent must be strictly between 9 and 10, close to 10. This is a scale check before using a decimal logarithm.

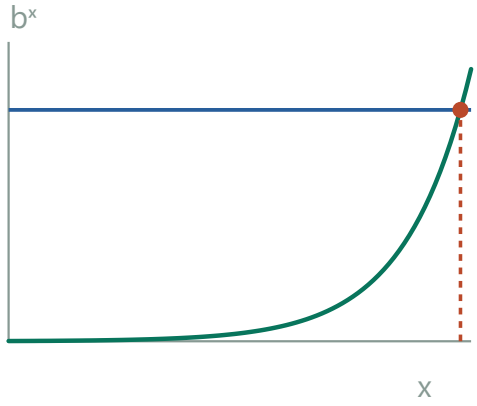


Set the target on the output axis

Heights 2^9=512 and 2^{10}=1024 bracket 1000.

2. Take a logarithm to unlock x

Taking natural logarithms on both sides gives x ln 2=ln 1000. The denominator when solving is ln 2 because 2 is the base.

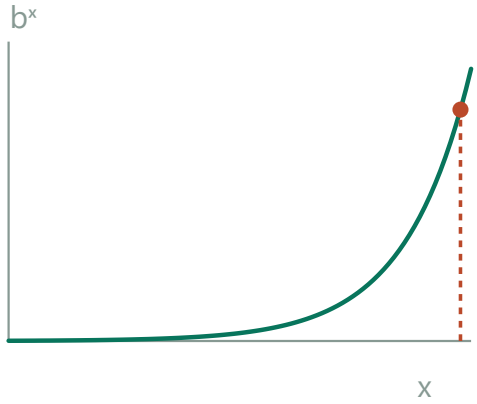


Identify the needed horizontal input

The target line reaches the curve just before x=10.

3. Evaluate the exponent

Take logs to obtain x=ln 1000/ln 2≈9.965784. A different common log base gives the same ratio. A decimal a little below 10 agrees with the fact that 1000 is a little below 1024.



Read its logarithmic coordinate

$\log_2 1000=\ln(1000)/\ln 2\approx 9.9658$.

Check. It must be slightly below 10 because $2^{10}=1024$.

Exercise 479 · Five logarithms by change of base

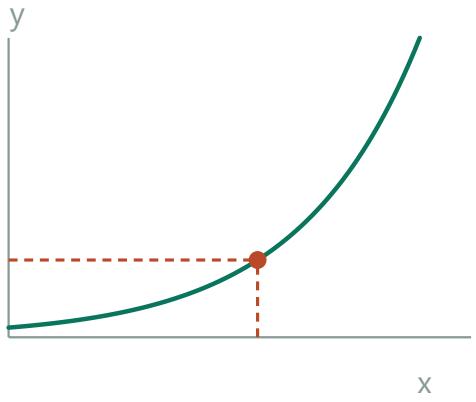
Printed page 50 · Change of base / Logarithms

You now know how to calculate logarithms by using 10 as a *common base*. Use this method to evaluate the following. Notice those for which a calculator is not necessary.

(a) $\log_8 5$ **(b)** $\log_5 8$ **(c)** $\log_5 \sqrt{5}$ **(d)** $\log_{1.005} 2.5$ **(e)** $\log_3(1/9)$

1. Change to a base the calculator knows

If x=log_b N, then $b^x=N$. Taking base-10 logarithms gives x log b=log N, hence x=log N/log b. This uses N>0 and b>0 with b≠1. All five examples meet those conditions.

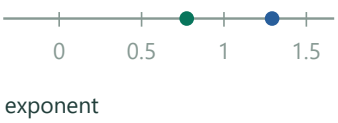


Read the exponent backward

A base-5 example: follow height 8 back to the exponent $\log_5 8$. Change of base computes that horizontal coordinate.

2. Evaluate the swapped base and argument

For (a), $\log_8 5=\log 5/\log 8\approx 0.773976$. This lies between 0 and 1 since 1<5<8. For (b), $\log_5 8=\log 8/\log 5\approx 1.292030$. Swapping the base and argument inverts the quotient, so the first two answers multiply to 1.



exponent

Compare reciprocal exponents

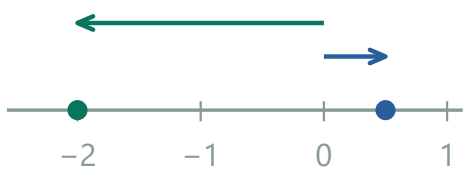
$\log_8 5 \approx 0.774$ and $\log_5 8 \approx 1.292$ surround 1; their product equals 1.

1. $\log_8 5=(\ln 5)/(\ln 8)$; $\log_5 8=(\ln 8)/(\ln 5)$
Change both logs to the same base.

2. $(\log_8 5)(\log_5 8)=1$
Both logarithms in the denominators are nonzero, so the quotients are reciprocals.

3. Recognize two exact powers

For (c), √5=5^(1/2), so $\log_5\sqrt{5}=1/2$ directly. For (e), $1/9=1/3^2=3^{-2}$, so $\log_3(1/9)=-2$. In each case you can work without a calculator by checking the defining power equation.



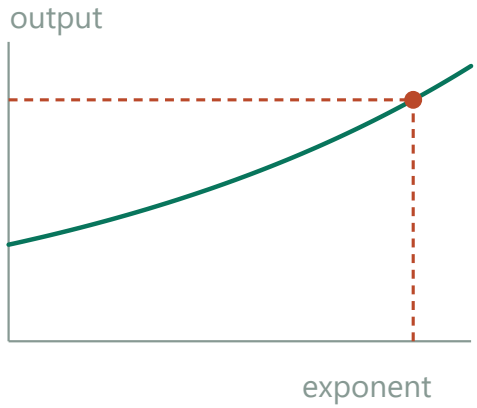
exponent

Recognize exact fractional and negative exponents

The right mark is 1/2 for √5 in base 5. The left is −2 for 1/9 in base 3.

4. Handle a base close to one

For (d), $\log_{1.005}2.5=\log 2.5/\log 1.005\approx0.3979400/0.00216606\approx183.7159$. Keep full calculator precision in the denominator. A large exponent makes sense because a factor of 1.005 grows by only 0.5% each time it is applied.



Compare a slowly growing base

A factor of 1.005 needs about 183.716 repetitions to reach 2.5. The guides mark this intersection.

Check. The first two values are reciprocals; $3^{-2}=1/9$ checks the last.

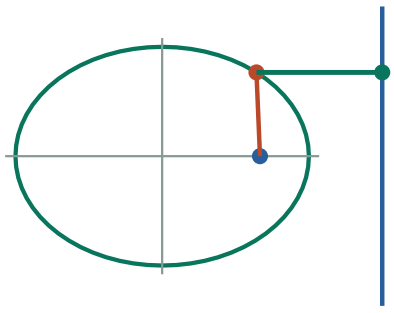
Exercise 480 · An ellipse to the left of a directrix

Printed page 50 · Ellipses / Foci and directrices / Eccentricity

Let $F=(0,0)$ be the focal point and λ (*Greek* "lambda") be the directrix $x=5$. Plot a point P so that the distance from P to F is two-thirds the distance from P to λ. The configuration of all such points P forms an ellipse. Find an equation for this curve, and make an accurate sketch of it, labeling key points (the vertices and the other focus) with their coordinates. Notice that the value of the eccentricity c/a for this ellipse is $2/3$, which equals the distance ratio used to draw the curve. It always works out this way (which can be proved as a supplementary exercise), thus there are two ways to think about eccentricity.

1. Set up the two distance measures

From P=(x,y) to F=(0,0) the distance is $\sqrt{(x^2+y^2)}$. Distance to the vertical line λ is abs(x−5). Multiply the stated ratio by 3 and square both nonnegative sides.

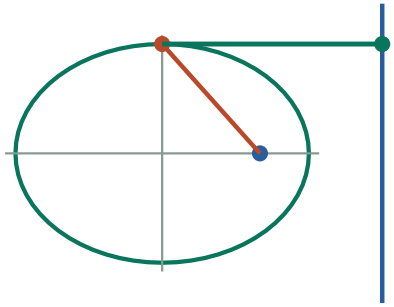


Compare focal and line distances

From a moving point measure to the origin and horizontally to x=5.

2. Derive the translated ellipse

Set $9(x^2+y^2)=4(x-5)^2$. Completing the square yields $(x+4)^2/36+y^2/20=1$.

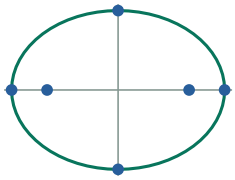


Maintain the two-thirds ratio

Each ellipse point has its focal distance equal to 2/3 of its perpendicular line distance.

3. Complete the square by hand

Expanding gives $5x^2+40x+9y^2=100$. Add 80 to both sides to make $5(x+4)^2+9y^2=180$. The center is four units left of the origin.



Complete the square around $x=-4$

The horizontal semilength is 6 and the vertical one is $\sqrt{20}$.

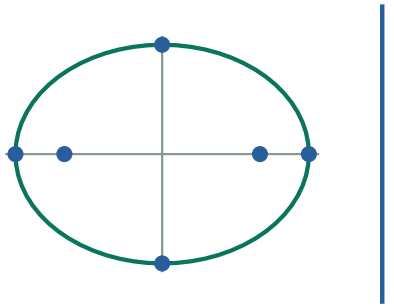
1. $9x^2+9y^2=4x^2-40x+100 \Rightarrow 5(x^2+8x)+9y^2=100$
Expand the directrix term and collect powers of x.

2. $x^2+8x=(x+4)^2-16 \Rightarrow 5(x+4)^2+9y^2=180$
Move the subtracted 80 across the equals sign.

3. $(x+4)^2/36+y^2/20=1$
Divide the completed-square equation by 180.

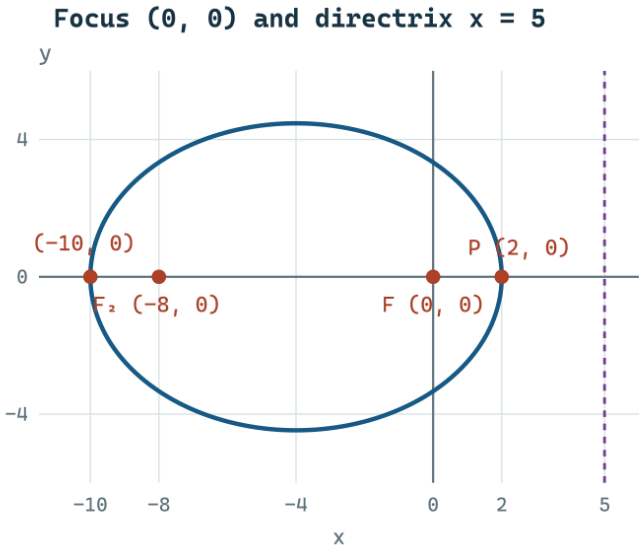
4. Mark all the landmarks

The vertices are $(-10,0),(2,0)$, minor endpoints $(-4,\pm\sqrt{20})$, other focus $(-8,0)$. Plot from these landmarks.



Mark the requested landmarks

The major vertices are $(-10,0),(2,0)$; the other focus is $(-8,0)$.



Check. At (2,0) the distances are 2 and 3. Here a=6,c=4, so c/a=2/3.

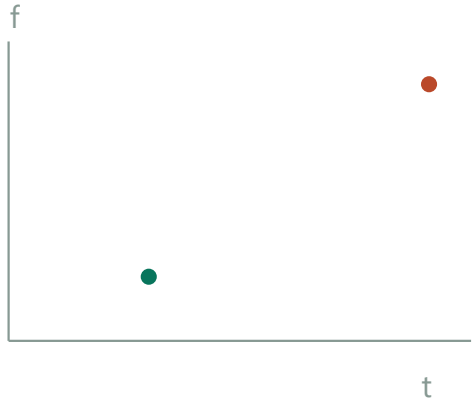
Exercise 481 · Fit two nonzero times

Printed page 50 · Exponential functions / Systems of equations

An exponential function f is defined by $f(t) = k \cdot b^t$. Its graph $y = f(t)$ contains the points $(1, 6)$ and $(3, 24)$. Find the constants k and b.

1. Read the change over two input units

From t=1 to t=3 the output changes from 6 to 24. In an exponential model, consecutive unit increases each multiply the output by the same b. Over two steps the ratio must equal b^2.

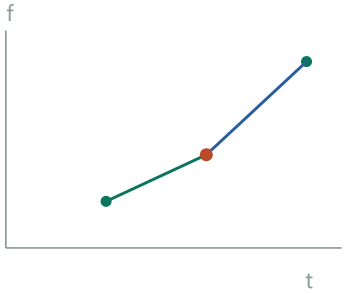


Mark the two observed points

From t=1 to t=3 the output grows from 6 to 24.

2. Remove the unknown initial factor

Dividing $kb^3=24$ by $kb=6$ gives $b^2=4$. Choose the positive base $b=2$.



Read the two-step multiplier

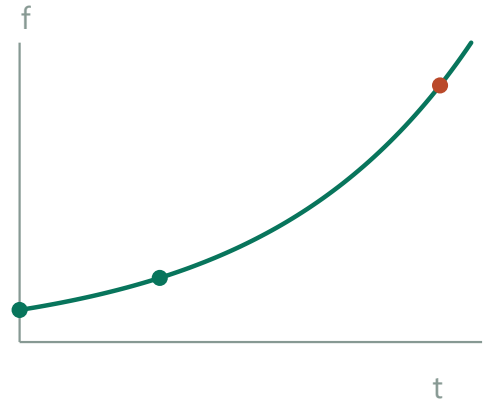
Moving two equal input units multiplies the output by 4, so the positive one-step factor is 2.

1. $kb^3/(kb)=24/6 \Rightarrow b^2=4$
The common factor kb equals 6, so cancellation is valid.

2. $b=2$; $k=6/2=3$
Discard the negative square root for a real exponential base, then use the first point.

3. Recover the intercept

Then $k=3$. Once b is known, dividing the value 6 at $t=1$ by one growth factor gives the value at $t=0$.

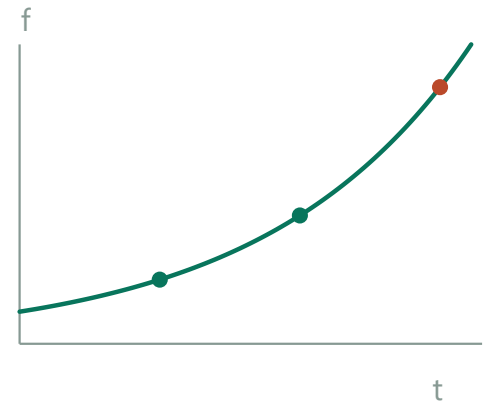


Step backward to the intercept

Halving 6 gives $k=3$ at $t=0$.

4. Check the intermediate visit

At $t=2$ the model gives $3\cdot2^2=12$. One doubling from 6 reaches 12, and one more reaches the given 24, as the two-unit separation requires.



Check the unmeasured midpoint

At $t=2$ the model gives 12, halfway between the known times in number of doublings.

Check. At $t=3$ the model $3\cdot2^3$ gives 24.

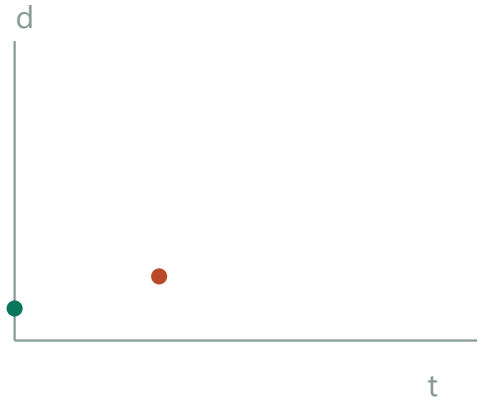
Exercise 482 · Doubling as percentage growth

Printed page 50 · Exponential growth and decay / Percentage change

You have come to associate a function such as $p(t) = 450(1.08)^t$ with the size of something that is growing (exponentially) at a fixed rate. Could such an interpretation be made for the function $d(t) = 450 \cdot 2^t$? Explain.

1. Match a multiplier to a percentage

A growth rate r as a decimal produces the factor 1+r. In 450(1.08)ᵗ the 0.08 beyond 1 represents an 8% gain. Compare the factor in the second expression the same way.

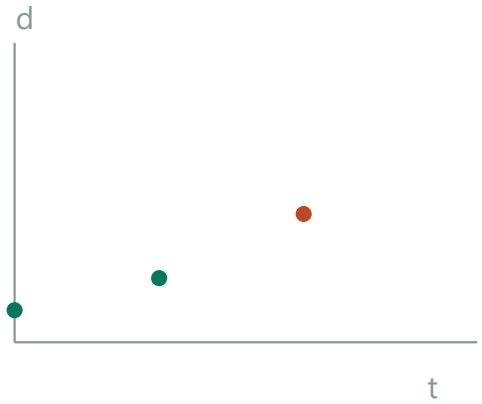


Interpret the leading coefficient

At t=0 there are 450 units; a factor of 2 gives 900 one period later.

2. Follow two time steps

At t=0 the value is 450, at t=1 it is 900 and at t=2 it is 1800. The amounts added differ but each is the entire previous amount.

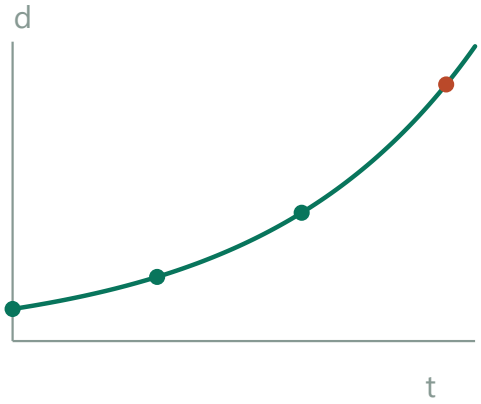


Add the first new amount again

The increment doubles as the current amount doubles.

3. State the fixed proportional rate

The initial amount is 450 and the one-period multiplier is 2=1+1. Thus the rate is 100% per unit time.



Read a proportional growth rate

Adding the entire current amount corresponds to 100% growth each period.

Check. The increase between periods is always equal to the amount at the start of that period.

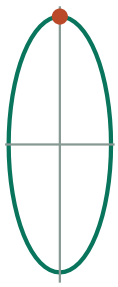
Exercise 483 · Clockwise around an ellipse

Printed page 50 · Ellipses / Parametric equations / Rates of change

An object is moving clockwise along the elliptical path $25x^2 + 4y^2 = 100$, making a complete tour every 20 seconds. The object is at $(0, 5)$ when $t = 0$.
(a) Write a parametric description of this motion, consistent with the given details.
(b) Do your equations describe an object that is moving with a constant speed? Explain.

1. Get the scale and period from the data

Dividing the ellipse equation by 100 gives $x^2/4+y^2/25=1$. The horizontal and vertical semiaxes are 2 and 5. A full 2π parameter change over 20 s gives π/10 radians per second.

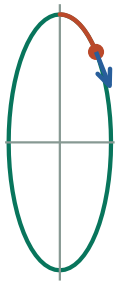


Start on the upper vertex

The ellipse has vertical semiaxis 5, horizontal semiaxis 2 and a 20-second tour.

2. Choose the start and direction

Set $\theta=\pi t/10$, $x=2\sin\theta$, $y=5\cos\theta$. At zero the point is (0,5) and moves to the right, hence clockwise.

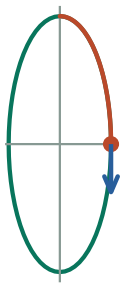


Move clockwise from the top

Use $x=2\sin(\pi t/10)$, $y=5\cos(\pi t/10)$; the initial tangent points right.

3. Check the path equation

Substitution gives $100(\sin^2\theta+\cos^2\theta)$. Substituting x and y into $25x^2+4y^2$ gives this expression. The trigonometric identity makes it exactly 100 at every time.

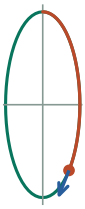


Check a quarter-turn

At five seconds the point reaches (2,0), still on $25x^2+4y^2=100$.

4. Test the speed

Differentiate the coordinates to get $x'=2(\pi/10)\cos\theta$ and $y'=-5(\pi/10)\sin\theta$. The speed is the length of this velocity vector, so the unequal coefficients allow it to change around the path.



Compare tangent velocities

The parameter turns at a fixed rate, but the coordinate velocities involve different semiaxis factors.

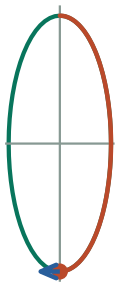
1. $x'=(\pi/10)\cdot2\cos\theta$; $y'=-(\pi/10)\cdot5\sin\theta$
Differentiate each coordinate, including $\theta'=\pi/10$.

2. $\text{speed}^2=(\pi/10)^2(4\cos^2\theta+25\sin^2\theta)$
Add the squares of the velocity components.

3. $4\cos^2\theta+25\sin^2\theta=4+21\sin^2\theta$
This ranges from 4 to 25, giving the speed extremes after taking square roots.

5. Compare the speeds around one tour

Speed is $(\pi/10)\sqrt{(4\cos^2\theta+25\sin^2\theta)}$, varying from $2\pi/10$ to $5\pi/10$. Angular parameter advances uniformly, although arc length does not.



Compare an upper/lower tip to a side tip

At a top or bottom tip the speed is $2\pi/10$. At either side tip it is $5\pi/10$.

Check. At 5 seconds the point is (2,0) and its speed is 2.5 times the speed at the top.

Exercise 484 · A falling ball's speed

Printed page 51 · Rates of change / Slopes and tangent lines

The table shown at right contains the results of an experiment in which a golf ball is dropped from the roof of the Library. For each of the displayed times, the corresponding height of the ball above the ground is given. Use this data to answer the following questions:

(a) What was the *average speed* of the ball during the first second?

(b) What was the average speed of the ball during the half-second interval from $t = 0.5$ to $t = 1.0$?

(c) What was the average speed of the ball during each of the short time intervals from $t = 0.9$ to $t = 1.0$ and from $t = 1.0$ to $t = 1.1$?

(d) Find an approximation for the speed of the ball when $t = 1.0$.

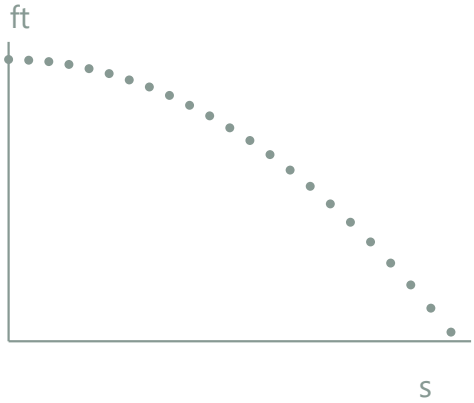
(e) Find an approximation for the speed of the ball when $t = 1.8$.

(f) Find an approximation for the speed of the ball when $t = 2.2$.

t sec	h ft
0.0	80.0
0.1	79.8
0.2	79.4
0.3	78.6
0.4	77.4
0.5	76.0
0.6	74.2
0.7	72.2
0.8	69.8
0.9	67.0
1.0	64.0
1.1	60.6
1.2	57.0
1.3	53.0
1.4	48.6
1.5	44.0
1.6	39.0
1.7	33.8
1.8	28.2
1.9	22.2
2.0	16.0
2.1	9.4
2.2	2.6

1. Choose signed height or positive falling speed

The table's height decreases, so use earlier height minus later height for the positive distance fallen. Divide by elapsed time for the average speed in ft/s.

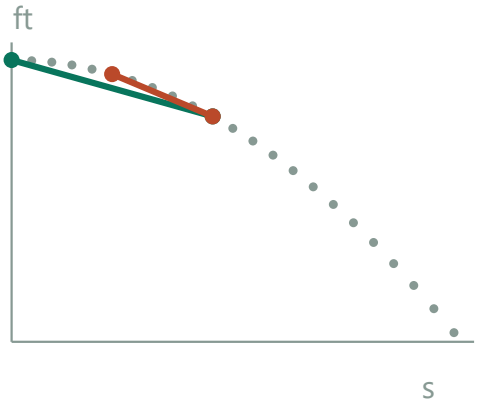


Plot height against elapsed time

Every new measurement has a lower height. Positive falling speed is the negative of this graph's slope.

2. Compute the two longer-interval means

Using positive distance fallen divided by time: (a) (80−64)/1=16 ft/s. (b) (76−64)/.5=24 ft/s.

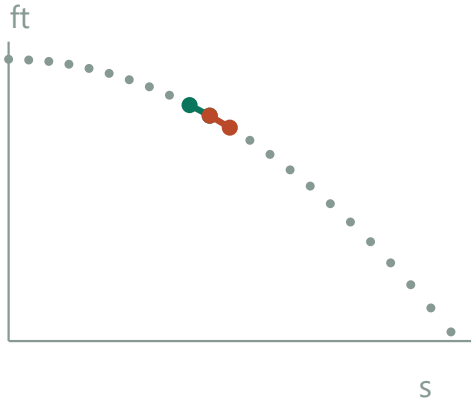


Compare the two longer secants

From 0 to 1 s the height falls by 16 ft. From 0.5 to 1 s the average speed is 24 ft/s.

3. Bracket the instantaneous speed at 1 s

(c) Left and right 0.1 s intervals give 30 and 34 ft/s. (d) Center their slopes to estimate 32 ft/s at 1 s.



Bracket a slope at 1 second

The two adjacent secants give positive falling speeds 30 and 34 ft/s, suggesting approximately 32 ft/s.

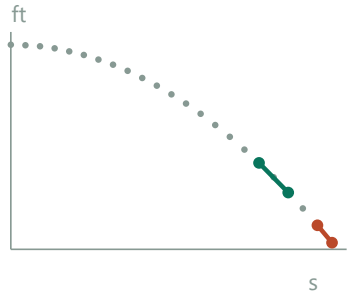
t sec	h ft
0.0	80.0
0.1	79.8
0.2	79.4
0.3	78.6
0.4	77.4
0.5	76.0
0.6	74.2
0.7	72.2
0.8	69.8
0.9	67.0
1.0	64.0
1.1	60.6
1.2	57.0
1.3	53.0
1.4	48.6
1.5	44.0
1.6	39.0
1.7	33.8
1.8	28.2
1.9	22.2
2.0	16.0
2.1	9.4
2.2	2.6

The rows bracketing one second

Three adjacent rows give two short intervals ending or starting at one second. Compare their falls before estimating the instantaneous rate.

4. Estimate at 1.8 s and at the endpoint

(e) At 1.8 s use (33.8−22.2)/.2=58 ft/s. (f) At 2.2 s the endpoint backward slope is (9.4−2.6)/.1=68 ft/s; allowing for increasing speed suggests about 70 ft/s at the instant.



Use the available neighboring measurements

A centered secant near 1.8 s gives 58 ft/s. The final 0.1 s gives a backward estimate of 68 ft/s.

1. (33.8−22.2)/(1.9−1.7)=11.6/0.2=58 ft/s
The times straddle 1.8 s. Divide the height lost by the width of that interval to estimate falling speed.

2. (9.4−2.6)/0.1=6.8/0.1=68 ft/s
At the last tabulated time only a backward interval is available. The result estimates speed over that last interval.

t sec	h ft
0.0	80.0
0.1	79.8
0.2	79.4
0.3	78.6
0.4	77.4
0.5	76.0
0.6	74.2
0.7	72.2
0.8	69.8
0.9	67.0
1.0	64.0
1.1	60.6
1.2	57.0
1.3	53.0
1.4	48.6
1.5	44.0
1.6	39.0
1.7	33.8
1.8	28.2
1.9	22.2
2.0	16.0
2.1	9.4
2.2	2.6

The final two rows

The table ends at 2.2 seconds, so the last short interval available for a slope approaches this time from the left.

Check. The table is consistent with $h \approx 80 - 16t^2$ after rounding; its derivative predicts speeds 32,57.6,70.4 ft/s.

Exercise 485 · Combine known powers of ten

Printed page 51 · Logarithm rules / Exponent rules

Given $10^{0.301} = 2$ and $10^{0.477} = 3$, solve without a calculator:
(a) $10^x = 6$; **(b)** $10^x = 8$; **(c)** $10^x = 2/3$; **(d)** $10^x = 1$.

1. Read the two given powers

The exercise gives 10^0.301≈2 and 10^0.477≈3. These short decimal exponents approximate the actual logarithms. We can use the rules for multiplying or dividing powers of the same base to obtain the requested decimal exponents without computing new logarithms.



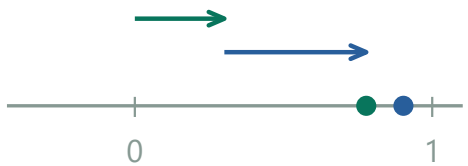
exponent

Register the two known exponents

The two positions are approximately 0.301 and 0.477, from the stated powers of ten.

2. Multiply to build 6 and 8

For (a), 6=2·3≈10^0.301·10^0.477=10^(0.301+0.477)=10^0.778. Thus x≈0.778. For (b), 8=2·2·2≈(10^0.301)^3=10^(3·0.301)=10^0.903, giving x≈0.903.



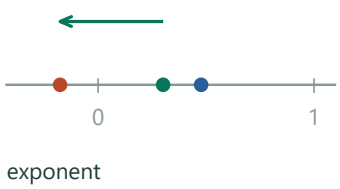
exponent

Walk along the exponent axis

Adding the two exponents produces 6. Three steps of size log 2 produce 8.

3. Divide to build two-thirds

For (c), 2/3≈10^0.301/10^0.477=10^(0.301−0.477)=10^(−0.176). Thus x≈−0.176. The sign agrees with 0<2/3<1, since a negative power of 10 is a reciprocal.



Reverse a step for division

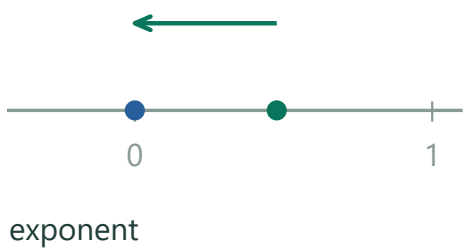
Starting at log 2 and moving left by log 3 lands below zero, as required for 2/3.

1. 2/3≈10^0.301/10^0.477=10^(0.301−0.477)
Division of powers with a shared base subtracts their exponents.

2. 0.301−0.477=−0.176
The negative sign agrees with the ratio lying between zero and one.

4. Identify the exact exponent for one

For (d), $10^0=1$, so x=0 exactly. You can also divide any power of 10 by itself: subtracting two equal exponents gives zero. Only the first three numerical answers depend on the rounded inputs.



Return to the zero exponent

Divide a power by itself to walk back to zero. The resulting number is exactly one.

Check. The sign for the quotient is negative because 2/3 lies below 1.

Exercise 486 · Combine known logarithms

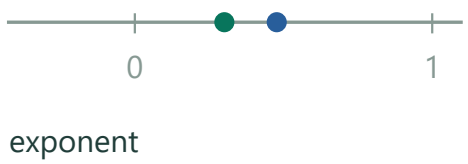
Printed page 51 · Logarithm rules

Given that $0.301 = \log 2$ and that $0.477 = \log 3$, you should not need a calculator to evaluate

(a) $\log 6$; **(b)** $\log 8$; **(c)** $\log(2/3)$; **(d)** $\log 1$.

1. Translate logarithms into powers

Let A=log 2 and B=log 3. Then $10^A=2$ and $10^B=3$. Their supplied values A≈0.301 and B≈0.477 let you calculate new base-10 logarithms by manipulating exponents.



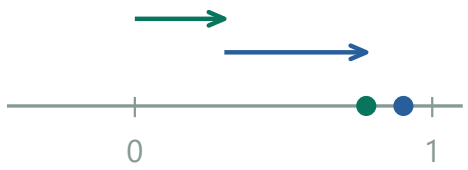
exponent

Locate A and B

The positions A and B denote log 2 and log 3, preserving the full-precision quantities.

2. Use the product and power rules

For (a), 6=2·3=10^(A+B), so log 6=A+B≈0.301+0.477=0.778. For (b), $8=2^3=10^{(3A)}$, so log 8=3A≈3·0.301=0.903. Multiplying an argument by another adds their logarithms, while cubing an argument triples its logarithm.



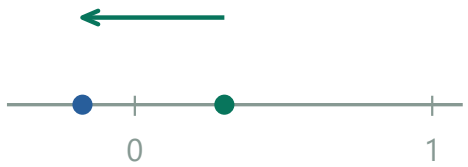
exponent

Add for log 6; triple for log 8

The two end positions are A+B and 3A, approximately 0.778 and 0.903.

3. Use the quotient rule

For (c), $2/3=10^A/10^B=10^{(A-B)}$, giving log(2/3)=A−B≈0.301−0.477=−0.176. The negative answer means it takes a reciprocal power of ten to obtain the fraction.



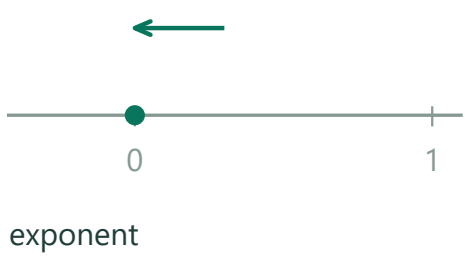
exponent

Subtract B for the quotient

Dividing by 3 moves the exponent left from A to A−B.

4. Evaluate the remaining exact power

For (d), log 1=0 because $10^0=1$. None of the rounded data are needed here. The four results also check the preceding exercise: a logarithm is precisely the unknown exponent in its corresponding exponential equation.



Place log one at its exact position

Every positive base other than one gives one at exponent zero.

Check. These are exactly the exponents sought by the preceding exercise, to the stated precision.

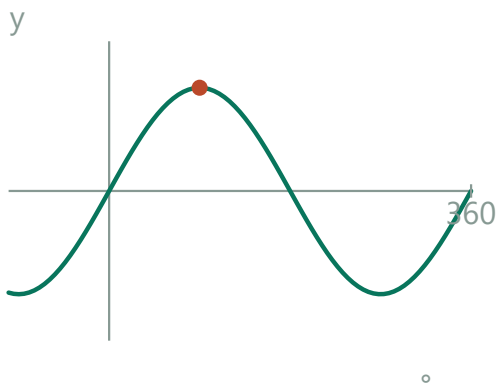
Exercise 487 · Shift a sine

Printed page 51 · Sinusoidal functions / Phase shifts / Trigonometric identities

What is the relationship between the graph of $y = \sin x$ and the graph of $y = \sin(x + 90)$? Work in degree mode.

1. Track when an old feature occurs

For y=sin(x+90°), the old sine crest at input 90° happens when x+90°=90°, namely x=0°. All the original points undergo the same horizontal translation.

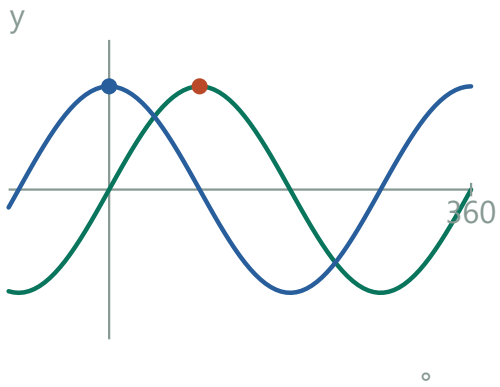


Locate a familiar sine crest

The original sine wave peaks at x=90°.

2. Recognize the phase relationship

At x=0° the new graph starts at 1 and at x=90° it reaches zero. That is the phase of cosine. The addition formula proves the match at every x.

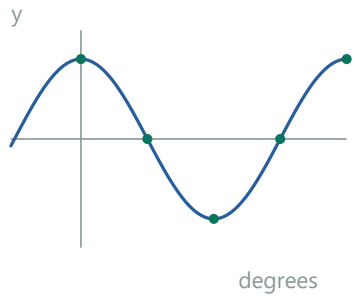


Move the crest left by 90°

Replacing x by x+90° reaches the original crest when x=0.

3. State the shift and identity

The new graph shifts 90° left. It equals cos x by the sine addition formula.



Compare the shifted graph with cosine

The shifted wave begins at its maximum, matching cosine.

1. sin(x+90°)=sin x cos 90°+cos x sin 90°
Use the addition identity on the shifted argument.

2. =0·sin x+1·cos x=cos x
The cosine-height curve is therefore the shifted sine.

Check. At x=0 its value is sin 90°=1, a cosine crest.

Exercise 488 · Prove the product law

Printed page 51 · Logarithm rules / Exponent rules

Given that $m = \log a$, $n = \log b$, and $k = \log(ab)$,
(a) express a, b, and ab as powers of 10;
(b) use your knowledge of exponents to discover a relationship among m, n, and k;
(c) conclude that $\log(ab) = \log a + \log b$.

1. (a) Rewrite the logarithms as powers

Throughout, log denotes base ten. The original log a and log b require a > 0 and b > 0. Their product is then positive as well.

1 **$m = \log a \iff a = 10^m$**
The logarithm is the exponent that produces a.

2 **$n = \log b \iff b = 10^n$**
Apply the same definition to the second positive input.

3 **$k = \log(ab) \iff ab = 10^k$**
Name the exponent that produces the product.

2. (b) Find the relation among the exponents

Compute ab again using the first two exponential forms. Compare that result with the third form.

1 **$ab = (10^m)(10^n)$**
Replace a and b by their equivalent powers.

2 **ab = 10^(m + n)**
Multiplying powers with the same base adds the exponents.

3 **$10^k = 10^{(m+n)} \Rightarrow k = m + n$**
Since powers of ten are strictly increasing, their equal outputs have equal exponents.

3. (c) Translate the exponent relation back

The proof covers every pair of positive real inputs, including numbers less than one. Their logarithms may be negative.

1 **$k = m + n$**
Use the relation established by multiplying the powers.

2 **$\log(ab) = \log a + \log b,\ a > 0, b > 0$**
Substitute the original meanings of k, m and n.

Check. Exponentiating both sides returns ab.

Exercise 489 · Power and quotient laws

Printed page 51 · Logarithm rules / Exponent rules

(Continuation) Justify the rules: **(a)** $\log(a^r) = r\log a$ **(b)** $\log(a/b) = \log a - \log b$

1. (a) Express the positive base as a power of ten

The right side contains log a, so a must be positive. For any real r, a positive base has a positive real power a^r.

1 **u = log a ⇔ a = 10ᵘ**
Name the unique base-ten exponent for a.

2 **aʳ = (10ᵘ)ʳ = 10^(ur)**
In a power of a power with positive base, multiply the exponents.

3 **log(aʳ) = ur = r log a**
The log of 10^(ur) is its exponent. Substitute u = log a.

2. (b) Divide two powers of the same base

The proposed right side contains both log a and log b. Thus each input must be positive. In particular b is nonzero and a/b is positive.

1 **u = log a, v = log b**
Write the two positive inputs as 10^u and 10^v.

2 **a/b = 10ᵘ/10ᵛ = 10^(u − v)**
Dividing powers of the same base subtracts their exponents.

3 **log(a/b) = u − v = log a − log b**
Apply log to recover the exponent and replace u and v.

Check. Raising the base to either resulting exponent gives the original argument.

Exercise 490 · Daily and hourly mold growth

Printed page 51 · Exponential growth and decay / Percentage change

The function F defined by $F(x) = 31416(1.24)^x$ describes the number of mold spores found growing on a pumpkin pie x days after the mold was discovered.
(a) How many spores were on the pie when the mold was first discovered?
(b) How many spores were on the pie two days before the mold was discovered?
(c) What is the daily rate of growth of this population?
(d) What is the hourly rate of growth?
(e) Let $G(x)$ be the spore count on the same pie, x hours after the mold was discovered. Write a description of the function G.

1. Read the time origin and model a past value

At discovery there are 31,416 modeled spores; two days earlier F(−2)≈20,432 spores.

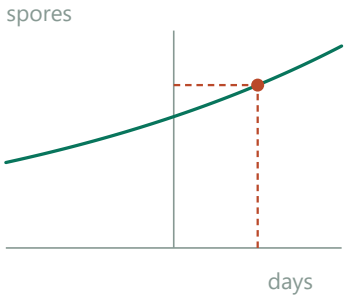


Move two days before discovery

Day zero is 31416 modeled spores. Two days earlier the curve gives roughly 20432.

2. Convert the daily multiplier to an hourly one

The daily increase is 24%. The hourly factor is 1.24^(1/24), giving about 0.9003% hourly growth.



Compare one full day

One daily step multiplies by 1.24. The hourly step must repeat 24 times to match it.

1. $g^{24}=1.24 \Rightarrow g=1.24^{(1/24)}$
The same positive hourly factor must compound to the daily factor.

2. $100(g-1)\approx 0.9003$
Remove the initial unit and express the gain per hour as a percentage.

3. Explain the root choice

Let g be the hourly multiplier. Twenty-four equal steps constitute one day, giving $g^{24}=1.24$. Taking the positive root produces the factor; subtract 1 to turn it into an hourly rate.

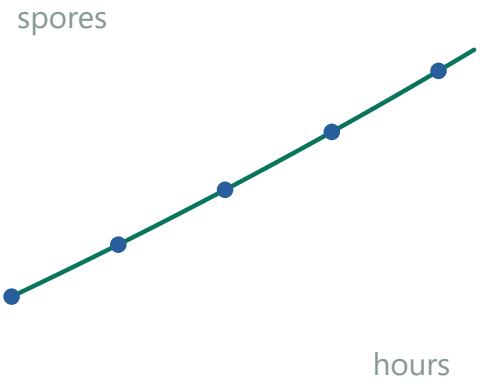


Divide the day into 24 factors

The 25 dots mark the start and ends of 24 one-hour intervals. The hourly multiplier is 1.24^(1/24).

4. Write the hourly-input function

With hours x, G(x)=31416(1.24)^(x/24). An elapsed x hours is x/24 days. Equivalently, apply the factor 1.24^(1/24) once per hour for x hours to obtain the same model.



Use hours as the input

At 24 hours the total matches one day. The exponent h/24 expresses elapsed time in days.

Check. At x=24, G(24)=F(1), and 24 hourly factors multiply to 1.24.

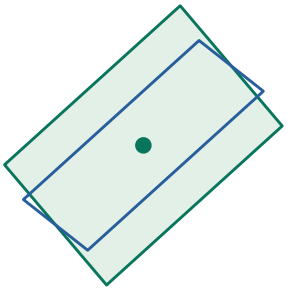
Exercise 491 · A second shared point

Printed page 51 · Intersections of planes / Parametric lines

The planes $x+3y+2z=3$ and $4x+y-2z=-5$ have the point $(2,-3,5)$ in common. Justify this statement. The challenge is now to find coordinates for another point that lies on both planes. Hint: Because there are infinitely many correct answers to this question, you will have to make an arbitrary choice during the solution process. For instance, you could start by assigning one of the unknowns a value, and then use the given equations to solve for the other two unknowns.

1. Check the common point

At (2,−3,5) the left sides are 2−9+10=3 and 8−3−10=−5.

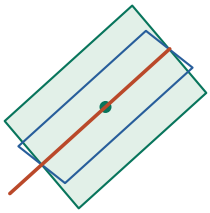


Start at the given common point

Both planes contain P=(2,−3,5).

2. Construct a shared direction

We need a displacement (u,v,w) satisfying 1u+3v+2w=0 and 4u+v−2w=0. Adding gives 5u+4v=0. Choosing u=−8 and v=10 makes w=−11 in the first equation.



Move along both surfaces

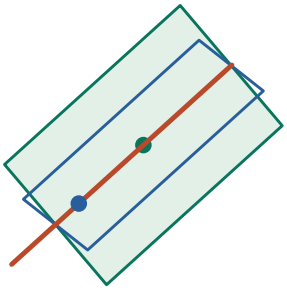
The displacement (−8,10,−11) is perpendicular to both coefficient vectors.

1. d=(3(−2)−2·1, 2·4−1(−2), 1·1−3·4)
Cross the two plane normals (1,3,2) and (4,1,−2).

2. d=(−8,10,−11)
Its dot products are −8+30−22=0 and −32+10+22=0.

3. Move along the line

A direction perpendicular to both normals is (−8,10,−11). Add this direction to get (−6,7,−6).



Mark a second point

P plus that displacement is (−6,7,−6), also on each plane.

Check. Substitution gives −6+21−12=3 and −24+7+12=−5.

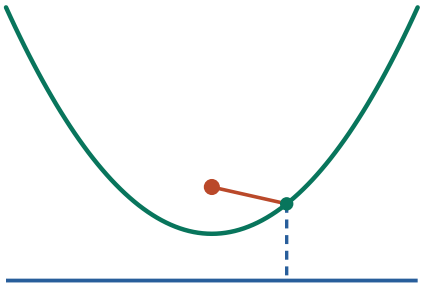
Exercise 492 · Nearest point on a parabola

Printed page 52 · Parabolas / Foci and directrices / Distance and loci

Of all the points on a parabola, which one is closest to the focus? How do you know?

1. Choose a coordinate model

Place the vertex at (0,0), focus at (0,p) and directrix $y=-p$, where $p>0$. Every point of the parabola satisfies $y=x^2/(4p)$, so its height is nonnegative.

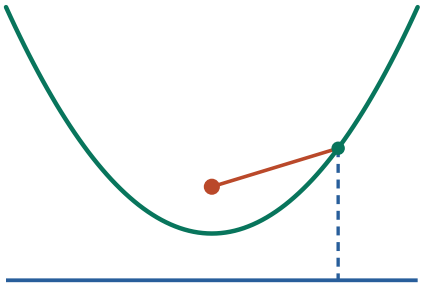


Place the focus and directrix

The vertex (0,0) is halfway between the focus (0,p) and line $y=-p$.

2. Translate distance to the directrix

For any such point the perpendicular distance to the horizontal directrix is $y+p$. This reaches its smallest value p precisely when $y=0$, which forces $x=0$.

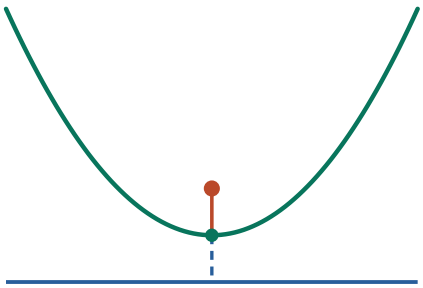


Read vertical distance to the line

At a point of height y the directrix distance is $y+p$, equal to its focal distance.

3. Identify the closest point by the focus property

At every parabola point, focus distance equals perpendicular distance to the directrix. The vertex is closest to the directrix, halfway along the perpendicular between focus and directrix, so it also minimizes focal distance.



Move to the vertex

The smallest possible y is zero, so the minimum distance p occurs uniquely at the vertex.

Check. For $y=x^2/(4p)$, $p>0$, the distance is $y+p\ge p$, with equality only at the vertex.

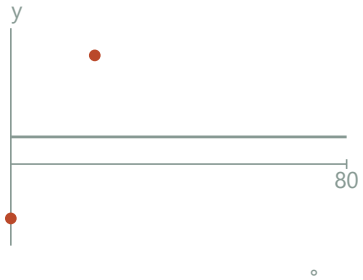
Exercise 493 · Build a cosine graph

Printed page 52 · Sinusoidal functions / Amplitude and period

Write an equation of the form $y = k + a\cos mx$ for a curve that has a maximum point at $(20, 4)$ and that has -2 as its y-intercept. Use a graphing tool to check that your equation fits the given description. When you compare your answers with your neighbor, is it expected that you will both have found the same equation?

1. Force the initial height and maximum

The initial cosine value is 1. Choosing it to occur at the trough makes −2 the minimum. The midpoint of heights −2 and 4 is 1 and their half-difference is 3. A negative coefficient supplies the trough.



Locate initial trough and desired crest

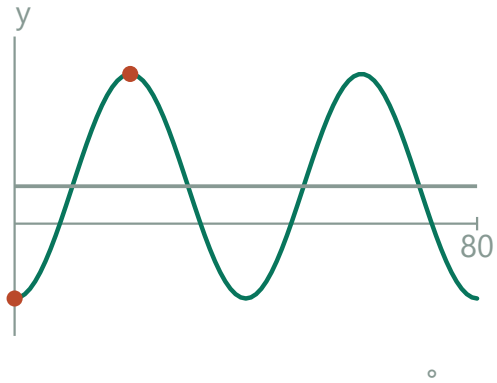
A centerline at 1 and signed amplitude −3 begin at −2 and allow a high point of 4.

1. k+a=−2; k−a=4
The cosine coefficient is negative, so x=0 is the trough and the crest has cos=−1.

2. 2k=2 ⇒ k=1; a=−2−1=−3
Add the equations, then replace k in the first equation.

2. Choose a frequency satisfying both points

Choose k=1,a=−3, making the initial value −2 and maximum 4. In degree mode let m=9: at x=20, mx=180°, so y=1−3cos(9x°)=4.

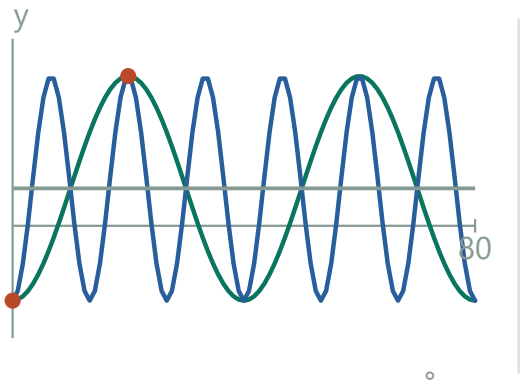


Reach the crest after 20 units

Choosing m=9 turns the inside angle to 180° when x=20.

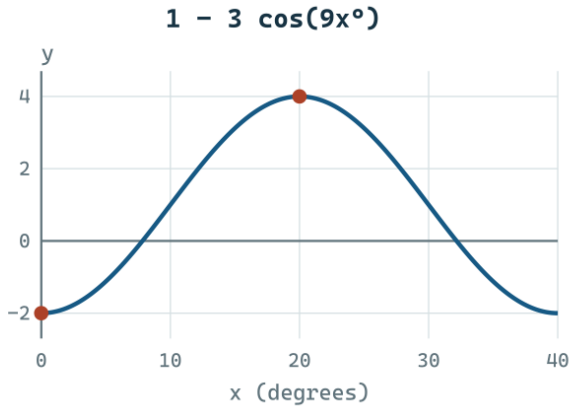
3. Build a second working model

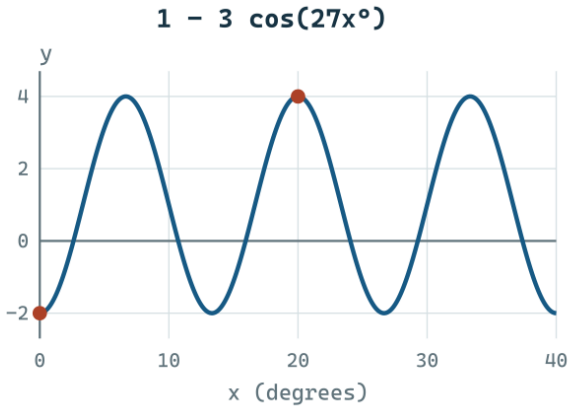
Other choices include m=27 since 540° also has cosine −1.



Allow extra cycles

Choosing m=27 gives 540° at x=20, also a crest. Both models match the supplied points.





Check. The distinct periods 40 and 40/3 demonstrate nonuniqueness.

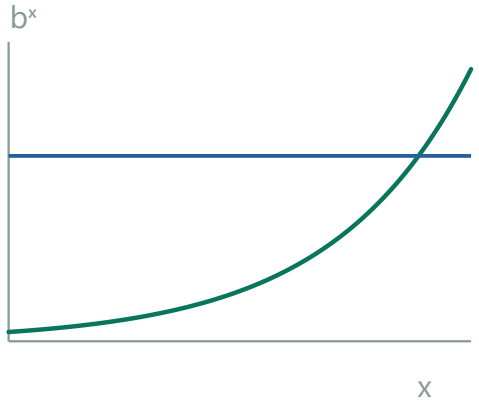
Exercise 494 · Solve 5^x=20

Printed page 52 · Logarithms / Change of base

Another approach to solving an equation like $5^x = 20$ is to calculate base-10 logarithms of both sides of the equation. Justify the equation $x \log 5 = \log 20$, then obtain the desired answer in the form $x = \dfrac{\log 20}{\log 5}$. Evaluate this expression. Notice that $\log_5 20 = \dfrac{\log 20}{\log 5}$.

1. Apply the same function to both sides

Both sides of $5^x=20$ are positive, so their base-10 logarithms exist. Because $\log_{10}$ is one-to-one this transformation preserves the solution.

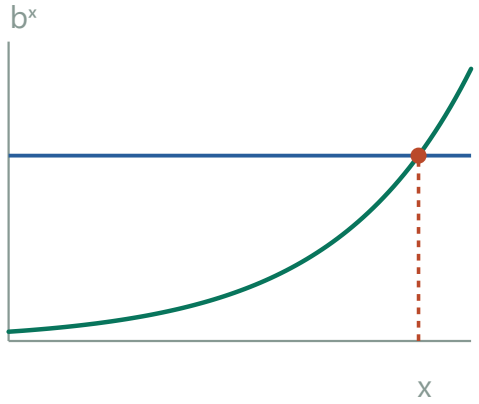


Set the target on the output axis

The curve 5^x reaches 20 between its first and second integer powers.

2. Bring the unknown exponent down

The logarithm power law gives $\log(5^x)=x\log 5$. Now the equation has x as a simple factor, which can be isolated by division.

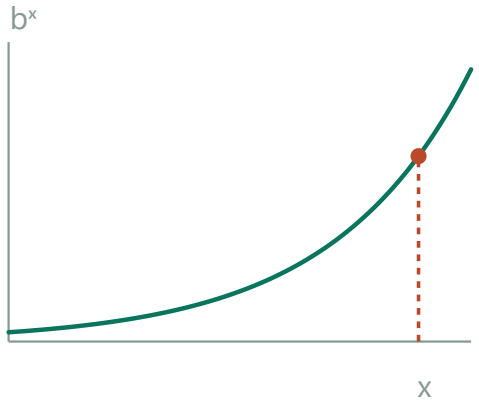


Identify the needed horizontal input

Applying base-ten log preserves equality and yields $x\log 5=\log 20$.

3. Evaluate the log quotient

The power rule gives $x\log 5=\log 20$, so $x=\log 20/\log 5\approx 1.861353$.



Read its logarithmic coordinate

Read $x=\log 20/\log 5\approx 1.86135$.

Check. It lies between 1 and 2 because $5<20<25$.

Exercise 495 · Change of base

Printed page 52 · Change of base

Write an expression for $\log_a N$ that refers only to base-10 logarithms, and explain.

1. Name the exponent in the unfamiliar base

A real logarithm requires N > 0 and a > 0 with a ≠ 1. Positive-base powers stay positive. Base 1 produces only 1 and cannot be inverted uniquely.

1 **$x = \log_a N$**
Name the real exponent sought.

2 **$a^x = N$**
By definition, x is the exponent on a that gives N.

2. Take common logs and isolate x

The notation log in this step means base ten, so a calculator with only common logs can evaluate the result.

1 **$\log(a^x) = \log N$**
Take base-ten logs of the two positive equal quantities.

2 **$x \log a = \log N$**
Use the power rule to move x in front.

3 **$x = \log N / \log a$**
Divide by log a. This is nonzero because $10^0 = 1$ and a ≠ 1.

4 **$\log_a N = \log N / \log a$**
Replace x with the original quantity. This also works for 0 < a < 1.

Check. The denominator is nonzero exactly when a≠1.

Exercise 496 · Division outside a logarithm

Printed page 52 · Logarithm rules / Change of base

Asked to simplify $\dfrac{\log 20}{\log 5}$, Brett replied "$\log 4$." What do you think of this answer?

1. Decode the quotient first

A quotient of logarithms is a change of base: the denominator supplies the new base. Here it asks for the exponent on 5 that makes 20. That exponent must exceed 1.

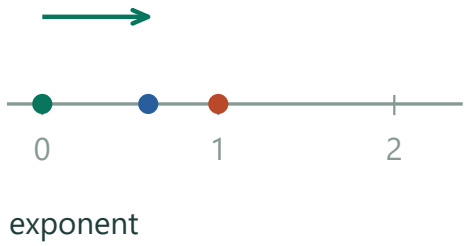


Bracket an exponent on 5

$5^1 < 20 < 5^2$. The exponent for 20 lies between 1 and 2.

2. Compare the claimed result

Since 4 is between 1 and 10, its common logarithm is between 0 and 1. This already contradicts the previous bound. The rule for the log of 20/5 instead involves subtraction.

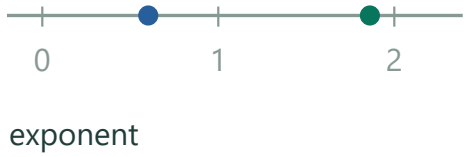


Bracket the common log of 4

As $1<4<10$, the common log of 4 lies strictly between 0 and 1.

3. Correct the two operations

The quotient is log base 5 of $20 \approx 1.86135$. It is the difference $\log 20 - \log 5$ that equals $\log 4 \approx 0.60206$.



Compare the two positions

The quotient of logs is about 1.86135. Subtracting the common logs gives the other point, about 0.60206.

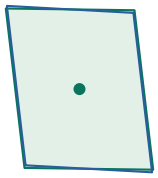
Check. 5 raised to 1.86135 is approximately 20; raising it to .60206 is not.

Exercise 497 · Parametrize a two-plane intersection

The intersection of the planes $3x + 2y + 5z = 22$ and $2x + y + 3z = 13$ is a line, one of whose points is $(1, 2, 3)$. Find another point on this line. Once you have found a second point, write a parametric description of *all* the points on the line.

1. Get a direction lying in both planes

A shared direction must be perpendicular to both plane normals. Cross products give such a vector; here its components come from (2·3−5·1,5·2−3·3,3·1−2·2).



A direction shared by both planes

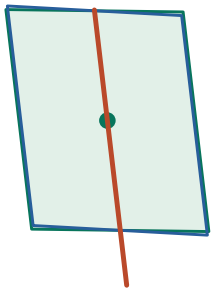
A cross product of the two normals gives (1,1,−1).

1. (3,2,5)×(2,1,3)=(6−5,10−9,3−4)
Use the two plane normals to construct their common perpendicular.

2. d=(1,1,−1)
The dot products with both normals vanish.

2. Use the known point as an anchor

Starting at (1,2,3), add that direction once for a second point. Then multiply it by any real number before adding it to the same anchor to reach every point on the intersection.

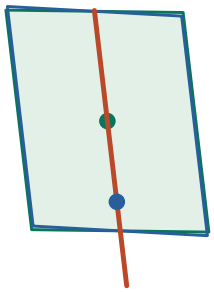


Anchor the common direction

Take P=(1,2,3). The red line passes through P and lies in both planes.

3. Write the second point and line

Their normals (3,2,5),(2,1,3) cross to (1,1,−1). A second point is (2,3,2); the entire line is (1,2,3)+t(1,1,−1).



Mark one full displacement

A unit step of the parameter reaches (2,3,2); varying t traces all intersections.

Check. The varying terms cancel in both plane equations, leaving constants 22 and 13.

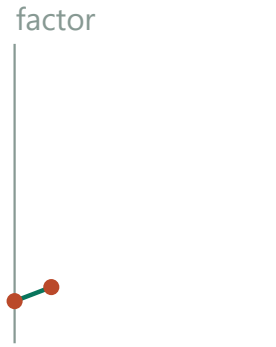
Exercise 498 · Monthly inflation accumulates

Printed page 52 · Compounding / Percentage change

Explain why compounding a monthly inflation rate of 1% is *not* equivalent to an annual inflation rate of 12%.

1. Follow a fixed price across two months

A $100 basket becomes $101 after the first 1% increase. Next month's 1% is $1.01, giving $102.01. Already this exceeds the result of two simple $1 additions.

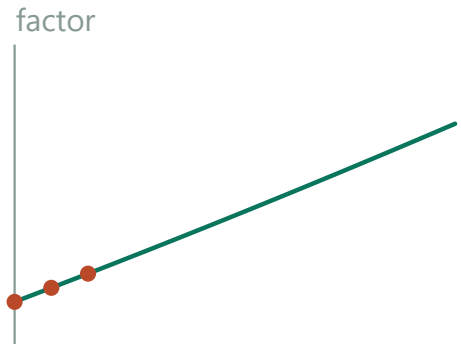


Begin with one monetary unit

The first step increases a unit price to 1.01.

2. Extend the model to twelve months

The multiplicative factor for each month is 1.01. After twelve months subtract 1 from the product of the twelve factors to isolate the fractional increase.

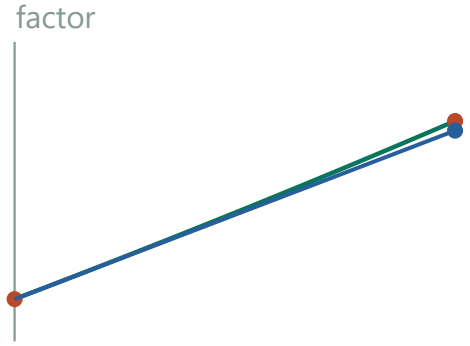


Accumulate equal intervals

A second step uses the increased price as its starting point.

3. Compare with one annual increase

The monthly plan multiplies prices by $1.01^{12} \approx 1.126825$, equivalent to about 12.68% annual inflation. Growth each month applies to the already-increased price.



Compare the equivalent rates

By month 12 compounding gives about 1.126825 per original unit, exceeding the simple 1.12 comparison.

Check. A $100 basket becomes about $112.68, whereas 12% once yields $112.

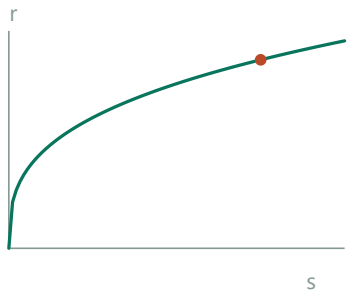
Exercise 499 · An expanding spherical balloon

Printed page 52 · Volume / Rates of change

A spherical balloon is being inflated by a machine. Starting from an initial volume of zero, the balloon's volume increases at a steady 1000 cc per second.
(a) Show that the balloon's radius after 6.0 sec of inflation is approximately 11.273 cm.
(b) What is the balloon's radius after 6.1 sec of inflation?
(c) Find approximately the rate (cm/sec) at which the radius is increasing at that instant when the volume reaches 6000 cc.

1. Relate the clock to the sphere radius

$V = 1000t = 4\pi r^3/3$, so $r = (750t/\pi)^{(1/3)}$. A steady 1000 cc/s for t seconds gives volume 1000t. Equating it with sphere volume, then dividing by $4\pi/3$ and taking a cube root, converts cubic centimeters to a radius in centimeters.



Use volume to find the radius

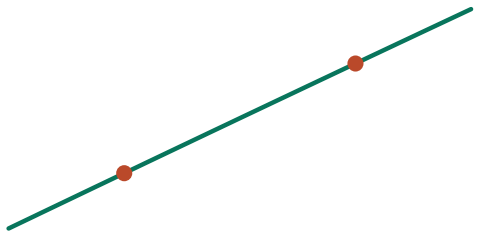
Constant added volume makes $r = (750t/\pi)^{(1/3)}$ increase at a decreasing rate.

1. $1000t = (4\pi/3)r^3 \Rightarrow r^3 = 3000t/(4\pi) = 750t/\pi$
Multiply by 3, then divide by 4π.

2. $r = \sqrt[3]{(750t/\pi)}$
The radius is the positive cube root once inflation begins.

2. Calculate the two nearby radii

At 6 s, r≈11.273 cm, and at 6.1 s r≈11.335 cm.

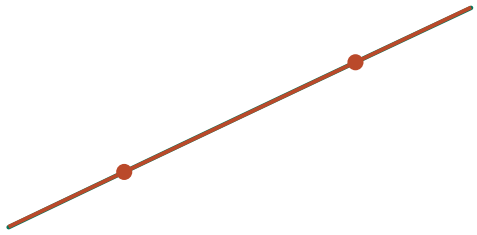


Magnify two nearby times

At 6 and 6.1 s the radii are about 11.273 and 11.335 cm.

3. Approximate from a small time interval

The forward secant gives about 0.62 cm/s. Subtract the two unrounded radii and divide by the elapsed 0.1 s. This is an average radial speed over the short interval beginning at exactly 6000 cc.

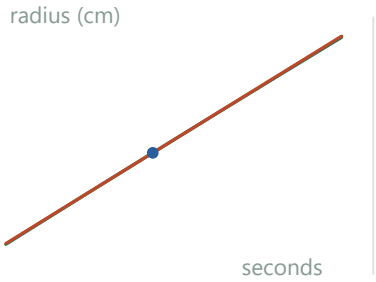


Connect the two radii on the graph

Rise over run for the secant is about 0.62 cm/s.

4. Check the instantaneous rate

Exactly $dr/dt = 1000/(4\pi r^2) \approx 0.6263$ cm/s at 6 s. Differentiating $V = 4\pi r^3/3$ gives $dV/dt = 4\pi r^2 \cdot dr/dt$. An additional thin layer fills the entire surface of area $4\pi r^2$, which explains physically why the radial speed equals incoming volume per second divided by this area.



Read the limiting rate

The instantaneous slope at 6 s equals $1000/(4\pi r^2) \approx 0.6263$ cm/s.

1. $dV/dt = 4\pi r^2\, dr/dt$
Differentiate the sphere volume, treating radius as a function of time.

2. $dr/dt = 1000/(4\pi r^2)$
Use the known volume flow and divide by positive surface area at t=6.

Check. Independently r∝t^(1/3), giving dr/dt=r/(3t).

Exercise 500 · Log rules in a common base

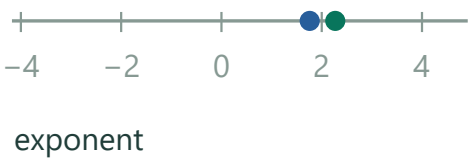
Printed page 52 · Logarithm rules

Given that $\log_c 8 = 2.27$ and $\log_c 5 = 1.76$, a calculator is *not needed* to evaluate

(a) $\log_c 40$ **(b)** $\log_c(5/8)$ **(c)** $\log_c 2$ **(d)** $\log_c(5^m)$ **(e)** $\log_c 0.04$

1. Read the supplied logarithms in their common base

Write U=log_c 8≈2.27 and W=log_c 5≈1.76. This means $c^U=8$ and $c^W=5$. Work in the same positive base c≠1 throughout. The short decimals are used at the precision given.

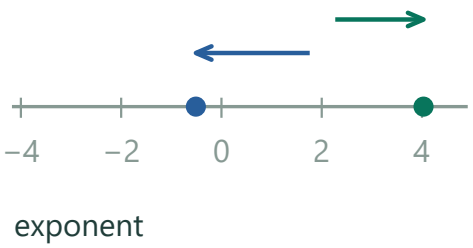


Mark the two supplied locations

The exponents U≈2.27 and W≈1.76 produce 8 and 5 in the same base.

2. Build a product and a quotient

For (a), 40=8·5=c^(U+W), giving log_c 40=U+W≈2.27+1.76=4.03. For (b), 5/8=c^(W−U), giving log_c(5/8)=W−U≈1.76−2.27=−0.51.

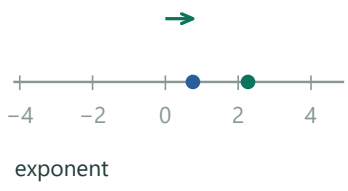


Add and subtract the displacements

The product gives U+W≈4.03. The quotient 5/8 gives W−U≈−0.51.

3. Extract a cube root

For (c), 2=$\sqrt[3]{8}$=(c^U)^(1/3)=c^(U/3). Thus log_c 2=U/3≈2.27/3≈0.7567. Three times the unrounded logarithm would return log_c 8.



Divide the exponent by three

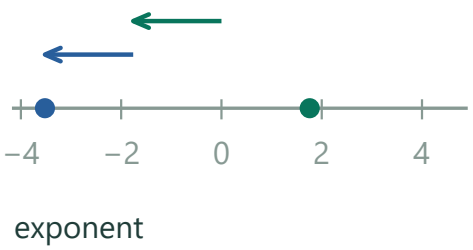
2 is the cube root of 8, so its logarithm is U/3≈0.7567.

1. 2=8^(1/3) ⇒ log_c 2=(1/3)log_c 8
Move the cube-root exponent in front of the logarithm.

2. log_c 2≈2.27/3≈0.7567
The source log is rounded, so the result is approximate.

4. Handle a variable exponent and a decimal

For (d), $5^m=(c^W)^m=c^{(mW)}$, so $\log_c(5^m)=mW\approx1.76m$. For (e), $0.04=4/100=1/25=5^{-2}$, so $\log_c 0.04=-2W\approx-2\cdot1.76=-3.52$. The factor -2 accounts for both the square and the reciprocal.



Move backward twice by W

Because $0.04=5^{-2}$, its logarithm is $-2W\approx-3.52$. In general the exponent of 5^m is mW.

Check. Products add log values, quotients subtract them, and powers multiply them.

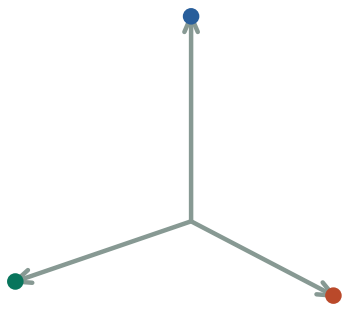
Exercise 501 · A region in the first octant

Printed page 52 · Planes in space / Inequalities

Consider the configuration of all points (x, y, z) for which x, y, and z are nonnegative and $x + y + z = 1$. This configuration is a familiar geometric object — what is it?

1. Locate the three axial extremes

If two coordinates vanish, the remaining one must be 1. This gives the intercepts on the three positive coordinate rays. Allowing one zero and two positive coordinates traces a segment joining two intercepts.

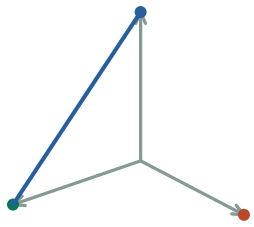


Start with three unit intercepts

Setting two coordinates to zero forces the remaining coordinate to equal 1.

2. Show that the interior is filled

The equation describes a plane, and nonnegativity cuts out the first-octant portion. Any weighted average of the three intercepts with nonnegative weights adding to 1 has coordinates equal to those weights.



Move along an edge

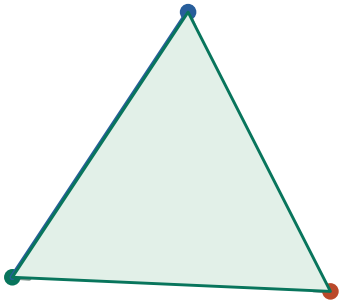
Nonnegative weights sum to 1. If one weight is zero the point lies on the opposite edge.

1. (x,y,z)=x(1,0,0)+y(0,1,0)+z(0,0,1)
Separate the coordinates into contributions from the three intercepts.

2. x+y+z=1; x,y,z≥0
Nonnegative weights summing to one give exactly the filled triangle.

3. Classify the bounded shape

It is the filled equilateral triangle in the plane with vertices (1,0,0),(0,1,0),(0,0,1). Each point is their convex combination with weights x,y,z.



Fill the triangle

Allowing all three weights to be positive fills the interior of this equilateral first-octant triangle.

Check. Each edge has length √2, and the nonnegative weights sum to 1.

Exercise 502 · Focus and directrix of a vertical ellipse

Printed page 52 · Ellipses / Foci and directrices / Eccentricity

Consider the ellipse whose equation is $25x^2 + 9y^2 = 3600$.

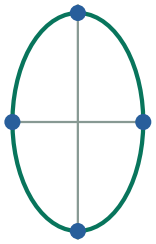
(a) For this ellipse, find a, b, c, and the eccentricity.

(b) Find coordinates for the focal point F that is on the positive y-axis. Also find coordinates for a point P on the ellipse that is not an axis intercept.

(c) Find an equation for the directrix that corresponds to the focus F.

1. Normalize and identify the axes

Divide by 3600: $x^2/144+y^2/400=1$. Thus $a=20, b=12, c=16, e=4/5$.

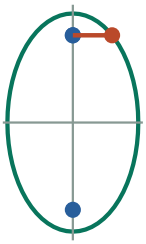


Read the semiaxis lengths

Normalize the equation to see horizontal extent 12 and vertical extent 20.

2. Place a focus and a non-axis point

Upper focus is (0,16). A non-axis point is (36/5,16), from normalized pair (3/5,4/5).

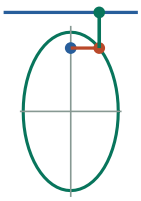


Place an off-axis point

P=(7.2,16) is horizontally aligned with the upper focus (0,16).

3. Explain the directrix calculation

The focus lies on the upward major axis, so its directrix is a horizontal line above the upper vertex. The vertex at $y=20$ is 4 from the focus and must therefore be $4/(4/5)=5$ from the directrix.



Compare the two distances

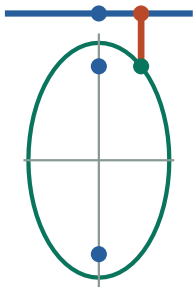
The red segment is 7.2. The perpendicular distance to the directrix is $7.2/(4/5)=9$.

1. $e=16/20=4/5$; $PF=20-16=4$ at (0,20)
Use the upper vertex and upper focus.

2. $4/d=4/5 \Rightarrow d=5 \Rightarrow$ directrix $y=20+5=25$
The focus-to-line ratio fixes the distance to the upper directrix.

4. Write its equation

The upper directrix is y=a/e=25. A horizontal directrix is described by its constant y-coordinate. The upper vertex at y=20 therefore sits 5 units below this line as required.



Locate the horizontal directrix

Nine units above P at height 16 gives the line y=25.

Check. At the chosen point the focus is 36/5 away and the line is 9 away, giving ratio 4/5.

Exercise 503 · A parabola straightened by logarithms

Printed page 53 · Data linearization / Logarithm rules / Power functions

Given the equation $y = 1000x^2$, fill in the missing entries in the table at right, and enter this data into a calculator.

x	y
0.1	
1.0	
2.0	
3.0	
4.0	

(a) Plot the data and notice that it is not linear.

(b) Create two new columns of transformed data, putting $\log x$ in one column and $\log y$ in the other.

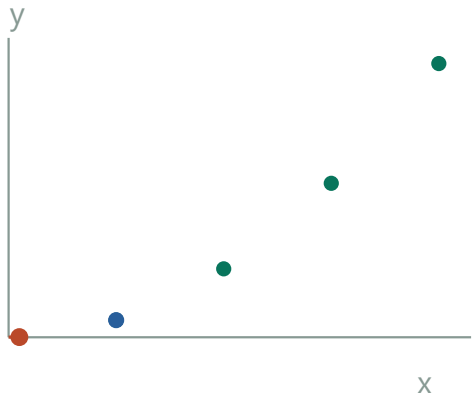
(c) Plot the new columns against each other, and verify that they are related linearly.

(d) Could you have predicted the slope?

(e) Algebraically justify the equation $\log y = 3 + 2\log x$. Notice: y is not linearly related to x, but $\log y$ *is* linearly related to $\log x$

1. Fill the blank output cells

Each row pairs an x from the left column with $y=1000x^2$ in the right column. At $x=0.1$, square first: $0.1^2=0.01$, then multiply by 1000 to get 10. The complete pairs are (0.1,10), (1,1000), (2,4000), (3,9000), and (4,16000).



Keep x and y paired

The five dots are the five table rows. At x = 0.1, the low orange point has y = 10.

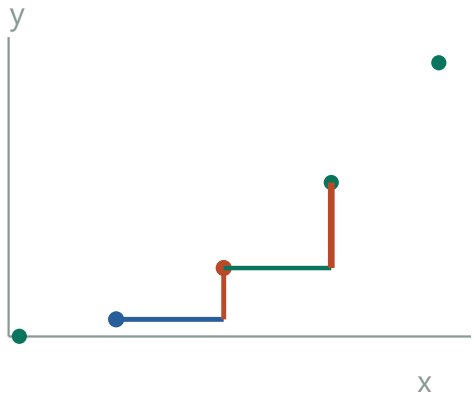
x	y
0.1	
1.0	
2.0	
3.0	
4.0	

The blank output cells

For each left-hand entry compute 1000 times its square and write the result across the same row before transforming either column.

2. Plot the original table

For (a), enter the five x inputs in one list and the five matching y-values in another. Use x on the horizontal axis and y on the vertical axis. From x=1 to 2, y rises by 3000. From 2 to 3 it rises by 5000. Equal horizontal changes give unequal rises, so no straight line fits these five points exactly.



Compare two equal runs

The rise from x=1 to 2 is 3000. From 2 to 3 it is 5000. The raw data bend upward.

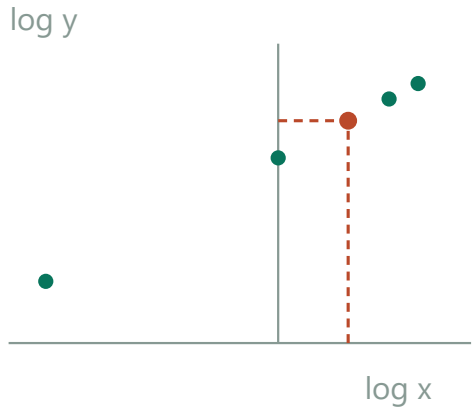
x	y
0.1	
1.0	
2.0	
3.0	
4.0	

The five x inputs

These five positive inputs make both the original power function and its later base-ten logarithms meaningful.

3. Create two common-log columns

For (b), calculate $X=\log_{10}x$ and $Y=\log_{10}y$ on each row without changing the pairings. The transformed pairs, rounded for display, are (−1,1), (0,3), (0.3010,3.6021), (0.4771,3.9542), and (0.6021,4.2041). For example, x=2 and y=4000 give the third pair.

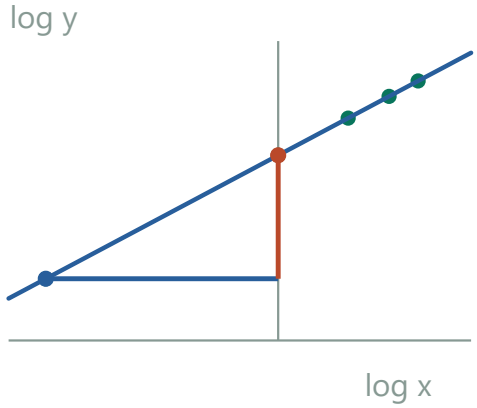


Move each pair onto log axes

The highlighted row (2,4000) becomes approximately (0.3010,3.6021).

4. Plot the changed coordinates and read their slope

For (c), plot X horizontally and Y vertically. The first two transformed points rise from 1 to 3 as X goes from −1 to 0, so their slope is (3−1)/(0−(−1))=2. At X=0 the height is 3. The other pairs agree with Y=3+2X to the displayed precision.

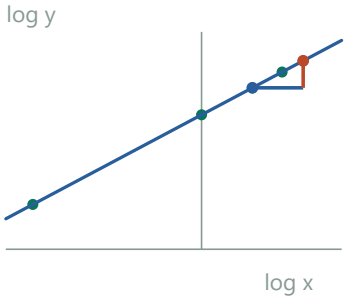


Measure a straight-line rise

From (−1,1) to (0,3), one horizontal unit gains two vertical units. The intercept is 3.

5. Predict the slope and prove the line equation

For (d), doubling x multiplies $y=1000x^2$ by 4. Thus X increases by log 2 while Y increases by log 4=2 log 2, predicting slope 2. For (e), log $y=\log(1000x^2)=\log 1000+\log(x^2)=3+2\log x$. The coefficient 1000 becomes the intercept 3 and the exponent 2 becomes the slope. This log-log statement requires x>0, which also makes y>0.

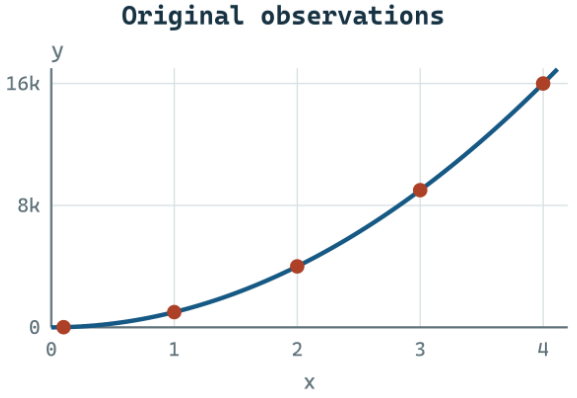


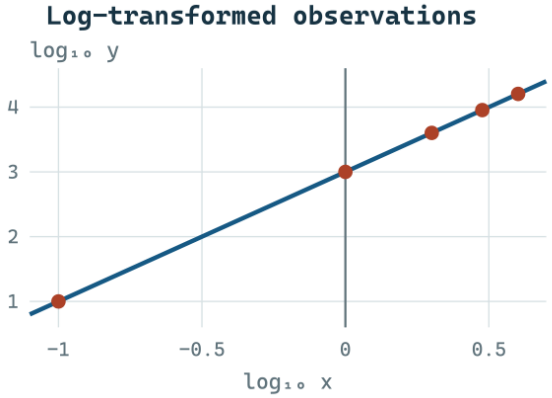
Double x on the original scale

From 2 to 4, y quadruples. On the log axes the rise log 4 is twice the run log 2.

1. log $y=\log(1000x^2)=\log 1000+\log(x^2)$
The input x is positive on these log–log axes.

2. log $y=3+2\log x$
The log of 1000 is 3; bring the exponent 2 down. Thus the plotted slope is 2.





Check. Each doubling of x multiplies y by 4, so the log change in y is twice the log change in x.

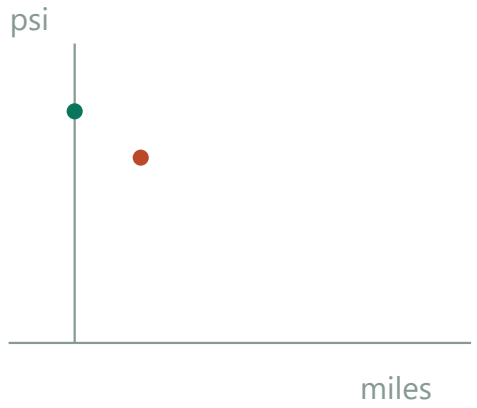
Exercise 504 · Altitude and pressure

Printed page 53 · Exponential growth and decay / Unit conversion

Our civilization lies at the bottom of an ocean of air. As we move through this ocean, pressure varies exponentially with altitude — each increase of one mile of altitude causes the pressure to drop by 20 percent. Given that air pressure at sea level is 14.7 pounds per square inch, write formulas for the following exponential functions:
(a) $M(x)$ is the air pressure at an altitude of x miles above sea level.
(b) $F(x)$ is the air pressure at an altitude of x feet above sea level.
Then calculate the air pressure
(c) on the top of Mt Everest, which is 29 141 feet above sea level;
(d) at the edge of the Dead Sea, which is 1300 feet below sea level.

1. Interpret the 20% decrease

One mile upward retains 80% of the pressure at the previous level. Sea level corresponds to exponent zero and provides the starting value 14.7 psi.

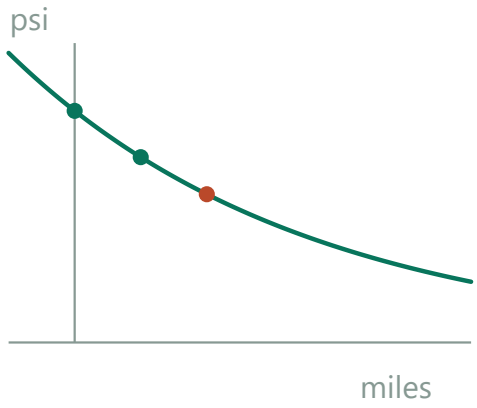


Move up one mile

At sea level pressure is 14.7 psi. One mile higher the model retains 80%.

2. Write formulas for miles and feet

In miles, M(x)=14.7(0.8)^x. In feet, F(x)=14.7(0.8)^(x/5280).

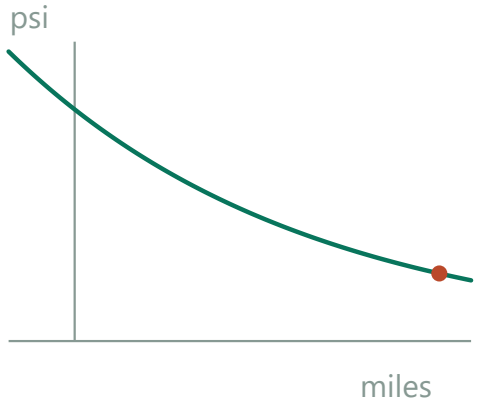


Relabel feet in miles

Convert z feet to z/5280 miles before applying the same curve.

3. Evaluate at the two elevations

At the exercise's 29,141 ft, pressure is about 4.29 psi. At −1300 ft it is about 15.53 psi.

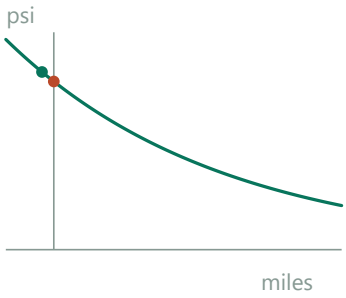


Mark the high-elevation site

29,141 feet is about 5.52 miles, where modeled pressure falls to about 4.29 psi.

4. Explain the below-sea-level case

The Dead Sea value uses a negative altitude. Its pressure moves in the reverse direction from an upward step, which means dividing by a power of 0.8 rather than multiplying by that same positive power.



Move below sea level

A negative altitude reverses the decrease, giving about 15.53 psi at −1300 ft.

1. −1300 ft=−1300/5280 miles
Keep the negative sign for below-sea-level altitude.

2. 0.8^(−1300/5280)=1/0.8^(1300/5280)>1
Reversing a pressure decrease raises pressure above 14.7 psi.

Check. The below-sea-level exponent is negative, giving a pressure above 14.7 as expected.

Exercise 505 · Many waves with one range and period

Printed page 53 · Sinusoidal functions / Phase shifts / Amplitude and period

Find a formula for a function f whose graph $y = f(x)$ has period 12 and y-values that vary between the extremes 2 and 8. How many such examples are there?

1. Place the centerline between the two extremes

The midpoint of 2 and 8 is (2+8)/2, and the distance from that midpoint to either extreme is (8−2)/2. These determine the vertical translation and amplitude of a sine model.



Bisect the extremes

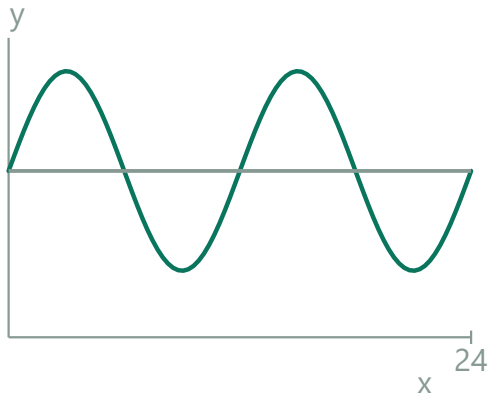
The heights 2 and 8 are three units below and above the midline 5.

1. M−A=2; M+A=8
The midline plus and minus the amplitude give the extremes.

2. 2M=10; 2A=6 ⇒ M=5, A=3
Add the equations for the midline; subtract for amplitude.

2. Choose a phase angle that cycles in 12 units

The change in a sine argument must be 2π radians over one period. Thus its coefficient multiplied by 12 must equal 2π.

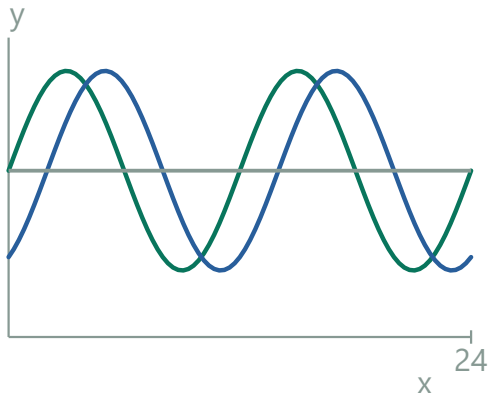


Measure one 12-unit cycle

A 30° inside change per x-unit gives 360° after 12 units. Equivalently use πx/6.

3. Write one model and vary the phase

Midline is 5 and amplitude is 3. One example is f(x)=5+3sin(πx/6). Every horizontal phase shift of this wave works, giving infinitely many examples.



Shift the phase freely

Shifting the wave horizontally preserves both the period and the range.

Check. The sine argument gains 2π in 12 units, and the extreme values are 5±3.

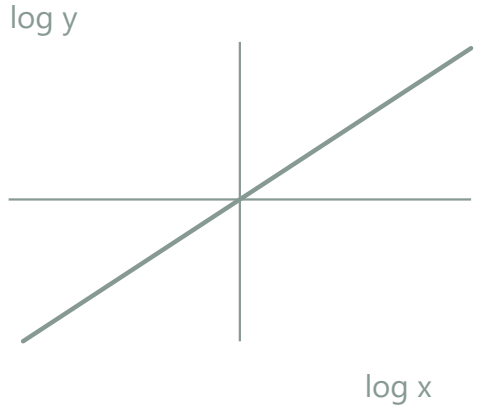
Exercise 506 · From a log line to a power law

Printed page 53 · Logarithm rules / Power functions

Given the equation $\log y = \log 3 + 4\log x$, justify first that $\log y = \log 3 + \log\left(x^4\right)$, then that $\log y = \log\left(3x^4\right)$. Finally, write an equation that relates x and y and that makes no reference to logarithms. Notice that this equation expresses a *nonlinear* relationship between y and x, whereas the original equation expresses a *linear* relationship between $\log y$ and $\log x$.

1. Recognize an exponent in the log coefficient

On the positive domain, raising x to the fourth power multiplies the logarithm by 4. Rewriting in that direction puts both terms into logarithm form.

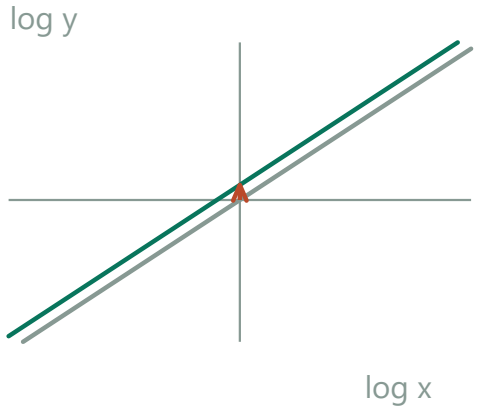


Multiply the log input by four

The line through zero represents 4 log x on log coordinates.

2. Combine the two logarithms

For $x>0$ use $4\log x=\log(x^4)$, then $\log 3+\log(x^4)=\log(3x^4)$.

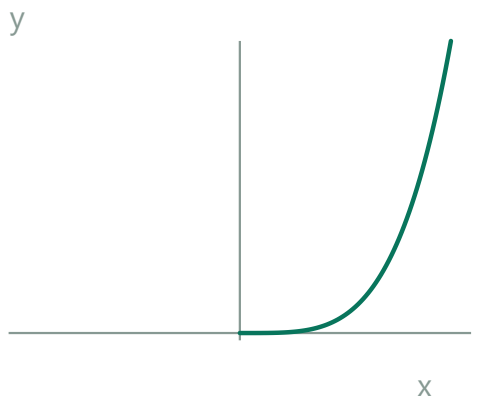


Add the coefficient in log form

Adding log 3 translates that line upward. It equals $\log(3x^4)$.

3. Remove equal logarithms

Injectivity yields $y=3x^4$, with the original positive domain retained.

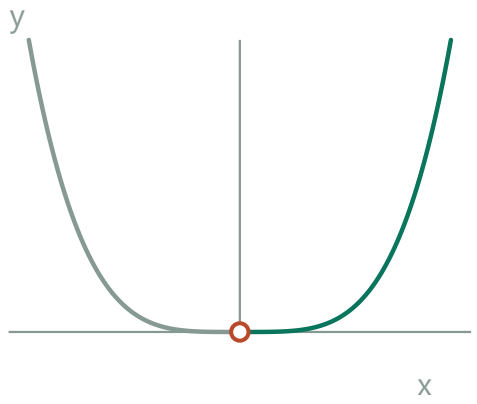


Recover the positive power curve

Exponentiating gives $y=3x^4$ for $x>0$.

4. Keep the original domain

The power formula $y=3x^4$ as an isolated equation makes sense at negative x or zero. The original problem's log x requires $x>0$, which also makes y positive.



Inspect the inherited domain

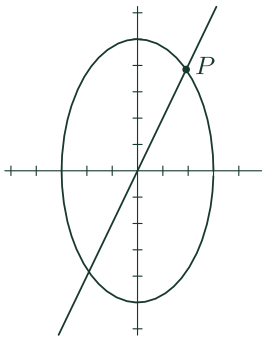
The gray extension and open zero belong only to the unrestricted polynomial. The original log relation allows the green branch for $x>0$.

Check. Taking logs of the resulting power law reverses both steps.

Exercise 507 · A line crossing an ellipse

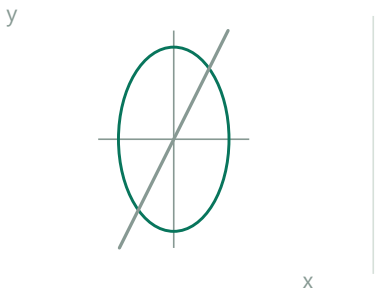
Printed page 53 · Ellipses / Parametric equations / Polar coordinates

Parametrized by $(x, y) = (3\cos t, 5\sin t)$, an object travels an elliptical path, crossing the line $y = 2x$ repeatedly. Describe all such t-values. For the intersection point P in the first quadrant, find the polar coordinates r and θ. Notice that θ is not equal to t.



1. Impose the line condition on the moving coordinates

The crossing equation is 5sin t=6cos t; cos t cannot vanish there, so t=arctan(6/5)+kπ in radians.



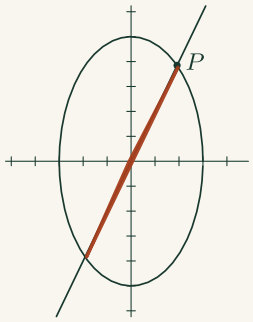
Intersect the ellipse with y = 2x

At a crossing, 5 sin t = 2(3 cos t). The line cuts opposite points.

1. 5sin t=2(3cos t)=6cos t
Substitute both coordinates of the ellipse into y=2x.

2. cos t=0 would imply sin t=0
This is impossible on the unit circle, so division by cos t loses no cases.

3. tan t=6/5 ⇒ t=arctan(6/5)+kπ
The direction repeats after every half-turn.

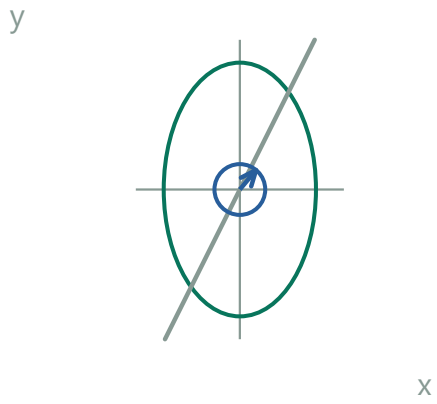


The line through the origin

The line fixes the slope of every crossing. Its angle measured at the origin is the polar direction of P.

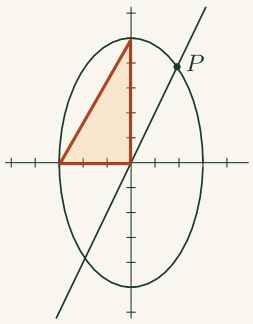
2. Explain why the parameter angle differs

The point (cos t,sin t) on a unit circle is stretched by unequal factors 3 and 5. Its slope after stretching is (5/3)tan t, while the polar angle of the resulting point must have tangent 2.



Stretch the unit-circle vector

The blue radius has angle t. Scaling x by 3 and y by 5 changes the direction.

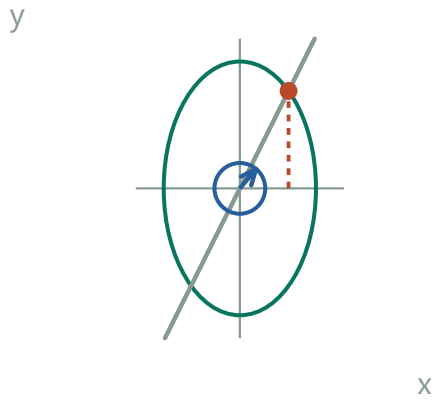


The unequal ellipse axes

The horizontal and vertical radii have different lengths. Stretching a circle unequally can change the direction of its position vector.

3. Find the first-quadrant point

At the first-quadrant crossing, (cos t,sin t)=(5,6)/√61, hence P=(15,30)/√61.

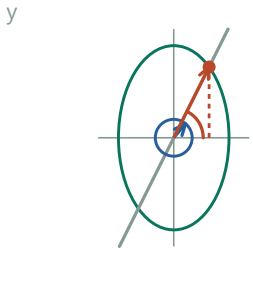


Mark the first-quadrant crossing

The stretched point has coordinates (15,30)/√61; its ordinate is twice its abscissa.

4. Convert that point to polar form

Its radius is 15√(5/61); its polar angle is arctan 2, unlike parameter t.



Measure the polar direction after stretching

The orange vector has angle arctan 2 and length 15√(5/61). The blue vector still has parameter angle arctan(6/5).

1. $r^2=(15^2+30^2)/61=(225+900)/61=1125/61$
Square the first-quadrant position coordinates and add.

2. r=15√(5/61); y/x=30/15=2
Take the positive radius and use the physical point's slope for its polar angle.

Check. The squared normalized coordinates add to 1 and y/x=2.

Exercise 508 · One vector, two perpendicularities

Printed page 53 · Cross products / Orthogonality

Find a nonzero vector that is perpendicular to both of the vectors $[2, 3, 2]$ and $[4, 9, 5]$. Because there are infinitely many correct answers to this question, you will have to make an arbitrary choice somewhere during the solution process.

1. Write the perpendicularity conditions

If the unknown vector is (u,v,w), perpendicularity gives 2u+3v+2w=0 and 4u+9v+5w=0. Subtract twice the first equation from the second to get 3v+w=0.



Place the two input vectors

The unknown vector must be perpendicular to both given arrows.

2. Choose a convenient scale

Set v=−2 and use the new relation to get w=6. Substitution in the first equation fixes u. Any common nonzero multiple preserves both dot products.



Choose a third perpendicular direction

Their cross product (−3,−2,6) gives the red arrow. A nonzero multiple also works.

1. 2x+3y+2z=0; 4x+9y+5z=0
Set both dot products equal to zero.

2. 3y+z=0 ⇒ z=−3y; 2x−3y=0
Subtract twice the first equation from the second, then substitute back.

3. y=−2 ⇒ x=−3, z=6
Choosing a nonzero scale yields a common perpendicular.

3. Confirm with the cross product

Take their cross product: (15−18,8−10,18−12)=(−3,−2,6). Every nonzero multiple works.



Check the common normal

The red vector is normal to the plane spanned by the green and blue arrows.

Check. The two dot products are −6−6+12=0 and −12−18+30=0.

Exercise 509 · Opposite investment rates

Printed page 53 · Compounding / Percentage change

All of Casey's money is tied up in two funds. On 1 April, each fund was worth \$10000, but the value of the first fund was gaining 0.8% per month while the value of the second fund was losing 0.8% per month. Using this data, forecast the total value of Casey's holdings on 1 April of the next year.

1. Treat the funds independently

For the increasing fund 0.8% adds 0.008 to the one-month multiplier. For the decreasing fund subtract 0.008 instead. The two initial balances are both \$10,000.

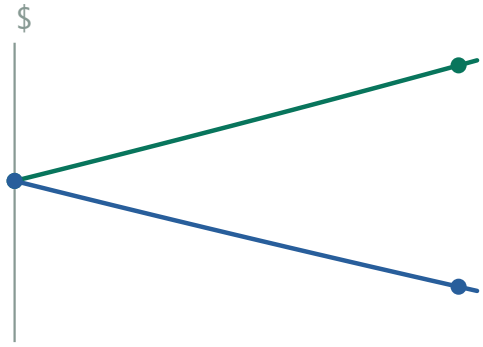


Start with equal balances

Each starts with \$10,000. Their first monthly changes have equal magnitudes and opposite signs.

2. Compound before combining

There are twelve monthly changes to each account. Apply each account's own factor twelve times before adding the resulting dollar amounts.

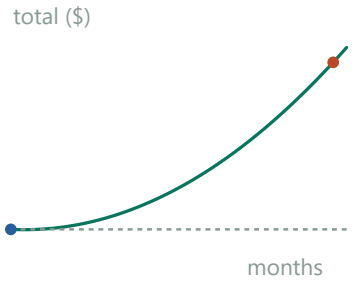


Apply the separate multipliers

The growing fund now earns on a larger balance while the shrinking fund loses on a smaller balance.

3. Calculate and interpret the total

After 12 months the total is $10{,}000[(1.008)^{12}+(0.992)^{12}]\approx\$20{,}084.52$. The initial equal-and-opposite rates act on increasingly unequal balances.



Compare accumulated gains and losses

At month 12 the sum is about $20,084.52, exceeding the original combined $20,000.

1. gain in month 2=0.008(10080)−0.008(9920)
After the first month the bases to which the rates apply differ.

2. =0.008(160)=1.28 dollars
The opposite rates do not cancel after the first month.

Check. In the binomial sum the odd powers of .008 cancel; positive even powers remain, so the total exceeds $20,000.

Exercise 510 · Kepler on log–log axes

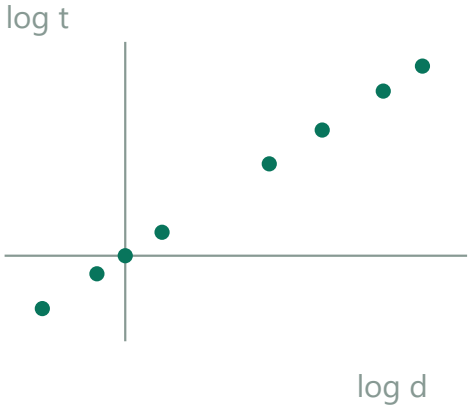
Printed page 54 · Kepler's third law / Data linearization / Logarithm rules

Logarithms would have changed Kepler's life (1571-1630). The table at right contains astronomical data for the eight planets of our solar system. Each entry in the d-column stands for the mean distance from the planet to the sun, measured using the earth-to-sun mean distance (93 million miles) as the astronomical unit. Each entry in the t-column is the amount of time needed for the planet to make one revolution of its orbit, measured using earth years. Create two new columns of transformed data, putting the logarithms of the d-entries in one column and the logarithms of the t-entries in the other. Use the linear relationship between these variables to write a simple equation that expresses Kepler's Third Law. Notice how the use of logarithms reveals the relationship between d and t, and how it allows you to deal comfortably with numbers of disparate sizes (such as 0.241 and 164.8).

d	t
0.387	0.241
0.723	0.615
1.000	1.000
1.524	1.881
5.203	11.86
9.539	29.46
19.19	84.07
30.06	164.8

1. Make the two log columns

In planet order, base-ten log distances are approximately −.412,−.141,0,.183,.716,.980,1.283,1.478. Log periods are −.618,−.211,0,.274,1.074,1.469,1.925,2.217.



Plot the logarithmic data

Earth lands at the origin since its distance and period are both 1 in the chosen units.

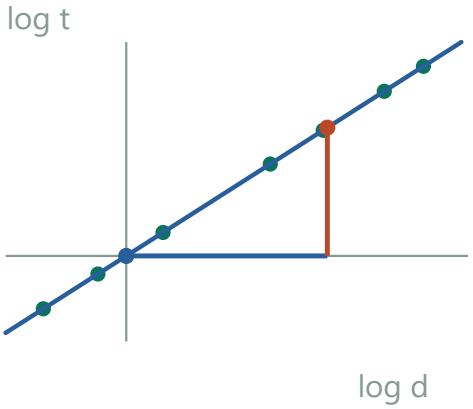
d	t
0.387	0.241
0.723	0.615
1.000	1.000
1.524	1.881
5.203	11.86
9.539	29.46
19.19	84.07
30.06	164.8

Orbital distance measurements

The distance column is measured in astronomical units. Transform its values to define the horizontal coordinates on the log plot.

2. Compare the transformed coordinates

The slope is near 1.5 and the intercept near zero, giving log t≈(3/2)log d.



Read the slope on log axes

Increasing log d by 1 increases log t by about 1.5. The line passes near (0,0).

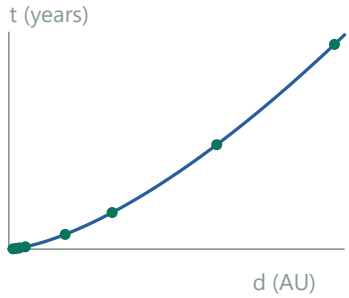
d	t
0.387	0.241
0.723	0.615
1.000	1.000
1.524	1.881
5.203	11.86
9.539	29.46
19.19	84.07
30.06	164.8

Orbital periods

Use the periods in years from the same rows as the distances. The paired logarithms determine the plotted points.

3. Return to the original variables

Exponentiating the straight-line relation means replacing 10^(log t) by t and 10^((3/2)log d) by d^(3/2). Rounding in the measured table explains small deviations.



Undo the logarithms

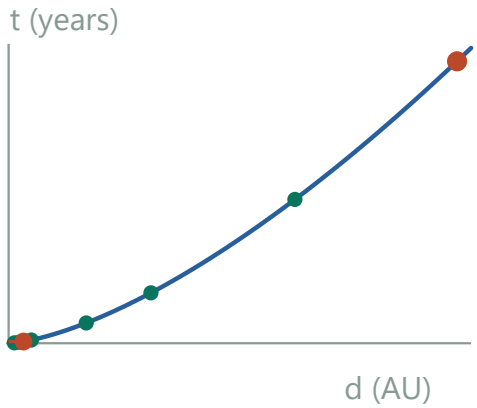
In distance and period coordinates, the line becomes the curved graph t = d^(3/2).

1. log t=(3/2)log d ⇒ 10^(log t)=10^(log(d^(3/2)))
Both d and t are positive, and the exponent can move inside the logarithm.

2. t=d^(3/2) ⇒ $t^2=d^3$
Exponentiating undoes the logs; squaring clears the fractional exponent.

4. State the ideal law in these units

The ideal law is t=d^(3/2), or $t^2=d^3$, in years and AU.



Compare the ideal law with the planets

Earth gives $t^2 = d^3 = 1$. Neptune is close to the same law at the opposite end of the scale.

Check. The outermost row has log ratio 2.217/1.478≈1.50; Earth gives the origin.

Exercise 511 · Linearize four power laws

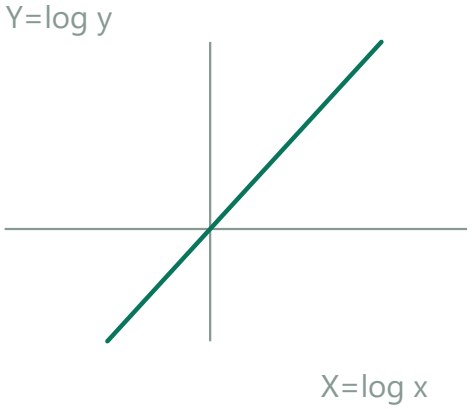
Printed page 54 · Data linearization / Logarithm rules / Power functions

Discover a linear relationship between $\log x$ and $\log y$ by applying logarithms to both sides of the following:

(a) $y = x^3$ **(b)** $y = 100\sqrt{x}$ **(c)** $xy = 3960$ **(d)** $y^{2/3} = 5x^{5/2}$

1. Identify the logarithm coordinates

Let X=log x and Y=log y. A positive constant factor becomes an added intercept, and an exponent on a variable becomes a numerical coefficient. Solve for Y to read each line.

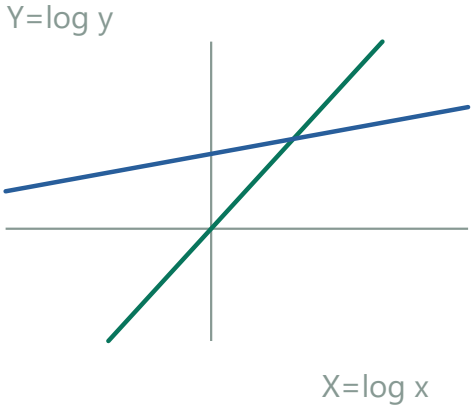


Place the logarithms on the axes

For $y=x^3$ the log coordinates obey Y=3X. A fixed exponent becomes a fixed slope.

2. Read the power and square-root cases

(a) log y=3log x. (b) log y=2+½log x. Move the exponent in x^3 to the multiplier outside log x. For 100√x, use log 100=2 and rewrite the square root as a power of one half.

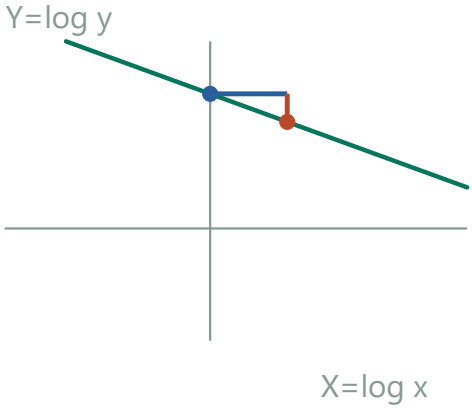


Compare two positive slopes

Green has slope 3 through the origin. Blue has slope 1/2 and vertical intercept 2.

3. Rearrange the reciprocal product

(c) log y=log3960−log x. Taking logs of xy=3960 gives log x+log y=log 3960. Subtract log x to isolate the vertical coordinate log y.

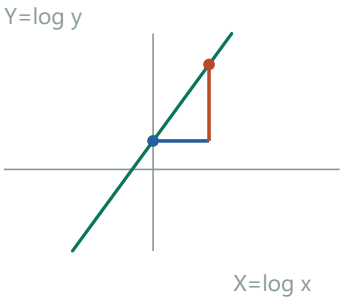


Watch a fixed product

For xy=3960, increasing X by one decreases Y by one. The intercept is log 3960.

4. Undo the factor on log y

(d) (2/3)log y=log5+(5/2)log x, hence log y=(3/2)log5+(15/4)log x.



Divide through by 2/3

The fourth line has intercept (3/2)log 5 and slope 15/4.

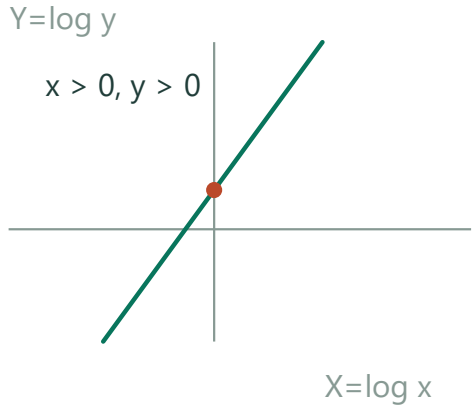
1. (2/3)log y=log 5+(5/2)log x
Apply the power and product rules on x,y>0.

2. log y=(3/2)log 5+(3/2)(5/2)log x
Multiply both sides, including both terms on the right, by 3/2.

3. slope=(3·5)/(2·2)=15/4
Read the coefficient of the transformed horizontal coordinate.

5. State the domain restriction

All four logarithmic identities describe the positive part of each original relation, where both x and y have real logarithms.



Check what the axes allow

Each plotted X and Y is the logarithm of a positive original input or output. X and Y themselves can be negative.

Check. Exponentiating the four lines recovers the original equations, including the reciprocal relation in (c).

Exercise 512 · Rewrite radicals as powers

Printed page 54 · Exponent rules / Radicals and rational powers

Use exponential notation to rewrite (a) $x^2\sqrt{x}$ (b) $\dfrac{x}{\sqrt{x}}$ (c) $\dfrac{\sqrt{x}}{\sqrt[3]{x}}$ (d) $\sqrt[3]{x^3y^4z^5}$

1. (a) Combine the square root with x^2

The real square root requires $x \geq 0$. No division occurs, so zero is allowed.

1 **$x^2\sqrt{x} = x^2 \cdot x^{(1/2)}$**
Express the square root as the one-half power.

2 **$= x^{(2 + 1/2)}$**
Add exponents on the common base.

3 **$= x^{(5/2)},\ x \geq 0$**
Convert 2 into four halves. At zero both sides equal zero.

2. (b) Divide by a square root

The root requires $x \geq 0$. The denominator $\sqrt{x}$ must also be nonzero, so x must be strictly positive.

1 **$x/\sqrt{x} = x^1/x^{(1/2)}$**
Write numerator and denominator as powers.

2 **$= x^{(1 - 1/2)} = x^{(1/2)},\ x > 0$**
Subtract the denominator exponent and retain the original domain.

3. (c) Divide a square root by a cube root

The numerator $\sqrt{x}$ requires $x \geq 0$. The cube root accepts every real x, but it is in the denominator and vanishes at zero.

1 **$\sqrt{x}/\sqrt[3]{x} = x^{(1/2)}/x^{(1/3)}$**
Replace each root by its fractional power.

2 **$= x^{(1/2 - 1/3)} = x^{(3/6 - 2/6)}$**
Subtract exponents and use a common denominator.

3 **$= x^{(1/6)},\ x > 0$**
Subtract the sixths. The original quotient still excludes zero.

4. (d) Take a real cube root term by term

A real cube root exists for positive, zero and negative inputs. Read powers with denominator 3 using this real cube-root convention.

1 $\sqrt[3]{(x^3y^4z^5)} = \sqrt[3]{(x^3)} \cdot \sqrt[3]{(y^4)} \cdot \sqrt[3]{(z^5)}$

Cube roots distribute over products of real numbers.

2 $= x \cdot y^{(4/3)} \cdot z^{(5/3)}$

The real cube root of x^3 is x. Write the other roots as powers.

3 $(x \cdot y^{(4/3)} \cdot z^{(5/3)})^3 = x^3y^4z^5$

Cubing recovers the radicand for any real x, y and z.

Check. Cubing the last expression reproduces $x^3y^4z^5$. The square roots in (a) require $x \geq 0$.

Exercise 513 · Admissible triangle sides

Printed page 54 · Triangle geometry / Inequalities

A triangle has side lengths of 10, x, and $x + 4$. What are the possible values for x?

1. Test each inequality

Positive sides must satisfy x+(x+4)>10, 10+x>x+4 and 10+(x+4)>x. The last two reduce to true numerical facts. The first is the one that can fail.



Compare the reach of the two shorter sides

For x=2, lengths 2 and 6 leave a gap against base 10. They cannot form a triangle.

2. Identify the boundary

At exact equality the two shorter sides would just meet along a straight line. A genuine triangle therefore needs a strict inequality.



Find the limiting straight configuration

At x=3 the endpoints just meet: 3+7=10. A straight segment encloses no area.

3. Solve the allowed range

Require the sum of any two sides to exceed the third. The only restrictive inequality is x+(x+4)>10, giving x>3; this ensures positivity and the other two inequalities.



Give the joined sides room to lift

At x=4 the lengths 4 and 8 meet above the base. In general x+(x+4)>10 requires x>3.

1. x+(x+4)>10 ⇒ 2x>6 ⇒ x>3
Add the shorter candidates and isolate x.

2. 10+x>x+4; 10+x+4>x
The other two triangle tests reduce to 10>4 and 14>0. At x=3 the triangle is flat.

Check. At x=3 the sides 3,7,10 just form a degenerate segment.

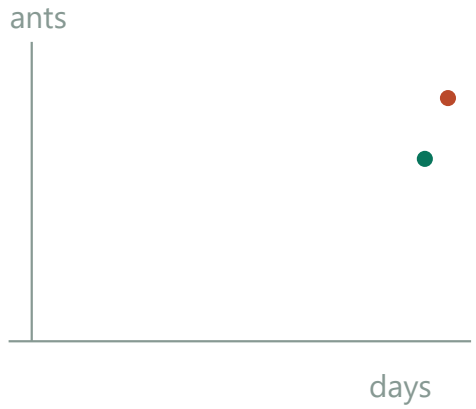
Exercise 514 · Ants before a vacation

Printed page 54 · Exponential growth and decay / Geometric sequences and series

Ryan spills some soda and neglects to clean it up. When leaving for spring break, Ryan notices some ants on the sticky mess but ignores them. Upon returning seventeen days later, Ryan counts 3960 ants in the same place. The next day there are 5280 ants. Assuming that the size of the ant population can be described by a function of the form $F(t) = a \cdot b^t$, calculate the number of ants that Ryan saw when leaving for spring break.

1. Use adjacent days to find the daily multiplier

Let t=0 be departure. The two known counts are at t=17 and t=18, so their ratio represents exactly one day of growth. Their common initial count cancels from that ratio.

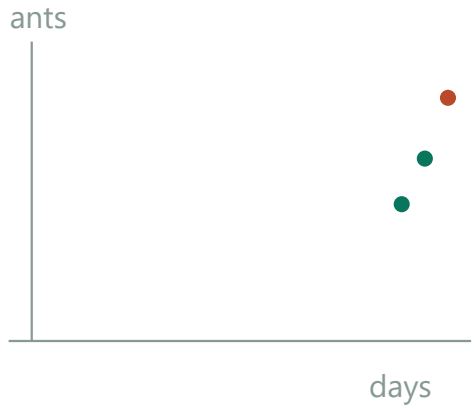


Compare the adjacent observed days

3960 then 5280 ants determine the change over one day.

2. Recover the daily factor

The daily factor is 5280/3960=4/3. This is a factor above 1, consistent with the observed increase in one day. Its reciprocal is the factor for backing up one day.

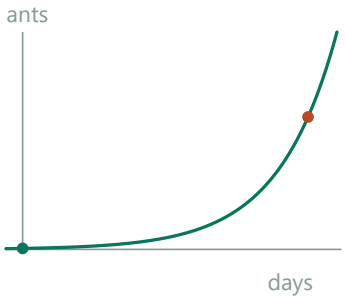


Find the one-day multiplier

5280/3960=4/3. Reversing one day multiplies by 3/4.

3. Run the model backward

Back up 17 days: initial count $=3960(3/4)^{17}\approx29.77$, approximately 30 ants under the fitted model.



Travel back seventeen daily steps

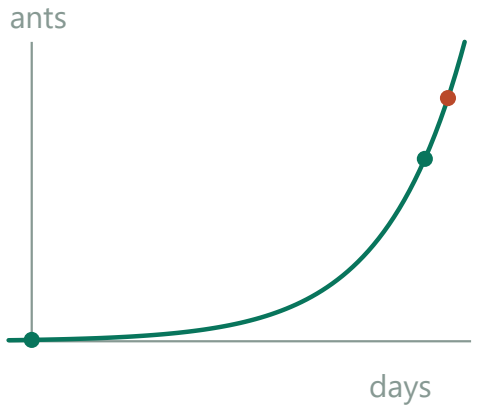
The start value is $3960(3/4)^{17}$.

1. $P_{17}=P_0(4/3)^{17}=3960$
Use the observed next-day ratio from day zero to day 17.

2. $P_0=3960/(4/3)^{17}=3960(3/4)^{17}$
Reverse all seventeen daily steps with the reciprocal factor.

4. Interpret the fractional estimate

An exponential fitted to two integer counts can produce a fractional estimate for an earlier date. Round after all seventeen inverse steps to express a number of ants.



Interpret the estimated start

The model gives roughly 29.77, suggesting about 30 ants when the soda spilled.

Check. Growing this unrounded estimate for 18 days yields 5280.

Exercise 515 · Remove logarithms

Printed page 54 · Logarithm rules / Exponential functions

Simplify the following equations by eliminating all references to logarithms

(a) $0.5 \log y + \log x = \log 300$ (b) $1.5 \log y = 2.699 - \log x$

1. (a) Move the coefficient inside the log

Both equations contain common logs of x and y. Each original argument must be positive, so retain x > 0 and y > 0 in both answers.

1 **0.5 log y + log x = log 300**
Start from the first equation.

2 **log(√y) + log x = log 300**
Use the power rule backwards with exponent one half.

3 **log(x√y) = log 300**
A sum of logs of positive quantities is the log of their product.

4 **x√y = 300, x > 0, y > 0**
Common log is one-to-one, so equal logs have equal arguments.

2. (b) Gather the logs on one side

Use the printed decimal 2.699 as given. The exact equation can be written without logarithms even though a decimal approximation of its right side is convenient.

1 **1.5 log y = 2.699 − log x**
Start with the second equation.

2 **log x + (3/2) log y = 2.699**
Add log x to both sides and replace 1.5 by three halves.

3 **log(x · y^(3/2)) = 2.699**
Use the power rule first, then combine by the product rule.

4 **x · y^(3/2) = 10^2.699 ≈ 500.035**
Raise ten to both sides. Keep x > 0 and y > 0; 500 is only a nearby value.

Check. Taking logs returns each original relation; replacing 10^(2.699) by exactly 500 would introduce rounding.

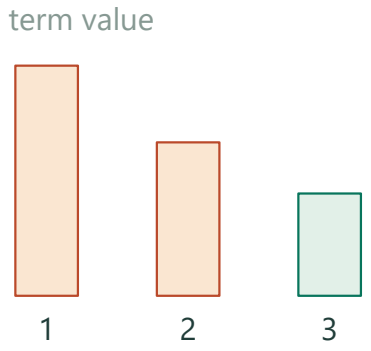
Exercise 516 · Terms of a geometric sequence

Printed page 54 · Geometric sequences and series

A *geometric sequence* is a list in which each term is obtained by multiplying its predecessor by a constant. For example, $81,\ 54,\ 36,\ 24,\ 16, \ldots$ is geometric, with constant multiplier $2/3$. The first term of this sequence is 81; what is the 40th term? the millionth term? the nth term? Check your formula for $n = 1$, $n = 2$, and $n = 3$.

1. Count jumps between terms

Term one is given before any multiplication. Moving from term one to term two uses one factor 2/3. Moving to term three uses two. This fixes the offset in the exponent.

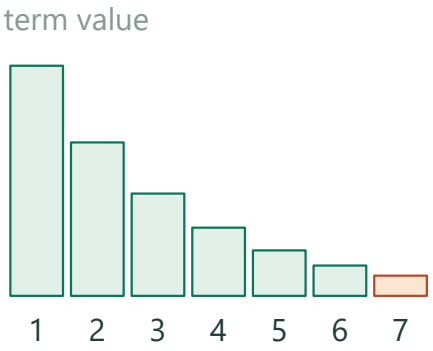


Follow adjacent terms

Moving from term 1 to term 2 reduces 81 to 54. Term 3 uses one more factor of 2/3.

2. Express any requested term

The nth term is 81(2/3)^(n−1). The 40th is 81(2/3)^39; the millionth is 81(2/3)^999999.

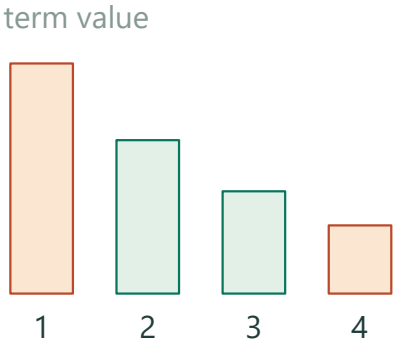


Apply the same factor repeatedly

The bars shrink by 2/3 at each index. At any n, the height is $81(2/3)^{n-1}$.

3. Explain the indexing

The exponent counts steps after term one. For example, the jump count from term one to term forty is 39, while the jump count to term one is zero.

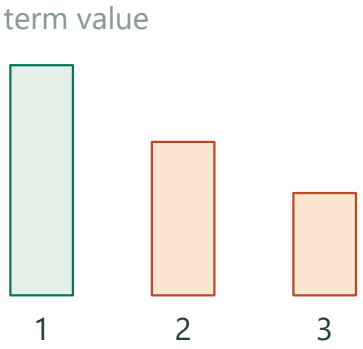


Count intervals instead of bars

From index 1 to index 4 there are three jumps. The exponent counts these spaces.

4. Check the first few values

When n=1 the power is zero, so the initial 81 is recovered. At n=2 it gives 81·(2/3)=54 and at n=3 it gives 81·(4/9)=36.



Check against the initial sequence

The first three heights are 81, 54, 36. Index 1 uses exponent zero.

Check. At n=1,2,3 the formula gives 81,54,36.

Exercise 517 · Build a Koch snowflake

In 1904 Helge von Koch invented his *snowflake*, which is probably the first published example of a *fractal*. It is the result of an endless sequence of stages: Stage 0 (the initial configuration) consists of an equilateral triangle, whose sides are 1 unit long. Stage 1 is obtained from stage 0 by replacing the *middle third* of each edge by a pair of segments, arranged so that a small equilateral triangle protrudes from that edge. In general, each stage is a polygon that is obtained by applying the middle-third construction to *every* edge of the preceding stage.

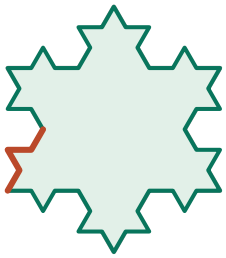


Stage	# of edges	# of vertices	edge length	perimeter
0	3	3	1	3
1				

(a) Stages 0, 1, and 3 are shown above. Make your own sketch of stage 2.
(b) Stage 0 has three edges, and stage 1 has twelve. How many edges do stages 2 and 3 have? How many edges does stage n have? Continue the table begun above.
(c) Stage 1 has twelve vertices. How many vertices does stage n have?
(d) How long is each edge of stage 1? of stage 2? of stage n?
(e) What is the perimeter of stage 1? of stage 2? of stage n?
(f) Does the snowflake have finite perimeter? Explain.
(g) Do you think the area enclosed by the snowflake is finite? Explain.

1. Construct the missing second stage

For stage 2 repeat the four-segment replacement on each of the 12 edges of stage 1. Stage 0 is the original equilateral triangle. For stage 2, keep the first and last thirds of each stage-1 edge; replace its middle third with the other two sides of a small outward triangle.



Replace each of twelve edges

The orange four-edge detour replaces one stage-1 edge. Doing this to every edge builds stage 2.

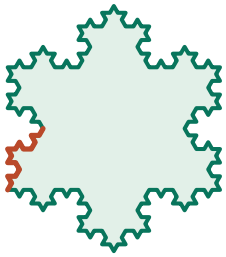


One replacement on a sloping side

A single edge of the original gives four shorter edges here. The little triangular bump is formed on its middle third.

2. Count segments and vertices

Edge and vertex counts are equal: $3 \cdot 4^n$, giving 12,48,192 for stages 1,2,3.

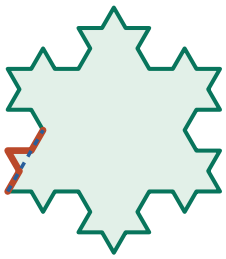


Follow a branch through two refinements

The highlighted stage-1 edge has sixteen stage-3 descendants. Every refinement multiplies the edge count by four.

3. Measure shrinking lengths and total perimeter

Individual edge lengths are 3^{-n}, giving 1/3,1/9,1/27. Perimeters are $3(4/3)^n$, giving 4,16/3,64/9. Thus the limiting perimeter is infinite.

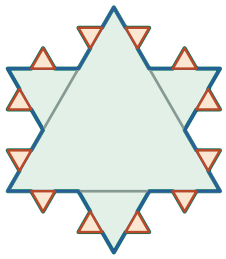


Compare old and new boundary lengths

The dashed chord is one old edge. Four new edges, each a third as long, replace it; perimeter gains a factor of 4/3.

4. Explain why area behaves differently

From one stage to the next there are four times as many new bumps, but each bump has one-third the previous side length and hence one-ninth its area. The added-area ratio is therefore below 1.

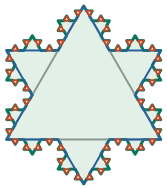


Compare successive bump areas

Orange triangles are the new bumps. A third of the side length produces a ninth of the area, with four times as many bumps.

5. Sum all added triangle areas

Let $A_0=\sqrt{3}/4$. The first added area is $A_0/3$ and later additions shrink by 4/9 each round. Limiting area is $A_0[1+(1/3)/(1-4/9)]=8A_0/5$, finite.



Accumulate the added areas

The original triangle, the first additions and the smaller orange additions fit inside the outline. Their geometric sum is 8/5 of the original area.

1. $\Delta A_1=3(A_0/9)=A_0/3$
Three new triangles each have side one third, hence one ninth of the starting area.

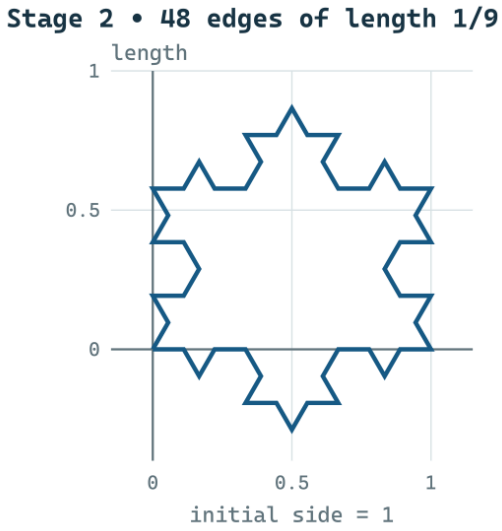
2. $\Delta A_{n+1}/\Delta A_n=4\cdot(1/3)^2=4/9$
Four times as many added triangles have one ninth of the area apiece.

3. $A\infty=A_0+(A_0/3)/(1-4/9)=A_0+(3/5)A_0$
Sum the additions as a convergent geometric series and then include the starting area.



The fine stage-3 boundary

The detailed rightmost outline emphasizes the simultaneous growth in number of edges and decrease in individual edge length.



Check. Multiplying edge count by edge length reproduces each perimeter; each new bump has one-ninth the preceding bump's area.

Exercise 518 · Triangle inequalities with x^2

Printed page 55 · Triangle geometry / Inequalities / Quadratic equations

A triangle has side lengths $x + 7$, x^2, and 13.

(a) If 13 is the largest side what are the possible values for x?

(b) If x^2 is the largest side what are the possible values for x?

1. Use the assumption about the largest side

If all sides are positive, only the sum of the two shorter sides needs testing against the largest one. Keep the order constraints as separate inequalities to intersect at the end.

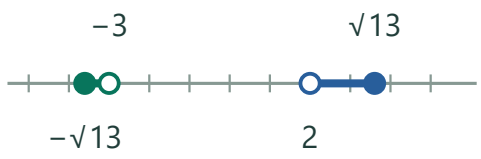


Check the largest side against the other two

As a trial, x=2 gives sides 4, 9, and 13. The shorter pair only just reaches the largest.

2. Take 13 as largest

If 13 is largest, require $x^2 \le 13$, $x+7 \le 13$ and $x^2+x+7 > 13$: the result is $[-\sqrt{13}, -3)$ or $(2, \sqrt{13}]$.



Let 13 be largest

Filled ends allow $x^2 = 13$. Open ends exclude equality in $x^2+(x+7) > 13$.

3. See where the inner endpoints arise

In this case $x^2+x-6=(x+3)(x-2)>0$ gives $x<-3$ or $x>2$. The bound $x^2 \le 13$ cuts these outer rays down to the reported intervals.



Factor the triangle test

The product $(x+3)(x-2)$ is positive on the two outer rays. Intersect these with $|x| \le \sqrt{13}$.

1. $x^2+x+7>13 \Rightarrow (x+3)(x-2)>0$
Subtract 13 and factor the strict triangle inequality.

2. $x<-3$ or $x>2$
A positive product requires same-sign factors.

3. $x^2 \le 13 \Rightarrow -\sqrt{13} \le x \le \sqrt{13}$
Intersect with the largest-side condition; ties at $\pm\sqrt{13}$ are possible.

4. Take x^2 as largest

If x^2 is largest, require $x^2 \ge 13$, $x^2 \ge x+7$ and $x^2 < x+20$; the result is $(-4, -\sqrt{13}]$ or $[\sqrt{13}, 5)$.



Let x^2 be largest

The central gap removes $x^2 < 13$. The outer ends -4 and 5 fail the strict triangle test.

5. Locate its outer endpoints

The strict triangle bound is $x^2-x-20=(x-5)(x+4)<0$, hence $-4<x<5$. Requiring $x^2\geq 13$ splits this interval into the two reported pieces.



Factor the outer bound

The product $(x-5)(x+4)$ is negative strictly between -4 and 5.

1. $x^2<13+(x+7) \Rightarrow x^2-x-20<0$
The sum of the other sides must strictly exceed x^2.

2. $(x+4)(x-5)<0 \Rightarrow -4<x<5$
A negative product occurs between the two roots.

3. $x^2\geq 13 \Rightarrow x\leq -\sqrt{13}$ or $x\geq\sqrt{13}$
Intersect those intervals with $(-4,5)$; equality in the strict sum is disallowed.

6. Handle ties explicitly

These ranges allow ties for largest. For a uniquely largest side exclude $\pm\sqrt{13}$ in each case.



Remove ties for a uniquely largest x^2

The two endpoints $\pm\sqrt{13}$ become open. Apply the same removal if 13 must be uniquely largest.

Check. At -4 or 5, x^2 equals the sum of the other two, so both endpoints must be excluded.

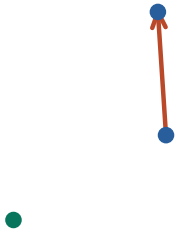
Exercise 519 · Plane perpendicular to a line

Printed page 55 · Planes in space / Orthogonality

Write an equation for the plane that contains the point $A = (4, 5, -3)$ and that is perpendicular to the line through $B = (5, -2, -2)$ and $C = (7, 1, 4)$.

1. Find the line's direction

Move from B to C and record each coordinate change in order. Since the plane is perpendicular to this line, those three changes give a normal vector to the plane.

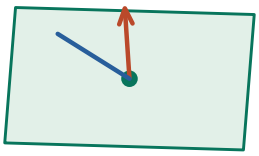


Subtract B from C

The line travels by (2,3,6); this is the plane's normal direction.

2. Anchor the plane at A

For a typical plane point subtract A to form $(x-4, y-5, z+3)$. Its dot product with the normal must vanish. Keep the plus sign in the third difference since A has $z=-3$.

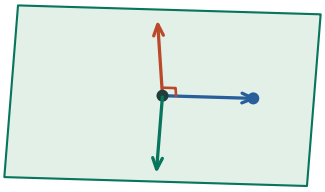


Translate that normal to A

The plane must pass through $A=(4,5,-3)$ and have the computed perpendicular.

3. Compute and simplify

$C-B=(2,3,6)$ is a normal. The plane is $2(x-4)+3(y-5)+6(z+3)=0$, or $2x+3y+6z=5$.



Keep sideways displacement perpendicular

The dot-product test simplifies to $2x+3y+6z=5$.

1. $2x-8+3y-15+6z+18=0$
Expand the translated dot product with the negative z-coordinate of A.

2. $2x+3y+6z=8+15-18=5$
Move the constants to the right.

Check. A gives $8+15-18=5$ and the line direction is parallel to the normal.

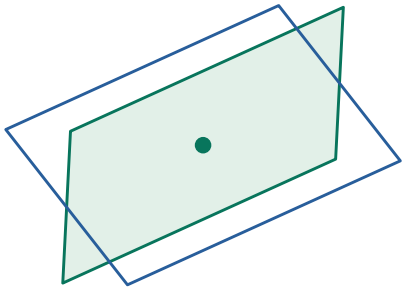
Exercise 520 · Parametrize a plane intersection

Printed page 55 · Intersections of planes / Parametric lines / Cross products

The planes $-x + 2y + 3z = 4$ and $2x + 3y - z = 4$ intersect to form a line. Find an equation for this line.

1. Choose an anchor for the line

For example set z=1. The remaining equations reduce to −x+2y=1 and 2x+3y=5. Solving gives x=y=1, providing a common starting point.

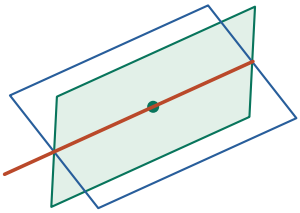


Mark the simultaneous solution

The anchor (1,1,1) satisfies the two equations.

2. Find the direction independently

A direction contained in the intersection must be perpendicular to both plane normals. Their cross product supplies such a direction, which is nonzero since they are not parallel.



Find an in-plane direction twice

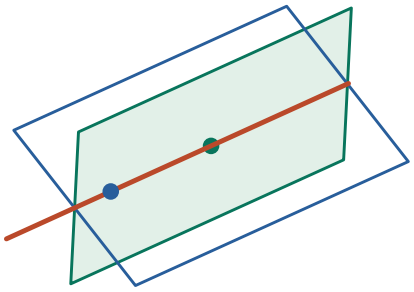
Crossing the normals yields (−11,5,−7), perpendicular to both.

1. d=(2(−1)−3·3,3·2−(−1)(−1),(−1)·3−2·2)
Cross the two normals to find a vector parallel to both planes.

2. d=(−11,5,−7)
It is nonzero, so the planes have independent normals and meet in a line.

3. Write the full parametric line

(1,1,1) satisfies both. Cross their normals (−1,2,3) and (2,3,−1) to obtain (−11,5,−7). The line is (1,1,1)+t(−11,5,−7).



Trace all common points

Every point on the red line is (1,1,1)+t(−11,5,−7).

Check. The direction has zero dot product with both normals; the normals are independent, so the intersection is exactly a line.

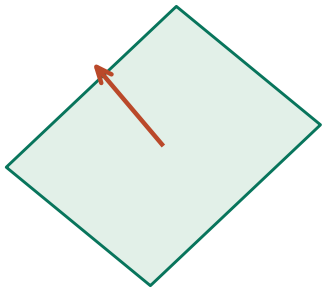
Exercise 521 · Angle between planes

Printed page 55 · Planes in space / The dot product

(Continuation) To the nearest tenth of a degree, find the size of the acute dihedral angle formed by the intersecting planes. Hint: You know an easy method of finding the angle formed by two vectors.

1. Convert a plane angle to an angle of normals

A perpendicular to each plane gives the two vectors $(-1,2,3)$ and $(2,3,-1)$. Choosing a different direction for a normal could produce the supplementary angle, so select the acute angle requested.

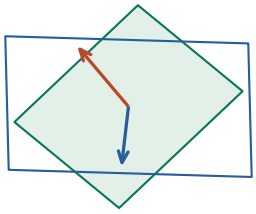


Compare surface orientations

Translate the normals to a common tail; the selected acute angle also measures the dihedral angle.

2. Set up the cosine ratio

The cosine of the included angle equals dot product divided by the product of lengths. Each squared length is $1+4+9=14$.



Choose both normals

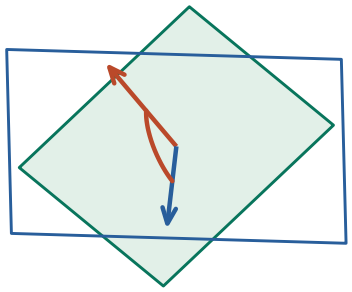
The normals have dot product 1 and squared lengths 14, giving $\cos\theta = 1/14$.

1. $n_1 \cdot n_2 = -2+6-3=1$; $\|n_1\|^2 = \|n_2\|^2 = 14$
Pair like coordinates of the normals; sum squares for lengths.

2. $\cos\theta = \text{abs}(1)/(\sqrt{14}\cdot\sqrt{14}) = 1/14$
Taking the absolute value selects the acute dihedral angle.

3. Evaluate in degree mode

Their normals have dot product $-2+6-3=1$, and each has magnitude $\sqrt{14}$. The acute angle is $\arccos(1/14)\approx 85.9°$.



Read the angle from the normals

The included angle is $\arccos(1/14)\approx 85.9°$.

Check. Taking the absolute dot product makes the result independent of the sign chosen for either normal.

Exercise 522 · Three inverse-power equations

Printed page 55 · Logarithms / Radicals and rational powers

Without a calculator, find x: **(a)** $\log_4 x = -1.5$ **(b)** $\log_x 8 = 16$ **(c)** $27 = 8(x-2)^3$

1. (a) Find a logarithm's positive argument

The argument of a real logarithm must be positive. Raising the valid base 4 to any real exponent automatically gives a positive answer.

1 **$\log_4 x = -1.5 \iff x = 4^{(-3/2)}$**

Rewrite in exponential form and convert the decimal to a fraction.

2 **$x = 1/(\sqrt{4})^3$**

The minus sign takes a reciprocal; the denominator 2 in the exponent takes a square root.

3 **$x = 1/2^3 = 1/8$**

This positive result lies in the required domain.

2. (b) Solve for a logarithm's base

Here x is the base. A real logarithm requires x > 0 and x ≠ 1. The given argument 8 is positive.

1 **$\log_x 8 = 16 \iff x^{16} = 8$**
Switch from logarithm to exponential form on the valid base domain.

2 **x = 8^(1/16)**
Take the positive sixteenth root; the negative root cannot be a logarithm base.

3 **$x = (2^3)^{(1/16)} = 2^{(3/16)}$**
Multiply the exponents. The result is greater than 1, so it is an allowed base.

3. (c) Undo a shifted cube

This part has no logarithm. Its polynomial expression is defined for every real x, and a real cube is one-to-one.

1 **$27 = 8(x - 2)^3 \iff (x - 2)^3 = 27/8$**
Divide both sides by the nonzero constant 8.

2 **$x - 2 = \sqrt[3]{(27/8)} = 3/2$**
Take the unique real cube root on both sides.

3 **$x = 2 + 3/2 = 7/2$**
Add 2. Substitution gives $8(3/2)^3 = 27$.

Check. Raising the answers in (a) and (b) according to their log definitions restores the given values.

Exercise 523 · Foci of a horizontal ellipse

Printed page 55 · Ellipses / Foci and directrices / Eccentricity

Find the focal points and the eccentricity of the ellipse $x^2 + 4y^2 = 16$.

1. Translate the equation into unit form

Divide by 16: $x^2/16+y^2/4=1$. On the right-hand side we want a 1 to expose the squared semiaxes. The factor 4 multiplying y^2 becomes division by 4 after the entire equation is divided by 16.

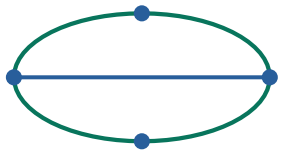


Put the equation in unit form

The denominators 16 and 4 give semiaxes 4 and 2.

2. Locate the long direction

The denominator on x^2 is larger. Thus the foci lie horizontally on opposite sides of the origin. Use the ellipse relation $c^2=a^2-b^2$ for their distance from the center.

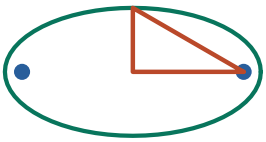


Find the long direction

The longer diameter runs horizontally, so the foci lie on the x-axis.

3. Compute the foci and eccentricity

Then $a=4, b=2, c=\sqrt{12}=2\sqrt{3}$. Foci are $(\pm 2\sqrt{3}, 0)$ and $e=\sqrt{3}/2$.



Use the focal triangle

$c=\sqrt{(16-4)}=2\sqrt{3}$; dividing by 4 gives eccentricity $\sqrt{3}/2$.

Check. At the top $(0,2)$, each focal distance is $\sqrt{(12+4)}=4=a$.

Exercise 524 · Speed of a light spot

Printed page 56 · The tangent function / Rates of change

The top view at right represents a lighthouse that is 30 feet from a seawall. The light L on top of the lighthouse turns counterclockwise, once every six seconds, casting a single moving beam of light onto the seawall. Let N be the point on the seawall nearest the light. Let M be the moving spot of light.

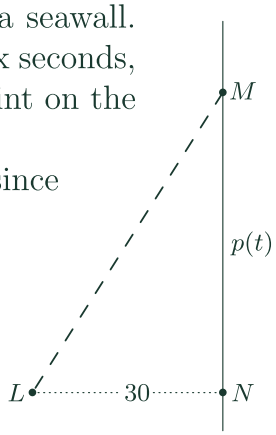
(a) The position p of M relative to N is a function of t, the time elapsed since the spot was at N. Write a formula for $p(t)$ and discuss its domain.

(b) Calculate $p(0.5)$.

(c) What is the average speed of M during the first half-second?

(d) A hundredth of a second after M passes N, how far from N is M?

(e) Find the average speed of M during this hundredth of a second.



1. Read the right triangle

In top view LN is perpendicular to the wall, NM lies along it, and the light beam follows LM. A turn of 2π in six seconds places the angular coordinate at πt/3. Dividing NM by LN uses tangent.

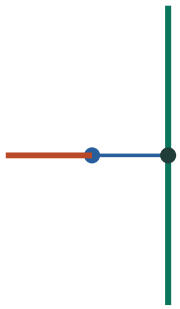


Locate the two triangle legs

The fixed 30-ft segment is adjacent to the beam angle. The signed opposite segment runs along the wall.

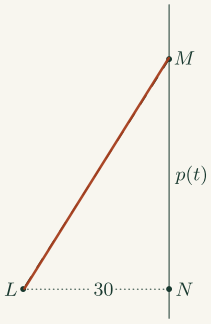
2. Write the position and restrict its domain

Signed position is $p(t)=30\tan(\pi t/3)$ ft when $\cos(\pi t/3)>0$, the half of each turn facing the wall.



Check the facing direction

Here the ray points away from the wall. A spot exists only when $\cos(\pi t/3)>0$.



The rotating beam LM

Only the forward ray emitted at L can illuminate M. Its angle with LN changes uniformly while the wall distance need not.

3. Interpret t since the most recent passage

For elapsed time since the most recent crossing of N use $0\le t<6$ and intersect with that condition ($0\le t<1.5$ or $4.5<t<6$).



Identify the returning interval

On the first revolution the spot reappears for $4.5<t<6$. Earlier it existed for $0\le t<1.5$.

4. Compute the half-second position and speed

At .5 s p=10√3 ft; mean speed from zero is 20√3≈34.64 ft/s.

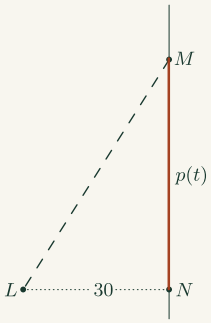


Take the first half-second

At a 30° turn the displacement is 10√3 ft. Divide by 0.5 s to get its mean speed.

1. p(0.5)=30tan(π/6)=30/√3=10√3 ft
At half a second the beam has rotated by π/6 radians.

2. vavg=(10√3−0)/0.5=20√3 ft/s
Divide the wall displacement by the elapsed time.

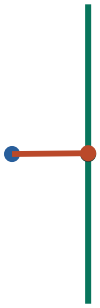


The signed wall position NM

Follow the wall from the nearest point N to the light spot M to measure its displacement along the wall.

5. Do the same on a hundredth of a second

At .01 s p=30tan(π/300)≈.31417 ft; mean speed ≈31.417 ft/s.



Shorten the time interval

At 0.01 s the spot is approximately 0.31417 ft from N; its mean speed is about 31.417 ft/s.

Check. The short-interval average approaches the initial speed 30·π/3=10π ft/s.

Exercise 525 · Digits from a decimal exponent

Printed page 56 · Digits and place value / Exponent rules

Determine by hand how many digits are found to the left of the decimal point when $10^{3.43429448}$ is evaluated? Use a calculator to check your answer and note the first three digits. What are the first three digits when $10^{9.43429448}$ is evaluated?

1. Use neighboring powers of ten

The exponent 3.43429448 lies between 3 and 4. Because powers of ten increase, that places the number between the corresponding whole powers of ten and fixes its integer digit count.



Bound the power before evaluating it

A position between 3 and 4 means a value between 10^3 and 10^4.

2. Determine the leading digits

The first lies between 1000 and 10000, giving four digits before the decimal point. Its value is about 2718.28, so its first three digits are 271.

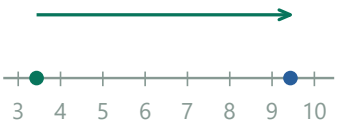


Locate the first number

The fractional exponent about 0.4343 supplies the significand 2.718… .

3. Shift the exponent by six

Raising the exponent by 6 shifts the decimal six places and leaves those same leading digits.



Add six full decades

Adding 6 preserves the fractional part, so the decimal shifts six places while its leading digits stay 271.

1. $10^{(9+0.43429448)} = 10^6 \cdot 10^{(3+0.43429448)}$
Split the exponent into the previous exponent plus six.

2. $10^6 \cdot (2.71828\ldots \times 10^3) = (2.71828\ldots) \times 10^9$
Multiplication by one million moves the decimal point six places to the right while preserving the leading digits.

Check. The common significand is $10^{(0.43429448)} \approx 2.71828$.

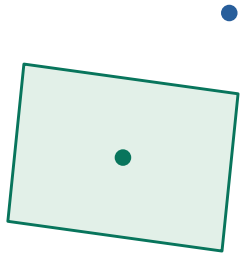
Exercise 526 · Check a proposed plane

Printed page 56 · Planes in space / Verifying equations

Asked to find an equation for the plane that contains points $A = (2, 3, 1)$, $B = (4, 1, 5)$, and $C = (3, 2, 4)$, Eugene answered, "$x + 3y + z = 12$." What do you think of Eugene's answer, and why?

1. Test the candidate using all three points

Substitute each coordinate into Eugene's left side. A single failure rules out the proposed plane, even if its equation passes through the other two points.

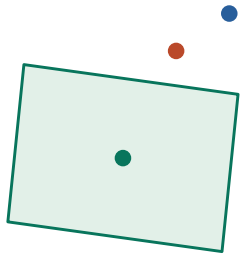


Test the proposed plane

A and B lie on the proposed plane x+3y+z=12.

2. Compare the substitutions

A and B each give 12, but C gives 3+6+4=13. The proposed plane therefore fails.

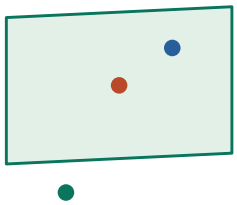


Inspect the third point

At C=(3,2,4) the proposed left side equals 13, so this plane misses C.

3. Find a replacement plane

The correct plane is x+y=5, satisfied by all three points.

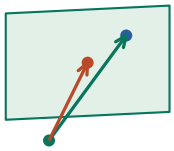


Rebuild using a common relation

All three points satisfy x+y=5, giving the vertical plane shown.

4. Show why three points suffice here

AB=(2,−2,4) and AC=(1,−1,3) are not parallel. They determine a unique plane, and both have horizontal components that sum to zero, making (1,1,0) perpendicular to them.



Check independence

The two displacements from A are not parallel. They determine a unique plane.

1. AB=(2,−2,4); AC=(1,−1,3)
Subtract A from each of the other points.

2. AB×AC=(−2,−2,0)≠0
The nonparallel displacements define a unique plane with normal parallel to (1,1,0).

3. x+y=2+3=5
Use A to find the constant.

Check. AB=(2,−2,4) and AC=(1,−1,3) are independent and both perpendicular to (1,1,0), giving uniqueness.

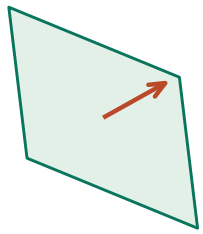
Exercise 527 · Parallel plane through a point

Printed page 56 · Planes in space / Vectors and components

Write an equation for the plane that is parallel to the plane $4x - 27y + 4z = 11$ and that contains the point $(6, 4, 95)$.

1. Keep the normal direction

Parallel planes have parallel normals. Use identical coefficients $4,\ -27$ and 4 for a convenient normal to the new plane. Only its offset from the origin must change.

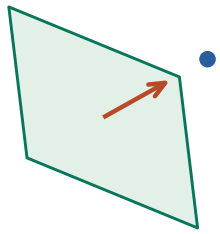


Keep the given normal

The coefficients $(4,-27,4)$ determine the fixed normal direction.

2. Insert the required point

Set the right side equal to $4\cdot 6-27\cdot 4+4\cdot 95$. This chooses exactly the member of the parallel family that contains the point.

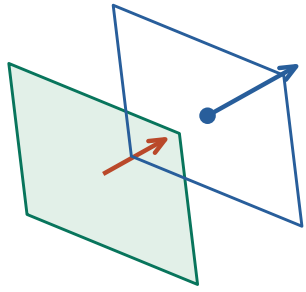


Mark the prescribed point

At $(6,4,95)$ the plane expression has value 296.

3. Write the equation

Evaluate the required left side there: $24-108+380=296$. Thus $4x-27y+4z=296$.



Translate the entire plane

The new constant 296 changes the location while preserving the orientation of the plane with constant 11.

Check. The original plane has the same normal and a different constant.

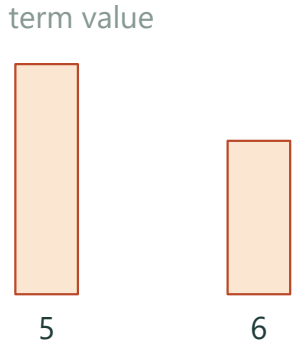
Exercise 528 · Recover a geometric sequence

Printed page 56 · Geometric sequences

The fifth and sixth terms of a geometric sequence are 2880 and 1920, respectively. Find the seventh and first terms of this sequence.

1. Find the constant step multiplier

For a geometric sequence, every consecutive pair has later term divided by earlier term equal to r. The given terms are adjacent, which avoids taking roots to recover r.

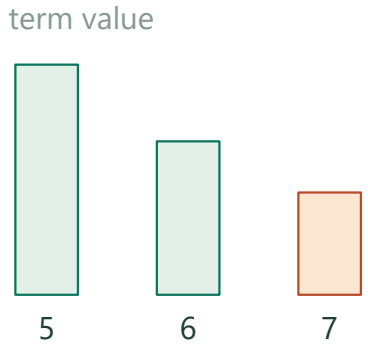


Compare the adjacent given heights

Term 5 is 2880 and term 6 is 1920. Their ratio gives the forward multiplier.

2. Advance one term

The factor is 1920/2880=2/3, so the seventh is 1280.

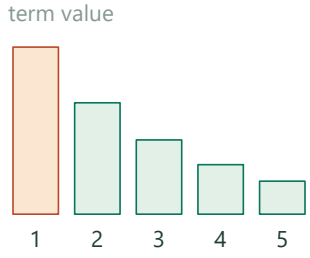


Extend one more bar

Multiplying 1920 by 2/3 gives 1280 at index 7.

3. Go back four steps

Divide the fifth by $(2/3)^4$ to get first term 14580.



Run the sequence backward

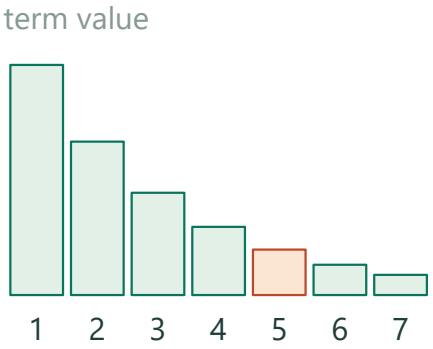
Recover each previous bar by multiplying by 3/2. Four backward jumps from term 5 give 14580.

1. $a_5 = a_1(2/3)^4 = 2880$
Four multiplier steps connect the first term to the fifth.

2. $a_1 = 2880(3/2)^4 = (2880/16)\cdot 81 = 14580$
Divide by the fourth power of the nonzero ratio.

4. Check the index

Reaching term five from term one takes four, rather than five, multiplications. Running those same four steps forward recovers the given value.



Check the four forward steps

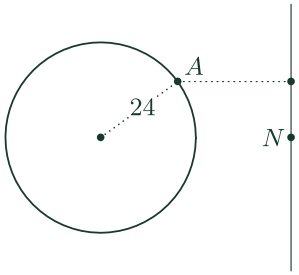
Starting at 14580, multiplying four times by 2/3 returns the orange fifth term, 2880.

Check. Four forward multiplications by 2/3 yield 2880 again.

Exercise 529 · Velocity of a rotating shadow

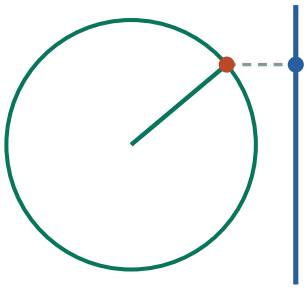
Printed page 56 · Circular motion / Rates of change / Vector projections

Illuminated by the rays of the setting sun, Andy rides alone on a merry-go-round, casting a moving shadow on a wall. The merry-go-round is turning 40 degrees per second. As the top view shows, Andy is 24 feet from its center, and the sun's rays are perpendicular to the wall. Let N be the point on the wall that is closest to the merry-go-round. What is the speed (feet per second) of Andy's shadow when it passes N? What is the speed of this shadow when it is 12 feet from N?



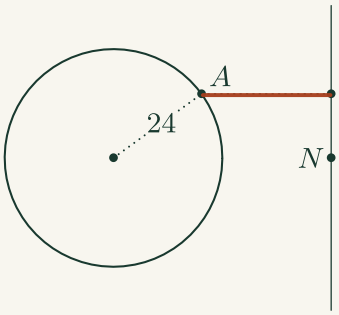
1. Project Andy onto the wall

The sunlight travels perpendicular to the wall, so the shadow shares Andy's coordinate parallel to the wall. Choose time zero when Andy's shadow passes N moving in the positive direction. A sinusoidal component of a 24-ft radius describes that coordinate.



Project the rider across to the wall

Parallel light rays make the shadow share the rider's along-wall coordinate.

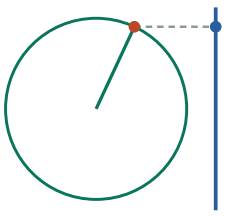


Perpendicular projection to the wall

The incoming rays project horizontally across this top view. The point on the wall has the same along-wall coordinate as Andy.

2. Convert the spin and differentiate

Signed displacement can be s(t)=24sin(ωt) with ω=2π/9 rad/s. Speed is 24ω abs(cos(ωt)).



Follow the changing component

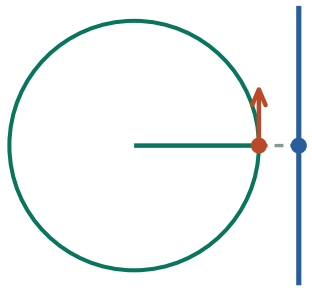
The signed shadow position is 24 sin(2πt/9). The wall projection travels at its derivative.

1. ω=40°/s·π/180°=2π/9 rad/s
The standard sine derivative uses a radian argument.

2. s′(t)=24 cos(ωt)·ω
Differentiate the outer sine and multiply by the inner rate.

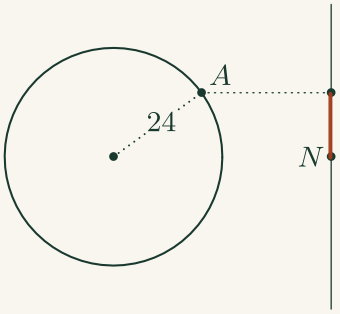
3. Evaluate at the midpoint

At N this is 16π/3≈16.76 ft/s. At the aligned point N, sin(ωt)=0 and the cosine magnitude is one. All the instantaneous circular speed is along the wall, so the projected speed is maximal there.



Look at the midpoint

Here the entire tangent velocity is parallel to the wall, so the speed is maximal.

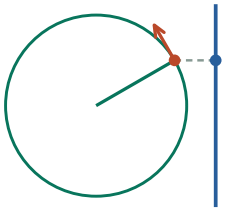


The reference point N

Measure the shadow's signed displacement along the wall from the aligned center level N up or down to the projected point.

4. Evaluate at half the radius

At 12 ft from N, abs(sin)=½, so speed is (8π√3)/3≈14.51 ft/s.



Move the shadow 12 feet from N

At sin θ=½, only √3/2 of the maximal along-wall speed remains.

1. abs(sin(ωt))=12/24=1/2 ⇒ $\cos^2(\omega t)=3/4$
Use the magnitude of the shadow displacement and the Pythagorean identity.

2. speed=24(2π/9)(√3/2)=8π√3/3 ft/s
Use the cosine magnitude because speed has no sign.

Check. Only the component of tangential velocity parallel to the wall contributes to shadow speed.

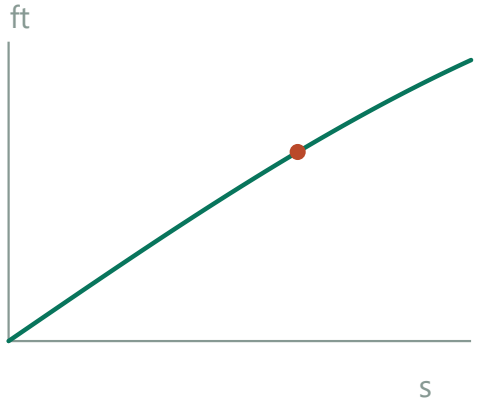
Exercise 530 · A short-time secant

Printed page 56 · Rates of change / Slopes and tangent lines

(Continuation) Working in degree mode, graph the equation $y = 24\sin 40x$. The point $P = (0.75, 12.00)$ is on this graph. What situation does P represent in the merry-go-round story? Find coordinates for another point Q on this graph that is very close to P, and find the slope of the line that goes through P and Q. What is the meaning of this slope in the story?

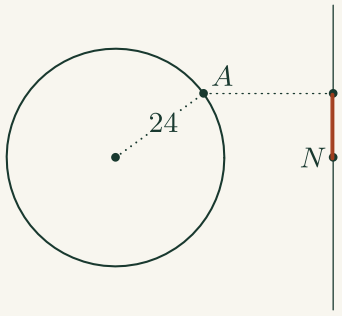
1. Relate P to the motion

At 0.75 s the spot is 12 ft from N in the positive direction.



Locate P on the shadow-position graph

At 0.75 s the signed position is 12 ft.

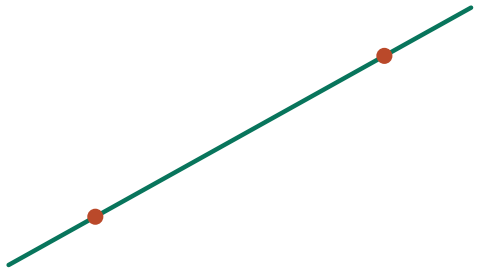


The reference point N · Figure from exercise 529

Measure the shadow's signed displacement along the wall from the aligned center level N up or down to the projected point.

2. Pick a close later time

Choose Q at .751 s, with $y=24\sin(30.04°)\approx12.0145075$ ft.

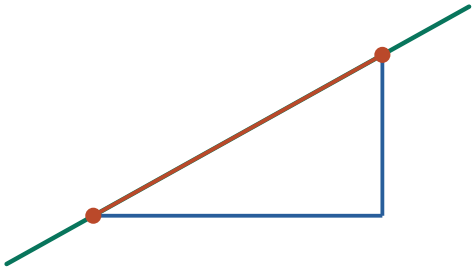


Zoom in on a close later point

At 0.751 s it is about 12.0145075 ft; the enlarged view separates the two points.

3. Divide a position change by elapsed time

The vertical difference is about 0.0145075 ft and the horizontal difference is 0.001 s. Their ratio has units ft/s. Over this tiny interval it closely tracks the slope of the tangent at P.

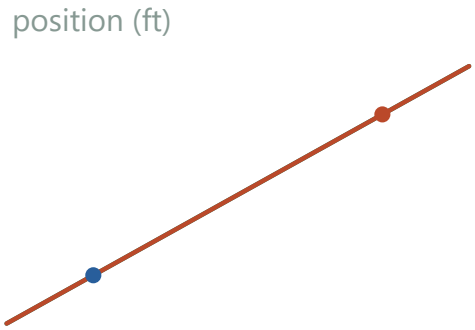


Compare rise and run

The rise is 0.0145075 ft over a 0.001 s horizontal interval.

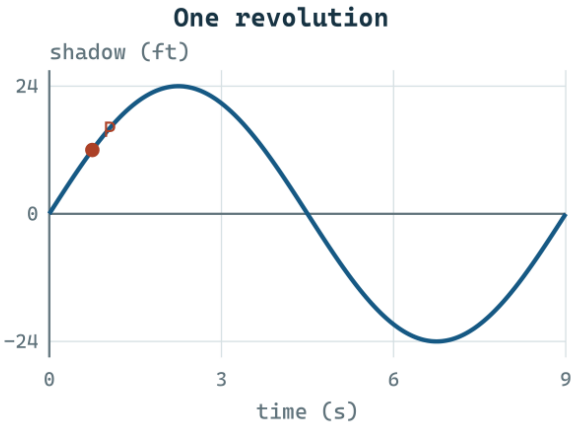
4. Interpret the measured secant

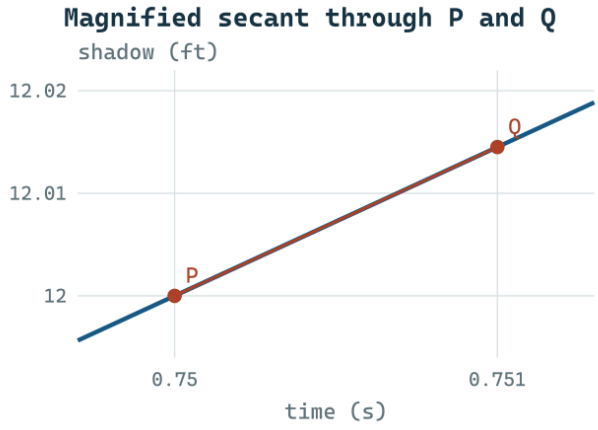
The secant slope is about 14.5075 ft/s, the mean signed velocity over that short interval.



Read the local secant

Their ratio is about 14.5075 ft/s. The short secant approximates the instantaneous tangent.





Check. Its limiting value from the previous exercise is 14.5104 ft/s.

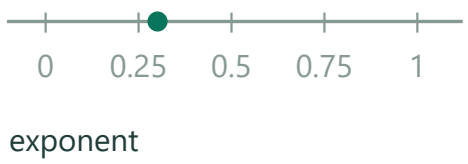
Exercise 531 · Count digits of a huge power

Printed page 56 · Digits and place value / Logarithm rules

Find the base-10 logarithm of the large number 2^{216091}. To write out 2^{216091} in full, how many digits are needed? What are the first three (most significant) digits? By the way, $2^{216091} - 1$ was at one time the largest number known to be prime.

1. Avoid constructing the enormous integer

The power law turns the requested logarithm into 216091 times the small number $\log_{10}2$. Keep enough digits of $\log_{10}2$ so rounding does not shift the fractional part too much.



exponent

Start with the exponent of a single two

Multiplying 216091 copies of 2 adds 216091 copies of log 2 ≈ 0.30103.

2. Get the number of decimal places

Use $216091\log_{10}2 \approx 65049.872793$. The integer part plus one gives 65,050 digits.

offset from 65049



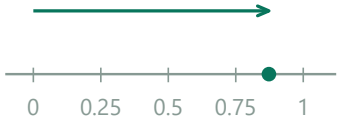
exponent

Find the enclosing decade

Subtract 65049 from the exponent to display its position between neighboring whole powers. An exponent in this decade gives 65050 digits.

3. Decode the fractional part

The fractional part gives significand $10^{0.872793} \approx 7.46093$, so the first three digits are 746.



exponent

Use the fractional distance within the decade

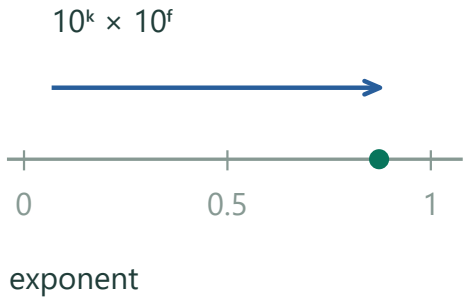
The marked fraction is about 0.872793. Raising 10 to that fraction gives about 7.46093, with leading digits 746.

1. $2^{216091} = 10^{65049} \cdot 10^{0.872793\ldots}$
Split off the whole-number decimal exponent without rounding it upward.

2. $10^{0.872793\ldots} \approx 7.46093 \Rightarrow$ first digits 746
The power 10^{65049} moves the decimal point without changing the significand.

4. Explain why the method works

Writing $\log_{10} N = k + f$ with integer k and $0 \le f < 1$ gives $N = 10^f \cdot 10^k$. The factor 10^f supplies the significant digits while k specifies their place value.



Keep scale and significand separate

A whole-number part k sets the power of ten. The fractional part f sets 10^f between 1 and 10.

Check. The full number lies between 7.46×10^65049 and 7.47×10^65049.

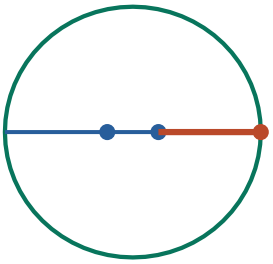
Exercise 532 · Mercury's extreme distances

Printed page 57 · Ellipses / Eccentricity

The planet Mercury follows an elliptical orbit of eccentricity 0.20, which — according to *Kepler's First Law* — has the sun at one focus. At their closest approach, Mercury and the sun are 48 million km apart. Make an accurate drawing of this orbit. What is the greatest distance between the sun and Mercury?

1. Split the closest distance into semiaxis and focal offset

At closest approach $a - c = .8a = 48$, giving $a = 60$ and $c = 12$ million km.

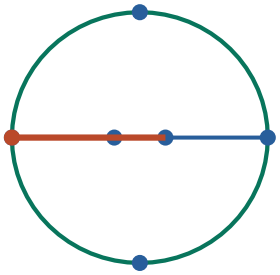


Locate the Sun at a focus

The closest 48 million km is $a - c$, and eccentricity .20 makes $c = .20a$.

2. Find the far vertex and construct a scaled drawing

The farthest distance is $a+c=72$ million km. For a scaled sketch mark vertices ± 60, foci ± 12, and vertical semiaxis $\sqrt{(60^2-12^2)}=24\sqrt{6}$ in the same units.

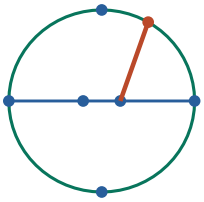


Read the far vertex

$a=60$ and $c=12$ million km give the opposite distance $a+c=72$ million km.

3. Prove the endpoints really are extreme

With the Sun at $(c,0)$ and $(x,y)=(a\cos t, b\sin t)$, its squared distance is $(a-c\cos t)^2$ because $b^2=a^2-c^2$.



Try a nonvertex position

The Sun-to-planet distance is $a-c\cos t$; the focus sits away from the ellipse center.

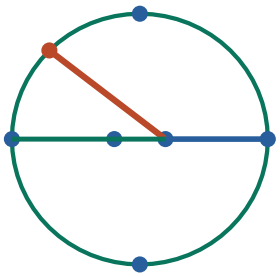
1. $d^2=(a\cos t-c)^2+(a^2-c^2)\sin^2 t$
Substitute the ellipse coordinates relative to the focus and eliminate b^2.

2. $d^2=a^2-2ac\cos t+c^2(1-\sin^2 t)=(a-c\cos t)^2$
Expand and use $\sin^2 t+\cos^2 t=1$.

3. $d=a-c\cos t$
Since $a>c\geq 0$, this expression is positive for every t.

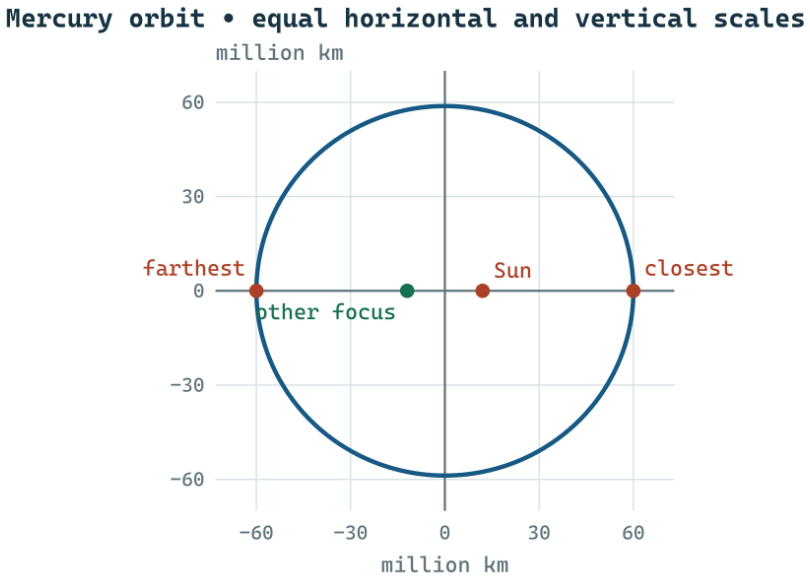
4. Use the full cosine range

Distance therefore ranges from $a-c$ to $a+c$, with both bounds attained at the vertices.



Use all possible cosine values

Since $-1\leq\cos t\leq 1$, the radius from the Sun stays between the two vertex distances.



Check. The average of 48 and 72 is a=60 and the focal offset is exactly 20% of that.

Exercise 533 · Fit pendulum data

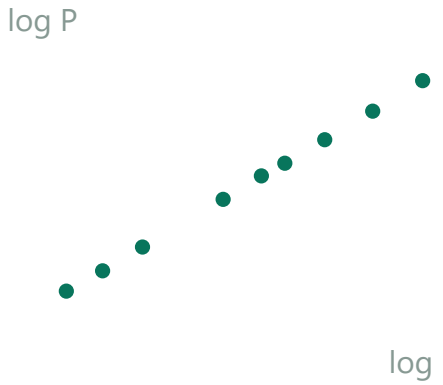
Printed page 57 · Linear regression / Data linearization / Power functions

The table contains data collected by measuring the oscillation of home-made pendulums. Each entry in the L-column is the length of a pendulum (in centimeters) and each entry in the P-column is the period of oscillation (in seconds). Apply logarithms to both columns of data to help you find an equation that relates L and P. There is a long pendulum at the Museum of Science and Industry in Chicago, IL. How long is that pendulum, given that its period is measured to be 8.95 seconds?

L	P
15.0	0.78
18.0	0.85
22.0	0.94
33.0	1.15
40.0	1.27
45.0	1.34
55.0	1.48
70.0	1.67
90.0	1.90

1. Turn a power law into a line

Assume $P \approx kL^m$. Taking base-ten logs gives $\log P \approx \log k + m \log L$. Fit the measured pairs using log L as the predictor and log P as the response; line slope estimates m and its intercept estimates log k.



Move the measurements to log axes

Each dot is a measured length and period. Their near alignment motivates a power model.

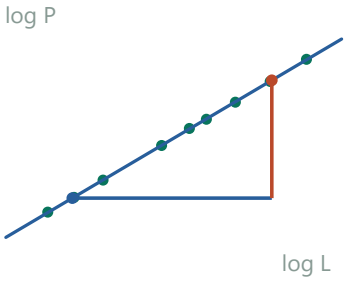
L	P
15.0	0.78
18.0	0.85
22.0	0.94
33.0	1.15
40.0	1.27
45.0	1.34
55.0	1.48
70.0	1.67
90.0	1.90

Pendulum lengths in centimeters

The left column supplies lengths to transform into horizontal coordinates. Its measurement unit is centimeters.

2. Fit the transformed data

Regressing $\log_{10} P$ on $\log_{10} L$ gives slope ≈ 0.49687 and intercept ≈ -0.69351. Hence $P \approx 0.2025\, L^{0.49687}$ with L in cm.



Fit a line to the transformed data

Its slope is about 0.49687 and its intercept about −0.69351. A power near one half emerges.

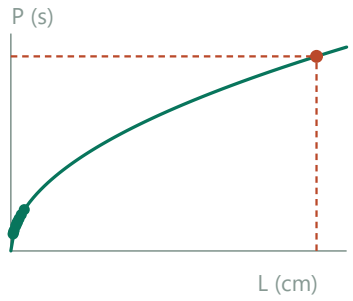
1. $x = \log_{10} L$, $y = \log_{10} P$; mean $x \approx 1.565253$, mean $y \approx 0.084220$
All measured lengths and periods are positive, so take both logs of each pair and then average each column.

2. $\Sigma (x - \text{mean } x)^2 \approx 0.568754$; $\Sigma (x - \text{mean } x)(y - \text{mean } y) \approx 0.282596$
Subtract the means and sum over the nine matched observations.

3. $m \approx 0.282596 / 0.568754 \approx 0.49687$; $b = \text{mean } y - m \cdot \text{mean } x \approx -0.69351$
Divide to get the least-squares slope. Substitute the means in $y = mx + b$ to recover the intercept using unrounded values.

3. Predict the museum length

Setting P=8.95 predicts L≈2048 cm, about 20.5 m.



Follow an 8.95-second period to the fit

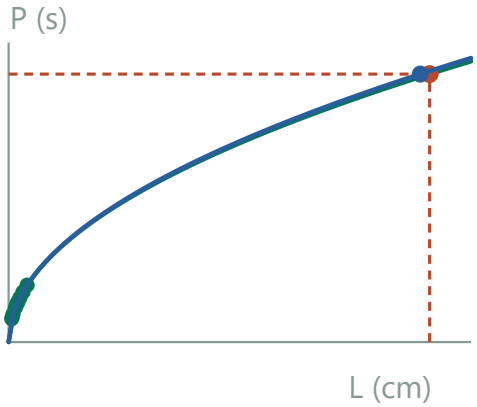
The inferred length is about 2048 cm. All observations cluster near the left edge on this scale.

1. log L=(log 8.95+0.69351)/0.49687
Set the fitted period equal to 8.95 and solve the linear equation in log L.

2. L≈10^((log 8.95+0.69351)/0.49687) cm
Undo the base-ten log before converting centimeters to meters.

4. Judge the extrapolation

A rounded square-root model P≈0.20√L predicts about 20.0 m; both extrapolate far beyond the data.



Compare the square-root approximation

The blue model gives about 2003 cm. Both predictions extend well beyond the 90 cm longest measurement.

L	P
15.0	0.78
18.0	0.85
22.0	0.94
33.0	1.15
40.0	1.27
45.0	1.34
55.0	1.48
70.0	1.67
90.0	1.90

Largest experimental period

The final row gives the longest period measured here. Compare its scale with the museum period before judging the extrapolation.

Check. For L=40 cm the fitted model gives about 1.27 s, close to the observed 1.27 s.

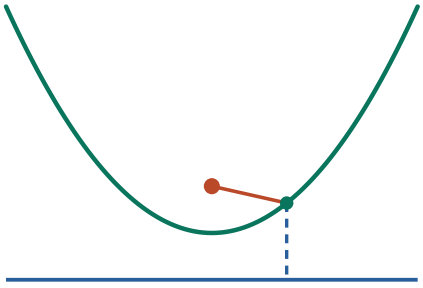
Exercise 534 · Prove a focus–directrix locus

Printed page 57 · Parabolas / Foci and directrices

Graph the curve $y = \frac{1}{4}(x+2)(x-6)$, and mark the point $F = (2,-3)$. Choose any point on the curve, and show that it is equidistant from F and the line $y = -5$. Do the same for three more points on the curve. These results suggest that the curve is the parabola whose *focus* is F and whose *directrix* is the line $y = -5$. What additional work is needed to prove this statement?

1. Sketch the curve and its distance markers

Expand $4y=(x+2)(x-6)=x^2-4x-12=(x-2)^2-16$, so $y=(x-2)^2/4-4$. Plot the vertex $(2,-4)$, draw the vertical symmetry axis $x=2$, and mark the intercepts $(-2,0)$ and $(6,0)$. Mark $F=(2,-3)$, one unit above the vertex, and the horizontal line $y=-5$ one unit below it.

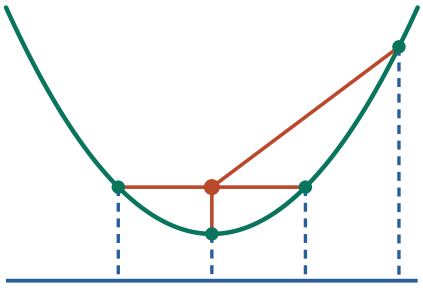


Locate the distance markers

The vertex $(2,-4)$ is halfway between $F=(2,-3)$ and directrix $y=-5$.

2. Compare the distances at four points

The shortest distance to $y=-5$ is the vertical distance $|y+5|$. At $(2,-4)$ the focus is 1 unit above and the line 1 unit below. At $(0,-3)$ and $(4,-3)$, horizontal distance to F is 2 and vertical distance to the line is also 2. At $(6,0)$, distance to F is $\sqrt{(4^2+3^2)}=5$ and distance to the line is 5. All four points satisfy the plotted equation.

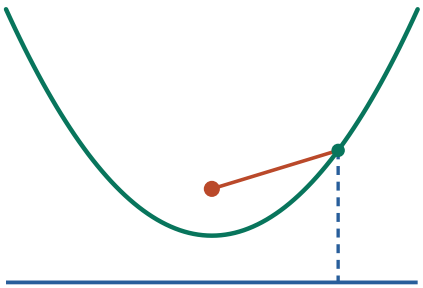


Test four different points

At $(2,-4),(0,-3),(4,-3),(6,0)$, the two distances match in each case.

3. Write the condition for an arbitrary point

Four agreements suggest a pattern. For $P=(x,y)$, its distance to F is $\sqrt{((x-2)^2+(y+3)^2)}$. Its perpendicular distance to the line is $|y+5|$. They are nonnegative, so equality is equivalent to $(x-2)^2+(y+3)^2=(y+5)^2$.

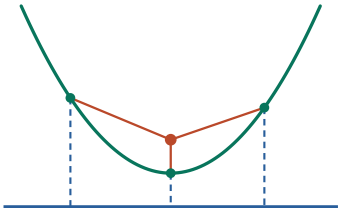


Replace the sample by an arbitrary P

The straight segment to F and the perpendicular segment to the line have equal length.

4. Prove both directions of the locus claim

Subtract $(y+3)^2$ from both sides. The difference of squares is $[(y+5)-(y+3)][(y+5)+(y+3)]=2(2y+8)=4(y+4)$. We recover $(x-2)^2=4(y+4)$, exactly the original curve. Every equidistant point therefore lies on it. Reversing the calculation proves every point on it is equidistant, which completes the focus-directrix proof.



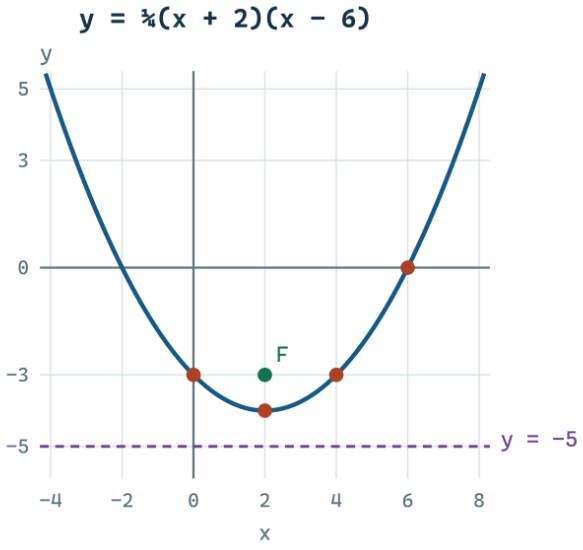
Recover the defining equation

Equating and simplifying their squared lengths gives $(x-2)^2=4(y+4)$, so the two loci coincide.

1. $(x-2)^2=(y+5)^2-(y+3)^2$
Subtract the shared squared vertical offset in the distance equation.

2. $=(5-3)(2y+8)=4(y+4)$
Use the difference of squares instead of expanding the quadratics.

3. $y=(x-2)^2/4-4=(x+2)(x-6)/4$
Recover the given curve. Both distances are nonnegative, so equality of squares suffices.



Check. Both distances are nonnegative, so equating their squares introduces no extraneous points.

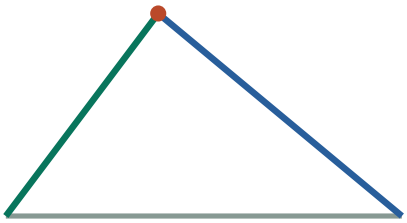
Exercise 535 · Geometric sides of a triangle

Printed page 57 · Geometric sequences and series / Inequalities / Triangle geometry

A triangle with a 32-inch side, a 40-inch side, and a 50-inch side is a curiosity, for its sides form a geometric sequence. Find the constant multiplier for this sequence. Find other such triangles. Are there any restrictions on the multipliers that can be used?

1. Read the multiplier from adjacent sides

The ratio is 5/4. Divide 40 by 32 and 50 by 40. Both give the same fraction, confirming that these are consecutive terms of a geometric sequence.



Compare the two scale factors

The given triangle has 32, 40, 50. Both successive ratios are 5/4.

2. Test the largest side for increasing order

For increasing sides a, ar, ar^2, require $1+r>r^2$, or $1 \le r < (1+\sqrt{5})/2$.



Push increasing ratios toward the boundary

At $r=1.6$ the triangle is thin. Equality $1+r=r^2$ flattens it, giving the excluded upper boundary.

1. $a+ar>ar^2 \Rightarrow 1+r>r^2$
For $r \ge 1$ the last side is largest; divide by the positive first side.

2. $r^2-r-1<0 \Rightarrow r<(1+\sqrt{5})/2$
The upward-opening quadratic is negative between its roots. The smaller root is negative.

3. Allow decreasing sequences

For decreasing order take reciprocals, so the complete positive range is $(\sqrt{5}-1)/2 < r < (1+\sqrt{5})/2$, including 1.



Reverse a decreasing side sequence

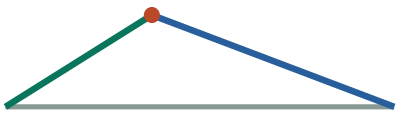
For ratio 0.7 the sides 1, 0.7, 0.49 still close. Reversing them uses the increasing ratio 1/0.7.

1. $0<r\le 1 \Rightarrow$ increasing reverse ratio $=1/r$
Reverse the order so the same largest-side result applies.

2. $1/r<(1+\sqrt{5})/2 \Rightarrow r>2/(1+\sqrt{5})=(\sqrt{5}-1)/2$
Take reciprocals of positive numbers and rationalize the denominator.

4. Construct other valid examples

For example 9,12,16 works with r=4/3. Another triangle is (16,24,36), with ratio 3/2. Scaling any valid triple by a common positive factor gives infinitely many further examples.



Check another increasing triple

Both neighboring ratios in (16,24,36) are 3/2; the shorter sides sum to 40, greater than 36.

Check. At the golden-ratio endpoints the longest side exactly equals the other two combined.

Exercise 536 · Fibonacci versus geometric growth

Printed page 57 · Sequences and recursion / Geometric sequences

Consider the famous *Fibonacci sequence* 1, 1, 2, 3, 5, 8, 13, 21, … . Describe the pattern, calculate the next three terms, and explain why the sequence is not geometric.

1. Identify the input pair for the next term

Start with 13 and 21, the final two terms shown. Once the next term is produced, move the pair one place forward and repeat with the most recent two terms.

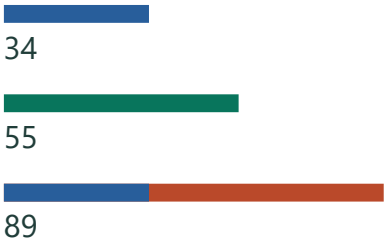


Select the latest two terms

The next length is formed by joining the 13-unit and 21-unit strips.

2. Extend the additive recurrence

Add the two preceding terms: $13+21=34, 21+34=55, 34+55=89$.

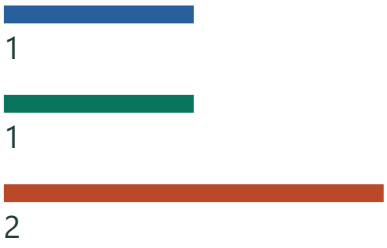


Join each latest pair

13+21 gives 34, then 21+34 gives 55, then 34+55 gives 89. The last bar splits at 34.

3. Test whether one multiplier works

Consecutive ratios vary (1/1 and 2/1 already differ), so it is not geometric.



Check early ratios

The first two bars have ratio 1. The next pair has ratio 2, ruling out a geometric sequence.

Check. Each new term minus the previous one returns the term two places earlier.

Exercise 537 · Four exact logs

Printed page 57 · Logarithms / Change of base

Without a calculator, find: (a) $\log_5 125$ (b) $\log_{0.125} 64$ (c) $\log_4 8$ (d) $\dfrac{\log 125}{\log 5}$

1. (a) Identify a whole-number exponent

All four logarithms here have positive arguments and valid bases. The logarithm gives the exponent required to reproduce its argument.

1 **$k = \log_5 125 \iff 5^k = 125$**

Use the definition with base 5.

2 **$125 = 5\cdot5\cdot5 = 5^3 \Rightarrow k = 3$**

Powers of base 5 are one-to-one.

2. (b) Turn the decimal base into a reciprocal

The base 0.125 lies between zero and one, so a negative exponent is needed to produce a value larger than one.

1 **$0.125 = 1/8 = 8^{-1}$**
Rewrite the decimal exactly.

2 **$(8^{-1})^{k} = 64 = 8^{2} \iff 8^{(-k)} = 8^{2}$**
Use k for the requested exponent and express 64 as a power of 8.

3 **$-k = 2 \Rightarrow \log_{0.125} 64 = -2$**
Match exponents, then change the sign.

3. (c) Express both numbers as powers of two

Although 8 is not a whole-number power of 4, the two numbers share the base 2.

1 **$k = \log_4 8 \iff 4^{k} = 8$**
Name the exponent.

2 **$(2^{2})^{k} = 2^{3} \iff 2k = 3$**
Multiply nested exponents and match the powers of 2.

3 **$k = 3/2$**
Divide by 2. In fact $(\sqrt{4})^{3} = 8$.

4. (d) Factor the numerator logarithm

Here log is the common logarithm. The denominator is nonzero because $5 \neq 1$.

1 **$\log 125 / \log 5 = \log(5^{3}) / \log 5$**
Replace 125 by an exact power of 5.

2 **$= (3 \log 5) / \log 5$**
The power rule brings the exponent 3 in front.

3 **$= 3$**
Cancel the nonzero factor log 5.

Check. Exponentiate each result with its indicated base to recover its argument.

Exercise 538 · Divide powers by their base

Printed page 57 · Exponent rules

What is half of 2^{40}? What is one third of 3^{18}?

1. Take half of a power of two

Taking half divides the value of the power by two. Each factor of the power is itself 2.

1 **$(1/2)\cdot 2^{40} = 2^{40}/2^{1}$**
Write the divisor as one factor of the same base.

2 **= 2^(40 − 1) = 2^{39}**
Cancel that one factor. Multiplying by 2 restores the original number.

2. Take one third of a power of three

Both denominators in this exercise are fixed nonzero integers, so there is no variable-domain exclusion.

1 **$(1/3)\cdot 3^{18} = 3^{18}/3^{1}$**
Write the divisor as a power of base 3.

2 **= 3^(18 − 1) = 3^{17}**
Remove one factor of 3. Three times the result is the original power.

Check. Multiplication by the corresponding base restores the inputs.

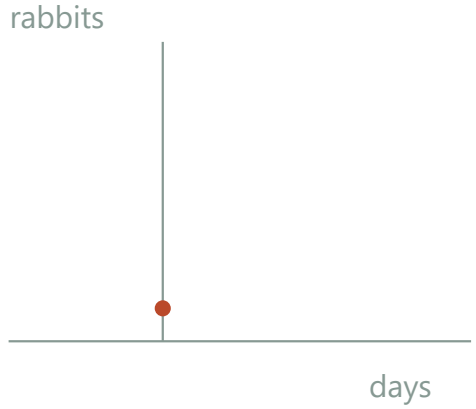
Exercise 539 · A rabbit-population graph

Printed page 57 · Exponential growth and decay / Graph interpretation

Let $R(t) = 55(1.02)^t$ describe the size of the rabbit population in the PEA woods (known as the Gillespie Tract) t days after the first of June. Use a graphing tool to make a graph of this function inside the window $-50 \le t \le 100$, $0 \le R(t) \le 500$. What is the R-intercept of the graph, and what does it signify? Does the graph show a t-intercept? Would it show a t-intercept if the window were enlarged?

1. Locate the time reference

June 1 is day zero. The variable t may be negative within the viewing window for dates before then. Evaluate a few well-separated points on either side of zero to place the curve.

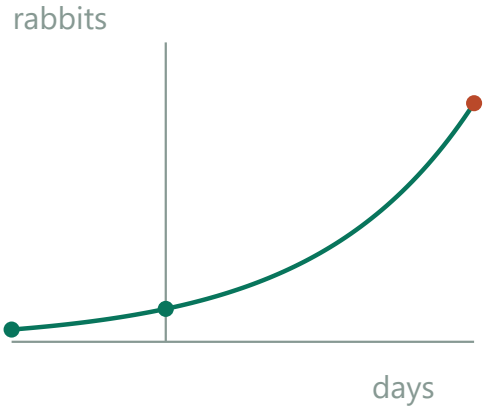


Use June 1 as day zero

The vertical-axis crossing represents 55 rabbits on the reference day.

2. Plot the requested window

Plot (0,55), approximately (−50,20.43), and (100,398.46), joining them by an increasing exponential.

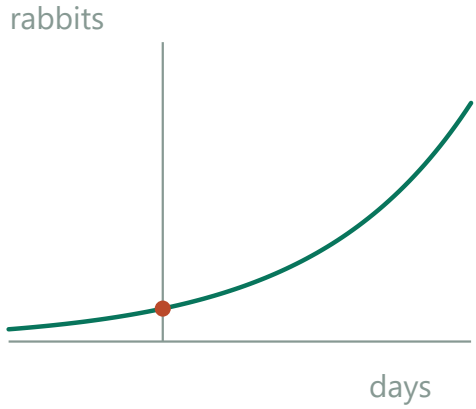


Plot the stated window

The curve rises from roughly 20.43 to 398.46 over the requested time span.

3. Interpret each possible intercept

The vertical intercept is 55 rabbits on June 1. No finite t-intercept exists, including outside the window, because 55 and 1.02^t are positive.

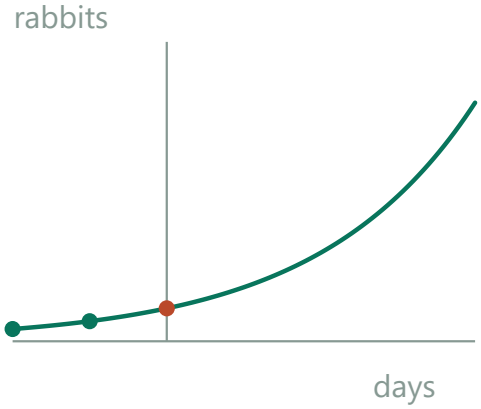


Inspect each axis

The model crosses the vertical axis at 55 but remains above the time axis.

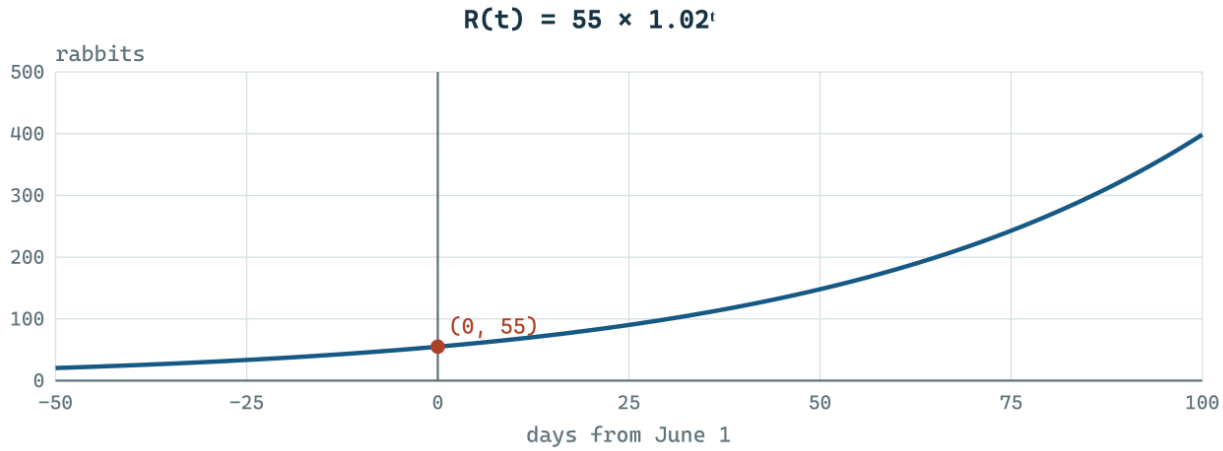
4. Explain why zooming cannot produce a zero

Every finite power of a positive base is positive, including reciprocal powers at negative t. The curve can get arbitrarily close to height zero while staying above it.



Move backward along the positive branch

Both 55 and every power of 1.02 are positive, so there is no finite zero even outside the window.



Check. The graph approaches zero only as t tends to negative infinity.

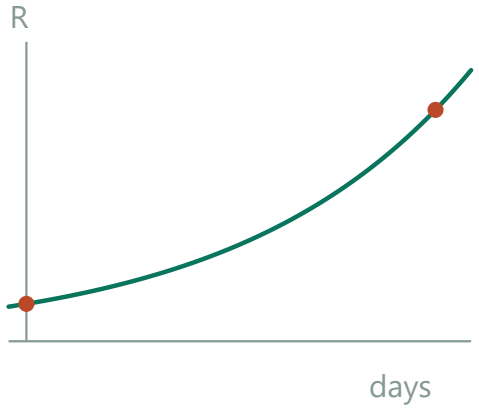
Exercise 540 · Instantaneous population growth

Printed page 57 · Exponential growth and decay / Rates of change

(Continuation) Choose a point on the graph that is very close to the R-intercept, then use these two points to estimate the rate (in rabbits per day) at which the population is growing on 1 June. In the same way, estimate the rate at which the population is growing on 1 September. Explain how your two answers relate to the given 2-percent growth rate.

1. Place the two dates on the time axis

June 1 is t=0, September 1 is t=92. June, July and August contain 30, 31 and 31 days. Their total puts September 1 at the specified elapsed time in days.

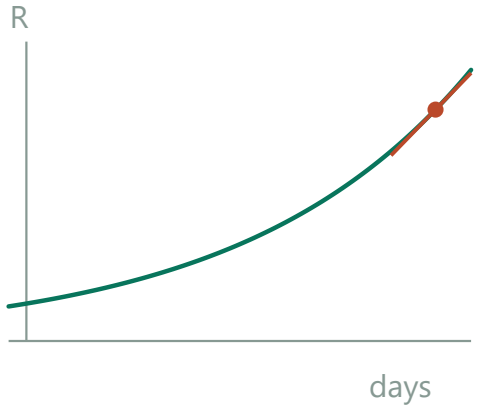


Mark June 1 and September 1

The dates correspond to t=0 and t=92 days.

2. Estimate slopes from nearby point pairs

A nearby secant of width .001 day gives [R(t+.001)−R(t)]/.001, about 1.089 rabbits/day on June 1 and 6.7345 on September 1.

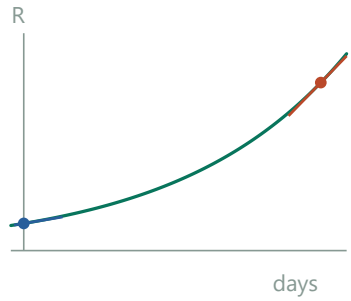


Use nearby times for a tangent estimate

Near September 1 the secant slope approaches about 6.73 rabbits/day.

3. Explain the scale difference

At any fixed small spacing h, the change quotient is R(t)[1.02^{h}−1]/h. The bracketed factor stays fixed across dates, so a larger September population produces a proportionally larger slope.



Compare the two local slopes

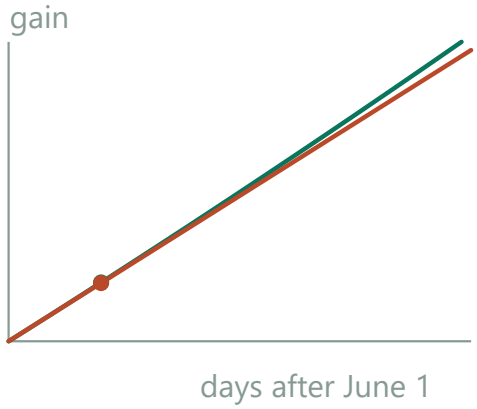
Near June 1 the smaller population gives a smaller slope, about 1.09 rabbits/day.

1. [R(t+h)−R(t)]/h=R(t)(1.02^{h}−1)/h
Factor the starting population from the secant numerator.

2. R′(t)=R(t) ln 1.02
As h tends to zero the second factor approaches the exponential's logarithmic rate.

4. Separate daily and instantaneous growth

In the limit $R'(t)=R(t)\ln 1.02$. The exact one-day gain is $.02R(t)$; the instantaneous rate is slightly smaller since growth continues during the day.



Distinguish one day's gain from the tangent

The tangent gives the limiting instantaneous rate $R\ln(1.02)$; the full-day gain is $0.02R$.

Check. The September-to-June growth-rate ratio is the same 1.02^{92} as the population ratio.

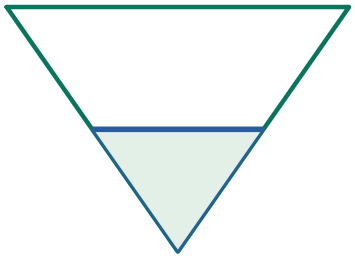
Exercise 541 · Half-full cone depth

Printed page 57 · Volume / Area and volume scaling

How deep is the water in a conical paper cup that is filled to half of its capacity?

1. Distinguish fill depth from fill fraction

Let H denote the full height and h the distance upward from the cone tip to the water surface. The water is a smaller copy of the full cone, with the same tip angle.

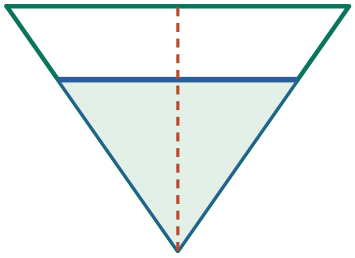


Try a half-height waterline

The halfway depth leaves only $(\tfrac{1}{2})^3=\tfrac{1}{8}$ of the cup's volume below it.

2. Scale lengths and volumes separately

The radius of the water surface is h/H times the rim radius. Thus two radius factors and one height factor in the volume are each scaled by h/H.



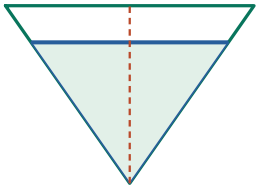
Move the waterline

Both the disk radius and the height change with the same scale factor; volume changes with its cube.

3. Solve the half-capacity condition

The water is a similar cone, so its volume fraction is $(h/H)^3$. Set it to ½ to get $h=H/\sqrt[3]{2}$, about 79.37% of the total height.

Water depth as a fraction of total height · 79.37% of H



Fill half the capacity

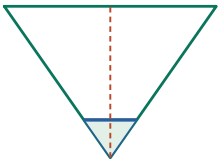
The required depth is $H\cdot\sqrt[3]{½}$, about 79.37% of the height.

1. $(h/H)^3=1/2 \Rightarrow h/H=\sqrt[3]{(1/2)}$
Take the positive cube root because the fraction is a depth ratio.

2. $h=H/\sqrt[3]{2}$
Multiply by the full height. This exceeds H/2 because volume scales cubically.

Compare settings · Water depth as a fraction of total height

25.00% of H



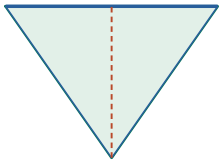
Change the depth fraction

A depth of 25.00% of the cup height gives 1.56% of its capacity. The radius and height scale together.

A depth of 25.00% of the cup height gives 1.56% of its capacity. The radius and height scale together.

Check. Cubing 0.7937005 gives approximately 0.5.

100.00% of H



Change the depth fraction

A depth of 100.00% of the cup height gives 100.00% of its capacity. The radius and height scale together.

A depth of 100.00% of the cup height gives 100.00% of its capacity. The radius and height scale together.

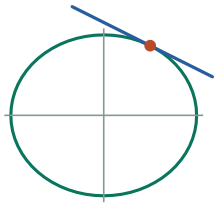
Exercise 542 · A line tangent to an ellipse

Printed page 58 · Ellipses / Tangent lines / Systems of equations

Verify that the line $x + 2y = 8$ meets the ellipse $3x^2 + 4y^2 = 48$ at exactly one point, namely $P = (2, 3)$. The line is said to be *tangent* to the ellipse. Verify also that the focal points for this ellipse are $F_1 = (-2, 0)$ and $F_2 = (2, 0)$.

1. Eliminate a coordinate

Solve the line equation for x. Substituting this into the ellipse gives $3(8-2y)^2+4y^2=48$. Expanding and grouping produces a perfect square, so there can be only one intersection value.



Impose both equations

The line $y=4-x/2$ meets the ellipse only where substitution satisfies the quadratic.

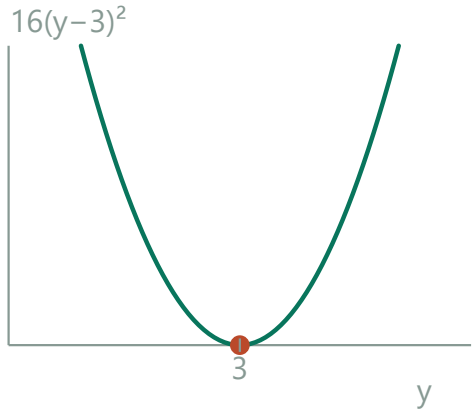
1. $3(8-2y)^2+4y^2=48$
Replace x by the expression from the line.

2. $192-96y+12y^2+4y^2-48=0$
Expand the square and subtract 48.

3. $16(y^2-6y+9)=16(y-3)^2=0$
Factor out 16, then recognize a perfect square.

2. Find the touching point

Substitute $x=8-2y$: $16(y-3)^2=0$. Thus $y=3, x=2$ is the only intersection.

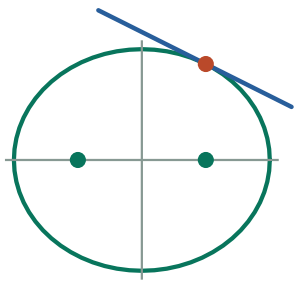


Find the double root

Elimination gives $16(y-3)^2=0$.
The one contact point is (2,3).

3. Use the semiaxes to find the foci

Divide the ellipse by 48 to get $x^2/16+y^2/12=1$, so $c=\sqrt{(16-12)}=2$ and the foci are $(\pm2,0)$.



Locate the foci

The semiaxes 4 and $\sqrt{12}$ give $c=2$, placing the foci at $(\pm2,0)$.

Check. At (2,3) the ellipse normal (6x,8y)=(12,24) is parallel to the line's normal (1,2).

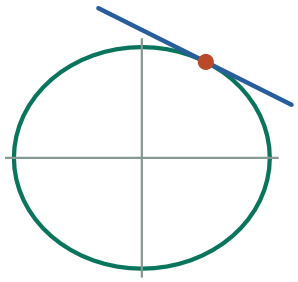
Exercise 543 · Reflection at an ellipse

Printed page 58 · Ellipses / Conic reflection properties

(Continuation) *The reflection property of the ellipse*: Use trigonometry to calculate the size of the acute angle formed by the tangent line and the focal radius F_2P. Do the same for the acute angle formed by the tangent line and the focal radius F_1P. How do your answers to these two calculations explain the title of this problem?

1. Use a vector along the tangent

A tangent direction is (2,−1). For x+2y=8 a change of +2 in x must be balanced by −1 in y. Thus the proposed vector lies along the tangent rather than across it.

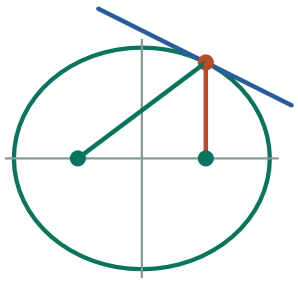


Read the tangent direction

The line x+2y=8 points in direction (2,−1).

2. Build both focus-to-contact vectors

The focal directions to P from (2,0) and (−2,0) are (0,3) and (4,3).

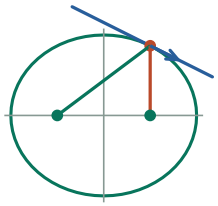


Connect each focus to the contact

The two directions at P=(2,3) come from the segments with components (0,3) and (4,3).

3. Find the acute angles

The direction vector along a line can be multiplied by −1. The absolute value of each dot product therefore selects the acute line angle. For example (2,−1)·(4,3)=5, and the lengths multiply to 5√5.



Measure each acute angle

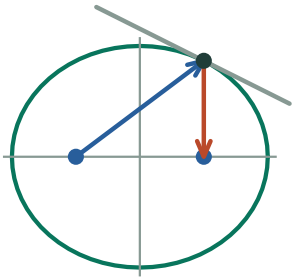
Taking the magnitude of the normalized dot product with the tangent gives 1/√5 in both cases.

1. abs((2,−1)·(0,3))/(√5·3)=3/(3√5)
Use absolute dot product because the requested angle is acute.

2. abs((2,−1)·(4,3))/(√5·5)=5/(5√5)
The second numerator is 8−3. Both cosines reduce to 1/√5.

4. Compare and interpret the angles

Taking absolute dot products divided by lengths gives cosine 1/√5 in both cases, or about 63.435°. Equal angles with the tangent mean a ray from one focus reflects toward the other.



Interpret the matching angles

A ray traveling from one focus and reflecting at P departs toward the other focus.

Check. The tangent direction is 26.565° below horizontal; it makes 63.435° with a vertical line.

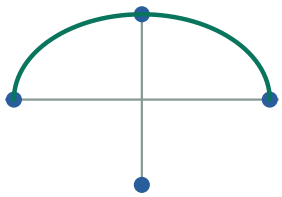
Exercise 544 · Split an ellipse into functions

Printed page 58 · Ellipses / Domain and range

Solve $4x^2 + 9y^2 = 36$ for y in terms of x. Graph both solutions.

1. Separate the two squared quantities

Subtract $4x^2$ from 36, then divide by 9. This gives $y^2=4(1-x^2/9)$. The quantity inside the parentheses must be nonnegative for a real y.

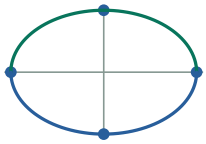


Take the positive square root

Solving for y^2 determines the magnitude of the height at each x.

2. Choose both signs

At each interior x the ellipse has an upper point and a lower point with the same squared height. Include both square roots; they meet at the two endpoints.



Keep the second root

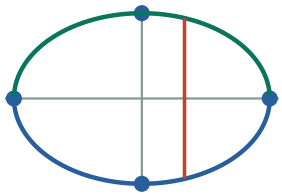
The upper graph uses the positive height and the lower graph its negative.

1. $9y^2=36-4x^2 \Rightarrow y^2=4(1-x^2/9)$
Subtract $4x^2$ and divide by 9.

2. $y=\pm 2\sqrt{(1-x^2/9)}$, $x^2\le 9$
Include both square-root branches and require a nonnegative radicand.

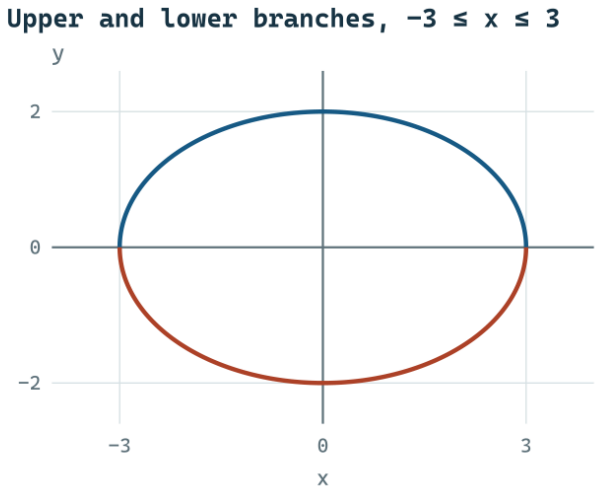
3. Write and sketch both branches

Move the x term, divide by 9 and take both roots: $y=\pm 2\sqrt{(1-x^2/9)}$ for $-3\le x\le 3$. Sketch the upper and lower halves through $(\pm 3,0)$ and $(0,\pm 2)$.



Trace both functions over their domain

Both branches lie over $-3\le x\le 3$ and meet at the two x-axis endpoints.

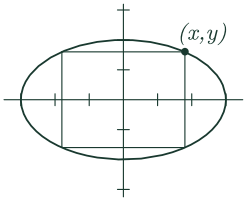


Check. Substituting either branch into the original equation gives 36.

Exercise 545 · Largest axis-aligned rectangle

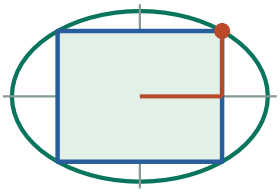
Printed page 58 · Ellipses / Optimization

(Continuation) The figure at right shows one of the many rectangles that can be inscribed in the ellipse $4x^2 + 9y^2 = 36$. What are the coordinates of point (x, y) that correspond to the rectangle with largest area?



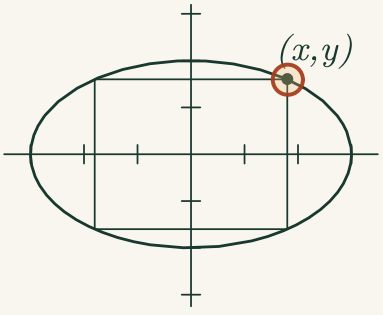
1. Translate the marked corner into an area

For a first-quadrant corner (x,y), area=4xy. Put u=x/3,v=y/2, so $u^2+v^2=1$.



Double each marked half-dimension

An axis-parallel rectangle stretching to (x,y) has width 2x and height 2y.

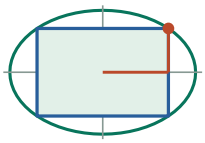


First-quadrant corner

The point labeled (x,y) lies on both the rectangle and the ellipse. Its horizontal and vertical distances from the two axes give the rectangle's half-dimensions.

2. Prove the best possible product

Since $(u-v)^2 \ge 0$, $2uv \le 1$ and equality requires $u=v=1/\sqrt{2}$. Thus the corner is $(3/\sqrt{2}, \sqrt{2})$, with maximum area 12.



Balance the normalized coordinates

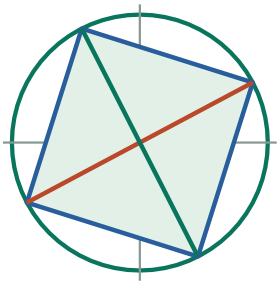
With $x/3=y/2=1/\sqrt{2}$, the normalized product is largest among axis-parallel rectangles.

1. $0 \le (u-v)^2 = u^2+v^2-2uv = 1-2uv$
The normalized corner lies on the unit circle.

2. $A = 4(3u)(2v) = 24uv \le 12$
Use $uv \le 1/2$. Equality requires $u=v$ and hence both equal $1/\sqrt{2}$.

3. Normalize a rectangle of any orientation

Even allowing arbitrary orientations cannot improve the bound. Sending (x,y) to $(x/3, y/2)$ takes any inscribed rectangle to a parallelogram in the unit circle and divides its area by 6.

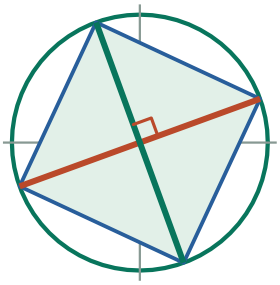


Transform ellipse to unit circle

Independent coordinate scaling turns any proposed rectangle into an inscribed parallelogram.

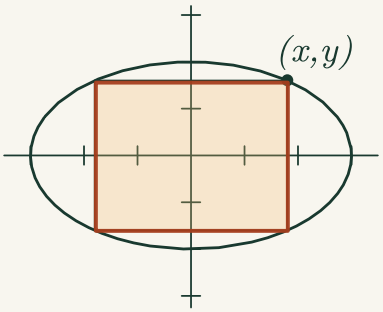
4. Bound its diagonals and area

Its diagonals are chords of length at most 2. The diagonal-area formula gives an area at most ½·2·2=2. The rectangle found above attains 12 in the original ellipse.



Bound the diagonal product

Both diagonals are chords of length ≤2. The area cannot exceed half their product, at most 2.

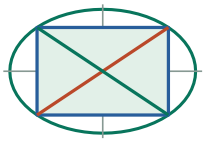


The inscribed rectangle

Each side spans twice the corresponding corner coordinate in this axis-aligned drawing.

5. Explain equality and uniqueness

Equality requires two perpendicular diameters as diagonals, so the normalized parallelogram must be a square. If a side makes angle φ with the horizontal, adjacent unit side directions become (3 cos φ,2 sin φ) and (−3 sin φ,2 cos φ) when scaled back. Their dot product is −5 sin φ cos φ. It vanishes only for axis-parallel sides, so the first-quadrant corner found above is unique.



Return the maximizing square to the ellipse

Equality gives a unit-circle square whose scaled sides remain perpendicular in the axis-aligned orientation.

1. (3cos φ,2sin φ)·(−3sin φ,2cos φ)
Stretch perpendicular side directions of the normalized square back to the ellipse.

2. =−9cos φ sin φ+4sin φ cos φ=−5sin φ cos φ
The stretched sides stay perpendicular only if one direction component vanishes.

Check. Parametrization gives area 12sin(2θ), also maximal at θ=π/4.

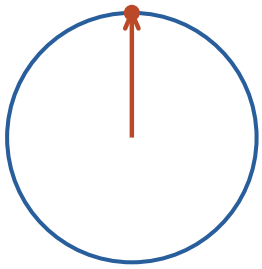
Exercise 546 · Tangent plane to a sphere

Printed page 58 · Sphere geometry / Planes in space / Tangent lines

Find an equation for the plane tangent to the sphere $x^2 + y^2 + z^2 = 81$ at $(1, 4, 8)$.

1. Mark the radius through the contact point

The sphere is centered at the origin. The radius to (1,4,8) has squared length 1+16+64=81, so the point is indeed on the sphere.

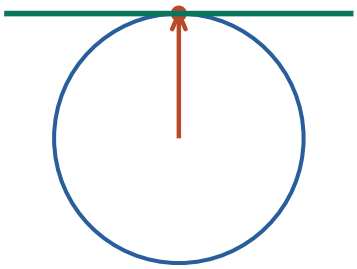


Locate P on the sphere

The radius OP points along (1,4,8) and has length √81=9.

2. Translate tangency into perpendicularity

A displacement within the tangent plane is perpendicular to the radius at contact. Use (x−1,y−4,z−8) for that displacement and make its dot product with the radius equal zero.

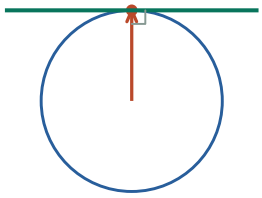


Cut through the normal at P

In this cross section the tangent plane meets the sphere at a single point, perpendicular to OP.

3. Write the plane equation

The radius vector (1,4,8) is perpendicular to the tangent plane. Set (1,4,8)·(x−1,y−4,z−8)=0, obtaining x+4y+8z=81.



Sweep out the full tangent plane

Every in-plane displacement has zero dot product with (1,4,8), giving x+4y+8z=81.

1. x−1+4y−16+8z−64=0
Expand the dot product of the plane displacement with the sphere radius.

2. x+4y+8z=1+16+64=81
Check the given contact point satisfies this equation.

Check. Distance from origin to this plane is 81/9=9, exactly one radius.

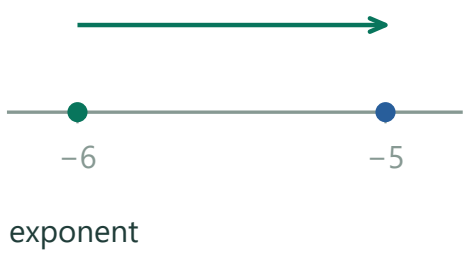
Exercise 547 · Leading decimal zeros

Printed page 58 · Digits and place value / Logarithms

The expansion of $\left(\frac{2}{3}\right)^{30}$ begins with $0.000\ldots$. How many zeros are there between the decimal point and the first nonzero digit? In the expansion of $\left(\frac{2}{3}\right)^{1\,000\,000}$ how many zeros are there between the decimal point and the first nonzero digit?

1. Relate leading zeros to neighboring powers

A number between 10^{-6} and 10^{-5} has exactly five zeros after the decimal before its first nonzero digit. More generally use consecutive negative powers of ten to locate the first significant digit.

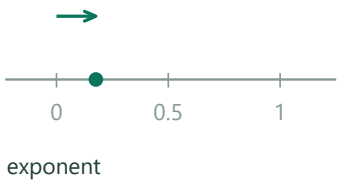


Place the first significant digit

Between 10^{-6} and 10^{-5}, the first nonzero digit sits in decimal place 6.

2. Take logs to count the decimal places

Write $\log_{10}((2/3)^n) = -n\log_{10}(3/2)$. For these noninteger logarithms, the number of zeros after the decimal before the first nonzero digit is $\lfloor n\log_{10}(3/2)\rfloor$.



Count exponent distance

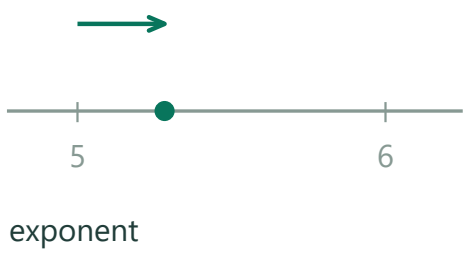
Each factor 2/3 moves left by $\log(3/2)$ decades. Its magnitude is about 0.176091.

1. $k < n\log_{10}(3/2) < k+1 \Rightarrow -k-1 < \log_{10}((2/3)^n) < -k$
Negation reverses the inequalities; the log is noninteger here.

2. $10^{(-k-1)} < (2/3)^n < 10^{(-k)}$
Exponentiating preserves order, placing the first nonzero decimal digit just after k zeros.

3. Apply the count at both exponents

At n=30 it is 5; at n=1,000,000 it is 176,091.



Take the whole part

Thirty factors travel about 5.283 decades: five leading zeros follow the decimal. A million factors give 176091.

Check. For n=30 the log is about −5.2827, placing the number between 10^{-6} and 10^{-5}.

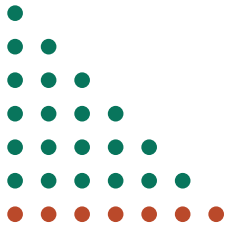
Exercise 548 · Triangular numbers

Printed page 58 · Triangular numbers / Sequences and recursion

For the list of *triangular numbers* 1, 3, 6, 10, 15, 21, . . . , describe the pattern, calculate the next three terms, and explain the terminology. Is this a geometric sequence? Explain.

1. Build the triangular arrays one row at a time

Each new triangular array adds a row one dot longer than the previous row. Add 7,8,9 to obtain 28,36,45.

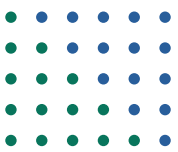


Add the next row

The orange seventh row adds 7 to 21, giving 28. Rows of 8 and 9 make 36 and 45.

2. Relate the dot drawing to a formula

Two copies of a triangle containing rows 1 through n can be paired into n rows each of length n+1. One copy therefore has half the rectangle's dot count.



Pair two copies

The green dots make rows 1 through 5. A rotated blue copy fills a rectangle of five rows and six columns.

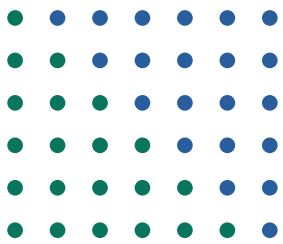
1. $T_n = 1+2+\cdots+n$; $T_n = n+(n-1)+\cdots+1$
Reverse the same sum to line up first and last terms.

2. $2T_n = (n+1)+\cdots+(n+1) = n(n+1)$
Each of the n columns adds to n+1.

3. $T_n = n(n+1)/2$
Divide by two because the triangular array was counted twice.

3. Compute a general term and classify the sequence

General term n(n+1)/2. Consecutive ratios change, so the sequence is not geometric.



Generalize the rectangle count

For n=6 the rectangle has 6×7 dots. One triangle always contains half of n(n+1).

Check. Two copies of the triangular array fit into an n-by-(n+1) rectangle.

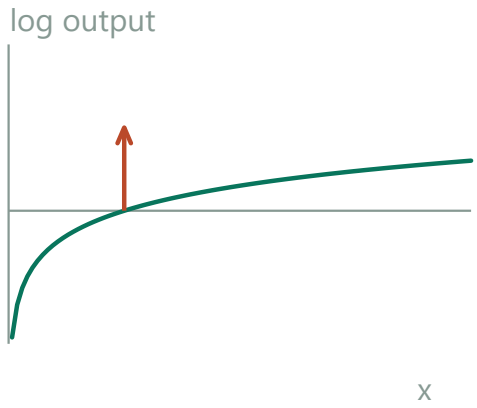
Exercise 549 · Scale the input of a log

Printed page 58 · Graph transformations / Logarithm rules

Compare the graph of $y = \log x$ and the graph of $y = \log(10x)$. How are they related?

1. Compare outputs at the same x

On the domain x>0, replacing the input x by 10x multiplies the positive argument by a ten. On a common-log scale a tenfold increase adds one to the output.

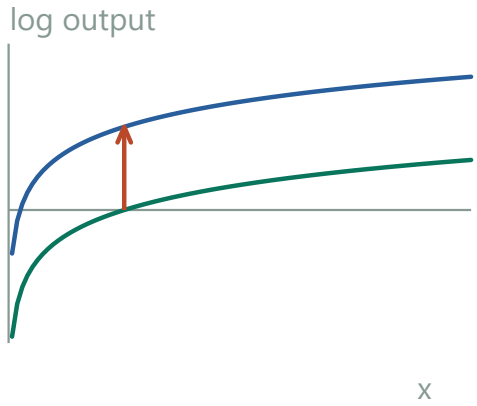


Choose a reference input

At x=1, log x is zero. Multiplying the input by 10 increases its log by one.

2. Use the log rule

For $x>0$ the product rule gives $\log_{10}(10x)=1+\log_{10}x$, a shift up one unit.

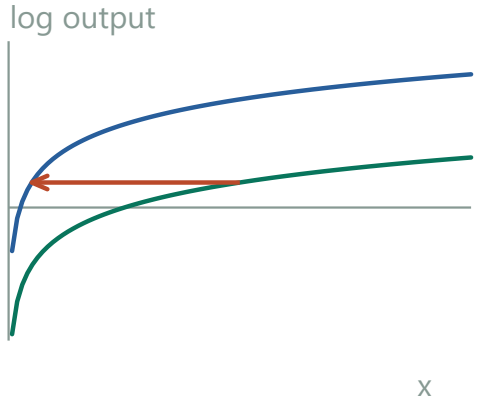


Draw the vertical translation

Every point of the green graph moves up one unit to the blue graph.

3. Describe the same change horizontally

Equivalently the input scaling compresses horizontal coordinates by 10.



Read the same change horizontally

At fixed height, the original x coordinate 2 becomes 0.2 on the blue curve.

Check. At $x=1$ the ordinates are 0 and 1.

Exercise 550 · Halley's ellipse

Printed page 58 · Ellipses / Kepler's third law / Eccentricity

For any object that orbits the sun, Kepler's Third Law relates the *period* — the time needed for one orbit — and the *mean distance* from the sun — the average of the least and greatest distances (recall that the sun is at a focus). Halley's comet has a period of 76 Earth years (it next returns in 2061), and the least distance from the comet to the sun is 0.59 *astronomical unit* (one a.u. is about 93 million miles). Use Kepler's Third Law to calculate the mean distance and the greatest distance from the comet to the sun, and the eccentricity of the comet's orbit.

1. Read Kepler's units

Use Earth years for period and astronomical units for distance, so the ideal law is period^2=semimajor axis^3. The period of 76 years thus fixes the geometric mean-distance parameter a.

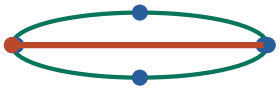


Use AU and years

For a 76-year period, Kepler's law $t^2=a^3$ determines a in AU.

2. Find the semimajor axis

Kepler gives mean distance a=76^(2/3)≈17.9422 AU.



Find the orbit's half-span

The semimajor axis is about 17.942 AU. The Sun sits near one tip.

3. Use the near endpoint to determine the far one

Since a−c=.59, aphelion is a+c=2a−.59≈35.2944 AU and eccentricity is c/a=1−.59/a≈0.96712.



Use the observed near distance

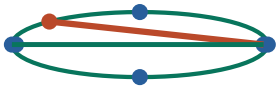
The closest distance is a−c=0.59 AU; the farthest one is 2a−0.59≈35.29 AU.

1. c=a−0.59 ⇒ a+c=2a−0.59
The focus is displaced toward the nearest end; substitute that offset into the far distance.

2. e=c/a=(a−0.59)/a=1−0.59/a
Divide both numerator terms by the positive semimajor axis.

4. Interpret the eccentricity

The distance c measures how far the Sun sits from the ellipse center along the major axis. Dividing by the much larger a gives a value below 1 but close to it, consistent with this highly elongated orbit.



Read eccentricity from the offset

The near-tip Sun gives c/a≈0.967, producing a strongly elongated orbit.

Check. The mean of the two extreme distances is a, and $a^3=76^2$ in these units.

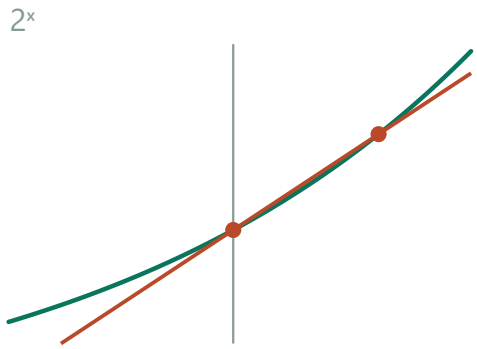
Exercise 551 · Estimate the slope of 2^x at zero

Printed page 58 · Exponential functions / Slopes and tangent lines / Rates of change

Estimate the slope of the curve $y = 2^x$ where it crosses the y-axis. The *slope of a curve* at a point P means the slope of the tangent line at P, and this line can be approximated by a *secant line* that goes through P and a nearby point on the curve.

1. Choose the point and a nearby input

The graph crosses the y-axis at (0,1). The slope to (h,2^h) is (2^h−1)/h.

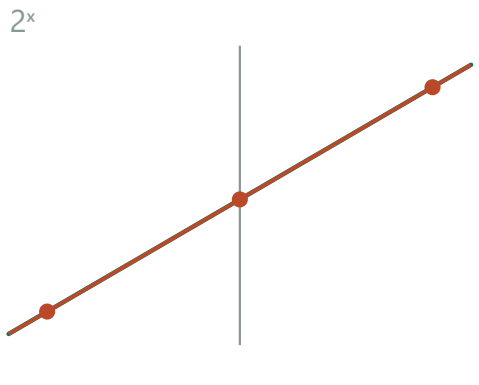


Start at (0,1)

A secant joins (0,1) to a second point of $y=2^x$.

2. Use two small steps with opposite signs

At h=.001 it is approximately .69339 and at h=−.001 approximately .69291.

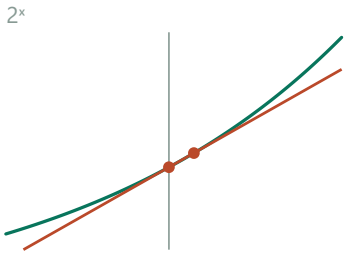


Approach from both sides

At h=±0.001 the neighboring slopes straddle about 0.69315; this view enlarges that interval.

3. Explain what the limiting slope means

Both secants pass through (0,1), one approaching it from the right and the other from the left. Their numbers close in as the input difference shrinks. The tangent is their common limiting direction.



Shrink the horizontal increment

The secant slope $(2^h-1)/h$ tends to a fixed value as h approaches zero.

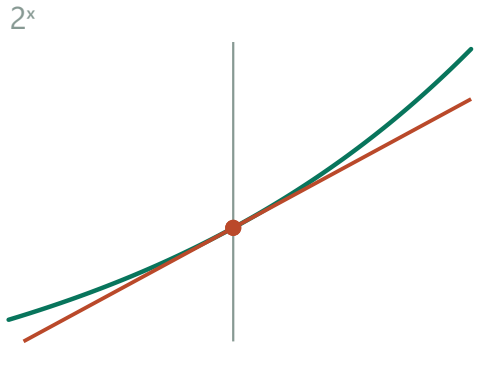
1. $2^h=e^{(h \ln 2)}$
Rewrite the base so the standard exponential limit applies.

2. $(2^h-1)/h=\ln 2\cdot(e^{(h \ln 2)}-1)/(h \ln 2)$
Multiply and divide by the same nonzero logarithm, for $h\neq 0$.

3. lim as h→0 = ln 2
The second factor tends to 1 as h ln 2 tends to zero.

4. Read the tangent estimate

The limiting slope is ln2≈.693147. The two slopes at small positive and negative h straddle this value; moving the second point still closer narrows the gap.



Read the tangent at zero

The limiting tangent slope is ln 2≈0.693147.

Check. The symmetric secant between ±.001 gives .693147 to six decimal places.

Exercise 552 · Inverse pairs in two tables

Printed page 59 · Logarithms / Inverse functions

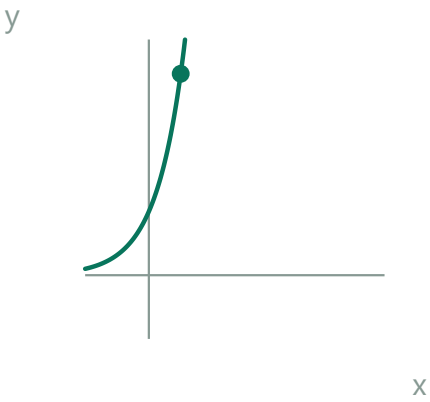
Fill in the missing entries in the two tables shown at right. Do this without a calculator.

x	10^x
-3	
	1
0.5	3.162
	100
3	
	1996

x	$\log x$
0.001	
	0
3.162	
	2
1000	
1996	3.300

1. Complete the exponential table

The exponential table pairs are $(-3,.001),(0,1),(.5,\sqrt{10}\approx 3.162),(2,100),(3,1000),(\log 1996\approx 3.300,1996)$.

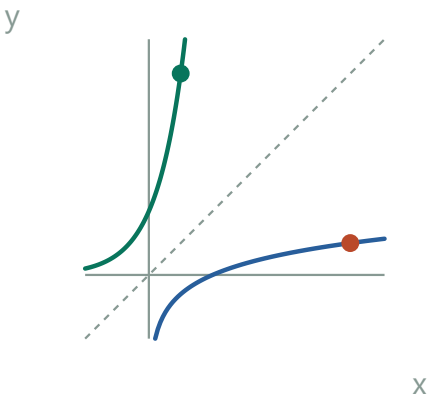


Plot an exponential-table pair

The highlighted visible entry is $(0.5,\sqrt{10})$. The same exponential rule supplies the full table.

2. Reverse each pair for the inverse table

Reverse each ordered pair for the log table. For instance, $10^{-3}=0.001$ becomes $\log(0.001)=-3$. Enter the same two numbers in the opposite column positions in the logarithm table.



Swap horizontal and vertical coordinates

The reflected point $(\sqrt{10},0.5)$ lies on the blue logarithm curve.

x	10^x
-3	
	1
0.5	3.162
	100
3	
	1996

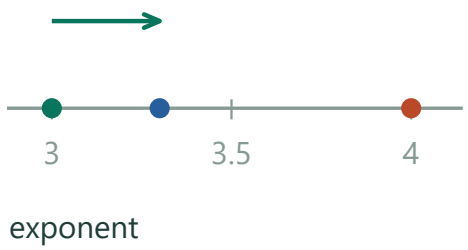
x	$\log x$
0.001	
	0
3.162	
	2
1000	
1996	3.300

Number and logarithm table

The right table reverses the roles of each left-hand pair. A known number becomes the logarithm problem's input.

3. Interpret the approximate entries

The exponent for 1996 is a little above 3 because 1996 is between 1000 and 10000. Its printed 3.300 and the decimal approximation for √10 carry limited precision.

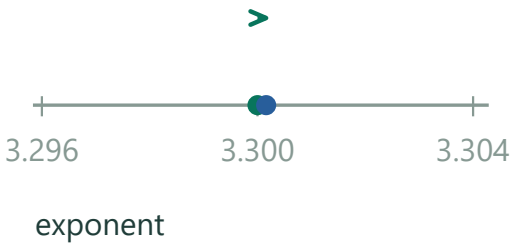


Bracket the exponent of 1996

It lies above 3 and below 4 since its output is between 1000 and 10000.

4. Distinguish exact from rounded data

The supplied 3.162 and 3.300 are rounded. The displayed √10 is a rounded irrational number. Likewise 3.300 is a rounded logarithm of 1996, so raising 10 to the displayed exponent reproduces the given value only approximately.



Magnify the rounded exponent

The marked values compare 3.300 with $\log_{10}1996 \approx 3.30016$. Their powers agree only to the displayed precision.

x	10^x
-3	
	1
0.5	3.162
	100
3	
	1996

x	$\log x$
0.001	
	0
3.162	
	2
1000	
1996	3.300

The rounded bottom pair

Use the row containing 1996 to understand how approximate finite-digit entries behave when input and output are interchanged.

Check. Reversing a pair expresses the same power equality by a logarithm.

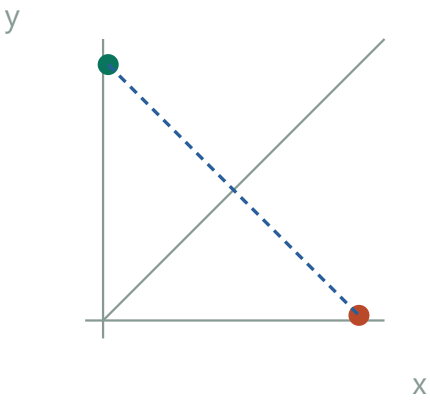
Exercise 553 · Exponential and log as inverses

Printed page 59 · Inverse functions / Exponential functions / Logarithms

(Continuation) What do your results tell you about the graphs of $y = 10^x$ and $y = \log x$? It is customary to call functions that are related in this way *inverse functions*. What do you notice about the domains and *ranges* of the functions?

1. Compare a pair from each table

For example, the exponential has (2,100) and the logarithm has (100,2). Reversing the roles of input and output exchanges the horizontal and vertical coordinates.

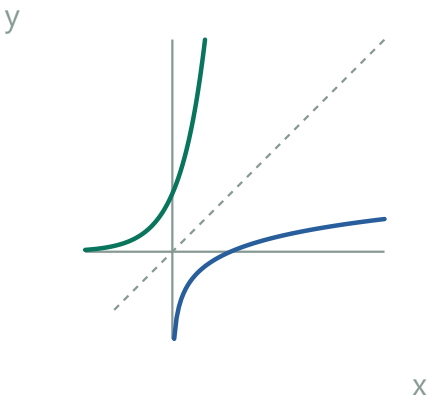


Exchange the two coordinates

(2,100) and (100,2) lie on opposite sides of the diagonal, at matching perpendicular distances.

2. Explain the symmetry

Interchanging (x,y) and (y,x) reflects a graph across y=x. Hence y=10^x and y=log x are reflections.

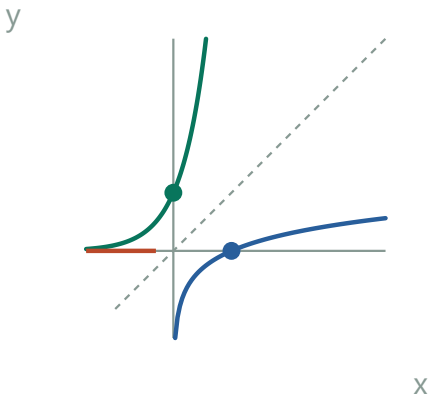


Reflect the entire exponential curve

The green exponential and blue log are mirror images across the dashed diagonal y=x.

3. Read the domain and range of each graph

Exponential takes all real numbers to all positive numbers; log takes all positive numbers to all real numbers.



Track admissible coordinates

Exponential outputs stay above the horizontal axis. After reflection, log inputs stay right of the vertical axis.

Check. Each composition returns its input on the appropriate domain.

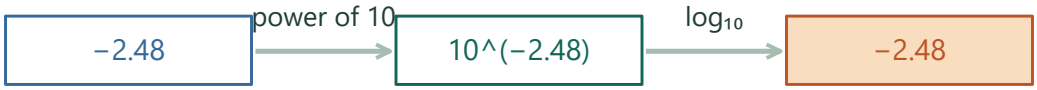
Exercise 554 · Undo two nested operations

Printed page 59 · Logarithms / Inverse functions

Without using a calculator, simplify **(a)** $\log\left(10^{-2.48}\right)$ **(b)** $10^{\log 4.8}$

1. Use the logarithm definition directly

The first expression asks for the exponent on 10 that produced 10^(−2.48). The indicated exponent is already visible, with its negative sign.



The exponent can be negative. The number inside the logarithm is positive.

Work from left to right. Taking the logarithm recovers the exponent used by the first arrow.

2. Reverse the order in the second expression

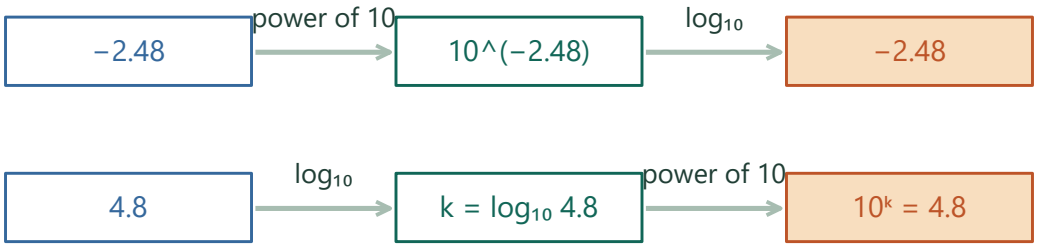
Call $\log_{10}4.8$ the number k. The defining statement is $10^k=4.8$, so using k as the exponent brings back exactly the input number.



Start with a positive number. The logarithm finds its exponent, and the power of ten recovers the input.

3. State the two exact results

Inverse operations give −2.48 and 4.8, respectively.



The two pairs of arrows have opposite orders. Each pair returns its own input.

Check. 10 raised to the first answer restores the first log argument, and log of the second restores log 4.8.

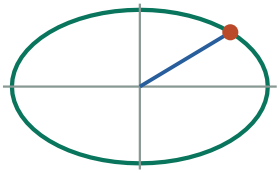
Exercise 555 · Eliminate a shifted ellipse parameter

Printed page 59 · Ellipses / Parametric equations / Translations

Justify the equivalence of $(x, y) = (2+5\cos t, -1+3\sin t)$ and $\dfrac{(x-2)^2}{25}+\dfrac{(y+1)^2}{9}=1$, by first solving the parametric equations for $\cos t$ and $\sin t$, then squaring both sides.

1. Undo the translation and the stretching

Subtract the center coordinate before dividing each coordinate by its semiaxis. For y this means $y-(-1)=y+1$. The normalized coordinates become a cosine and a sine of the same angle.

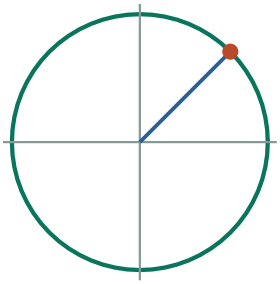


Measure from $(2,-1)$

Subtract the center before dividing by the semiaxes 5 and 3.

2. Eliminate the parameter

Solve $\cos t=(x-2)/5$ and $\sin t=(y+1)/3$; square and add to obtain $(x-2)^2/25+(y+1)^2/9=1$.

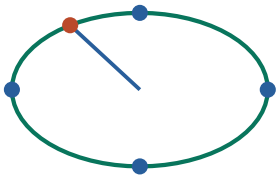


Normalize to a unit circle

The two normalized components become $\cos t$ and $\sin t$; their squares sum to 1.

3. Show that all points are recovered

Conversely every solution gives a normalized pair on the unit circle, so some t has those sine and cosine values.



Reverse the scaling and shift

Every normalized unit vector has an angle t, hence every ellipse point is reached.

Check. The four quarter turns yield $(7,-1),(2,2),(-3,-1),(2,-4)$, also the axis endpoints.

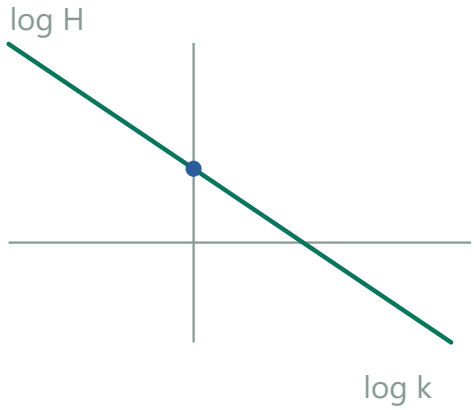
Exercise 556 · Undo a line on log axes

Printed page 59 · Data linearization / Power functions

Suppose that the logarithms of two quantities H and k are related by the equation $\log H = 1.48 - 2.5 \log k$. Relate these two quantities by an equation that makes no reference to logarithms. Sketch a graph of $\log H$ versus $\log k$. On a separate set of axes sketch a graph of H versus k. Recall that a graph of a versus b means a is plotted on the vertical axis and b is plotted on the horizontal axis.

1. Undo the logarithms while retaining the coefficient

Apply a power of 10 to both sides. The expression 10^(1.48−2.5 log k) separates into 10^1.48 times (10^(log k))$^{-2.5}$. This requires k>0.

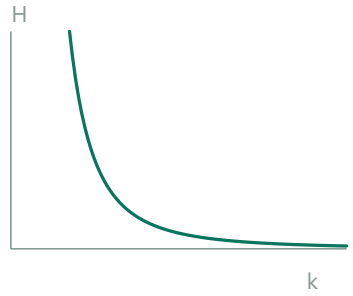


Read the log intercept

A log-input of 0 corresponds to k=1. The intercept 1.48 therefore makes H=$10^{1.48}$ there.

2. Write the raw-variable curve

Exponentiating base 10 gives H=10^(1.48) k^(−2.5) for k>0.



Return to original variables

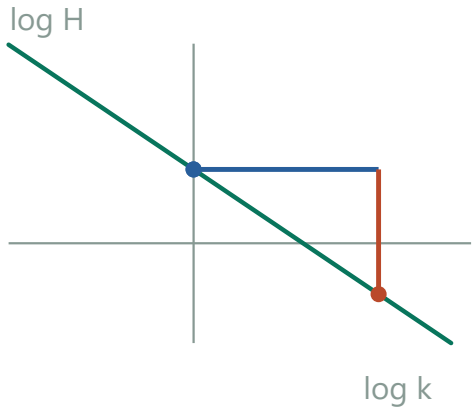
The power −2.5 produces a decreasing curve for positive k.

1. 10^(log H)=10^1.48/10^(2.5 log k)
A difference in the exponent becomes a quotient. The original log requires k>0.

2. H=10^1.48/k^2.5=10^1.48·k^(−2.5)
Use 10^(log k)=k and invert the denominator.

3. Draw the logarithmic plot

With horizontal log k, the graph is a straight line of slope −2.5 and intercept 1.48.

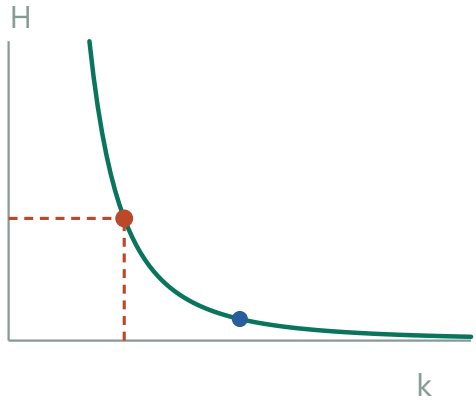


Measure the log slope

A one-unit increase in log k drops log H by 2.5, from 1.48 to −1.02.

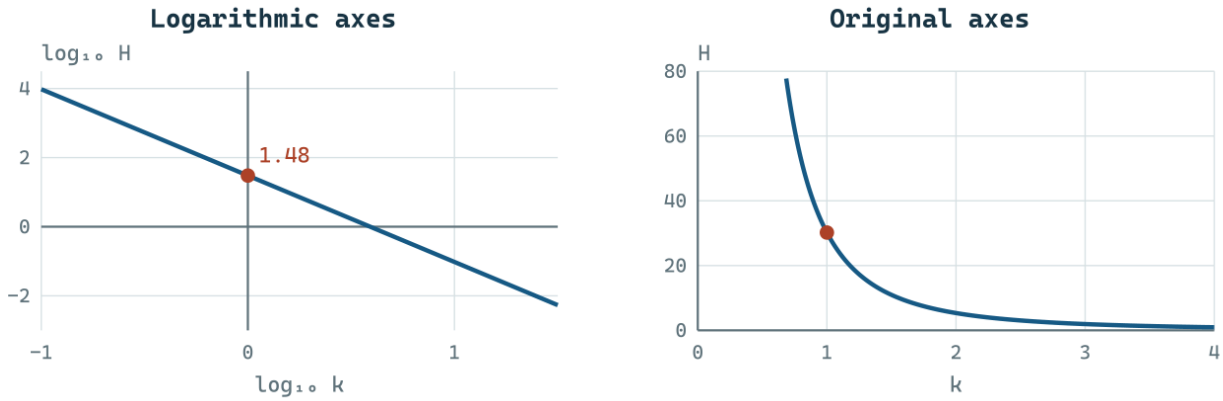
4. Read the untransformed axes

With horizontal k, H decreases steeply from infinity toward zero, passing through (1,10^(1.48)).



Mark a reference point

At k=1, H≈30.20. The positive branch rises without bound near zero and approaches zero to the right.



Check. Doubling k multiplies H by 2^(−2.5), equivalent to a drop of 2.5log 2 in log H.

Exercise 557 · Test two linear relationships

Printed page 59 · Data linearization / Logarithm rules

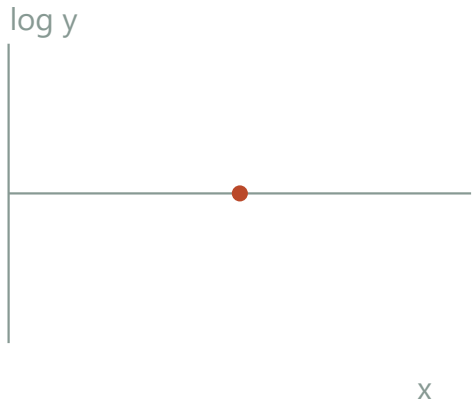
If there exist constants a, b and c such that $a \cdot p + b \cdot q = c$, then the variable quantities p and q are *linearly related*. For example, the equation $2.5 \log k + \log H = 1.48$ shows that $\log k$ and $\log H$ are linearly related with $a = 2.5$, $b = 1$ and $c = 1.48$.

(a) Are x and $\log y$ linearly related if $x = 2.5 + \dfrac{\log y}{3}$?

(b) Are x and y linearly related if $6xy = 10$?

1. Use the definition on the named variables

In (a) treat x and log y as the two coordinates. Their coefficients must be constants and one may clear denominators to expose an ordinary line equation.

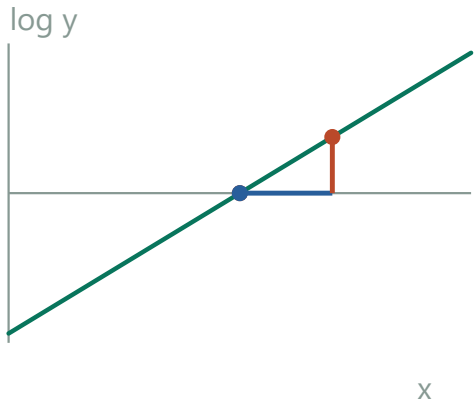


Name the coordinate variables

Choose x horizontally and log y vertically. The relation has a zero vertical coordinate when x=2.5.

2. Test the printed logarithmic relation

(a) The printed fraction is (log y)/3. Multiply x=2.5+(log y)/3 by 3 and rearrange to obtain 3x−log y=7.5, so x and log y are linearly related for y>0.

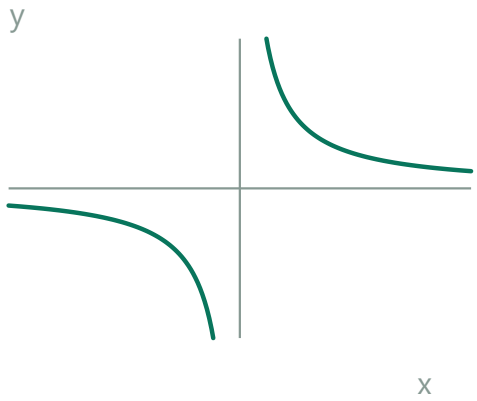


Expose the line equation

The rise is 3 for a run of 1 because log y=3x−7.5.

3. Examine the reciprocal relation

(b) 6xy=10 gives y=5/(3x) for x≠0. It is a reciprocal curve.

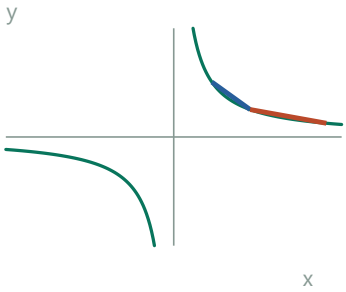


Plot the reciprocal relation

On the original x,y axes the product xy=5/3 makes two branches.

4. Prove one line cannot fit it

For example the slopes between the points with x=1,2 and x=2,4 are −5/6 and −5/24, respectively, so no single line contains these points.



Compare adjacent secants

The slopes on [1,2] and [2,4] are −5/6 and −5/24. A straight line could not have both.

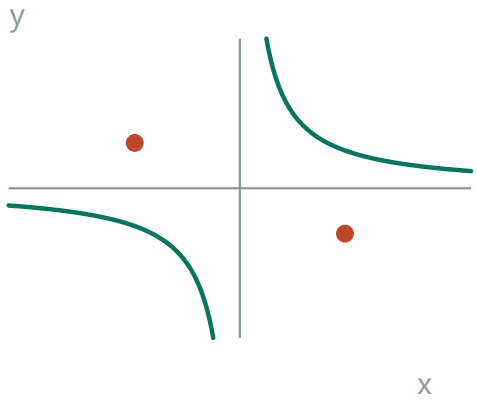
1. y(1)=5/3; y(2)=5/6; y(4)=5/12
Insert three positive x-values into y=5/(3x).

2. $m_1=(5/6-5/3)/(2-1)=-5/6$
Measure the change over the first interval.

3. $m_2=(5/12-5/6)/(4-2)=-5/24$
The unequal slopes rule out a single straight line.

5. State the nontriviality condition

The definition of a linear relation implicitly requires at least one of its two coefficients to be nonzero; the identity 0=0 would otherwise describe every set of data.



Test a vacuous equation

The orange off-curve points would also satisfy 0=0. That identity cannot distinguish the reciprocal graph.

Check. Doubling x halves y in (b); secant slopes differ for different intervals.

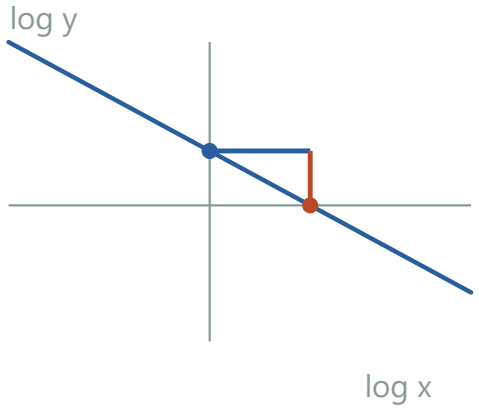
Exercise 558 · General log–log line

Printed page 59 · Data linearization / Power functions / Logarithm rules

Suppose that $\log y$ and $\log x$ are linearly related such that $a \log y + b \log x = c$. Show that y can be expressed as a *power function* of x. In terms of a, b, and c, find this function.

1. Separate the dependent log

Assuming the coefficient a is nonzero, subtract b log x from both sides and then divide by a. The result gives a straight-line equation in the two logarithmic coordinates.

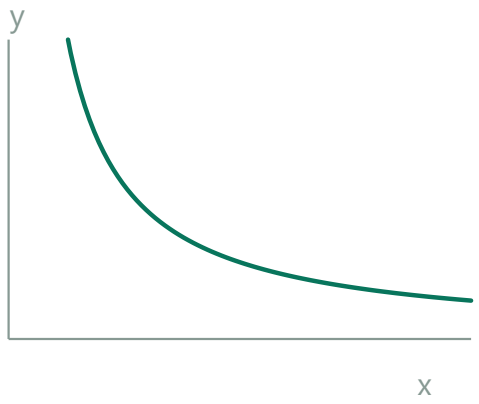


Read slope and intercept in log coordinates

For illustration choose a=b=c=1. The log line has intercept 1 and slope −1.

2. Undo the logarithms

For positive x,y and a≠0, solve log y=c/a−(b/a)log x. Exponentiating yields y=10^(c/a)x^(−b/a).

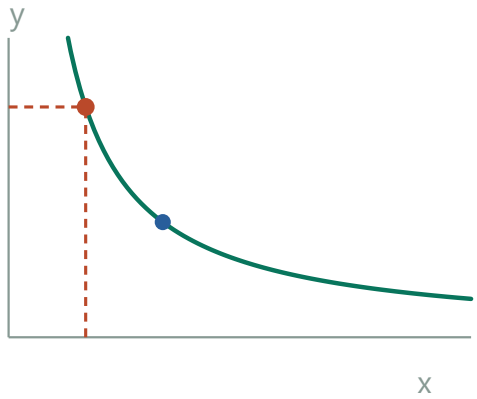


Undo the sample relation

The sample line log y=1−log x becomes y=10/x on positive axes.

3. Identify the constant and exponent

The coefficient of the power function is 10^(c/a), and its exponent is −b/a. The two logarithms in the original relationship require positive x and y.

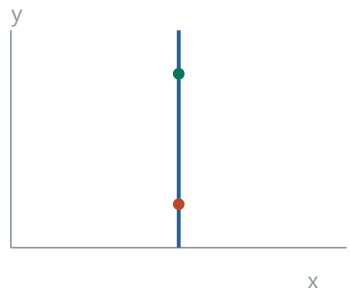


Extract the constants

In the example, y=10 at x=1, and doubling x halves y. In general the coefficient is 10^(c/a) and exponent −b/a.

4. Handle the zero-coefficient case

If a=0 the relation does not in general define y as a function of x.



Test a=0

If only b log x=c remains, x can be fixed while any positive y fits. The illustrated vertical line x=2 is one example.

1. a=0, b≠0 ⇒ b log x=c ⇒ x=10^(c/b)
The equation then fixes one positive x, leaving every positive y possible.

2. a=b=0 ⇒ 0=c
If c=0 every positive pair works; otherwise there are no solutions.

Check. Taking logs of the power-law expression recovers the original coefficients.

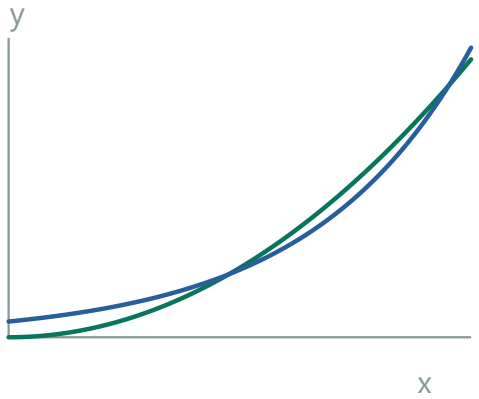
Exercise 559 · Power versus exponential

Printed page 59 · Power functions / Exponential functions

Ming is wondering how to distinguish a power function from an exponential function. How would you explain the difference?

1. Identify the variable's job

A power function has fixed exponent and variable base, for example $3x^2$. An exponential has fixed positive base and variable exponent, for example 3·2^x.

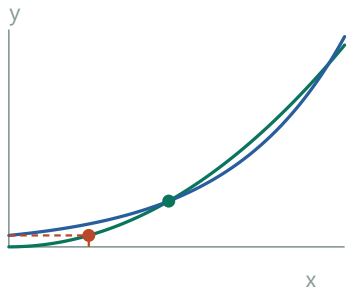


Compare two rules on the same axes

Green is $3x^2$ and blue is $3\cdot 2^x$. The variable is a base in green and an exponent in blue.

2. Test by changing the input in two ways

For a power law, multiplying x by a fixed factor multiplies y by a fixed factor. For exponential growth, adding a fixed amount to x multiplies y by a fixed factor.



Double a power-law input

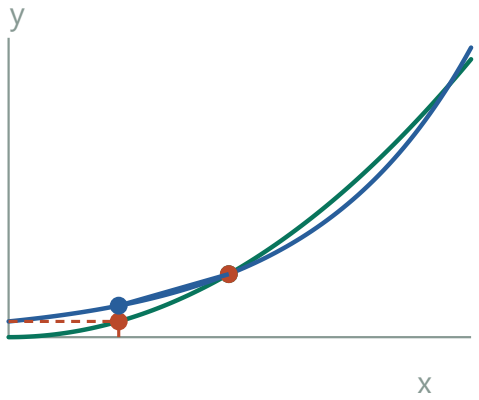
On green, x=1 gives 3 and x=2 gives 12. The doubled input multiplies the output by four.

1. $f(x)=Cx^p \Rightarrow f(kx)=k^p f(x)$
Multiplying a positive input by k>0 produces a fixed output multiplier.

2. $g(x)=Cb^x \Rightarrow g(x+h)=b^h g(x)$
For an exponential the corresponding multiplier comes from adding h to the input.

3. Try the same two sample rules

With $y=3x^2$, replacing x by 2x gives $12x^2$, four times the original. With $y=3\cdot 2^x$, replacing x by x+1 gives $3\cdot 2^x\cdot 2$. The first multiplier follows an input ratio, the second an input difference.



Advance an exponential input

On blue, x=1 gives 6 and x=2 gives 12. One unit of input doubles the output.

Check. Doubling x quadruples $3x^2$; adding 1 to x doubles 3·2^x.

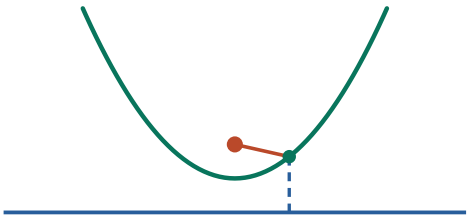
Exercise 560 · Find a quadratic's focus

Printed page 59 · Parabolas / Foci and directrices

Show that the graph of the quadratic equation $y = x^2$ is a parabola, by finding coordinates for its focus and an equation for its directrix.

1. Choose a focus and directrix symmetrically

The graph has its vertex at the origin and opens upward. Put the focus at (0,p) and the directrix at y=−p. Equal squared distances give $x^2+(y-p)^2=(y+p)^2$, hence $x^2=4py$.

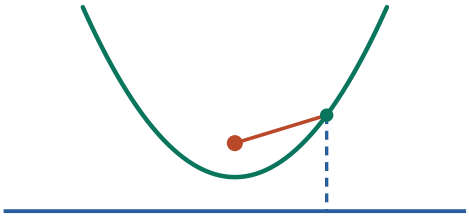


Place symmetric focus and directrix

For $y=x^2$, try a focus (0,p) and directrix y=−p equally far above and below the vertex.

2. Match the given coefficient

To obtain $y=x^2$ rather than a wider or narrower curve, set 4p=1. This determines the distance from the vertex to both the focus and the directrix.

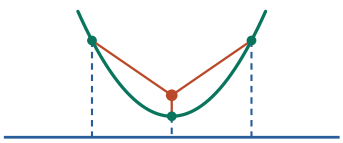


Determine p

The equal-distance condition simplifies to $x^2=4py$, so p=¼ matches the given graph.

3. Confirm the entire locus

Take F=(0,¼) and directrix y=−¼. Equality of squared distances is $x^2+(y-¼)^2=(y+¼)^2$. Expanding leaves $x^2=y$. Thus the locus equals the entire quadratic graph.



Test the entire locus

Every point on $y=x^2$ has equal focal and directrix distances; reversing the algebra proves the converse.

1. $\mathbf{x^2=(y+¼)^2-(y-¼)^2}$
Subtract the squared vertical focal displacement from both sides.

2. $\mathbf{x^2=[(y+¼)-(y-¼)][(y+¼)+(y-¼)]=(½)(2y)=y}$
Factor the difference of squares. Nonnegative distances make the reverse implication valid.

Check. At the vertex the two distances are each ¼, and both sides of the distance equation are nonnegative.

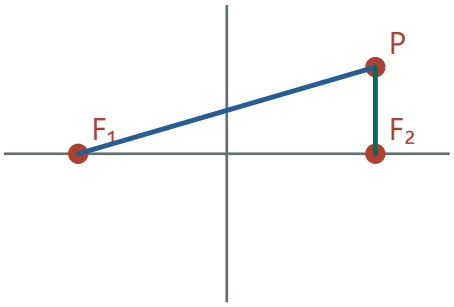
Exercise 561 · Four ellipse intercepts

Printed page 59 · Ellipses / Foci and directrices

The focal points of an ellipse are $(12, 0)$ and $(-12, 0)$, and the point $(12, 7)$ is on the ellipse. Find the points where this curve intersects the coordinate axes.

1. Draw the two focal radii

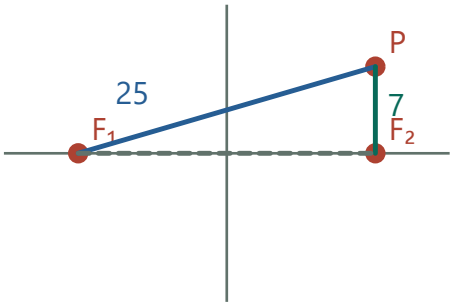
Mark $F_1=(-12,0)$, $F_2=(12,0)$ and $P=(12,7)$. PF_2 runs straight up 7 units. PF_1 crosses 24 horizontal units and 7 vertical ones, so Pythagoras gives $\sqrt{(24^2+7^2)}=25$.



Start at the two foci and measure to the same P.

2. Use the defining sum

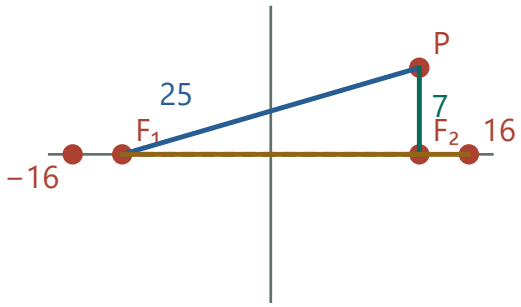
On an ellipse, adding the distances to the two foci gives the same total at every point. Here the total is 25+7=32.



A 7–24–25 triangle gives the longer focal distance.

3. Locate the horizontal intercepts

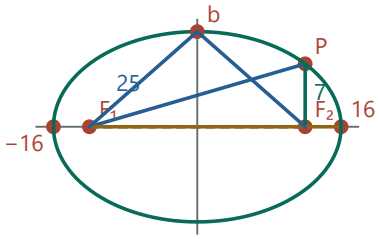
At the rightmost point (a,0), the two distances are a+12 and a−12. Their sum is 2a=32, so a=16. Reflection across the vertical axis gives the other intercept: (16,0) and (−16,0).



On the major axis, the focal lengths add to twice a.

4. Locate the vertical intercepts

At (0,b) the two distances are equal by symmetry. Each must be 16. The triangle from a focus to this point has legs 12 and b, so $b^2=16^2-12^2=112$. The vertical intercepts are (0,4√7) and (0,−4√7).



On the vertical axis, each focal segment has length 16.

1. $b^2 = 16^2 - 12^2 = 256 - 144 = 112$
Subtract the squared horizontal leg from the squared focal distance.

2. √112 = √(16·7) = 4√7
The height is a positive length. Reflect it across the x-axis for the lower intercept.

Check. At either minor vertex each focal distance is √(144+112)=16.

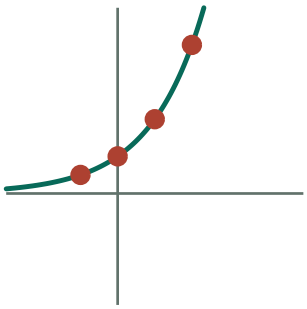
Exercise 562 · Inverse graphs

Printed page 59 · Inverse functions / Exponential functions / Logarithms

Draw the graph of the equation $y = \log_2 x$. How does this graph compare to the graph of the equation $y = 2^x$? What is the domain and range of each function?

1. Build the exponential from powers

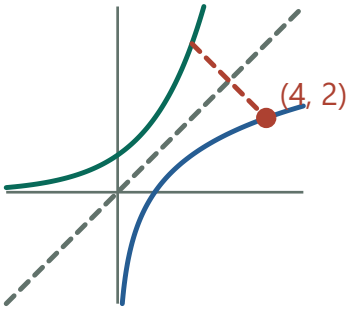
Use the exponent as the horizontal coordinate. Since $2^{-1}=\tfrac{1}{2}$, $2^{0}=1$, $2^{1}=2$ and $2^{2}=4$, plot $(-1,\tfrac{1}{2}),(0,1),(1,2),(2,4)$. Join them with a smooth increasing curve.



Each power of 2 supplies one point.

2. Reverse the question

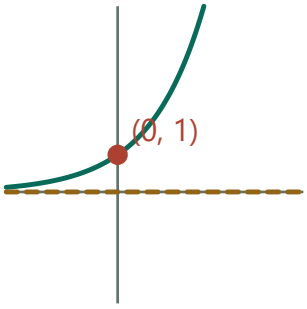
The logarithm asks which exponent produced an output. For instance, $2^{2}=4$ means $\log_{2}4=2$. Swap each pair: $(\tfrac{1}{2},-1),(1,0),(2,1),(4,2)$. Swapping coordinates reflects the graph across $y=x$.



Exchange the coordinates across y = x.

3. Read the allowable values

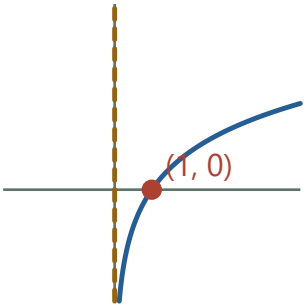
Every real exponent is permitted. An exponential output is always positive, and any positive number occurs. Hence the exponential has domain $\mathbb{R}$ and range $(0,\infty)$.



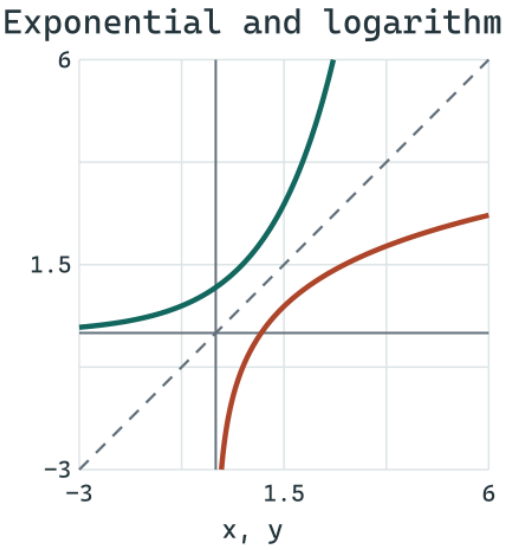
Every output of the exponential is above y = 0.

4. Exchange domain and range

For the logarithm the positive numbers become the inputs, and every real exponent becomes a possible output. Its domain is $(0,\infty)$ and range is $\mathbb{R}$. The exponential approaches $y=0$ on the left, so the reflected logarithm approaches $x=0$ from the right.



The positive outputs become the allowed logarithm inputs.



Green: exponential. Orange: logarithm. The dashed line is y = x.

Check. Applying $\log_2$ to 2^x returns x, and raising 2 to $\log_2 x$ returns x for $x>0$.

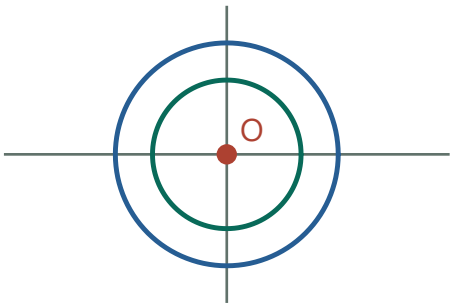
Exercise 563 · Common centers and foci

Printed page 59 · Ellipses / Foci and directrices / Circle geometry

Give equations for two circles that are *concentric*. Give equations for two ellipses that are *confocal* (which means that their focal points are shared). Are confocal ellipses necessarily concentric? Why?

1. Choose a shared center

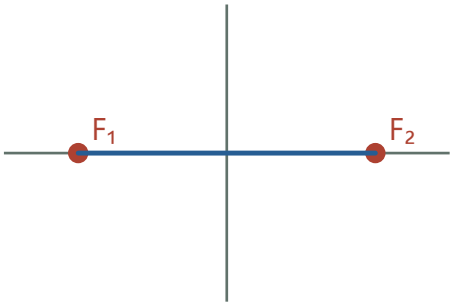
Concentric means sharing a center. Put two circles at the origin and choose radii 2 and 3. Squaring the distance from (x,y) to the origin gives $x^2+y^2=4$ and $x^2+y^2=9$.



Changing a radius can leave the center in place.

2. Choose shared focal points

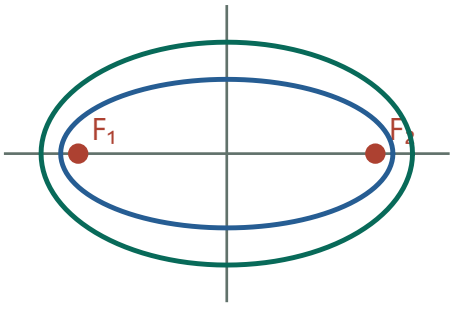
For a horizontal ellipse centered at the origin the foci are $(\pm c,0)$, where $c^2=a^2-b^2$. Fix $c=4$. We are free to vary a and b provided their squared difference stays 16.



Hold the foci at -4 and 4.

3. Construct two different ellipses

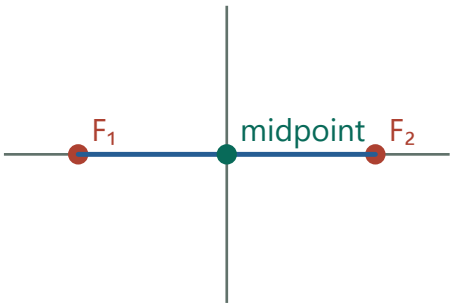
The pairs (25,9) and (20,4) both differ by 16. Thus $x^2/25+y^2/9=1$ and $x^2/20+y^2/4=1$ share $(\pm4,0)$ as foci.



Both ellipses have $a^2 - b^2 = 16$.

4. Explain why the centers coincide

Reflection through the center of an ellipse exchanges its two foci. The center is therefore their midpoint. If the foci are fixed, their midpoint is fixed too.



The two foci fix their midpoint as the common center.

Check. At each major vertex the two focal distances sum to 2a.

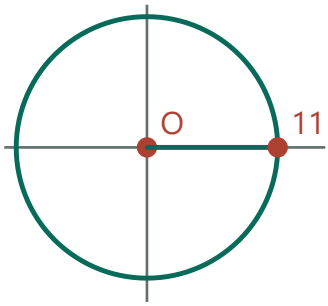
Exercise 564 · Translate a circle

Printed page 59 · Graph transformations / Circle geometry

How does the graph of $(x-3)^2+(y+1)^2 = 121$ compare with the graph of $x^2+y^2 = 121$?

1. Read the original distances

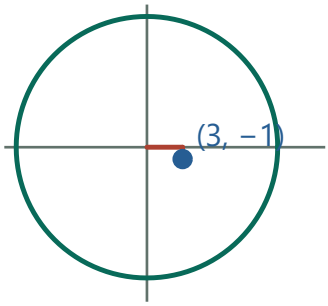
In $x^2+y^2=121$, the terms x and y measure displacement from (0,0). The radius is $\sqrt{121}=11$.



Read the original reach from its center.

2. Find the new reference point

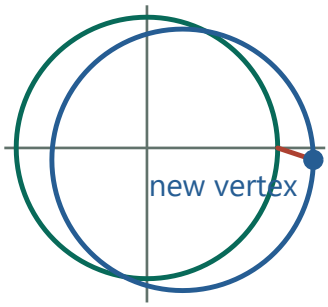
The displacements x−3 and y+1 both vanish at (3,−1). Distances are now measured from that point.



The center moves three right, then one down.

3. Move the whole curve

Send any original (u,v) to (u+3,v−1). Then x−3=u and y+1=v, so the same sum of squares is 121. The whole circle shifts 3 right and 1 down with radius 11.



The shape and its radii travel with the center.

Check. The original rightmost point (11,0) becomes (14,−1), which satisfies the new equation.

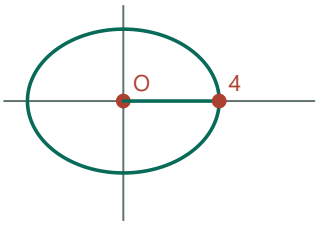
Exercise 565 · Translate an ellipse

Printed page 60 · Graph transformations / Ellipses

$$\text{How does the graph of } \frac{(x-3)^2}{16}+\frac{(y+1)^2}{9}=1 \text{ compare with the graph of } \frac{x^2}{16}+\frac{y^2}{9}=1?$$

1. Find the ellipse's original reaches

The horizontal and vertical denominators are squared lengths. At y=0 the horizontal reach is √16=4. At x=0 the vertical reach is √9=3.



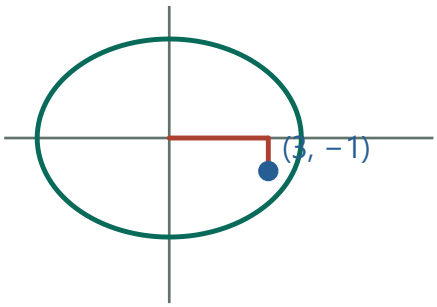
Read the original reach from its center.

1. $x^2/16 = 1 \Rightarrow x^2 = 16 \Rightarrow x = \pm 4$
Set y=0 and multiply by 16. Both signs correspond to intercepts.

2. $y^2/9 = 1 \Rightarrow y^2 = 9 \Rightarrow y = \pm 3$
Set x=0 and multiply by 9. The semiaxis is the positive length 3.

2. Identify what changed inside the squares

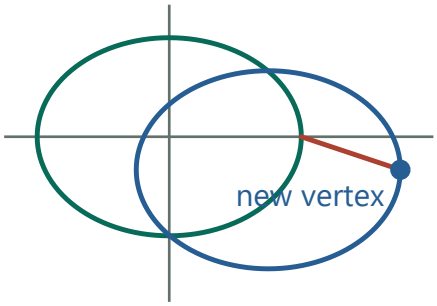
An original displacement u horizontally becomes x−3, and v vertically becomes y+1. Solving for the new coordinates gives x=u+3 and y=v−1.



The center moves three right, then one down.

3. Locate the moved ellipse

Every point shifts 3 right and 1 down. Its center is (3,−1), its horizontal semiaxis is 4 and its vertical semiaxis is 3. For example the right vertex moves from (4,0) to (7,−1).



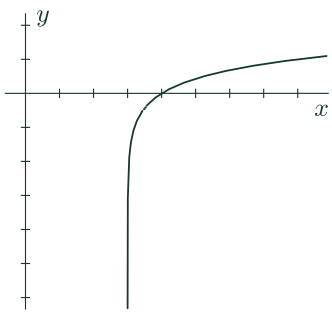
The shape and its radii travel with the center.

Check. Its extreme x-values are −1 and 7, each four units from 3.

Exercise 566 · A shifted base-five logarithm

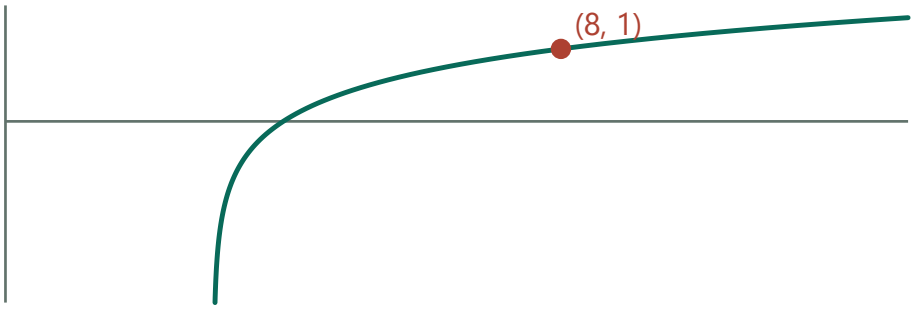
Printed page 60 · Logarithms / Graph transformations / Domain and range

The equation graphed at right is $y = \log_5(x - 3)$. What is the x-intercept of this graph? There are many vertical lines that do not intersect this graph; which one of them is farthest to the right? For what x-values does the equation make sense? What x-value corresponds to $y = 1$? to $y = 2$? to $y = 3$?



1. Translate from log to exponent

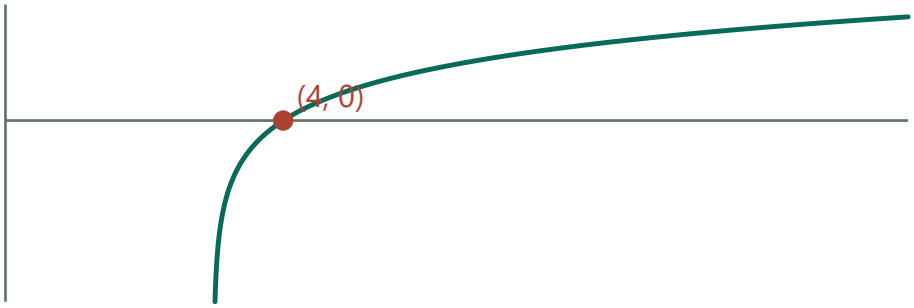
The statement $y=\log_5(x-3)$ means $5^y=x-3$, so each chosen height y corresponds to $x=3+5^y$.



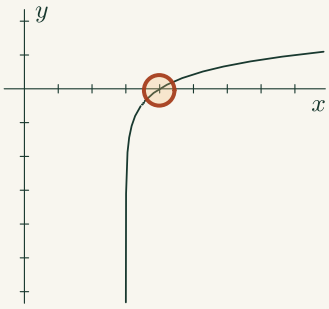
The gap x − 3 is the value of 5^y.

2. Find the crossing

On the x-axis y=0. Then $5^0=1$ gives x=4, so the crossing is (4,0).



At height zero the gap equals 1.

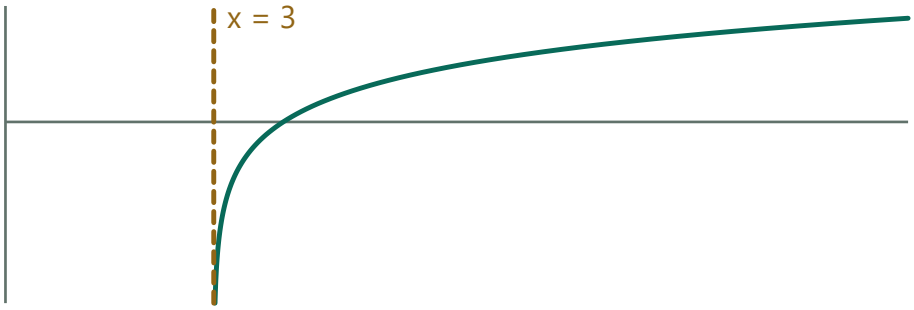


Crossing at height zero

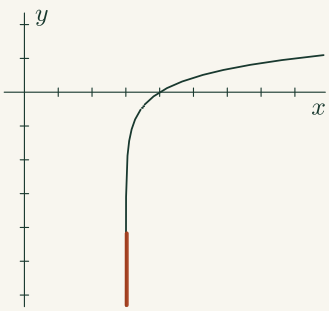
Read the input at the intersection with the horizontal axis. The exponent that produces one explains why this point exists.

3. Locate the boundary

Every 5^y is positive, so x must exceed 3. Conversely every x>3 gives a positive argument and a real logarithm. All lines x≤3 miss the curve; the rightmost is its vertical asymptote x=3.



The curve approaches the boundary from the right.

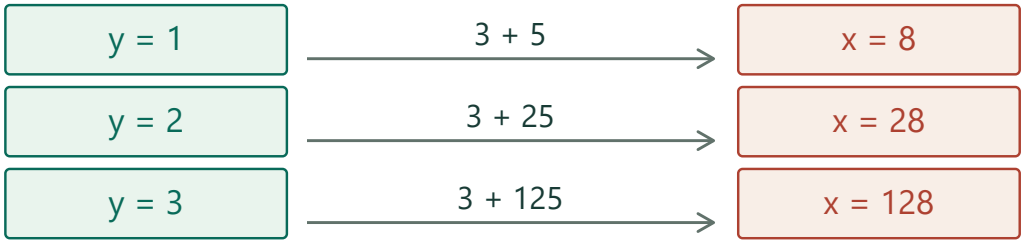


Boundary of the logarithm

Trace the vertical line the curve approaches as its positive input shrinks. Measure horizontal distance from this line when the exponent changes.

4. Match three indicated heights

At y=1,2,3 the positive terms are 5,25,125. Adding 3 gives the x-values 8,28,128. Moving up by 1 multiplies the distance to x=3 by 5.



Each added unit of height multiplies the gap by five.

Check. For every x>3 there is exactly one exponent giving x−3.

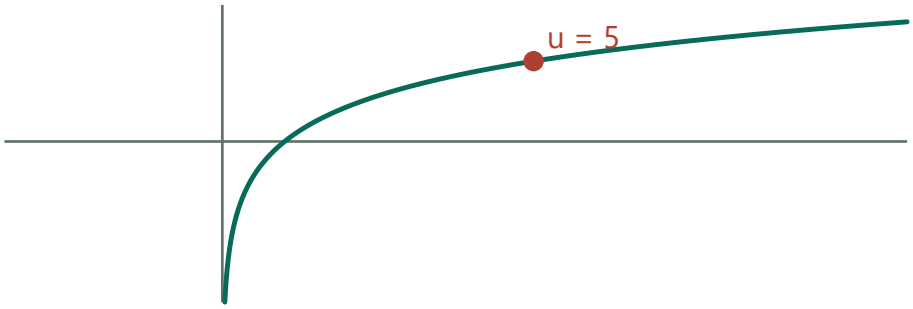
Exercise 567 · Compare three logarithms

Printed page 60 · Logarithms / Graph transformations / Domain and range

(Continuation) How does the given graph compare to the graph of $y = \log_5 x$? How does the given graph compare to the graph of $y = \log_5(x + 2)$? What is the domain for each function, $y = \log_5(x - 3)$, $y = \log_5 x$, and $y = \log_5(x + 2)$?

1. Track one logarithm input

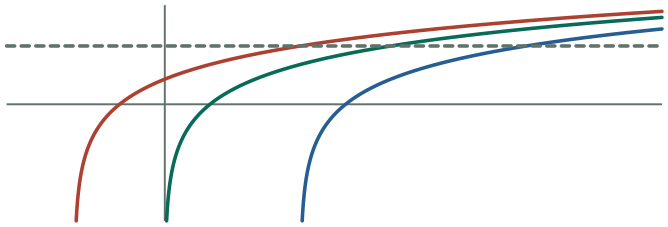
Choose any positive number u. The three curves all have height $\log_5 u$ when their arguments equal u. Solve each argument equation for its horizontal coordinate.



Choose one positive input, here u = 5.

2. Compare horizontal positions

For $\log_5 x$ the coordinate is u. For $\log_5(x-3)$ it is u+3, and for $\log_5(x+2)$ it is u−2. The given curve is 3 right of the first and 5 right of the last.



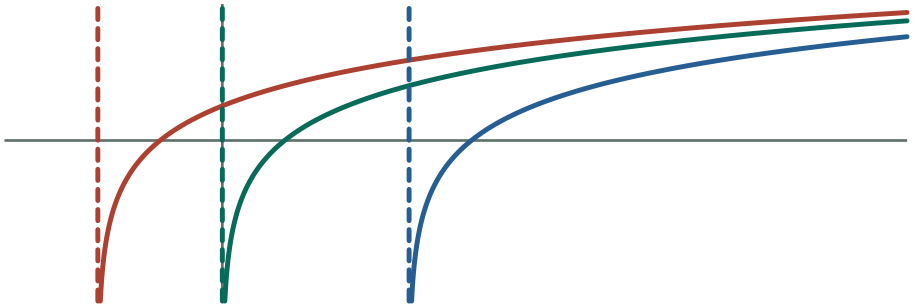
At height 1 the curves pass through x = 3, 5 and 8.

1. $x-3=u \Rightarrow x=u+3$; $x+2=u \Rightarrow x=u-2$
Use the same positive argument u to compare points of equal height.

2. $(u+3)-(u-2) = u+3-u+2 = 5$
Subtract the two horizontal coordinates to measure the translation.

3. Read each domain

The argument must exceed zero. In the order asked the domains are $(3,\infty),(0,\infty),(-2,\infty)$. Their left boundaries are also their vertical asymptotes.



Each horizontal translation carries its input boundary along.

Check. Their zeroes occur at x=4,1,−1, with the same separations.

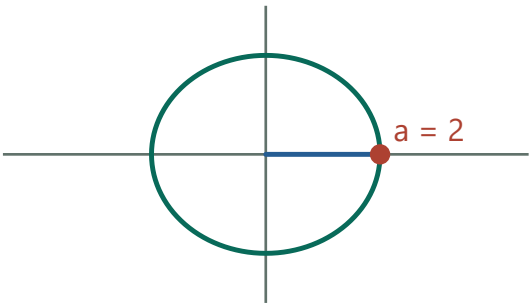
Exercise 568 · Directrices of an ellipse

Printed page 60 · Ellipses / Foci and directrices

Find equations for the directrices of the ellipse $3x^2 + 4y^2 = 12$.

1. Put the ellipse in standard form

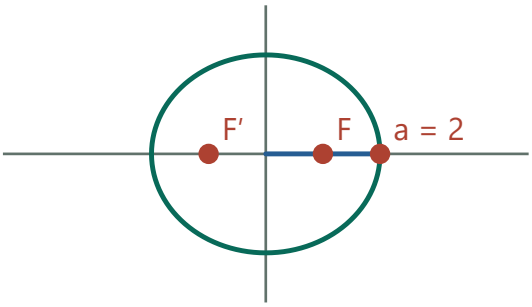
Divide $3x^2+4y^2=12$ by 12 to obtain $x^2/4+y^2/3=1$. The longer semiaxis is horizontal and has length $a=2$. The other has $b=\sqrt{3}$.



Read the longer semiaxis from the intercept.

2. Locate the foci

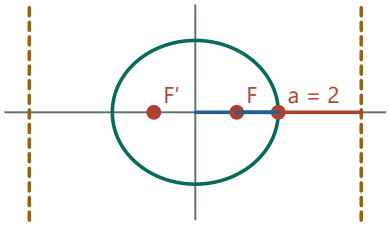
The squared focal distance from the center is $c^2=a^2-b^2=4-3=1$. Thus $c=1$ and the eccentricity is $e=c/a=½$.



The focal displacement satisfies $c^2 = a^2 - b^2$.

3. Locate the directrices

A directrix is perpendicular to the major axis. Its distance from the center is a/e, so the two vertical lines are $x=4$ and $x=-4$. At the right vertex, the nearer focus and directrix are 1 and 2 units away, giving the required ratio ½.



At the near vertex, compare its distances to focus and directrix.

1. $e = c/a = 1/2$; $a/e = 2/(1/2)$
Use the major semiaxis and the focal ratio.

2. $2/(1/2) = 2\cdot 2 = 4 \Rightarrow x = \pm 4$
Divide by a fraction by multiplying by its reciprocal, then use symmetry.

Check. At (2,0), distance to (1,0) is 1 and distance to $x=4$ is 2, consistent with $e=1/2$.

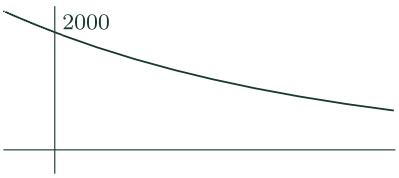
Exercise 569 · Rates and absolute loss

Printed page 60 · Exponential growth and decay / Rates of change

Let $f(x) = 2000(0.96)^x$. Find a context to interpret your answers to each of the next two questions:

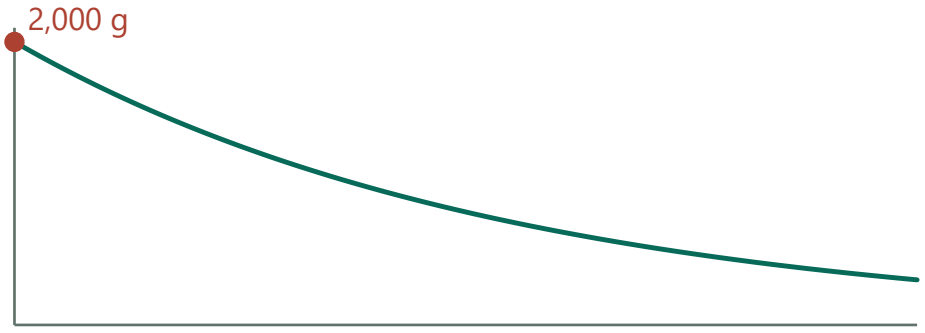
(a) Calculate $\dfrac{f(2.01) - f(2.00)}{0.01}$.

(b) Solve the equation $f(x + 1) = f(x) - 25$.

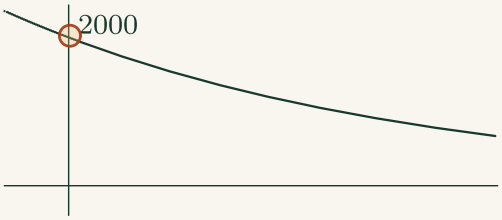


1. Give the expression units

Imagine an original 2000 g sample that retains 96% of its mass per day. Then x is elapsed time in days and f(x) is mass in grams. Fractional days use the same exponential model.



Each day retains 96% of the previous mass.



Mass at time zero

The marked vertical intercept gives the initial amount. Its units set the units for every later height on the curve.

2. Measure the short-interval rate

Subtract the mass at day 2 from the mass at day 2.01, then divide by 0.01 day. Factor the common mass at day 2 to obtain $2000(0.96)^2[(0.96)\text{^}0.01-1]/0.01 \approx -75.23$ g/day. The minus sign indicates loss.

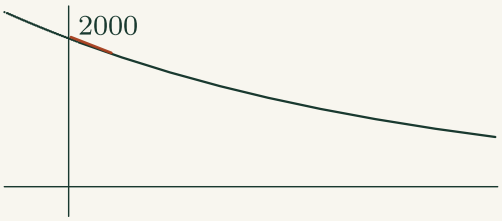


A short-interval loss divided by 0.01 day gives the rate.

1. $[f(2.01)-f(2)]/0.01 = 2000[0.96\text{^}2.01-0.96^2]/0.01$
Insert the two masses in the change-in-mass quotient.

2. $0.96\text{^}2.01 = 0.96^2 \cdot 0.96\text{^}0.01$
Split 2.01 into 2+0.01 to expose the shared factor.

3. $2000 \cdot 0.96^2(0.96\text{^}0.01-1)/0.01 \approx -75.23$ g/day
Factor before evaluating; the remaining difference is negative.



A short falling interval

Read the earlier height first, then the slightly lower later height. The signed change divided by elapsed time describes average loss rate.

3. Interpret the one-day difference

Over one day the sample loses 4% of its starting mass. A 25 g decrease therefore occurs when $0.04f(x)=25$, or $f(x)=625$ g.



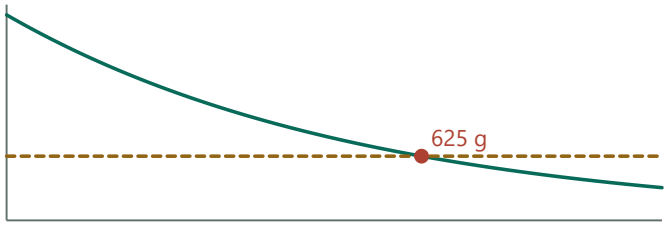
A loss of 25 g must be 4% of the starting mass.

1. $f(x)-f(x+1) = f(x)-0.96f(x) = 0.04f(x)$
Express the next day's mass using the retention factor.

2. $0.04f(x)=25 \Rightarrow f(x)=25/0.04=625$
Divide by the positive loss fraction.

4. Find when that mass occurs

From $2000(0.96)^x=625$, divide by 2000 and take logarithms. The starting day is $x=\ln(625/2000)/\ln(0.96)\approx 28.49$. One day later the mass is $0.96 \cdot 625=600$ g.



Read the time when the exponential falls to 625 g.

1. $0.96^x = 625/2000 \Rightarrow x \ln 0.96 = \ln(625/2000)$
Divide by 2000 and take logs of positive quantities.

2. $x = \ln(625/2000)/\ln 0.96$
Divide by the nonzero logarithm. Both logs are negative, giving a positive time.

Check. $625 \times 0.96=600$, a 25 g decrease.

Exercise 570 · Index a geometric sequence

Printed page 60 · Geometric sequences and series

The first two terms of a geometric sequence are 2000 and 1920. What is the forty-third?

1. Find the repeated multiplier

A geometric sequence multiplies the preceding term by the same ratio. Dividing the second term by the first gives $r = 1920/2000 = 0.96$.



Divide one term by the preceding term.

2. Count the transitions

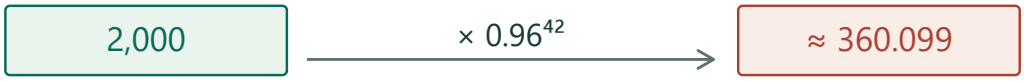
To move from term 1 to term 2 takes one multiplication. To move from term 1 to term 43 takes $43 - 1 = 42$ multiplications.



Count the gaps between indices.

3. Evaluate the term

Use $a_{43}=2000(0.96)^{42}\approx 360.099$. Its magnitude is consistent with a sequence that shrinks by 4% on each transition.



Every gap contributes another factor of 0.96.

Check. The formula gives 1920 when the term index is 2.

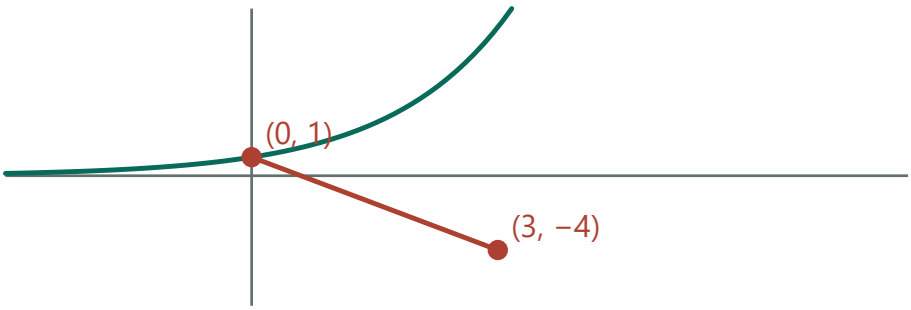
Exercise 571 · Three exponential shifts

Printed page 60 · Exponential functions / Graph transformations

For each of the transformations, write an equation for the curve obtained by shifting the curve $y = 2^x$ and identify x- and y-intercepts and other significant features:
(a) 3 units to the right; **(b)** 5 units down; **(c)** 3 units to the right *and* 5 units down.

1. Follow the two kinds of shift

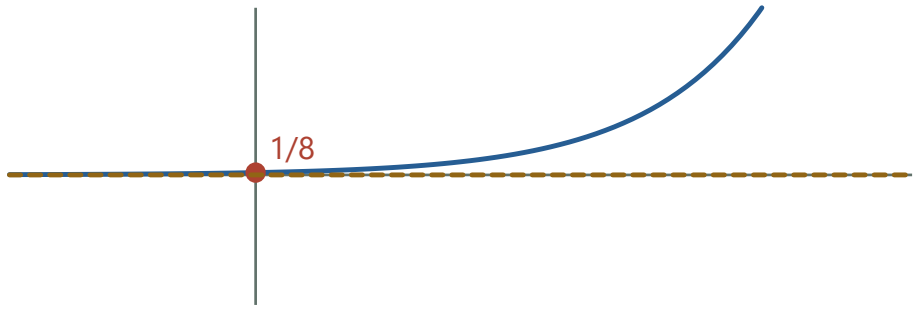
Use u as a point's old horizontal coordinate. Moving right 3 makes x=u+3, so write u=x−3 inside the exponent. Moving down 5 subtracts 5 from the output.



Track a single point by the requested displacement.

2. Shift only right

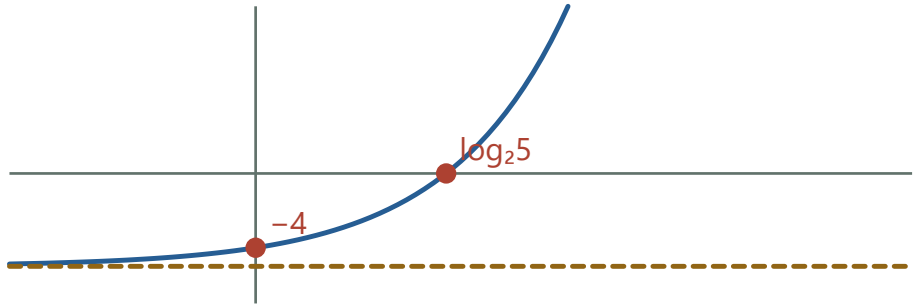
The curve y=2^(x−3) crosses the y-axis at $2^{-3}=\frac{1}{8}$. All its values are positive, so there is no x-intercept and the horizontal asymptote is y=0.



Only the input changes on the right-shifted curve.

3. Shift only down

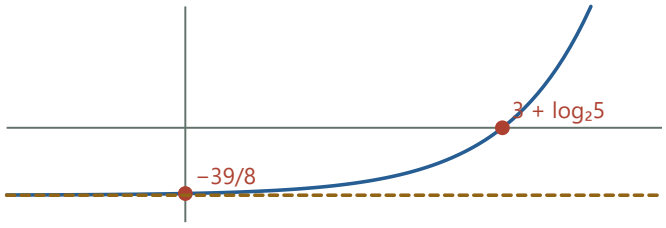
For $y=2^x-5$, x=0 gives y=−4. At y=0 solve $2^x=5$, so $x=\log_2 5$. Its asymptote is y=−5.



Lowering the outputs moves the horizontal asymptote.

4. Combine both moves

For y=2^(x−3)−5, the y-intercept is $\frac{1}{8}-5=-\frac{39}{8}$. The equation 2^(x−3)=5 gives $x=3+\log_2 5$ at the x-intercept. The asymptote stays y=−5. Each curve increases on $\mathbb{R}$ and takes every value strictly above its own asymptote.



Combine the rightward and downward changes.

1. $0 = 2^{(x-3)}-5 \Rightarrow 2^{(x-3)} = 5$
At the x-axis add 5 to isolate the exponential.

2. $x-3 = \log_2 5 \Rightarrow x = 3 + \log_2 5$
Take the base-two logarithm, then add 3.

3. $2^{-3}-5 = \frac{1}{8}-\frac{40}{8} = -\frac{39}{8}$
At the y-axis, x=0. Use a common denominator.

Check. Substitute x=0 for each vertical intercept; setting y=0 in (c) gives 2^(x−3)=5.

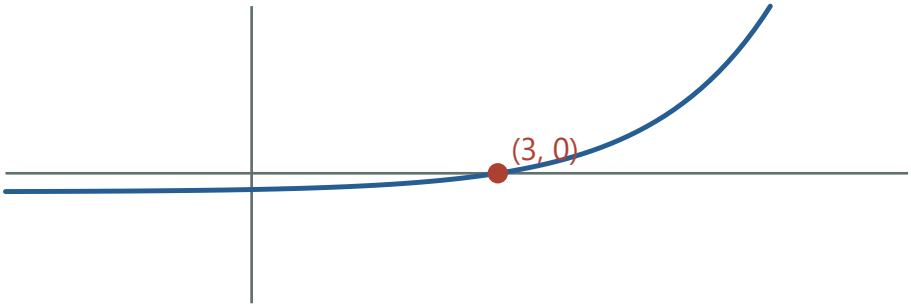
Exercise 572 · Equivalent exponential equations

Printed page 60 · Exponential functions / Graph transformations / Domain and range

Find the y-intercept of the graph of $y+1=2^{x-3}$. How does the graph of $y+1=2^{x-3}$ compare with the graph of $y=2^x$? How about the graph of $y=2^{x-3}-1$? What is the range of $y+1=2^{x-3}$ and $y=2^x$?

1. Isolate the output

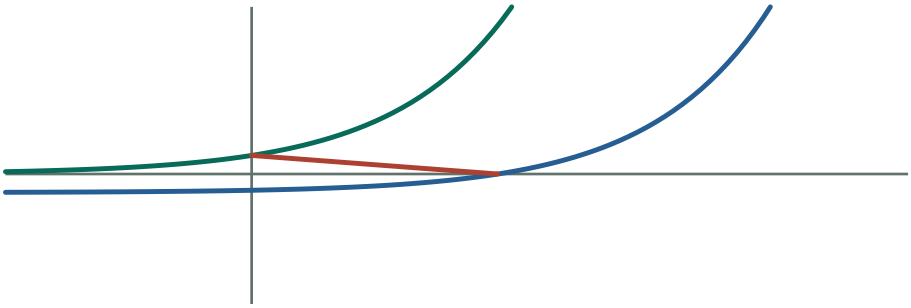
Subtract 1 from y+1=2^(x−3) to obtain y=2^(x−3)−1. These two equations specify precisely the same point pairs.



Subtract 1 from both sides to isolate the height.

2. Follow a point of the original exponential

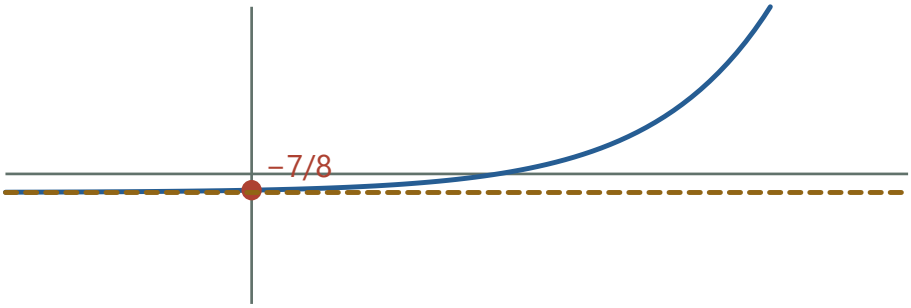
For a fixed original exponent u, the new exponent equals u at x=u+3, while the output is 2^u-1. Every point moves right 3 and down 1.



The same exponent is reached three units farther right.

3. Read the intercept and range

At x=0 the height is $2^{-3}-1=-7/8$. Because 2^(x−3) takes every positive value, subtracting 1 changes its range from (0,∞) to (−1,∞), with asymptote y=−1.



The output can approach −1 but cannot reach it.

Check. The horizontal asymptote shifts from 0 to −1.

Exercise 573 · Translate a general graph

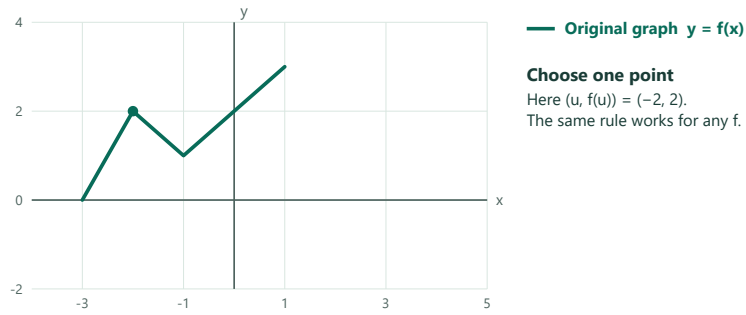
Printed page 60 · Graph transformations / Translations

How does the graph of $y = f(x)$ compare to the graph of $y + 1 = f(x - 3)$?

1. Name an arbitrary point

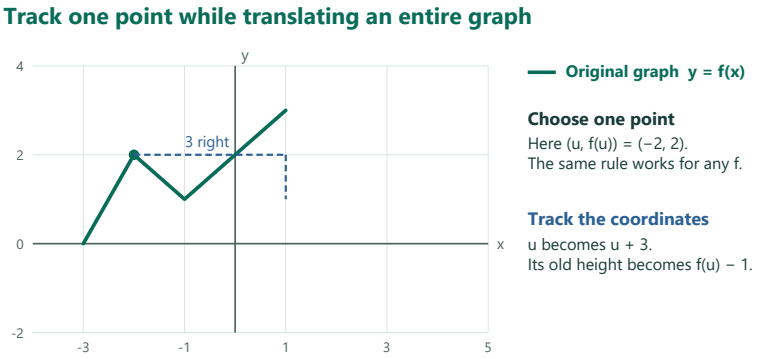
Choose any allowed input u of f. Its point on the old graph is (u,f(u)); this works even when no formula for f is given.

Track one point while translating an entire graph



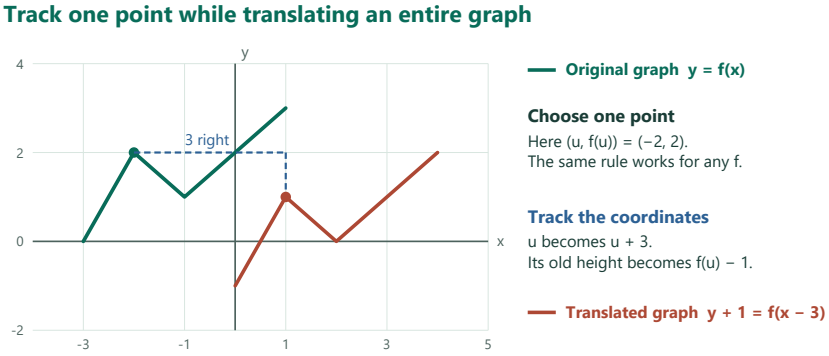
2. Recover the same function value

To evaluate f at u on the new graph set x−3=u, so x=u+3. The equation then says y+1=f(u), so y=f(u)−1.



3. State the motion for every point

The resulting correspondence is (u,f(u))→(u+3,f(u)−1), a translation 3 right and 1 down. Reversing those changes returns the old point.



The bends and endpoints move by the same displacement as the selected point.

Check. Plug the image into y+1=f(x−3) to get f(u)=f(u).

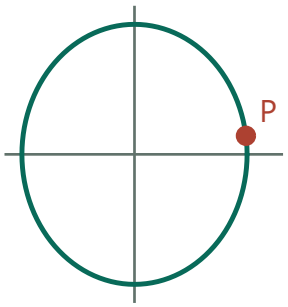
Exercise 574 · Equal angles at an ellipse

Printed page 60 · Ellipses / Tangent lines / Conic reflection properties

Verify that the line $8x + y = 98$ is tangent to the ellipse $4x^2 + 3y^2 = 588$ at $P = (12, 2)$. Verify also that the focal points for this ellipse are $F_1 = (0, -7)$ and $F_2 = (0, 7)$. Calculate the size of the acute angle formed by the tangent line and the focal radius F_2P. Do the same for the acute angle formed by the tangent line and the focal radius F_1P.

1. Test the point of contact

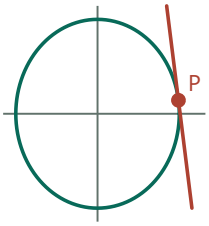
At P=(12,2), $4\cdot12^2+3\cdot2^2=588$ and $8\cdot12+2=98$. The point lies on both curves. Sharing one displayed point alone would also be possible for a secant.



Substitute (12, 2) in both equations.

2. Test for a second intersection

On the line $y=98-8x$. Substituting into the ellipse and collecting terms gives $196(x-12)^2=0$. The only intersection is the double root $x=12$, so the line is tangent there.



The steep straight line has one double intersection.

1. $4x^2 + 3(98-8x)^2 - 588 = 0$
Replace y using the line and move 588 to the left.

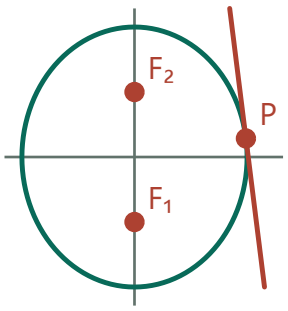
2. $4x^2 + 3(9604-1568x+64x^2) - 588 = 0$
Expand the square with its negative middle term.

3. $196x^2-4704x+28224 = 196(x-12)^2 = 0$
Collect like terms and recognize a perfect square.

4. $x=12 \Rightarrow y=98-8\cdot12=2$
Only zero has square zero, so there is just one intersection.

3. Identify the correct focal axis

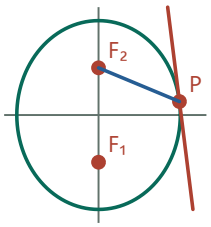
Divide by 588: $x^2/147+y^2/196=1$. The greater squared semiaxis 196 belongs to y, so the focal axis is vertical. Its focal distance is $\sqrt{(196-147)}=7$, giving $F_1=(0,-7)$, $F_2=(0,7)$.



The larger squared radius belongs to the vertical axis.

4. Measure the angle to F_2P

The tangent slope is -8; F_2P has slope $(2-7)/12=-5/12$. For two lines, tan of their acute angle is the absolute value of $(m_2-m_1)/(1+m_1m_2)$. Here it is $(91/12)/(13/3)=7/4$.



Compare the upper focal ray with the tangent.

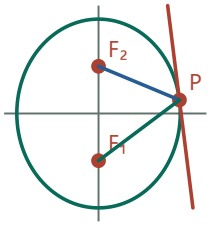
1. $m_2-m_1 = -5/12-(-8) = 91/12$
Subtract the tangent slope from the focal slope.

2. $1+m_1m_2 = 1+40/12 = 13/3$
The slopes have the same sign, so their product is positive.

3. $(91/12)/(13/3) = (91/12)(3/13) = 7/4$
Multiply by the reciprocal to find the acute angle's tangent.

5. Repeat for F_1P

The other focal slope is $(2+7)/12=3/4$. The same angle formula gives $(35/4)/5=7/4$. Both acute angles measure $\arctan(7/4)\approx 60.26°$.



The lower focal ray gives the same acute angle.

1. $3/4-(-8) = 35/4$; $1+(-8)(3/4) = -5$
Use the other focal radius in the same angle formula.

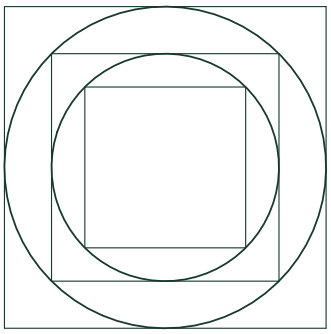
2. $\text{abs}[(35/4)/(-5)] = 35/20 = 7/4$
The absolute value selects the acute angle.

Check. The normal from the ellipse gradient $(96,12)$ is proportional to the line normal $(8,1)$.

Exercise 575 · Nested square side lengths

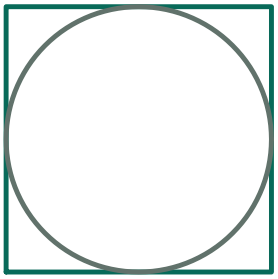
Printed page 61 · Geometric sequences and series / Similarity and scaling

The figure at right shows an outermost 1×1 square, within which appears an inscribed circle, within which appears an inscribed square, within which appears another inscribed circle, within which appears another inscribed square. Although the figure does not show it, this process can be continued indefinitely. Let $L_1 = 1$ be the length of a side of the first (largest) square, L_2 be the length of a side of the second square, L_3 be the length of a side of the third square, and so on. Show that the numbers L_1, L_2, L_3, $\ldots$ form a geometric sequence, and calculate L_{20}.



1. Identify the link between successive squares

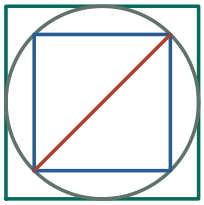
The circle touches opposite sides of the outer square, so its diameter equals that square's side L_n. Opposite vertices of the next square lie at the ends of a diameter.



The first circle spans the 1-unit side of the outer square.

2. Relate a diagonal to a side

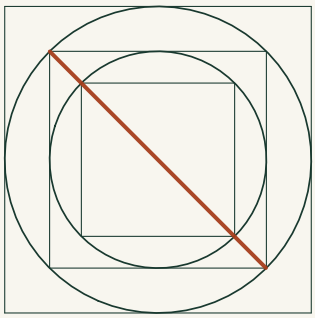
Its diagonal is L_n but its two perpendicular sides are L_{n+1}. Pythagoras gives $2L_{n+1}^2=L_n^2$, hence $L_{n+1}=L_n/\sqrt{2}$.



The new square has a diagonal equal to the old side.

1. $2L_{n+1}^2=L_n^2 \Rightarrow L_{n+1}^2=L_n^2/2$
Divide the Pythagorean equation by 2.

2. $L_{n+1}=\sqrt{(L_n^2/2)}=L_n/\sqrt{2}$
The square's side lengths are positive.

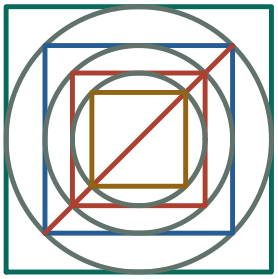


Diagonal of the next square

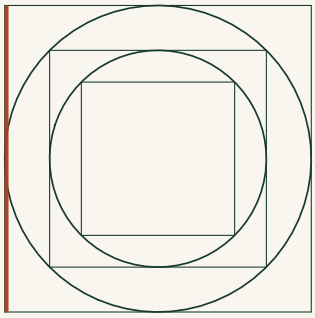
This construction joins opposite corners of the next square. The midpoint is the shared center and both ends touch the same circle.

3. Iterate from the first square

The factor $1/\sqrt{2}$ is independent of n, which makes the sequence geometric. From $L_1=1$ to L_{20} there are 19 reductions, so $L_{20}=(1/\sqrt{2})^{19}=2^{(-19/2)}$.



Each smaller square repeats the same side and area ratios.



Side of the outer square

The two horizontal tangencies of the largest circle are one diameter apart vertically. That distance also equals the square's side.

Check. Squaring consecutive lengths shows that the area halves.

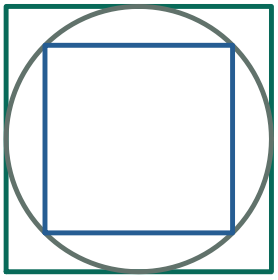
Exercise 576 · Nested square areas

Printed page 61 · Geometric sequences and series / Area and volume scaling

(Continuation) Let A_n be the area of the nth square. How is A_2 related to A_1? How is A_3 related to A_2? How is A_{n+1} related to A_n? This last equation, together with the equation $A(1) = 1$, is said to be a *recursive description* of the sequence A_n.

1. Start with the length reduction

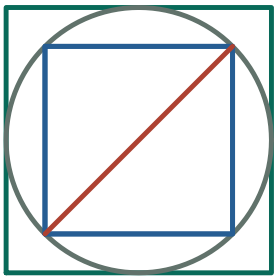
From the inscribed-circle construction each new side equals the preceding side divided by $\sqrt{2}$. Its area is the square of its side length.



The first circle spans the 1-unit side of the outer square.

2. Square the ratio

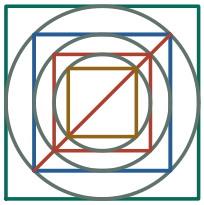
$A_{n+1}/A_n=(L_{n+1}/L_n)^2=(1/\sqrt{2})^2=1/2$. In particular $A_2=A_1/2$ and $A_3=A_2/2$.



The new square has a diagonal equal to the old side.

3. Give the recursive rule

$A_1=1$ fixes the starting square and $A_{n+1}=A_n/2$ describes each successive square. It produces 1,½,¼,..., agreeing with $A_n=2^{(1-n)}$.



Each smaller square repeats the same side and area ratios.

1. $A_n = 1\cdot(1/2)^{(n-1)}$
Starting at index 1 requires n−1 halvings.

2. $(1/2)^{(n-1)}=2^{(-(n-1))}=2^{(1-n)}$
Replace the reciprocal by a negative exponent.

Check. A closed form $2^{(1-n)}$ satisfies the starting value and recurrence.

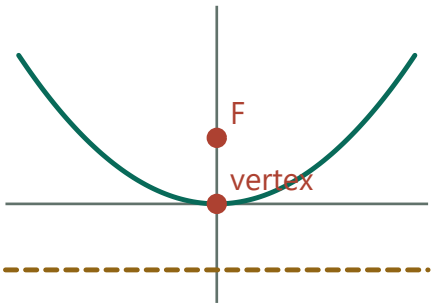
Exercise 577 · Focal parameter of a parabola

Printed page 61 · Parabolas / Foci and directrices

Find the focus and the directrix for the parabola $y = ax^2$. What special meaning does the expression $\dfrac{1}{4a}$ have?

1. Place a focus and a line symmetrically

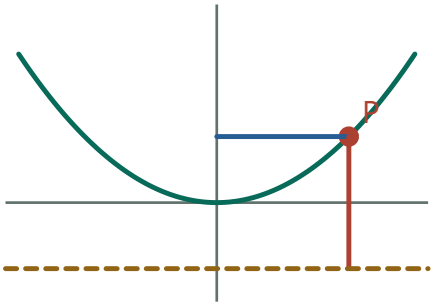
For a nondegenerate parabola take $a \neq 0$. Its vertex is (0,0) and its symmetry axis is the y-axis. Write the focus (0,p) and the directrix $y = -p$ so the vertex is equally distant from both.



Example with p = 1: vertex midway between focus and line.

2. Express the two distances

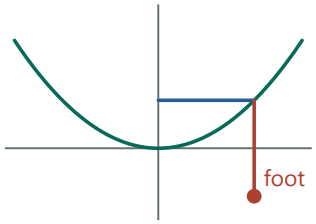
A point (x,y) on a parabola lies equally far from the focus and directrix. The squared focal distance is $x^2+(y-p)^2$; the squared perpendicular distance to the horizontal directrix is $(y+p)^2$.



The focal segment and the perpendicular drop are equal.

3. Compare coefficients

Equate and expand. The y^2 and p^2 terms cancel, leaving $x^2=4py$. Comparing with $y=ax^2$ gives $p=1/(4a)$.



Squaring these lengths cancels y^2 and p^2, leaving $x^2 = 4py$.

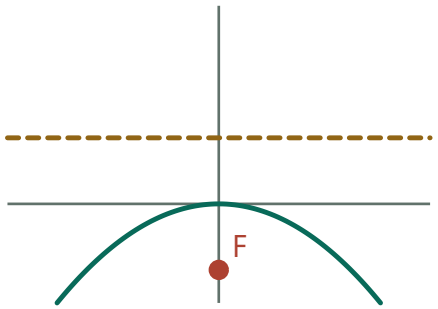
1. $x^2+y^2-2py+p^2 = y^2+2py+p^2$
Expand both squares in the equal-distance equation.

2. $x^2-2py=2py \Rightarrow x^2=4py$
Cancel y^2 and p^2 and add 2py.

3. $y=x^2/(4p) \Rightarrow a=1/(4p) \Rightarrow p=1/(4a)$
Compare coefficients with $y=ax^2$. Both p and a are nonzero.

4. Interpret the sign

The focus is (0,1/(4a)); the directrix is y=−1/(4a). The expression 1/(4a) is the signed vertical displacement from vertex to focus. Its absolute value is the ordinary distance.



Negative a places the focus below the vertex.

Check. If a<0 the focus lies below the vertex, consistent with downward opening.

Exercise 578 · Translate a common logarithm

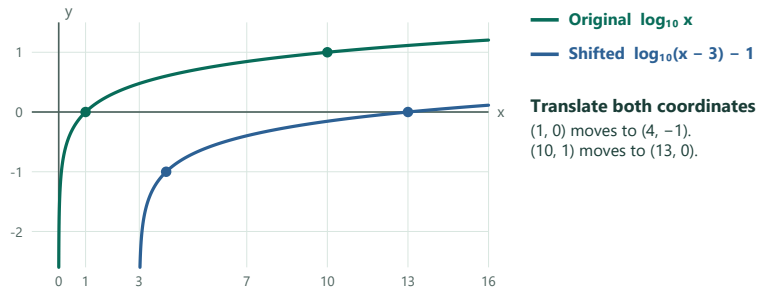
Printed page 61 · Logarithms / Graph transformations / Domain and range

What is the x-intercept of the graph of $y = \log(x - 3) - 1$? How does the graph of $y = \log(x - 3) - 1$ compare with the graph of $y = \log x$? Remember to consider domain and range.

1. Track the shifts

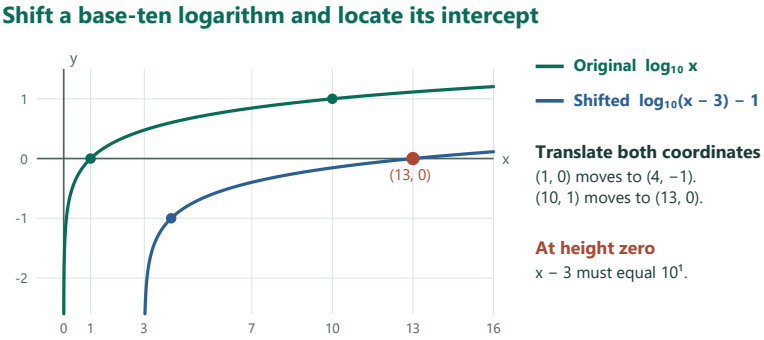
Use common logarithms. Subtracting 3 inside $\log_{10}(x-3)$ moves each old point right 3; subtracting 1 outside moves it down 1.

Shift a base-ten logarithm and locate its intercept



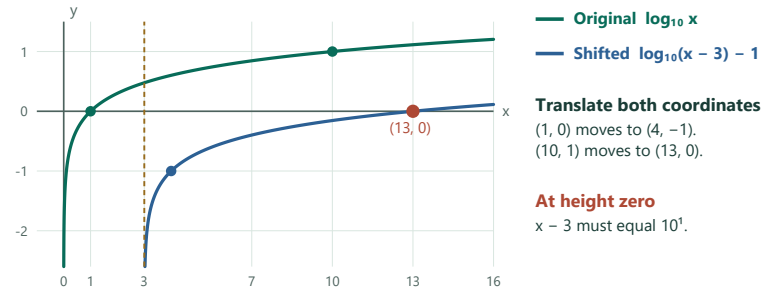
2. Solve for the x-axis crossing

At an x-intercept the output is zero, so $\log_{10}(x-3)=1$. The exponent yielding $x-3$ is 1: $x-3=10^{1}$. Thus the crossing is (13,0).



3. Read the permissible inputs and heights

The original log requires $x>0$; the translated log requires $x-3>0$, or $x>3$. A logarithm can equal any real number, so subtracting 1 still leaves all real heights available.



Vertical boundary moves from x = 0 to x = 3. Both logarithms take every real height.

Check. At $x=4$ the argument equals 1 and the height is -1.

Exercise 579 · Change an exponential base

Printed page 61 · Logarithms / Change of base

Write 10^{1997} as a power of 2.

1. Find the unknown exponent

A power of 2 has a fixed base and an unknown exponent. The target is positive, so the real base-two logarithm exists. The base 2 is greater than 1, so it gives exactly one exponent.

1 **$2^k = 10^{1997}$**
Name the exponent that would produce the target.

2 **$\log_2(2^k) = \log_2(10^{1997})$**
Apply the same logarithm to the two positive sides.

3 **$k = \log_2(10^{1997})$**
The logarithm returns the exponent of 2.

2. Write and check the power of 2

Use the power law to bring 1997 out of the logarithm. Substituting this exponent back into the requested form also checks the change of base.

1 **$k = 1997 \log_2 10$**
The input 10 is positive, so the log power law applies.

2 **$10^{1997} = 2\text{^}(1997 \log_2 10)$**
Insert the exponent into 2^k. This is the requested form.

3 **$2\text{^}(1997 \log_2 10) = (2\text{^}(\log_2 10))^{1997}$**
Group the exponent using the power-of-a-power law.

4 **$(2\text{^}(\log_2 10))^{1997} = 10^{1997}$**
The exponential and base-two logarithm undo one another.

Check. Using $2\text{^}(\log_2 10)=10$ and raising both sides to 1997 recovers the original.

Exercise 580 · Five exponent simplifications

Printed page 61 · Exponent rules / Radicals and rational powers

Simplify by hand:

(a) $(3^{-1}+4^{-1})^{-1}$ (b) $\dfrac{6^{4000}}{12^{2000}}$ (c) $7^u 7^u$ (d) $\sqrt{64x^{16}}$ (e) $2^m 3^{-m}$

1. (a) Add before taking the outer reciprocal

The outer exponent applies to the entire parenthesis. The two inner denominators are nonzero, and their positive sum can also be inverted.

1 **$(3^{-1} + 4^{-1})^{-1} = (1/3 + 1/4)^{-1}$**
Replace each inner negative power by its reciprocal.

2 **$(1/3 + 1/4)^{-1} = (4/12 + 3/12)^{-1}$**
Use the common denominator 12.

3 **$(4/12 + 3/12)^{-1} = (7/12)^{-1}$**
Add the numerators over the shared denominator.

4 **$(7/12)^{-1} = 12/7$**
Invert the nonzero sum.

2. (b) Match the exponents

Rewrite the numerator with exponent 2000. The denominator is positive, so the quotient of powers with the same exponent can be grouped into one power.

1 **$6^{4000} = (6^2)^{2000}$**
Multiplying 2 by 2000 recovers the original exponent.

2 **$6^{4000}/12^{2000} = (36/12)^{2000}$**
Evaluate 6^2 and combine equal exponents.

3 **$(36/12)^{2000} = 3^{2000}$**
Simplify just the base inside the parenthesis.

3. (c) Collect powers with a shared base

Both factors have the same base. Since 7 is positive, the exponent rule applies to every real value of u.

1 **$7^u \cdot 7^u = 7\text{^}(u + u)$**
Multiplication of like bases adds their exponents.

2 **$7\text{^}(u + u) = 7\text{^}(2u)$**
Collect the two copies of u.

4. (d) Respect the sign of the square root

The principal square root is nonnegative. The expression under the radical is a square for every real x, including zero.

1 **$64x^{16} = (8x^{8})^{2}$**
Squaring 8 gives 64, and squaring x^{8} doubles its exponent.

2 **$\sqrt{(64x^{16})} = |8x^{8}|$**
The principal square root of a real square is its absolute value.

3 **$|8x^{8}| = 8x^{8}$**
An even power of a real number is nonnegative.

5. (e) Move the negative exponent to the denominator

The bases 2 and 3 are positive. The denominator 3^{m} never vanishes, so this rewrite works for every real m.

1 **$3^{(-m)} = 1/3^{m}$**
A negative exponent gives a reciprocal.

2 **$2^{m} \cdot 3^{(-m)} = 2^{m}/3^{m}$**
Substitute that reciprocal into the product.

3 **$2^{m}/3^{m} = (2/3)^{m}$**
Use the shared exponent on the quotient of the bases.

Check. Squaring (d) returns $64x^{16}$ with a nonnegative root for every real x.

Exercise 581 · Deuces and complements

Printed page 61 · Probability

A six-sided die is to be rolled once. What is the probability of obtaining a "2" (a *deuce*)? What is the probability of obtaining a non-deuce? The same die is to be rolled three times. What is the probability of obtaining three deuces? three non-deuces?

1. Count one roll

Assume a fair die. One of its six equiprobable faces shows 2, so P(deuce)=1/6. The other five faces give P(non-deuce)=5/6.



One equally likely face is a deuce.

2. Describe three independent rolls

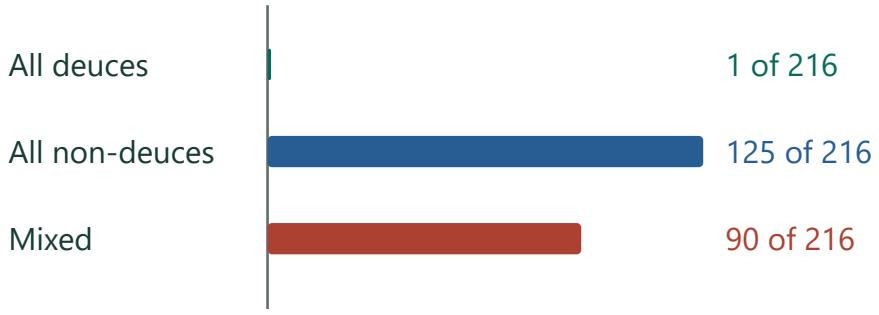
Rolling a 2 now leaves all faces equally possible next time. To require a condition at every roll, multiply its three individual probabilities.



Independent rolls give three separate choices.

3. Compare the two events

Three deuces have probability $(1/6)^3=1/216$. Three non-deuces have probability $(5/6)^3=125/216$. The other ordered outcomes mix deuces and non-deuces. A separate count gives $6^3=216$ equally likely ordered triples. Just one is (2,2,2), while each of three non-deuce positions has five choices, accounting for 125 triples.



Count each set of ordered outcomes.

Check. Among 216 equiprobable ordered triples exactly one is all twos and 5^3 avoid twos.

Exercise 582 · Two phases of one sinusoid

Printed page 61 · Sinusoidal functions / Phase shifts

There are many examples of graphs that have period 60 and whose y-values vary between -8 and 8. First, find a formula for such a function f, given that $f(0) = 8$. Next, find a formula for such a function g, given that $g(0) = 0$.

1. Choose the vertical scale

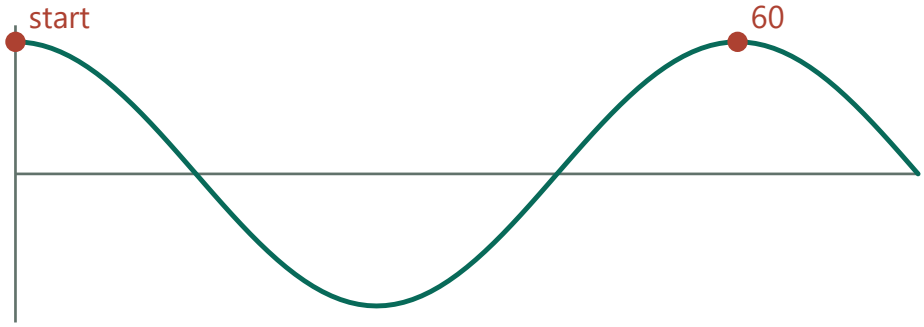
The halfway height between −8 and 8 is zero and the displacement from that midline to an extreme is 8. Multiply sine or cosine by 8.



The midline bisects the two extreme heights.

2. Choose the angular speed

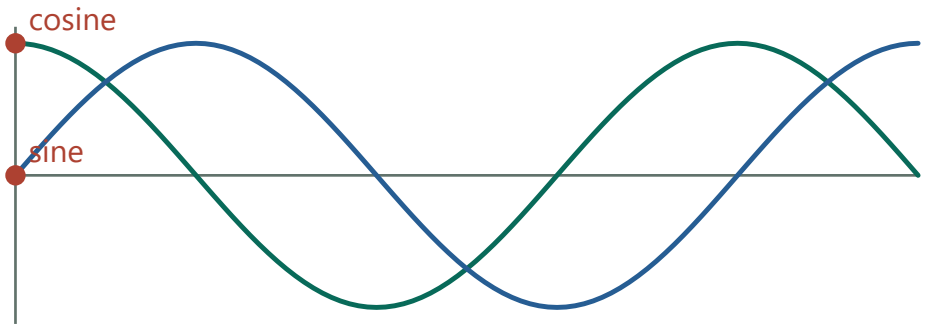
One complete turn of 2π radians must take 60 horizontal units. The phase therefore advances at 2π/60=π/30 radians per unit.



A full cosine cycle occupies 60 horizontal units.

3. Match the initial heights

Cosine starts at 1, so f(x)=8cos(πx/30) starts at 8. Sine starts at 0, so g(x)=8sin(πx/30) starts at zero. Both reach ±8 and return after 60.



Two phases share the same amplitude and period.

Check. A quarter period later f(15)=0 and g(15)=8.

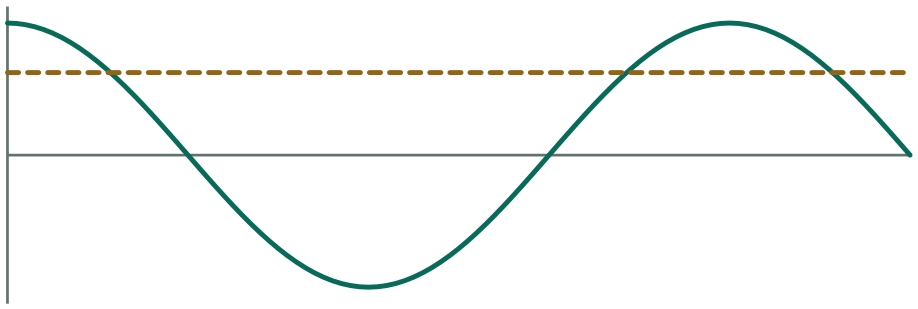
Exercise 583 · First three threshold times

Printed page 61 · Trigonometric equations / Periodic functions

(Continuation) For your f, find the three smallest positive x that make $f(x) = 5$.

1. Use the cosine model from 582

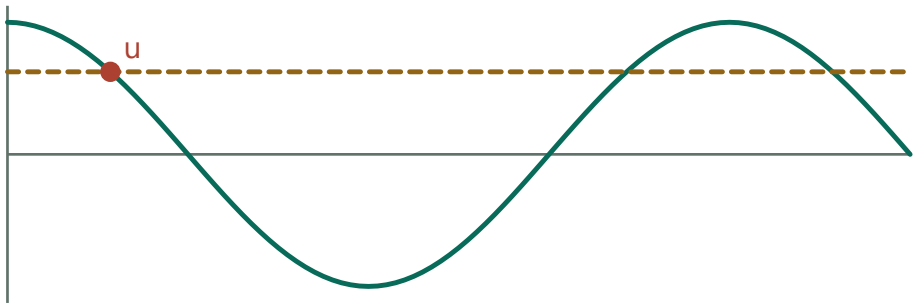
With f(x)=8cos(πx/30), the target height 5 corresponds to cos(πx/30)=5/8. The first positive angle having this cosine is arccos(5/8).



Ask when the cosine curve meets height 5.

2. Convert the angle to an input

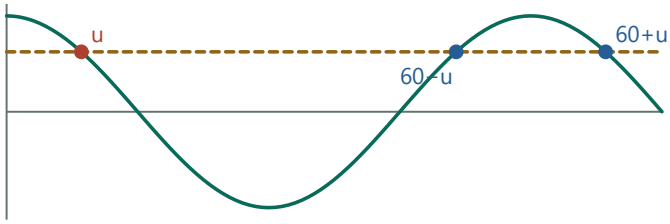
Multiply the angle by 30/π to get u=(30/π)arccos(5/8). This lies strictly between 0 and 15, in the falling quarter of the first cycle.



The first meeting lies in the falling quarter-cycle.

3. Locate the next two occurrences

Reflection of the cosine wave about x=30 gives height 5 at 60−u. The next cycle repeats the first hit at 60+u. Thus the smallest three positive inputs are u,60−u,60+u for this choice of f.



Reflection gives the next hit and periodicity the third.

1. $\theta_1=\arccos(5/8)$, $\theta_2=2\pi-\theta_1$, $\theta_3=2\pi+\theta_1$
Cosine takes the same value at reflected angles and after a full turn.

2. $(30/\pi)\cdot 2\pi=60 \Rightarrow x_2=60-u$, $x_3=60+u$
Convert each phase to horizontal distance by the same scale.

Check. Cosine takes the same value at opposite angles and after addition of 2π. Other valid choices in 582 may yield different times.

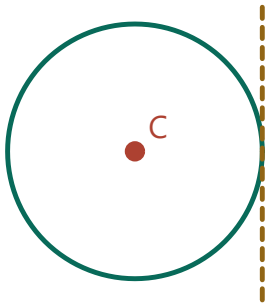
Exercise 584 · A tangent plane to a sphere

Printed page 61 · Sphere geometry / Planes in space / Tangent lines

Show that the sphere $(x-5)^2+(y+2)^2+(z-4)^2=36$ and the plane $2x+y+2z=34$ have exactly one point in common. Hint: vectors, hmm…

1. Read the center and the radius

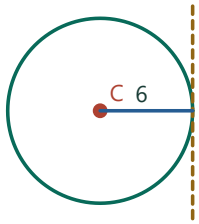
The squared differences show center C=(5,−2,4) and radius √36=6. The plane has normal n=(2,1,2), perpendicular to every displacement within it.



Normal cross section through the sphere and plane.

2. Find the perpendicular travel

At C the expression 2x+y+2z equals 16; on the plane it must be 34. A unit displacement along n changes the expression by ‖n‖=3. The shortest distance is therefore (34−16)/3=6.



The plane is one sphere radius from the center.

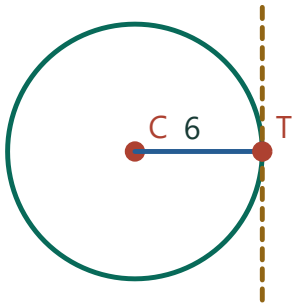
1. n·C = 2·5−2+2·4 = 16; n·n = 4+1+4 = 9
Evaluate the plane's linear expression and the squared normal length.

2. n·(C+tn)=16+9t=34 ⇒ t=2
Use a point on the normal through C and solve for its parameter.

3. ‖tn‖ = 2√9 = 6
The foot of the perpendicular is exactly one sphere radius away.

3. Construct the contact point

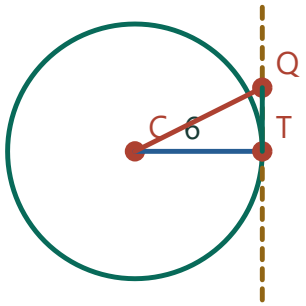
Move 6 units along n/3. Adding (4,2,4) to C gives T=(9,0,8), on the sphere and plane.



Travel along the unit normal to reach first contact.

4. Prove uniqueness

For any other point Q on the plane, CT is perpendicular to TQ. Hence $CQ^2=6^2+TQ^2>6^2$, putting Q strictly outside the sphere.



Any other Q adds a perpendicular leg, lengthening CQ.

Check. This point satisfies both equations; any other plane point has a right-triangle distance from C greater than 6.

Exercise 585 · Recover an exponential from two readings

Printed page 61 · Exponential functions / Systems of equations

A function of the form $H(x) = a \cdot b^x$ has the property that $H(1) = 112$ and $H(3) = 63$. Find the values $H(0)$ and $H(4)$.

1. Cancel the unknown scale factor

Divide $H(3)=ab^3$ by $H(1)=ab$. The factor a cancels, leaving $b^2=63/112=9/16$.



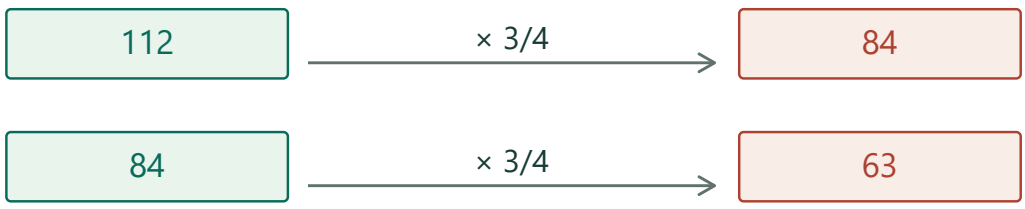
1. $H(3)/H(1) = ab^3/(ab) = b^2$
Cancel a and b; H(1)=112 is nonzero.

2. $63/112 = 9/16 \Rightarrow b=\sqrt{(9/16)}=3/4$
Reduce by 7 and use the positive base of an exponential model.

Two steps multiply the height by 63/112.

2. Choose the exponential base

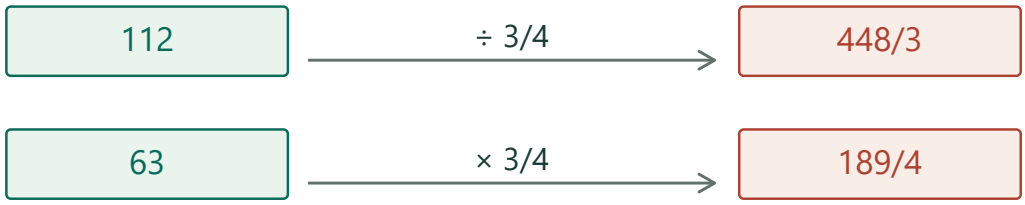
A real exponential function uses b>0, so b=3/4. Each increase of one in x multiplies the value by 3/4.



The positive square root gives the one-step multiplier.

3. Step once in each direction

Moving backward from H(1) divides by 3/4: H(0)=112·4/3=448/3. Moving forward from H(3) gives H(4)=63·3/4=189/4.



Use division backward and multiplication forward.

Check. $112(3/4)^2=63$.

Exercise 586 · Invert bounds on a logarithm

Printed page 61 · Logarithms / Inequalities

Given that $\log_4 x$ is somewhere between -1.0 and 0.5, what can be said about x?

1. Translate the logarithm inequality

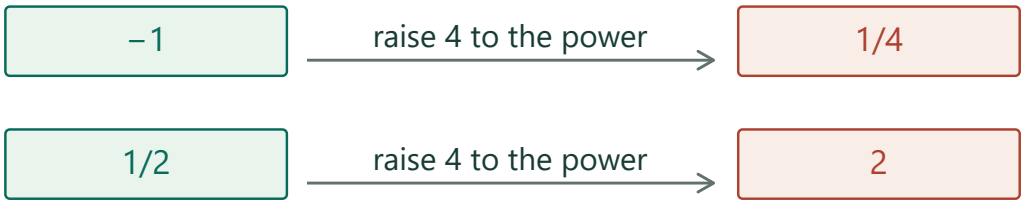
The given information is $-1<\log_4 x<0.5$. A base-four logarithm represents the exponent that produces x.



The logarithm is an exponent strictly between two values.

2. Exponentiate the endpoints

Raising 4 to each part preserves the order because powers of a base greater than 1 increase with the exponent. Thus $4^{-1}<x<4^{0.5}$.



Base 4 is increasing, so the endpoints keep their order.

3. Express the interval

The lower endpoint is 1/4 and the upper endpoint is $\sqrt{4}=2$, so x lies strictly between 1/4 and 2. Both endpoints correspond to excluded equalities.



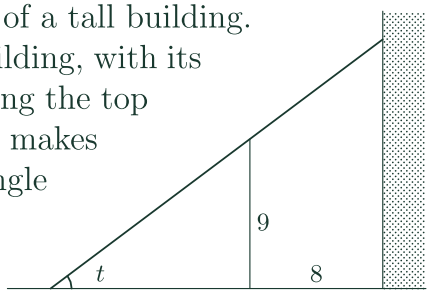
The open circles exclude the two equality cases.

Check. At the two excluded endpoints the logarithm equals −1 and 0.5.

Exercise 587 · Shortest ladder over a fence

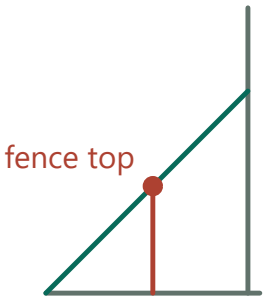
Printed page 62 · Optimization / Right-triangle trigonometry

A fence that is 9 feet tall is situated 8 feet from the side of a tall building. As the figure at right shows, a ladder is leaning against the building, with its base outside the fence. It so happens that the ladder is touching the top of the fence. Find the length of the ladder, given that **(a)** it makes a 60-degree angle with the ground; **(b)** it makes a t-degree angle with the ground. **(c)** Use a graphing tool to graph the function from part (b) and find the length of the shortest ladder that reaches the building from outside the fence.

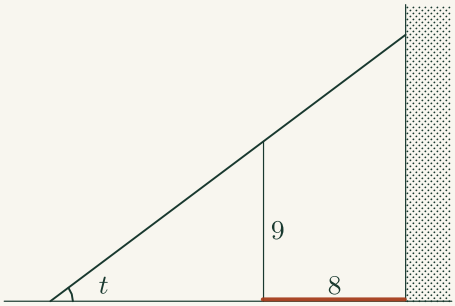


1. Split the ladder at the fence

The top of the fence divides the ladder into two segments. Complete a right triangle around each segment so the segment is its hypotenuse. The lower triangle has a vertical leg of 9 ft. The upper triangle, between the fence and the building, has a horizontal leg of 8 ft. The ladder makes the same angle t with either horizontal side.



At 45°, the 9-ft fence splits the ladder into two pieces.

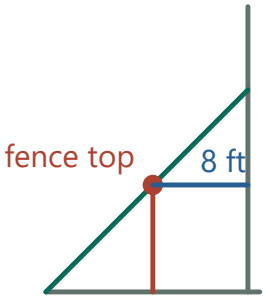


Gap from fence to building

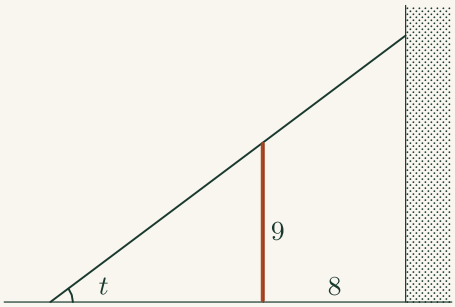
The horizontal gap is the run for the upper portion of the ladder. Project its endpoints straight down to the ground to see that run.

2. Convert the projections to lengths

For the lower portion sin t=9/lower; for the upper portion cos t=8/upper. Adding gives L(t)=9/sin t+8/cos t, with 0°<t<90°. At 60° this gives 9/(√3/2)+8/(1/2)=6√3+16 ft.



Each piece is the hypotenuse of its own right triangle.

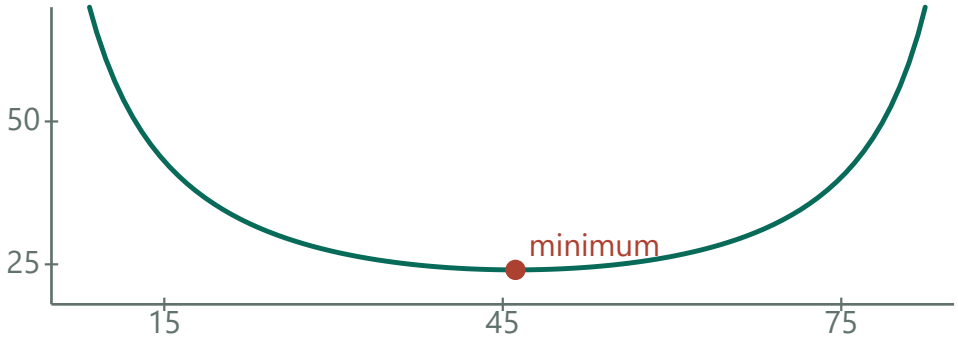


Fence height

The fence top is where the ladder separates into two portions. The vertical distance from ground to that point belongs to the lower right triangle.

3. Read the low point of the graph

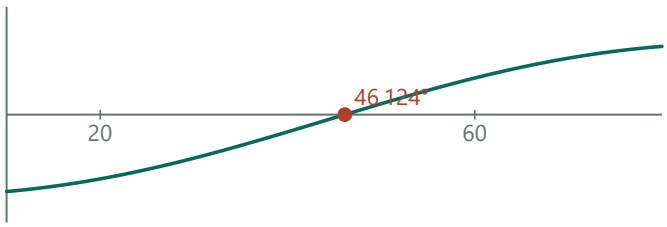
Near 0° the lower portion is very long; near 90° the upper portion is very long. The graph dips between them to about 24.028 ft at 46.124°.



Both ends rise sharply, so inspect the interior low point.

4. Verify the minimum

In radians $L'=-9\cos t/\sin^2 t+8\sin t/\cos^2 t$. Clearing positive denominators gives the sign of $8\sin^3 t-9\cos^3 t$. Its sign changes from negative to positive when $\tan^3 t=9/8$. The single solution in this interval is the global minimum. For its exact length, take legs 9^(1/3) and 8^(1/3) in a reference triangle with this tangent. Its hypotenuse is H=√(9^(2/3)+8^(2/3)). Substituting the resulting sine and cosine gives L=H[9^(2/3)+8^(2/3)]=(9^(2/3)+8^(2/3))^(3/2) ft.



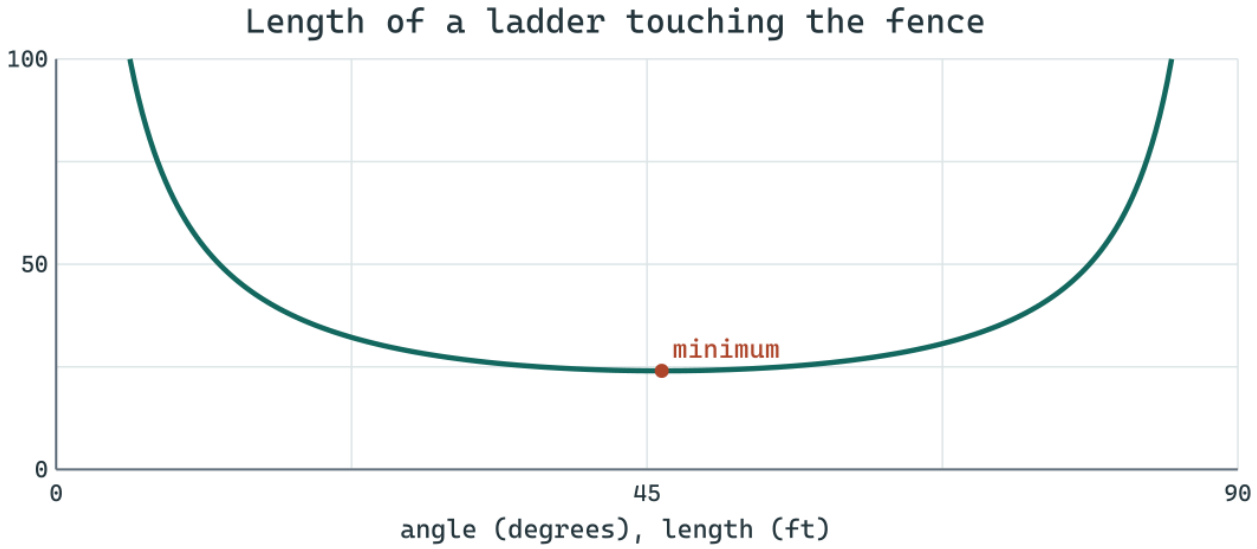
The derivative changes sign once, from negative to positive.

1. $L'\sin^2 t\cos^2 t = -9\cos^3 t+8\sin^3 t$
Both squared factors are positive on the permitted angle interval, so signs agree.

2. $8\sin^3 t=9\cos^3 t \Rightarrow \tan^3 t=9/8$
Divide by $8\cos^3 t$. Tangent increases from 0 to infinity here.

3. $\sin t=\sqrt[3]{9}/H$, $\cos t=\sqrt[3]{8}/H$
Use the legs $\sqrt[3]{9}$ and $\sqrt[3]{8}$ in the right triangle with hypotenuse H.

4. $L=9H/\sqrt[3]{9}+8H/\sqrt[3]{8} = H(9^{(2/3)}+8^{(2/3)}) = H^3$
At the minimizing angle the bracket equals H^2 by Pythagoras.



Check. L tends to infinity at both endpoints and its only stationary point gives (9^(2/3)+8^(2/3))^(3/2).

Exercise 588 · Three parabolas

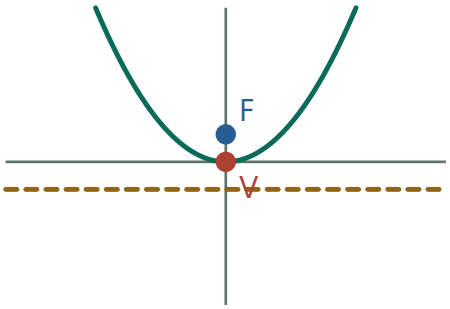
Printed page 62 · Parabolas / Foci and directrices

Show that the following equations are parabolas by finding foci and directrices for each.

(a) $y = 2x^2$ **(b)** $y = 2x^2 + 5$ **(c)** $y = 2x^2 + 12x + 5$

1. Find the focal displacement

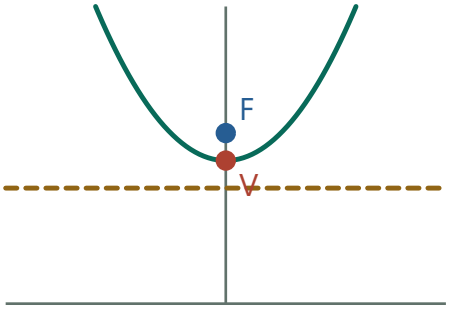
For $y=2(x-h)^2+k$ rewrite $(x-h)^2=\tfrac{1}{2}(y-k)$. Comparing with $(x-h)^2=4p(y-k)$ gives $p=\tfrac{1}{8}$. Points on that equation are equidistant from $(h,k+p)$ and the line $y=k-p$.



Focus and directrix lie |p| from vertex (0, 0).

2. Use the first two vertices

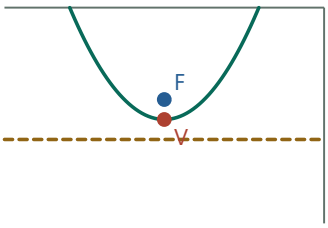
For (a) $(h,k)=(0,0)$. This gives focus $(0,\tfrac{1}{8})$, directrix $y=-\tfrac{1}{8}$. For (b) adding 5 moves the entire construction up 5, giving focus (0,41/8) and directrix $y=39/8$.



Focus and directrix lie |p| from vertex (0, 5).

3. Find the hidden vertex in (c)

Complete the square: $2(x^2+6x)+5=2[(x+3)^2-9]+5=2(x+3)^2-13$. The vertex is $(-3,-13)$. Add $\tfrac{1}{8}$ vertically for the focus $(-3,-103/8)$; subtract $\tfrac{1}{8}$ for the directrix $y=-105/8$.



Focus and directrix lie |p| from vertex (-3, -13).

1. $x^2+6x = x^2+6x+9-9 = (x+3)^2-9$
Add and subtract the square of half the linear coefficient.

2. $2[(x+3)^2-9]+5=2(x+3)^2-18+5$
Distribute 2 to both terms before shifting the vertex.

3. $-13\pm1/8 = (-104\pm1)/8$
From vertex height −13, move up or down the focal distance.

Check. The vertex always lies halfway between focus and directrix.

Exercise 589 · Parallel plane separations

Printed page 62 · Planes in space / Distance and loci

What is the distance between the parallel planes
(a) $y = -1$ and $y = 5$?
(b) $4x + 3y = 7$ and $4x + 3y = 27$?
(c) $4x + 7y + 4z = 33$ and $4x + 7y + 4z = 87$?

1. Move along a common normal

Two planes $n\cdot X=d_1$ and $n\cdot X=d_2$ have the same perpendicular direction n. Moving one unit along $n/\|n\|$ changes $n\cdot X$ by $\|n\|$. Their separation is therefore $|d_2-d_1|/\|n\|$.



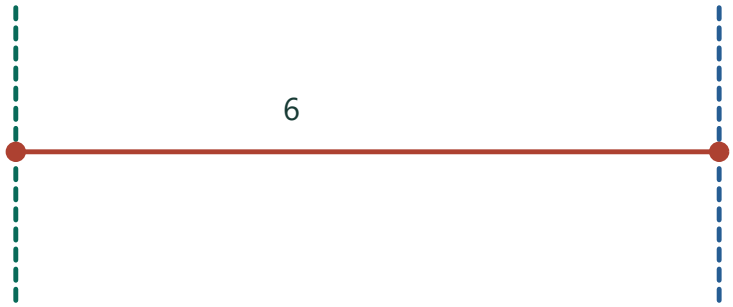
1. $Q=P+s\,n/\|n\| \Rightarrow n\cdot Q=n\cdot P+s\|n\|$
Parametrize the perpendicular through a point on the first plane.

2. $s=(d_2-d_1)/\|n\| \Rightarrow$ distance=abs(s)
Use the second plane's level and take the magnitude of the signed displacement.

Travel perpendicular to the planes to measure their spacing.

2. Read the horizontal slices (a)

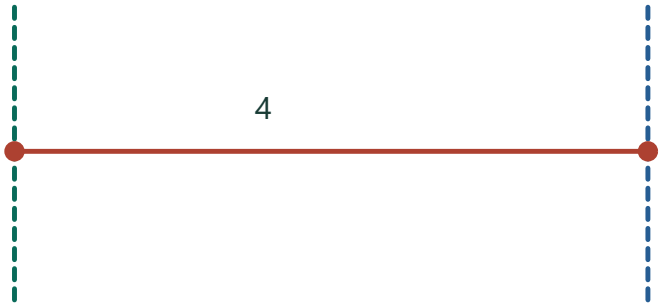
Only y changes, from −1 to 5, so the two planes are 6 units apart.



Parallel y-planes have separation 5 − (−1).

3. Use the norm in (b)

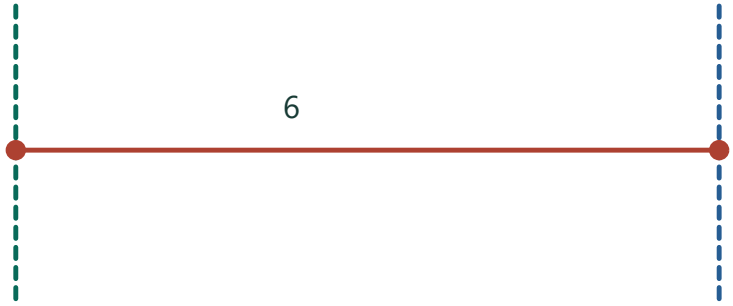
The normal (4,3,0) has length 5. The right side increases by 20, so the distance is 20/5=4.



A level change of 20 is four normal-length units.

4. Use the norm in (c)

For (4,7,4) the norm is √(16+49+16)=9. The increase is 87−33=54, giving 54/9=6.



The change 87 − 33 divided by 9 gives six units.

Check. Shifting a point by the stated distance along the unit normal increases its dot product by the required Δd.

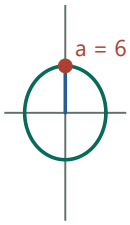
Exercise 590 · Focus and directrix of a vertical ellipse

Printed page 62 · Ellipses / Foci and directrices

Find a focus and a directrix for the ellipse $4x^2 + 3y^2 = 108$.

1. Locate the major axis

Divide by 108 to obtain $x^2/27+y^2/36=1$. The longer radius is vertical, a=6, and $b^2=27$.



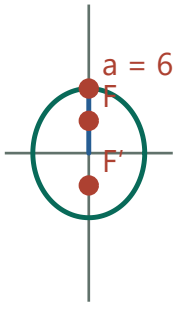
Read the longer semiaxis from the intercept.

1. $(4x^2+3y^2)/108=1$
Divide each side of the original ellipse equation by the same positive constant.

2. 4/108=1/27; 3/108=1/36
Reduce the two coefficients separately. The larger squared semiaxis is the denominator 36 under y^2.

2. Locate the focus

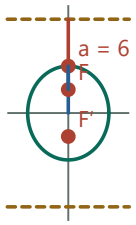
The focal distance satisfies $c^2=a^2-b^2=36-27=9$. The upper focus is (0,3) and e=c/a=½.



The focal displacement satisfies $c^2 = a^2 - b^2$.

3. Locate the matching line

The upper directrix is horizontal at y=a/e=12. At the upper vertex (0,6) the distances to (0,3) and y=12 are 3 and 6. Reflection supplies the other matched pair, (0,−3) and y=−12.



At the near vertex, compare its distances to focus and directrix.

1. d=a/e=6/(1/2)=12
The center-to-directrix distance equals the semimajor axis divided by the eccentricity.

2. PF=6−3=3; distance to directrix=12−6=6
At the upper vertex the focus and directrix are on its vertical line, so subtract the y-coordinates.

3. PF/6=3/6=1/2
The vertex produces the specified focus-to-directrix distance ratio.

Check. At (0,6) the focal distance 3 is half the distance 6 to y=12.

Exercise 591 · Logarithms of a geometric progression

Printed page 62 · Geometric sequences and series / Arithmetic sequences and series / Logarithm rules

The numbers 3, 12, 48, 192, ... form a geometric sequence. What can be said about the sequence of *logarithms* of these numbers? Does it have any special property? Does it make any difference what base is used for the logarithms?

1. Find the geometric term being logged

Index the first term by n=1. Every term is positive, so every logarithm below is defined.

1 **12/3 = 48/12 = 192/48 = 4**
Successive ratios reveal the common multiplier.

2 **$a_1 = 3,\quad a_{n+1} = 4a_n$**
Each new term uses one more factor of 4.

3 **$a_n = 3 \cdot 4^{(n-1)}$**
Starting at 3 takes n−1 multiplications to reach term n.

2. Take the logarithm of each factor

Fix one base $B>0$ with $B\neq1$. A positive base other than 1 gives a one-to-one exponential and hence a real logarithm. Use this same base for the whole sequence.

1 **$L_n = \log_B(3 \cdot 4^{(n-1)})$**
Call the logarithm of term n by the name L_n.

2 **$L_n = \log_B 3 + \log_B(4^{(n-1)})$**
Expand a positive product with the log product rule.

3 **$L_n = \log_B 3 + (n-1)\log_B 4$**
Bring out the exponent using the log power rule.

3. Compare adjacent logarithms

The log sequence is arithmetic in every valid base. A base above 1 gives a positive common difference here, while a base between 0 and 1 gives a negative one.

1 **$L_{n+1} = \log_B 3 + n \log_B 4$**
Replace n by n+1 in the term formula.

2 **$L_{n+1} - L_n = [n - (n-1)]\log_B 4$**
The two copies of $\log_B 3$ cancel.

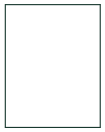
3 **$L_{n+1} - L_n = \log_B 4$**
The difference depends on B and never on the index n.

Check. Subtract consecutive terms; the difference is $\log(4a)-\log a=\log 4$.

Exercise 592 · Fifty ideal folds

Printed page 62 · Geometric sequences and series / Exponent rules

A rectangular sheet of paper (such as the one shown in the figure at right) has thickness 0.003 inches. Suppose that it is folded in half, then folded in half again, then folded in half again — fifty times in all. How thick is the resulting wad of paper?



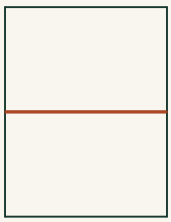
1. Model one fold

Folding in half stacks the two halves atop each other. Assuming constant thickness, no gaps and no compression, the number of layers doubles each time.

2^1 layers



One fold places one half-sheet over the other.



A possible halfway crease

Imagine bringing the top edge down to the bottom edge along this horizontal crease. Matching halves occupy the same footprint in two layers.

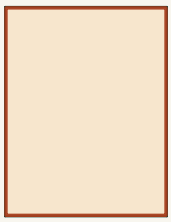
2. Track the starting index

Before folding there is one layer of thickness 0.003 in. After one fold there are two layers, after two there are four, and after 50 there are 2^{50}.

2^2 layers



After two folds, four layers occupy the same footprint.

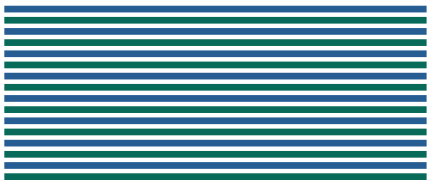


One sheet before any fold

The outline encloses the original sheet. Counting layers starts before the first fold has been made.

3. Convert layers to thickness

Multiply thickness per layer by layer count: $0.003 \cdot 2^{50} \approx 3.378 \times 10^{12}$ in. This is the consequence of the idealized folding rule.



2^{50} layers

partial stack shown

Multiply the idealized layer count by 0.003 inch.

Check. After two folds four layers give 0.012 inches, consistent with the indexing.

Exercise 593 · Recursive paper thickness

Printed page 62 · Sequences and recursion / Geometric sequences and series

(Continuation) Give a recursive description for the thickness of the paper.

1. Define the state of the model

Let T_n denote thickness in inches after n folds. Zero folds means one original sheet, so $T_0 = 0.003$.

2^0 layers

The starting sheet contributes one layer, even at n = 0.

2. Prescribe the update

The next fold stacks the current wad onto itself. The two equal thicknesses add: $T_{n+1}=T_n+T_n=2T_n$.

2^1 layers



Stacking two equal thicknesses doubles the previous total.

3. Test the recursion

It returns 0.006 and then 0.012 in for the first two folds. Repeating the update n times gives $T_n=0.003\cdot 2^n$, agreeing with the 50-fold calculation.

2^2 layers



A second doubling gives 0.012 inch before the pattern continues.

Check. T_{50} matches the explicit calculation in 592.

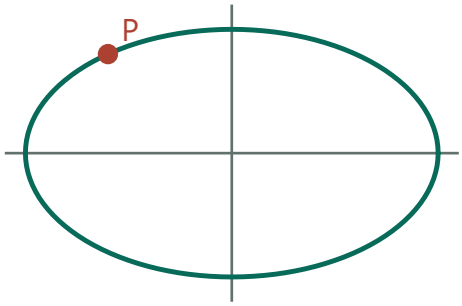
Exercise 594 · Tangent slope at an ellipse point

Printed page 62 · Ellipses / Slopes and tangent lines

The point $P=(-3,2.4)$ is on the ellipse $9x^2+25y^2=225$. Verify this, then come as close as you can to finding the slope of the line that is tangent to the ellipse at P.

1. Verify the marked point

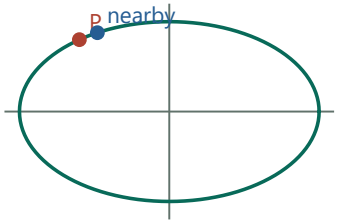
Substitute x=−3 and y=2.4. The two terms are 9·9=81 and 25·5.76=144, whose sum is 225.



The marked point satisfies the ellipse equation.

2. Relate the two coordinate changes

Near P, y varies as x changes along the ellipse. Differentiate $9x^2+25y^2=225$ with respect to x. The chain rule gives $18x+50y\cdot dy/dx=0$.



Vary x and y together along the upper branch.

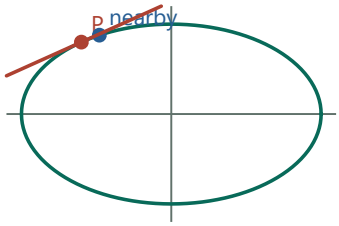
1. $d(y^2)/dx=2y\cdot dy/dx$
Along the ellipse y depends on x, so differentiating its square requires the chain rule.

2. $d(9x^2)/dx=18x$; $d(25y^2)/dx=50y\cdot dy/dx$
Differentiate the two summands separately.

3. $18x+50y\cdot dy/dx=0$
The constant 225 on the opposite side has derivative zero.

3. Evaluate the tangent slope

Solving gives dy/dx=−18x/(50y). At P it is 54/120=9/20=0.45. Moving a little right therefore moves the curve slightly upward there.



The rising tangent has slope 9/20.

1. $50y(dy/dx)=-18x \Rightarrow dy/dx=-18x/(50y)$
Subtract 18x and divide by 50y, nonzero at this point.

2. $-18(-3)/(50\cdot 2.4)=54/120=9/20$
Substitute the point before simplifying the signs.

Check. The normal vector (−54,120) is perpendicular to the direction (20,9).

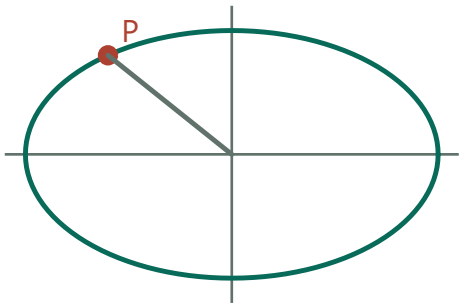
Exercise 595 · Parametrize nearby ellipse points

Printed page 62 · Ellipses / Parametric equations

(Continuation) Points on the ellipse can be described parametrically by the equation $(x, y) = (5\cos t, 3\sin t)$. Find a t-value that produces P. Then find a t-value that produces a point on the ellipse that is very close to P.

1. Read the required sine and cosine

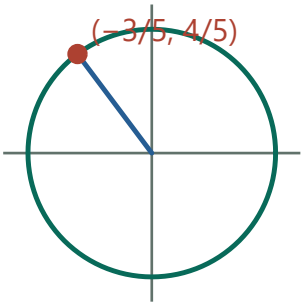
At P=(−3,2.4) the parameter equations give cos t=−3/5 and sin t=2.4/3=4/5. Negative cosine and positive sine place the angle in quadrant II.



Negative cosine and positive sine select quadrant II.

2. Choose the correct angle

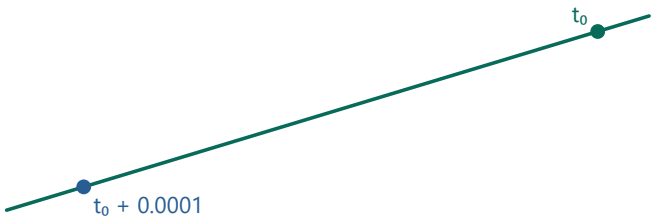
The reference triangle has side ratio 4:3, giving reference angle arctan(4/3). Thus $t_0=\pi-\arctan(4/3)\approx2.21430$ radians produces P.



On the unit circle the reference triangle has legs 3 and 4.

3. Move just along the curve

Take $t=t_0+0.0001$ radians. The coordinates become $(5\cos(t_0+0.0001),3\sin(t_0+0.0001))$. Continuity makes them close to P, and using the same parameterization keeps them on the ellipse.



Magnify the same ellipse near P to resolve the small change.

1. Let h=0.0001. Then x=−3cos h−4sin h
Use the cosine addition formula on $5\cos(t_0+h)$, substituting cos $t_0=-3/5$ and sin $t_0=4/5$.

2. y=2.4cos h−1.8sin h
Apply the sine addition formula to $3\sin(t_0+h)$.

3. x≈−3.000400; y≈2.399820
Evaluate in radians. Squaring the normalized coordinates and adding gives $\cos^2(t_0+h)+\sin^2(t_0+h)=1$.

Check. The local parametric slope 3cos t/(−5sin t) at t_0 equals 9/20.

Exercise 596 · Swapping coordinates

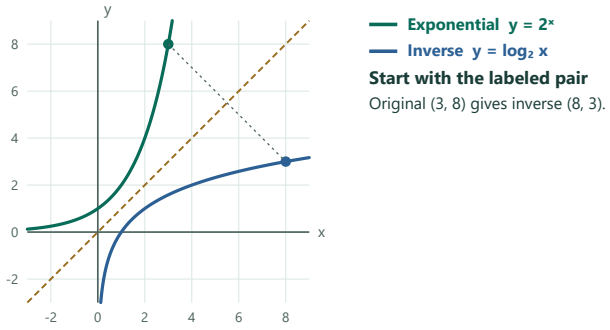
Printed page 62 · Inverse functions / Logarithms

The point $(3, 8)$ is on the graph $y = 2^x$. What is the corresponding point on the graph of the inverse function? Find four more pairs of points like these.

1. Reverse the input and output

The equation $2^3=8$ associates input 3 with output 8. On the inverse graph, input 8 produces 3, so (3,8) becomes (8,3).

Exchanging exponential coordinates reflects the graph

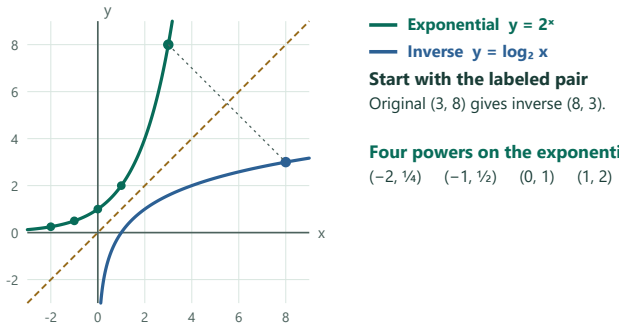


Exponential $y = 2^x$
Inverse $y = \log_2 x$
Start with the labeled pair
Original (3, 8) gives inverse (8, 3).

2. Use four more powers

The original exponential also contains $(-2,¼),(-1,½),(0,1),(1,2)$. These follow from $2^{-2},2^{-1},2^0$ and 2^1.

Exchanging exponential coordinates reflects the graph

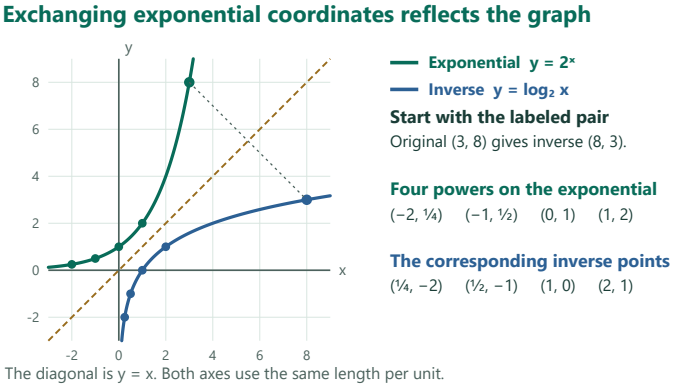


Exponential $y = 2^x$
Inverse $y = \log_2 x$
Start with the labeled pair
Original (3, 8) gives inverse (8, 3).

Four powers on the exponential
(−2, ¼) (−1, ½) (0, 1) (1, 2)

3. Match each inverse point

In the same order the points become (¼,−2),(½,−1),(1,0),(2,1). All satisfy $y=\log_2 x$, the inverse relation.

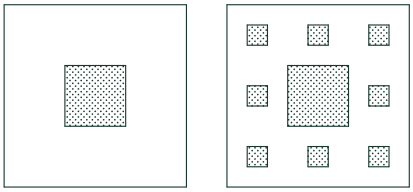


Check. In every pair $\log_2$ of the new x-coordinate is the new y-coordinate.

Exercise 597 · An unpainted fractal area

Printed page 63 · Fractals and iteration / Geometric sequences and series / Limits and convergence

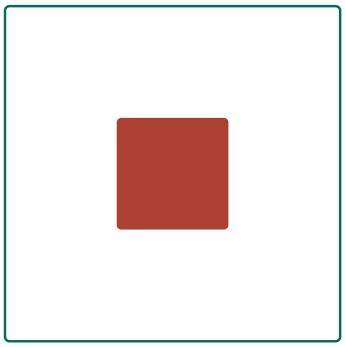
A blank, square sheet of paper is painted as follows: Step 1 consists of ruling the square into nine smaller congruent squares, and painting the central one. Step 2 consists of applying this process to each of the remaining eight squares (a small central square is painted in each). Step 3 consists of applying this process to each of the 64 remaining small squares, etc. The first two steps are shown.



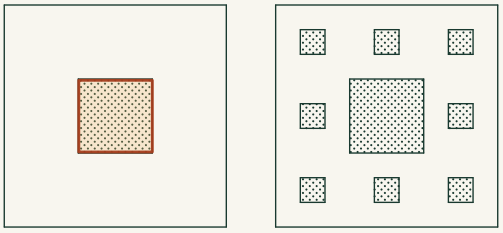
Let A_0 be the area of the original square, A_1 be the area that is unpainted after Step 1, A_2 be the area that is unpainted after Step 2, and so on. Show that the decreasing sequence A_0, A_1, A_2, ... is geometric. Is it ever true that $A_n < 0.01A_0$? Justify your answer.

1. Look inside any unpainted square

At the next stage it is split into nine equal areas. Its middle ninth is painted, leaving eight ninths. The same fraction survives in every unpainted piece, whatever its size.



Paint the middle ninth of each still-unpainted square.



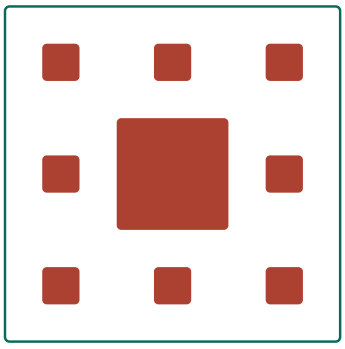
Middle ninth in the first square

Dividing each side into thirds creates nine equal small squares. Identify the one newly painted in this first picture.

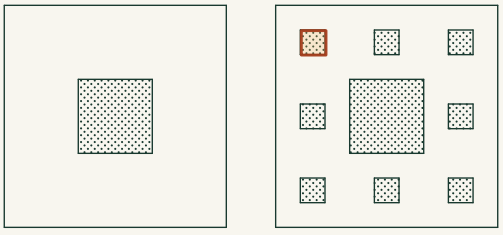
2. Update the total unpainted area

Adding the surviving pieces gives $A_{n+1}=(8/9)A_n$. Repeating from A_0 yields $A_n=A_0(8/9)^n$, a decreasing geometric sequence.

Subdivision count · Stage 2



Paint the middle ninth of each still-unpainted square.

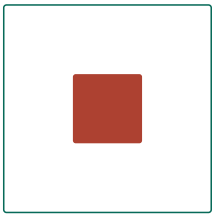


A middle ninth at the next scale

Compare this smaller painted square with its parent in the second drawing. The area fraction is measured relative to the parent square at each stage.

Compare settings · Subdivision count

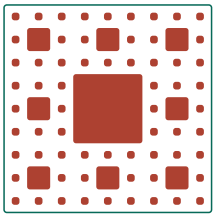
Stage 1



Paint the middle ninth of each still-unpainted square.

8/9 of the starting area is still unpainted.

Stage 3

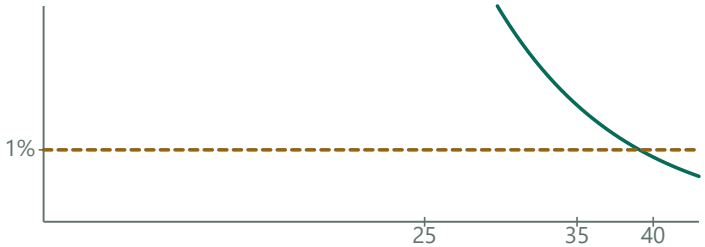


Paint the middle ninth of each still-unpainted square.

512/729 of the starting area is still unpainted.

3. Solve the 1% condition

Divide $A_n < 0.01A_0$ by $A_0 > 0$. Taking logs gives $n \ln(8/9) < \ln(0.01)$. Since $\ln(8/9) < 0$, divide and reverse the inequality to obtain $n > \ln(0.01)/\ln(8/9) \approx 39.10$.



Zoom in around the threshold 0.01 of the initial area.

1. $(8/9)^n < 0.01 \Rightarrow \ln((8/9)^n) < \ln 0.01$
Both quantities are positive. Natural log is increasing.

2. $n \ln(8/9) < \ln 0.01 \Rightarrow n > \ln 0.01/\ln(8/9)$
Bring the exponent down; division by a negative reverses the inequality.

4. Select a whole stage

The first whole stage greater than this threshold is 40. At stage 39 more than 1% remains; at stage 40 less remains.



Only the second of these integer stages is below 1%.

Check. Comparing $(8/9)^{39}$ and $(8/9)^{40}$ to .01 brackets the crossing.

Exercise 598 · Inverse of a cubic

Printed page 63 · Inverse functions / Power functions

Consider the function $f(x) = x^3$.

(a) Write a formula for $g(x)$, the inverse of $f(x)$.

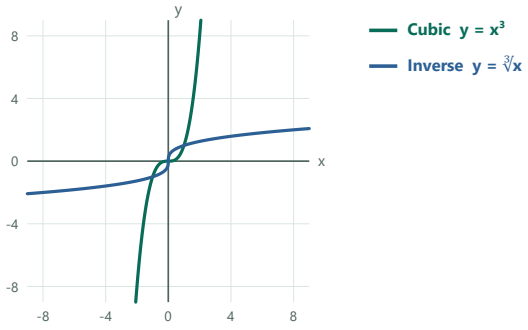
(b) Choose three points on the graph of $y = f(x)$, at least one of them with a negative x-coordinate. What are their corresponding points on the graph of $y = g(x)$?

(c) What geometrical transformation can be applied to the graph of $y = f(x)$ in order to generate the graph of the inverse function?

1. Undo a cube

Cubing is one-to-one on the real numbers, including negative values. To undo $y=x^3$, take a real cube root. The inverse is $g(x)=\sqrt[3]{x}$.

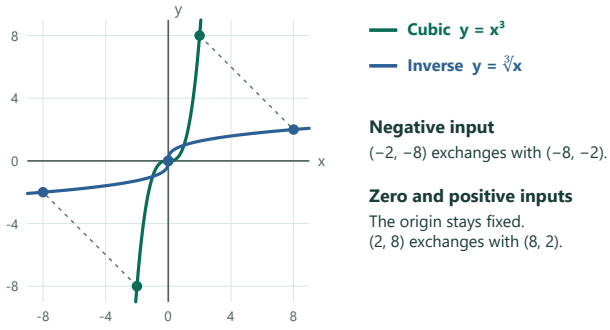
A cubic and its inverse across the diagonal



2. Switch the coordinate roles

Some points on the original cubic are $(-2,-8),(0,0),(2,8)$. Reading each output as the input to the inverse produces $(-8,-2),(0,0),(8,2)$.

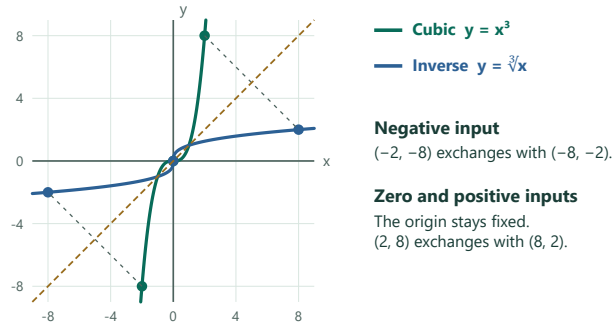
A cubic and its inverse across the diagonal



3. Identify the geometry

Exchanging horizontal and vertical coordinates reflects a point across y=x. The entire inverse graph is that reflection of the cubic, including its negative branch.

A cubic and its inverse across the diagonal



The diagonal y = x bisects each joining segment at a right angle. Both axes have equal scales.

Check. Both compositions give x for all real x.

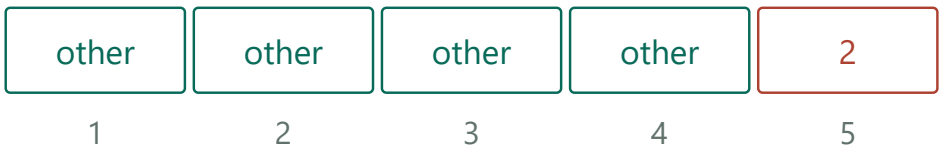
Exercise 599 · One designated deuce

Printed page 63 · Probability

Morgan is going to roll a six-sided die five times. What is the probability that only the final roll will be a deuce?

1. Fix the pattern before counting

Only the final roll can be a 2. Positions one through four must be non-deuces, and position five must be a deuce. The desired position is already specified.



The event specifies all five positions.

2. Assign one-roll probabilities

On each fair roll five of the six faces are non-deuces. The final face has just one acceptable outcome, 2.



Five faces work in each of the first four positions.

3. Multiply independent chances

The probability is $(5/6)^4 \cdot (1/6) = 625/7776$. Counting ordered sequences also gives 5^4 favorable out of 6^5 equally likely outcomes.



Multiplying choices gives 625 favorable strings among 6^5.

Check. Exactly 5^4 of the 6^5 ordered outcomes satisfy the restriction.

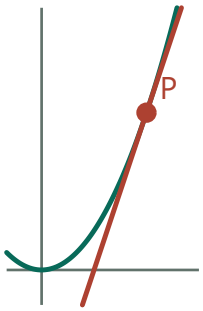
Exercise 600 · A parabola's equal reflection angles

Printed page 63 · Parabolas / Conic reflection properties / Tangent lines

Verify that the line $y = 3x - 9$ is tangent to the parabola $4y = x^2$ at $P = (6, 9)$. Verify also that the focal point for this parabola is $F = (0, 1)$. Calculate the size of the acute angle formed by the tangent line and the segment FP. Do the same for the acute angle formed by the tangent line and the line $x = 6$.

1. Check contact algebraically

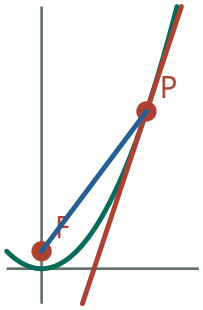
Insert $y=3x-9$ into $x^2=4y$. It gives $x^2-12x+36=(x-6)^2=0$. The two graphs meet only at $x=6$, where $y=9$; the double root verifies tangency.



The tangent and parabola share a double root at $x = 6$.

2. Identify the focal radius

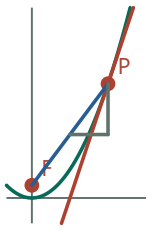
The standard equation $x^2=4py$ has focus $(0,p)$. Here $p=1$. From $F=(0,1)$ to $P=(6,9)$, the segment rises 8 and runs 6, giving slope $4/3$.



The focal segment rises 8 while running 6.

3. Compare the focal segment and tangent

The tangent slope is 3. The acute angle between two slopes satisfies $\tan\theta=|(3-4/3)/(1+3\cdot4/3)|=1/3$. Thus $\theta\approx18.435°$.



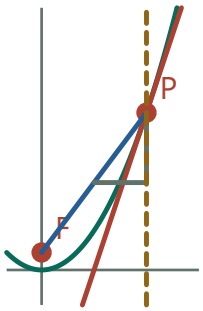
Compare the focal segment and the tangent at P.

1. $3-4/3=5/3$; $1+3\cdot4/3=5$
Compute the numerator and denominator of the angle formula separately.

2. $(5/3)/5=(5/3)(1/5)=1/3$
Take the positive ratio for the acute angle.

4. Compare with the vertical line

A direction vector of the tangent is (1,3). Its acute angle from the vertical has horizontal-to-vertical ratio 1/3, so it too is arctan(1/3).



The vertical direction makes the same acute angle.

Check. The derivative of $x^2/4$ at x=6 is 3, independently confirming the tangent.

Exercise 601 · Recursive population growth

Printed page 63 · Sequences and recursion / Exponential growth and decay

The equations $U_{1950} = 25\,791\,017$ and $U_n = U_{n-1} + 0.027U_{n-1}$ provide a recursive description of a population that has an annual growth rate of 2.7%, where U_n is the size of the population at the beginning of year n. Note that U_{1950} represents the population of Mexico in 1950.

(a) Explain why the numbers U_n form a geometric sequence.

(b) Write explicit equations for U_{1951}, U_{1952} , U_{2000}, and U_{1950+k}. An explicit description for f_n is different from a recursive description because it does not require the evaluation of another f_k. For example, $f_n = 1.05f_{n-1}$, with $f_0 = 10$ is recursive while $f_n = 10(1.05)^n$ is explicit.

(c) Find the first integer n for which $100\,000\,000 \leq U_n$.

1. Factor the yearly update

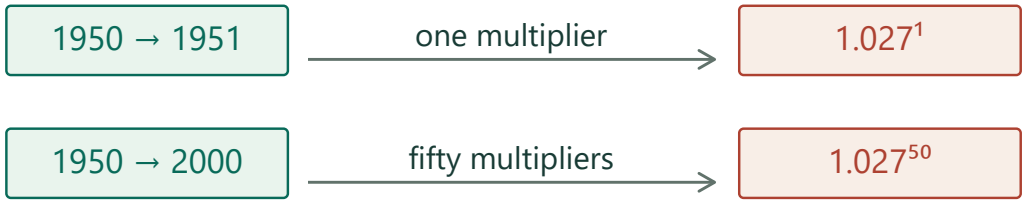
The existing population and its 2.7% growth add: $U_n = U_{n-1} + 0.027U_{n-1} = 1.027U_{n-1}$. This constant ratio makes the model geometric.



Every year adds the same percentage of the current count.

2. Count elapsed years

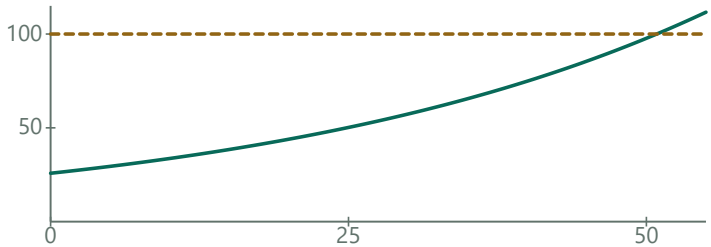
Starting at the beginning of 1950, one factor applies by 1951 and two by 1952. Thus $U_{1951}=25{,}791{,}017(1.027)$ and $U_{1952}=25{,}791{,}017(1.027)^2$. The year 2000 uses 50 factors, so $U_{2000}=25{,}791{,}017(1.027)^{50}$. For any year $1950+k$ the same count gives $U_{1950+k}=25{,}791{,}017(1.027)^k$.



Use elapsed years for the exponent.

3. Locate the threshold

Solve $25{,}791{,}017(1.027)^k \geq 100{,}000{,}000$. Division and logs give $k \geq \ln(100{,}000{,}000/25{,}791{,}017)/\ln(1.027) \approx 50.8651$.



Millions of people: find where the model reaches 100.

1. $1.027^k \geq 100{,}000{,}000/25{,}791{,}017$
Divide by the positive initial population.

2. $k \ln 1.027 \geq \ln(100{,}000{,}000/25{,}791{,}017)$
Take the increasing natural log and bring the exponent down.

3. $k \geq \ln(100{,}000{,}000/25{,}791{,}017)/\ln 1.027$
This time the logarithm of the growth factor is positive, so the order stays the same.

4. Return to the calendar

The question samples the beginning of integer years. Round up to $k=51$, giving 2001 as the first year at or above 100 million.



Compare the two adjoining calendar-year counts.

Check. The exponents 50 and 51 respectively give a population below and above the target.

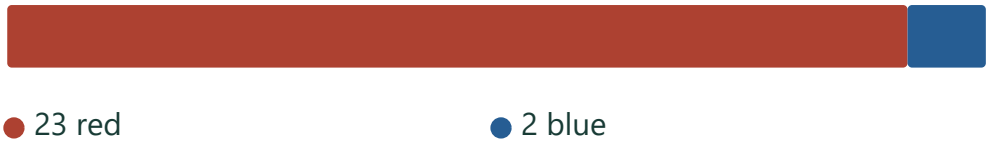
Exercise 602 · Draws with replacement

Printed page 63 · Sampling with replacement / Probability

There are twenty-three red marbles and two blue marbles in a box. A marble is randomly chosen from the box, its color noted, then put back in the box. This process is repeated. What is the probability that **(a)** the first marble is blue? **(b)** the first four marbles are red? **(c)** the first four marbles are red and the fifth is blue?

1. Rebuild the sample space each time

The box has 23+2=25 marbles. Replacing the marble after every draw restores those same 25 possibilities and leaves the draws independent.



Replacement restores the box before each new draw.

2. Blue first (a)

Two of the 25 marbles are blue. The first-draw probability is therefore 2/25.



Choose the blue portion for the first draw.

3. Require a run of reds (b)

For each of the first four draws, the required color has probability 23/25. Multiply four equal factors to obtain $(23/25)^4$.



Four independent red draws multiply four red factors.

4. Append one more specified color (c)

The four-red run must now be followed by blue. Multiply its probability by 2/25, giving $(23/25)^4(2/25)$.



The required fifth draw contributes a blue factor.

Check. Replacement restores 25 choices and the same color probabilities each time.

Exercise 603 · A short secant interval

Printed page 63 · Exponential functions / Rates of change

$$\text{Let } f(x) = 5^x. \text{ Calculate } \frac{f(1.003) - f(1.000)}{0.003} \text{ and then explain its significance.}$$

1. Identify the two points

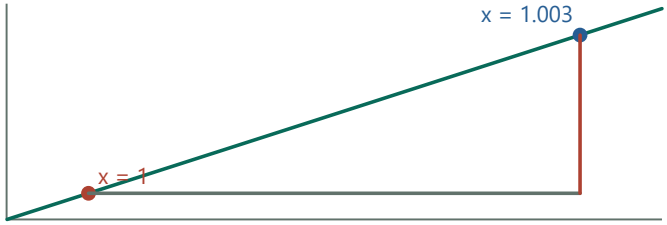
For f(x)=5ˣ, the quotient compares x=1 and x=1.003, a horizontal separation of 0.003. Subtract their outputs in that same order.



Magnified view of the two locations on y = 5ˣ.

2. Factor before evaluating

Since 5^1.003=5·5^0.003, the mean slope is 5(5^0.003−1)/0.003≈8.06665.



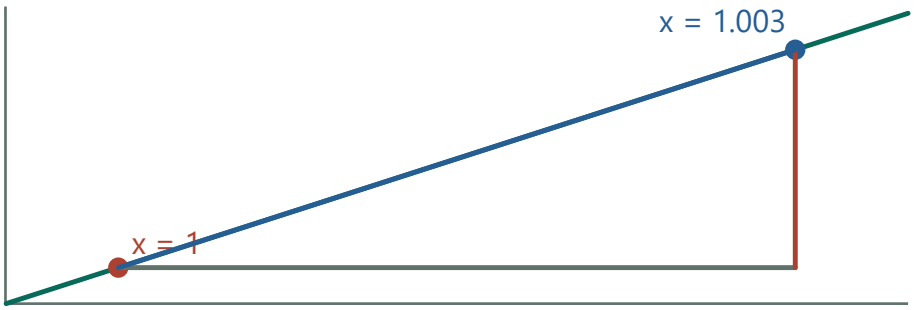
Divide the vertical change by the run of 0.003.

1. $s=[5^{(1+.003)}-5^1]/.003$
Subtract the starting output and divide by the change in input.

2. s=[5·5^.003−5]/.003=5(5^.003−1)/.003
Apply the exponent addition law and factor the shared 5.

3. Read the geometric meaning

This is the slope of the secant line through the two points: average rise for one unit of x over that small interval. It estimates the tangent slope at x=1, which is 5 ln 5≈8.047.



The straight chord is the secant for this short interval.

Check. The exact derivative 5 ln 5≈8.047 is slightly smaller, as expected for upward curvature.

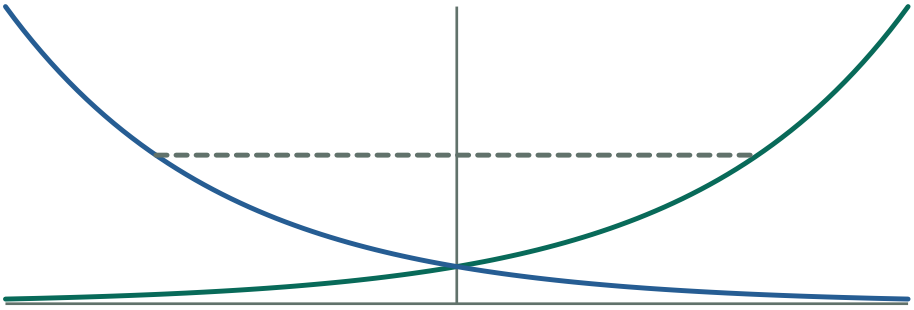
Exercise 604 · Reflect an exponential

Printed page 64 · Exponential functions / Reflections / Slopes and tangent lines

How does the graph of $y = \left(\frac{1}{2}\right)^x$ compare with the graph of $y = 2^x$? What features do these curves have in common? How are the slopes of these curves related at their common y-intercept?

1. Rewrite the base

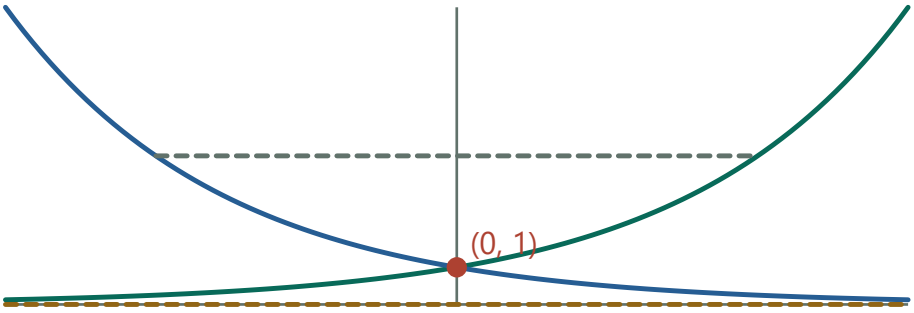
The identity (½)ˣ=2^(−x) means that the value at x on one curve equals the value at −x on the other. This is reflection across the vertical axis.



At opposite x-values the two graphs have the same height.

2. Compare the shared features

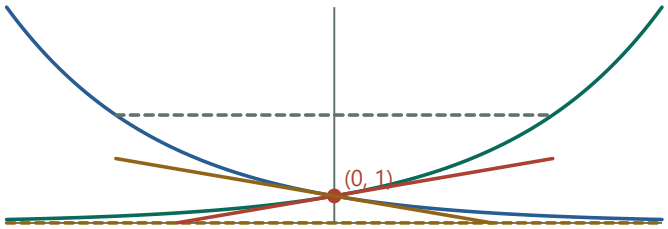
Both accept every real x, have positive range, approach the horizontal line y=0 at one end and pass through (0,1). Reflection makes the ½-base curve decrease as the base-two curve increases.



They share an intercept and a horizontal asymptote.

3. Compare tangent directions

Reflecting a secant at x=0 reverses its horizontal component and preserves its vertical component. The limiting tangent slopes are opposites, ln 2 for 2^x and −ln 2 for $(½)^x$.



Reflecting the tangent reverses the sign of its slope.

1. g(x)=2^(−x) ⇒ g′(x)=2^(−x) ln(2)·(−1)
The derivative of the inner exponent −x is −1.

2. g′(0)=−ln 2, f′(0)=ln 2
The same height 2^0=1 multiplies both rates.

Check. Corresponding secants have opposite horizontal increments and equal vertical increments.

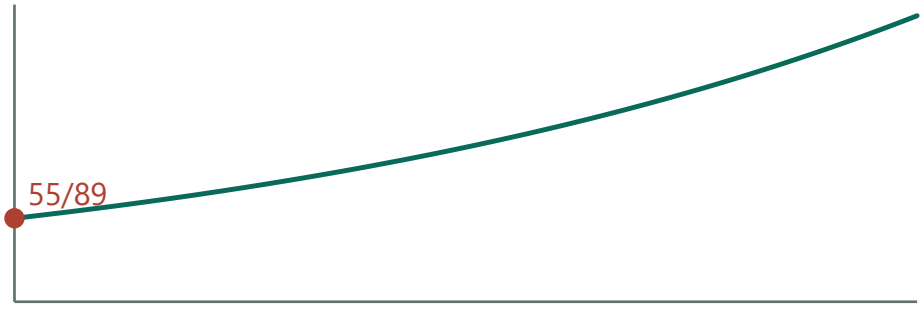
Exercise 605 · Two growing populations

Printed page 64 · Exponential growth and decay / Logarithms / Rates of change

Let $R(t) = 55(1.02)^t$ describe the size of the rabbit population in the PEA woods t days after the first of June. Let $B(t) = 89(1.01)^t$ describe the size of the beaver population in the same area t days after the first of June.
(a) When are there the same number of rabbits as beavers? At what rate (rabbits per day) is the rabbit population increasing then? At what rate is the beaver population increasing?
(b) Solve the equation $R(t) = 2B(t)$. Interpret your answer.

1. Track the rabbit-to-beaver ratio

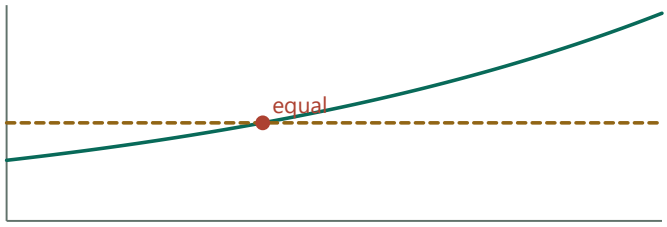
Divide the two models: R(t)/B(t)=(55/89)(1.02/1.01)ᵗ. The starting ratio is below one, but it rises because 1.02/1.01>1.



Read rabbits divided by beavers at each time.

2. Find the crossing

Set this ratio to 1 and take logs. The time is $t=\ln(89/55)/\ln(1.02/1.01)\approx48.852$ days after June 1. Substitution gives about 144.710 of each animal in the continuous model.



Equal populations give a quotient of 1.

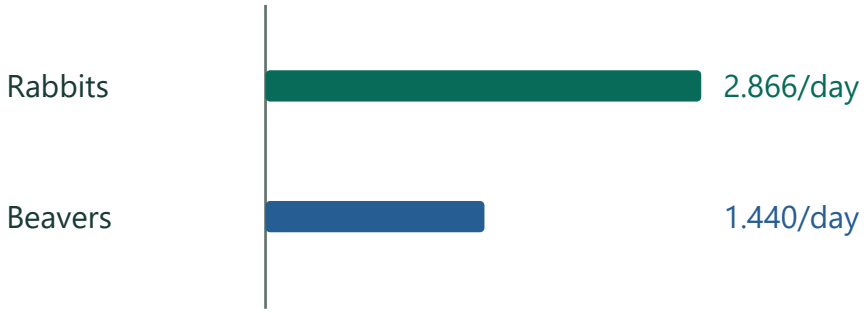
1. $(55/89)(1.02/1.01)^t=1 \Rightarrow (1.02/1.01)^t=89/55$
Isolate the factor raised to t.

2. $t\ln(1.02/1.01)=\ln(89/55)$
Both sides are positive before taking logs.

3. $t=\ln(89/55)/\ln(1.02/1.01)$
Divide by the positive log of the relative growth factor.

3. Interpret rate at that instant

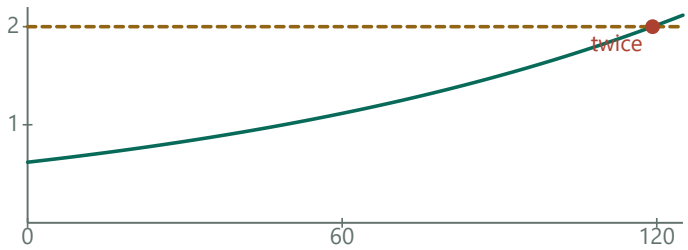
For $A(t)=Cb^t$, differentiating gives $A'=A\ln b$. Thus the rates are about $144.710\ln(1.02)=2.866$ rabbits/day and $144.710\ln(1.01)=1.440$ beavers/day. Over the following complete day, the modeled gains instead equal 2% and 1% of 144.710, about 2.894 and 1.447.



Compare instantaneous gains at the crossing.

4. Find when rabbits are twice as numerous

Set the ratio to 2. Replacing 89/55 by 178/55 in the logarithm calculation yields $t\approx119.206$ days after June 1.



Doubling the other population means a quotient of 2.

1. $(55/89)(1.02/1.01)^t=2 \Rightarrow (1.02/1.01)^t=178/55$
Multiply the new target ratio by 89/55.

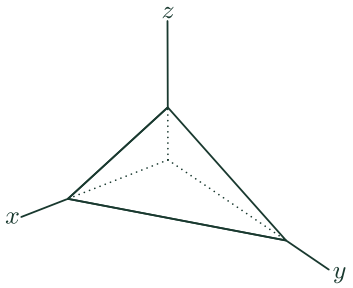
2. $t=\ln(178/55)/\ln(1.02/1.01)\approx119.206$ days
Take logs, then divide by the same nonzero log.

Check. R/B grows by the constant daily factor 1.02/1.01; at the two solutions it equals 1 and 2.

Exercise 606 · A plane by two methods

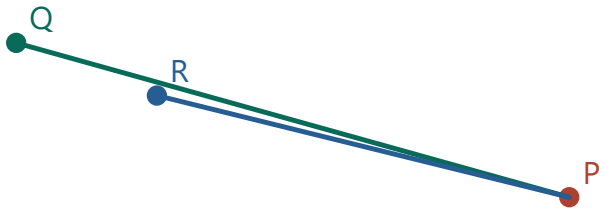
Printed page 64 · Planes in space / Cross products / Systems of equations

The plane that contains the points $(8, 3, 1)$, $(2, 6, 3)$, and $(4, 6, 2)$ has an equation of the form $ax + by + cz = d$. Find coefficients for this equation, trying *two* different approaches to the problem. One method uses vectors, another does not.

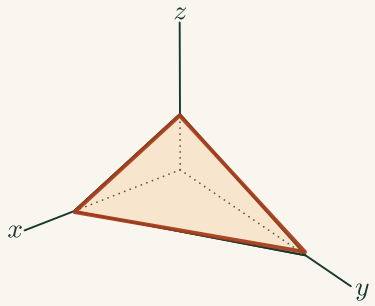


1. Build two directions in the plane

Subtract (8,3,1) from the other points to get u=(−6,3,2) and v=(−4,3,1). Both point along the desired plane.



Two displacement vectors begin at the same point.

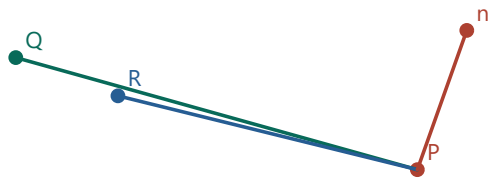


Triangular patch of the plane

The triangle marks a piece of the infinite plane. Two independent directions within that patch determine how the plane tilts.

2. Use the vector method

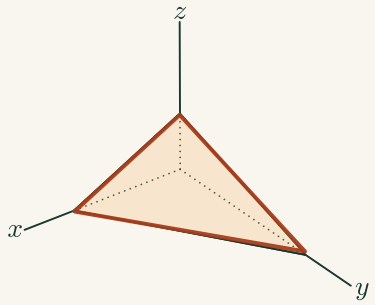
The cross product u×v=(−3,−2,−6) is perpendicular to both. Use the opposite normal (3,2,6), then substitute (8,3,1) to obtain d=24+6+6=36. The equation is 3x+2y+6z=36.



The cross product gives a direction normal to the plane.

1. u×v=(3·1−2·3, 2·(−4)−(−6)·1, (−6)·3−3·(−4))
Form each component from the other two coordinate directions.

2. u×v=(−3,−2,−6) ⇒ n=(3,2,6)
Reversing the perpendicular vector preserves its normal direction.

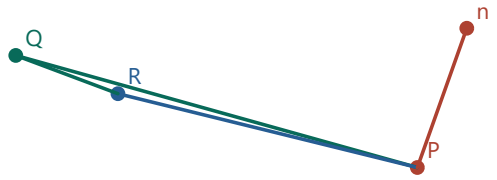


Two directions sharing an endpoint

These two edges are examples of nonparallel directions within the patch. A normal must be perpendicular to both directions at once.

3. Use equations alone

Write 8a+3b+c=2a+6b+3c=4a+6b+2c=d. Comparing the last two gives c=2a. Comparing the first two gives 6a−3b=2c, hence 3b=2a. Choose a=3 to recover b=2, c=6 and d=36.



Every point in this triangular patch has n · X = 36.

1. 2a+6b+3c=4a+6b+2c ⇒ c=2a
Cancel 6b and subtract 2a+2c.

2. 8a+3b+c=2a+6b+3c ⇒ 6a−3b=2c
Subtract common terms in the other pair.

3. 6a−3b=4a ⇒ b=2a/3
Replace c with 2a, collect terms, and divide by 3.

Check. Direct substitution gives 36 for all three points.

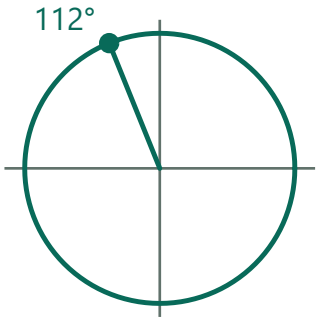
Exercise 607 · Two equal sines

Printed page 64 · Reference angles and symmetry / Trigonometric equations

Working in degree mode, find both values of n between 0 and 360 for which $\sin 7672 = \sin n$. A calculator is not needed.

1. Remove whole turns

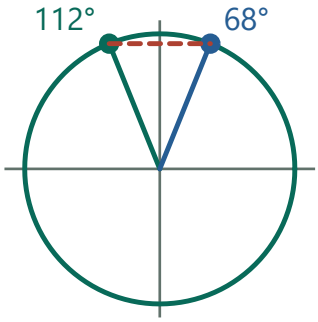
Sine is the vertical coordinate on the unit circle, so adding or subtracting 360° leaves it unchanged. Subtract $21 \cdot 360° = 7560°$ from 7672° to get 112°.



Remove 21 full turns to locate 7672° at 112°.

2. Find the mirror point

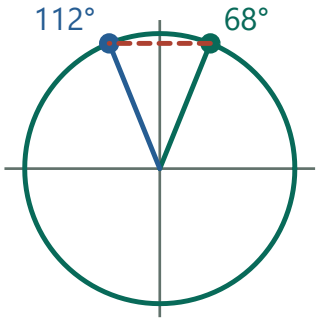
Reflection in the vertical diameter preserves height. The paired angle is $180° - 112° = 68°$.



Vertical reflection preserves the sine height.

3. Check completeness over one turn

Positive height away from the top is attained exactly twice between 0° and 360°, once on each side of the vertical diameter. The solutions are 68° and 112°.



A horizontal line below the top meets this circle twice.

Check. They sit symmetrically about the vertical diameter and have equal heights on the unit circle.

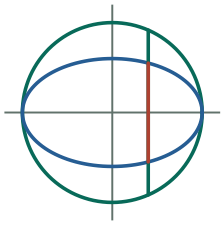
Exercise 608 · Ellipse area by vertical slices

Printed page 64 · Ellipses / Area and volume scaling / Cross sections

On the same system of coordinate axes, graph the circle $x^2 + y^2 = 25$ and the ellipse $9x^2 + 25y^2 = 225$. Draw the vertical line $x = 2$, which intersects the circle at two points, called A and B, and which intersects the ellipse at two points, called C and D. Show that the ratio $AB : CD$ of chord lengths is 5:3. Choose a different vertical line and repeat the calculation of the ratio of chord lengths. Finally, using the line $x = k$ (with $|k| < 5$, of course), find expressions for the chord lengths and show that their ratio is 5:3. Where in the diagram does the ratio 5:3 appear most conspicuously? Because the area enclosed by the circle is known to be 25π, you can now deduce the area enclosed by the ellipse.

1. Find both vertical heights at x=2

For the circle, $y^2=25-4=21$, so A and B have heights $\pm\sqrt{21}$ and $AB=2\sqrt{21}$. Rewrite the ellipse as $y^2=(9/25)(25-x^2)$; its heights are $\pm(3/5)\sqrt{21}$ and $CD=(6/5)\sqrt{21}$.



At x = 2 compare the two vertical chords.

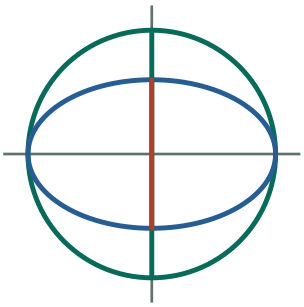
1. $x^2/25+y^2/9=1 \Rightarrow y^2=9(1-x^2/25)$
Isolate the vertical coordinate squared.

2. $y^2=(9/25)(25-x^2) \Rightarrow y=\pm(3/5)\sqrt{(25-x^2)}$
Factor the constant and take both real roots for abs(x)≤5.

3. $x=2 \Rightarrow CD=(3/5)\sqrt{21}-(-3/5)\sqrt{21}$
Subtract the lower height from the upper height.

2. Compare a second pair

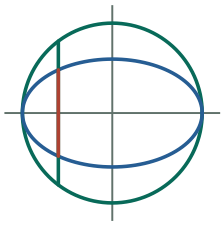
The first pair's length ratio is 5:3 because both lengths contain the same factor $2\sqrt{21}$. At x=0 the chords are the full vertical diameters 10 and 6. Here the 5:3 ratio is easiest to see.



At the center their full lengths are 10 and 6.

3. Prove one formula covers every slice

At $x=k$, $|k|<5$, the circle's chord has length $2\sqrt{(25-k^2)}$, and the ellipse's chord has length $(6/5)\sqrt{(25-k^2)}$. The common positive square root cancels in their ratio.



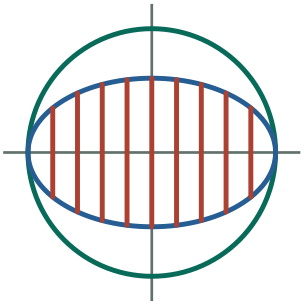
The same vertical ratio holds at any interior x.

1. $L(\text{circle})=\sqrt{(25-k^2)}-(-\sqrt{(25-k^2)})$
The chord length is its upper y-coordinate minus its lower y-coordinate.

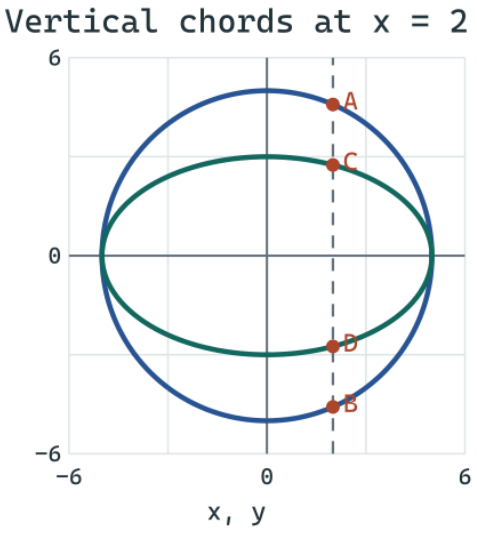
2. $L(\text{ellipse})=(3/5)L(\text{circle})$ ⇒ $L(\text{circle})/L(\text{ellipse})=5/3$
The same vertical scaling applies to both endpoints. At $k=\pm5$ both chords have length zero and their ratio is undefined.

4. Turn chord scaling into area scaling

Compress every vertical coordinate of the disk by 3/5 and leave horizontal coordinates fixed. Every vertical strip keeps its width and loses the same fraction of height. The ellipse area is therefore $(3/5)\cdot25\pi=15\pi$.



Every vertical slice keeps its width and scales by 3/5.



Blue: circle. Green: ellipse. At x = 0 the lengths are their vertical diameters.

Check. The area determinant of the vertical scaling is 3/5.

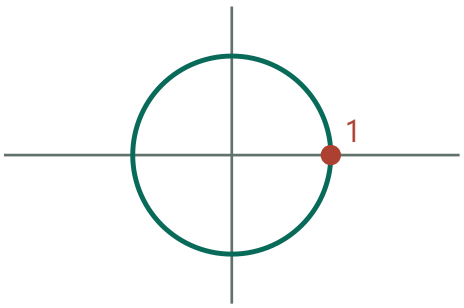
Exercise 609 · Area of a general ellipse

Printed page 64 · Ellipses / Area and volume scaling

(Continuation) What is the area enclosed by the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$?

1. Use a circle whose area is known

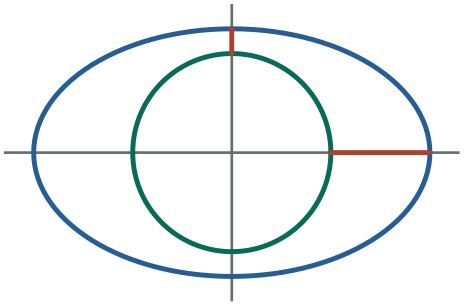
Start with the unit disk $u^2+v^2\le 1$, of area π. To obtain horizontal radius a and vertical radius b, send its point (u,v) to (x,y)=(au,bv), for positive a,b.



Start with a disk of radius 1 and area π.

2. Check the new boundary

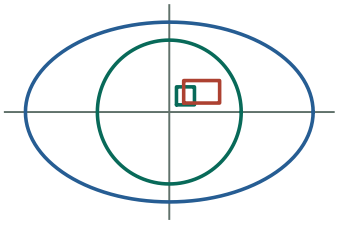
Substituting u=x/a and v=y/b in the circle equation gives $x^2/a^2+y^2/b^2=1$, precisely the given ellipse.



Example: scale horizontal distances by 2 and vertical ones by 1.25.

3. Track area under the two stretches

The horizontal stretch multiplies widths by a; the vertical stretch multiplies heights by b. A small rectangle's area scales by ab, so the transformed disk has area πab.



A sample rectangle scales by the product of both length factors.

1. $\Delta A=\Delta u\cdot\Delta v \Rightarrow \Delta A'=(a\Delta u)(b\Delta v)=ab\Delta A$
An axis-aligned rectangle in the disk stretches horizontally and vertically by the indicated factors.

2. $A'=ab\cdot\pi$
The area factor applies to each rectangle in increasingly fine approximations to the unit disk and therefore to their limiting area.

Check. For a=b=r the formula reduces to πr².

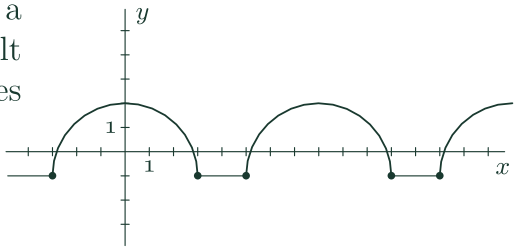
Exercise 610 · Read a periodic diagram

Printed page 64 · Periodic functions / Graph transformations

The figure at right shows the graph $y = f(x)$ of a periodic function. The graph, whose period is 8, is built from segments and semicircular arcs. Notice the values $f(3) = -1$ and $f(5) = -1$.

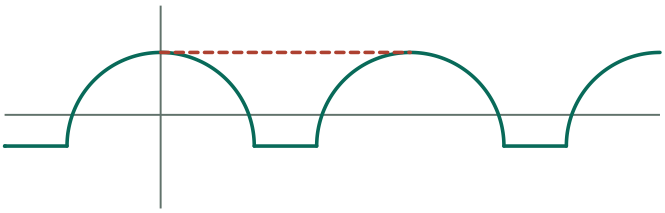
(a) Calculate $f(320)$, $f(323)$, and $f(558)$.

(b) What does the graph of $y + 1 = f(x - 3)$ look like?



1. Use the horizontal repetition

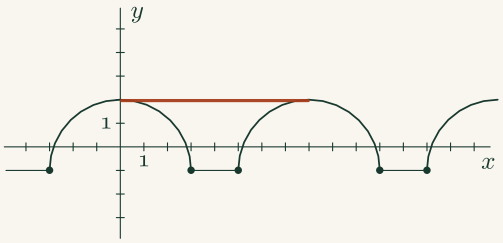
Period 8 means f(x+8)=f(x). Thus f(320)=f(0), f(323)=f(3) and f(558)=f(−2), since each pair differs by a multiple of 8.



Identical peaks are separated by 8 horizontal units.

1. 320=40·8, 323=40·8+3, 558=70·8−2
Write each input as a known graph coordinate plus whole periods.

2. f(8n+u)=f(u) for integer n
Removing a whole number of periods preserves height, including when u is negative.

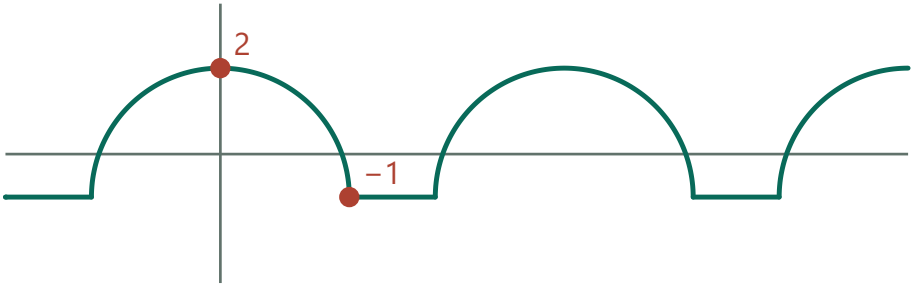


One peak to the next

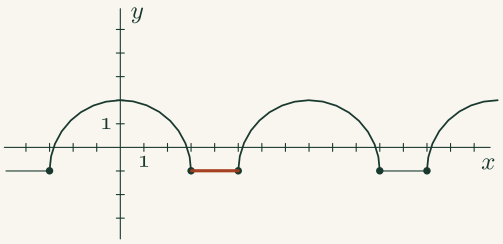
The peak-to-peak horizontal displacement measures a full repeat. Corresponding points have the same height.

2. Read two heights directly

At x=0 the displayed upper arc reaches 2. At x=3 its endpoint has height −1. These give the first two requested values.



Read the center peak and right end of the first arc.

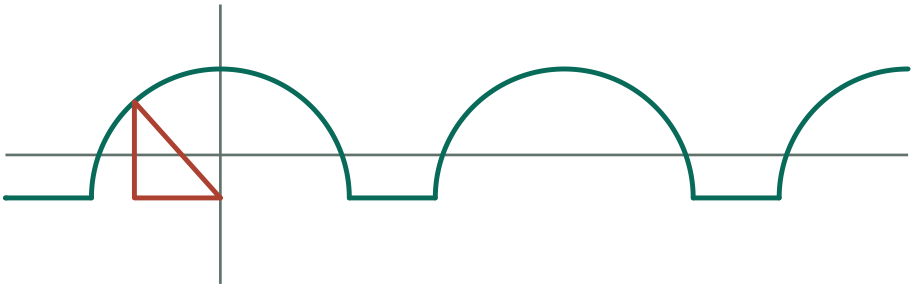


Constant segment between arches

On this horizontal interval all inputs share the same output. The next arch begins at its right edge.

3. Use the arc to get the third height

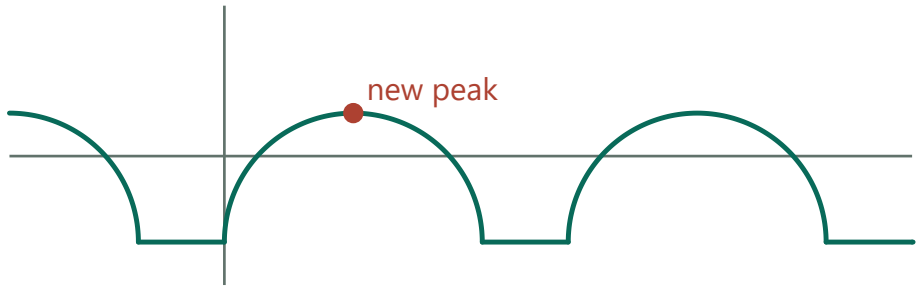
At x=−2 the upper semicircle has center (0,−1) and radius 3. The squared horizontal displacement is 4, leaving $(y+1)^2=9-4=5$. Select the upper sign: f(558)=√5−1.



The radius is 3 and the horizontal leg is 2.

4. Move the periodic pattern

Solving $y+1=f(x-3)$ for y shows that both the segments and the arcs shift 3 right and 1 down. Their horizontal period stays 8.



Translate each arc and flat segment three right and one down.

Check. At $x=-2$, $(-2)^2+(y+1)^2=9$, selecting the upper root.

Exercise 611 · Logs or a fractional power

Printed page 64 · Radicals and rational powers / Logarithms

Show that it is possible to use logarithms to solve the equation $x^{2.5} = 1997$. Then show that it is not necessary to do so.

1. Determine which x can solve the equation

Here 2.5 means 5/2. With real radicals, the even denominator calls for a square root. Before taking logarithms, also check whether zero could solve the equation.

1 **$x^{(2.5)} = (\sqrt{x})^5$**

Write the rational power as an odd power of the principal square root.

2 **$x \geq 0$**

A real square root requires a nonnegative input.

3 **$0^{(5/2)} = 0 \neq 1997$**

Exclude zero as a solution, leaving $x>0$.

2. Solve by taking logarithms

Both sides are now positive, so taking natural logs preserves the equation. The power rule turns the exponent into a coefficient.

1 **ln(x^(5/2)) = ln 1997**
Apply the natural logarithm to the original equality.

2 **(5/2) ln x = ln 1997**
Move the exponent outside the logarithm.

3 **ln x = (2/5) ln 1997**
Multiply both sides by the reciprocal of 5/2.

4 **x = exp((2/5) ln 1997)**
Exponentiate both sides to recover x.

3. Solve directly and compare

A reciprocal power also undoes raising a positive quantity to 5/2. The resulting positive value passes the original equation and is unique because that power increases for x>0.

1 **(x^(5/2))^(2/5) = 1997^(2/5)**
Apply the same positive, increasing power to both sides.

2 **x = 1997^(2/5)**
The exponents multiply to 1.

3 **(1997^(2/5))^(5/2) = 1997**
Substitution restores the original right-hand side.

4 **exp((2/5) ln 1997) = 1997^(2/5)**
The logarithmic method gives the same real power.

Check. Raising the positive answer to 5/2 returns 1997.

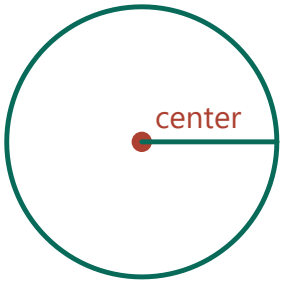
Exercise 612 · A one-day satellite orbit

Printed page 64 · Kepler's third law / Power functions

Just as it governs the satellites of the sun, Kepler's Third Law governs the satellites of Earth; the largest one is the moon. The center of the moon is about 239000 miles from the center of the earth, and it takes 27.3 days to complete one orbit. How far from the center of the earth must a satellite be, if it takes exactly *one day* to complete one orbit?

1. Use the same central body

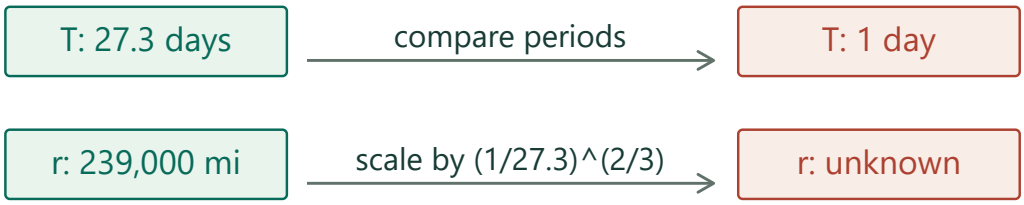
For orbits about Earth, Kepler's third law keeps r^3/T^2 constant, where r is the orbital semimajor axis (the radius for a circular orbit) and T is the period.



Read the lunar orbital radius from the center of Earth.

2. Form a ratio to cancel the constant

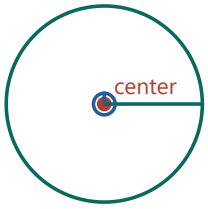
Use the Moon as the reference. For a one-day orbit, $(r/239000)^3=(1/27.3)^2$. Both periods are in days and both distances are measured from Earth's center.



Keep the central body and the distance origin fixed.

3. Take the cube root

The required distance is 239000(1/27.3)^(2/3)≈26,361 miles from the center. Its smaller value matches the shorter period.



The smaller one-day orbit has radius about 26,361 miles.

1. $(r/239000)^3=(1/27.3)^2 \Rightarrow r/239000=(1/27.3)^{(2/3)}$
Take a cube root of positive physical distances.

2. r=239000·(1/27.3)^(2/3) miles
Multiply by the reference radius to recover the orbital distance.

Check. The period is shorter, so the required radius must be smaller than the Moon's.

Exercise 613 · Daily helium loss

Printed page 65 · Sequences and recursion / Exponential growth and decay

A balloon is losing its helium at 12 percent per day. Given that the balloon initially held 5000 cc of helium, write a complete recursive description of the balloon's volume.

1. Choose the starting time

Let V_n be helium volume in cubic centimeters after n complete days. No helium has escaped at n=0, so $V_0=5000$.



Start before any helium has escaped.

2. Translate a percentage into an update

Each day's loss is 0.12 of the helium present at the start of that day. Subtracting leaves $V_n-0.12V_n=0.88V_n$.



The loss is measured against the current volume.

3. State and test the recursion

For n=0,1,2,..., take $V_{n+1}=0.88V_n$. It predicts 4400 cc after one day and 3872 cc after two. The changing loss in cc is built into the rule.



Apply the same fraction to successively smaller volumes.

Check. The first two volumes are 4400 and 3872 cc.

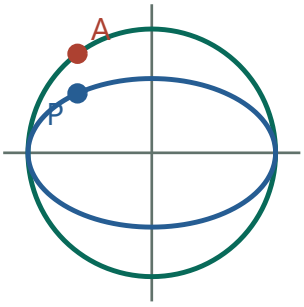
Exercise 614 · Scale a circle's tangent

Printed page 65 · Ellipses / Tangent lines / Area and volume scaling

Verify that $A=(-3,4)$ is on the circle $x^2+y^2=25$ and $P=(-3,2.4)$ is on the ellipse $9x^2+25y^2=225$. Find an equation for the line tangent to the circle at A. Then show how this can be used to find an equation for the line tangent to the ellipse at P.

1. Verify both plotted points

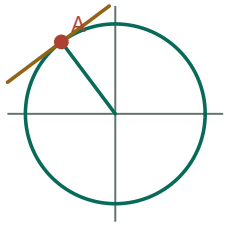
A=(−3,4) gives 9+16=25 on the circle. P=(−3,2.4) gives 9·9+25·5.76=225 on the ellipse. Notice the x-coordinate stays fixed while 4 is scaled by 3/5.



Both points have x = −3; only their heights differ.

2. Find the circle tangent

The radius vector OA=(−3,4) is a normal to the tangent. Hence (−3,4)·(x+3,y−4)=0, or −3x+4y=25.



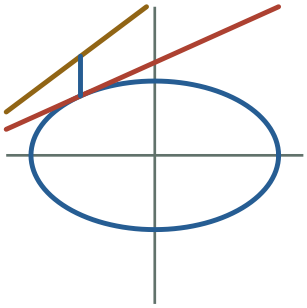
The radius is normal to the original circle tangent.

1. −3(x+3)+4(y−4)=0
Take the dot product of the radius and a tangent displacement.

2. −3x−9+4y−16=0 ⇒ −3x+4y=25
Distribute both coefficients and move the constants.

3. Transport the line and point together

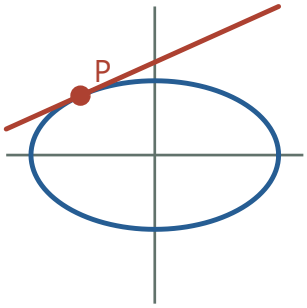
The vertical compression (u,v)→(u,3v/5) maps the whole circle to the ellipse and A to P. Invert this coordinate change in the old tangent: −3x+4(5y/3)=25. Multiplication by 3 gives −9x+20y=75.



Apply the same vertical scale to every point on the line.

4. Explain why contact is preserved

The scaling is invertible. A second intersection on the new line would give another intersection of the original circle with its tangent, which is impossible.



The mapped line contacts the ellipse at the mapped point.

Check. Its slope 9/20 agrees with implicit differentiation.

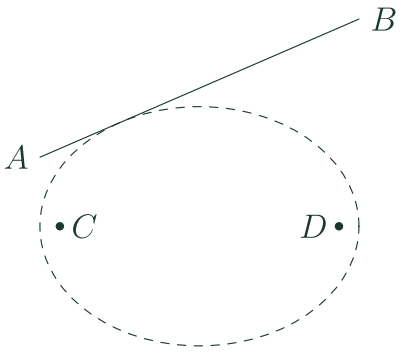
Exercise 615 · Shortest reflected path

Printed page 65 · Reflections / Inequalities / Optimization

It can be shown that there is a unique point P on line AB that makes the sum $CP + PD$ as small as possible and simultaneously makes the angles APC and BPD the same size. To show this reflect C across AB to C'. Draw $C'D$ and label P the intersection of $C'D$ and AB. Let Q be any point on AB other than P.

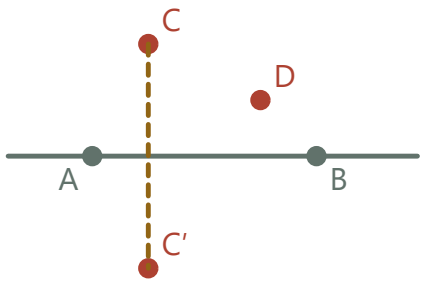
(a) Show that $CQ + QD > CP + PD$.

(b) Show that angles AQC and BQD are not congruent.

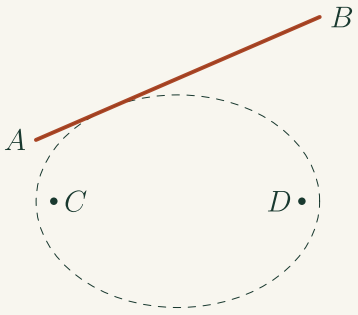


1. Reflect the broken path

Reflect C in line AB to obtain C′. Every Q on AB has QC′=QC, so the path length CQ+QD equals C′Q+QD. This replaces the bend with a comparison to the straight segment C′D.



Reflect C across the mirror to preserve distance from it.

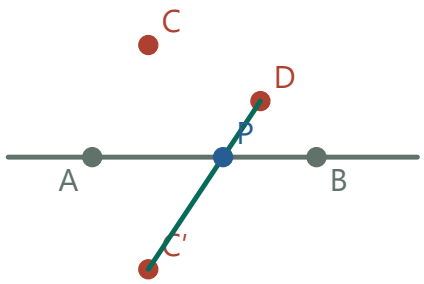


Mirror line AB

Follow the sloping line between its named ends. Every point on it has the same distance to C as to the reflection of C.

2. Prove the shortest route

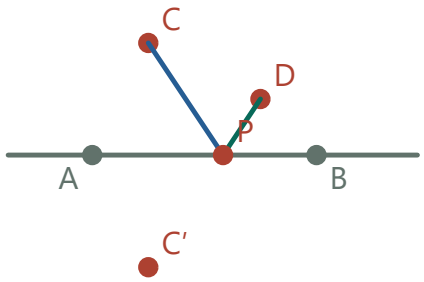
C′ and D are on opposite sides of the line; their segment crosses at a single P. There C′P+PD=C′D. At every other Q the triangle inequality is strict. Thus CP+PD is the unique minimum on the whole line.



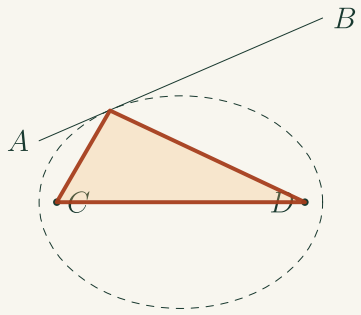
The reflected path is shortest when its two pieces are straight.

3. Compare the rays inside segment AB

At interior P the rays PA and PB point in opposite directions along the mirror. Reflection sends PC to PC′, which points opposite PD. This gives equal angles. At another interior Q, equality would likewise make QC′ and QD opposite rays, requiring Q on the segment C′D and hence Q=P.



At the interior contact the mirror exchanges the relevant rays.

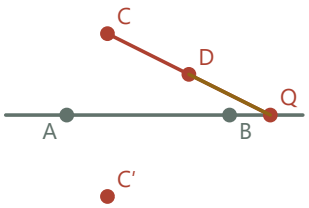


Tangency and the two path segments

Trace from C up to the contact and back to D. Compare the incoming and outgoing directions against the same straight mirror.

4. Test the claim outside the segment

Take A=(−2,0), B=(2,0), C=(−1,2), D=(1,1). Reflection gives C′=(−1,−2) and P=(1/3,0). At Q=(3,0), the rays QA,QB coincide, and QC=(−4,2) is twice QD=(−2,1). Both named angles are equal there too.



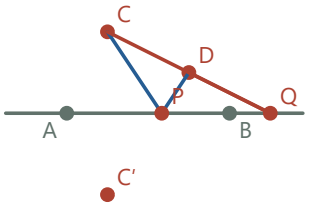
Example: both exterior rays QA and QB point left.

1. $C'+t(D-C')=(-1+2t,-2+3t)$
A segment from C′ to D uses their displacement (2,3).

2. $-2+3t=0 \Rightarrow t=2/3 \Rightarrow x=-1+4/3=1/3$
Set height to zero to find the mirror crossing.

5. State the conclusion with its conditions

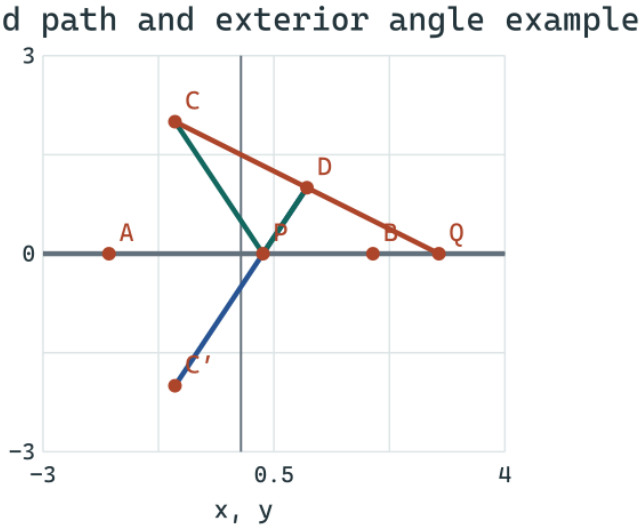
Part (a) holds everywhere on line AB. Part (b) holds for Q strictly between A and B. The exterior example has path length $3\sqrt{5}>\sqrt{13}$, despite its angle equality. At A or B one of those angles has no defined ray.



The exterior equal-angle route is longer than the interior route.

1. $CQ+QD=\sqrt{20}+\sqrt{5}=3\sqrt{5}$
At Q=(3,0), one displacement is twice the other.

2. $C'D=\sqrt{(2^2+3^2)}=\sqrt{13}$; $45>13$
Square the positive lengths to see the exterior path is longer.



Green: shortest path through P. At the exterior point Q both target angles agree.

Check. At the exterior test point the distances sum to $3\sqrt{5}$, strictly greater than $C'D=\sqrt{13}$. Part (a) still holds on the entire line.

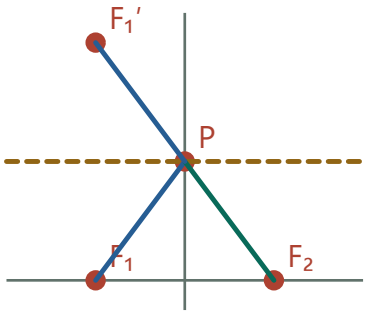
Exercise 616 · Reflection at an ellipse

Printed page 65 · Ellipses / Conic reflection properties

Reflection property of the ellipse. Suppose that P is a point on an ellipse whose focal points are F_1 and F_2. Draw the intersecting lines PF_1 and PF_2, as well as the bisectors of the four angles they form. Consider the bisector that does not separate F_1 and F_2. Prove that given any point Q other than P on this line, $QF_1 + QF_2 > PF_1 + PF_2$. Explain why the line meets the ellipse *only* at P. Justify the title of this problem.

1. Pick the exterior angle bisector

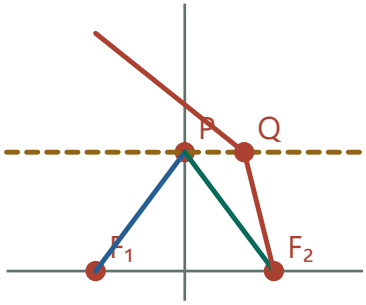
At P draw both lines to the foci. Choose the bisector that leaves both foci on the same side. Reflect F_1 in that line to F_1'. The reflected ray points opposite PF_2, so P lies between F_1' and F_2.



Example: reflect the lower focus ray across the horizontal tangent.

2. Straighten the focal path

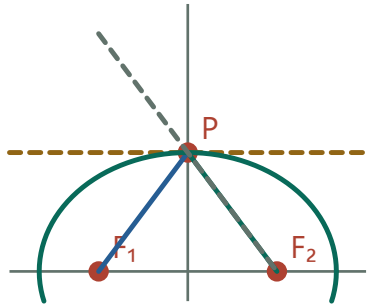
For any Q on the bisector, $QF_1 = QF_1'$. At P the sum equals $F_1'F_2$, since both pieces then make a straight path. Any $Q \neq P$ on the bisector makes a strict detour by the triangle inequality.



A second point on the same line forces a bent reflected route.

3. Identify the tangent and reflected ray

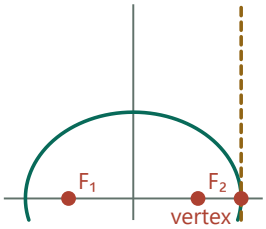
The ellipse requires the same focal distance sum as at P. Every other Q on the bisector has a larger sum, and the filled ellipse, being a convex sublevel set of that sum, lies on one side. Thus the bisector is a tangent. A ray arriving at P from one focus leaves toward the other at an equal angle to that tangent.



Every other point of the line has a larger focal sum.

4. Handle an axis vertex separately

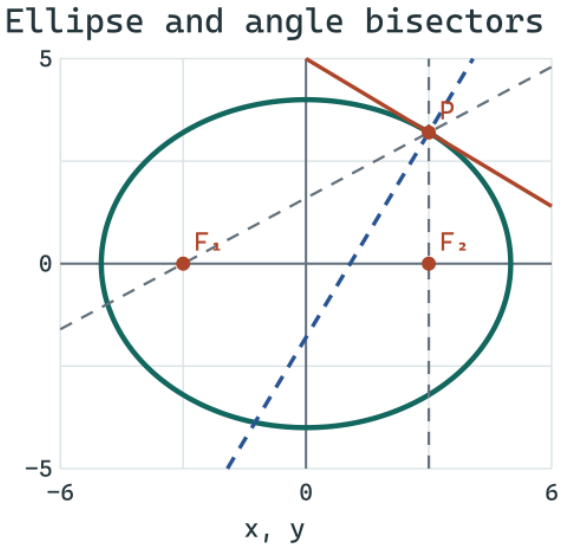
At a major-axis vertex the focal lines coincide. Use the line through the vertex perpendicular to the axis. Moving along it by any nonzero distance increases both focal distances by Pythagoras. The same contact conclusion follows.



At an axial vertex the tangent is perpendicular to the focal axis.

1. $QF_i^2 = PF_i^2 + PQ^2 > PF_i^2$ for $Q \neq P$
At the axis vertex the displacement PQ is perpendicular to each focal radius.

2. $QF_1 + QF_2 > PF_1 + PF_2$
Square roots preserve strict order for nonnegative distances. Both addends increase.



Gray: focal lines. Orange: tangent. Dashed blue: normal.

Check. The triangle inequality becomes equality precisely at the straight path through P; major-axis vertices follow directly by symmetry.

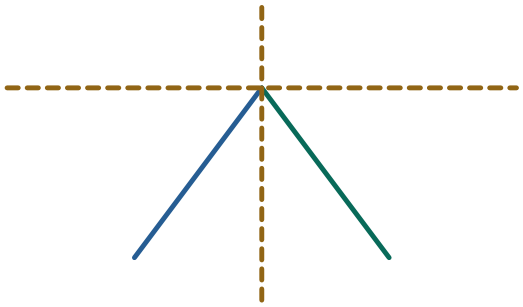
Exercise 617 · Two perpendicular angle bisectors

Printed page 65 · Ellipses / Tangent lines / Orthogonality

Two intersecting lines form four angles. The bisectors of these angles have a special property. Show how this property can be helpful when finding lines tangent to ellipses.

1. Name two neighboring angles

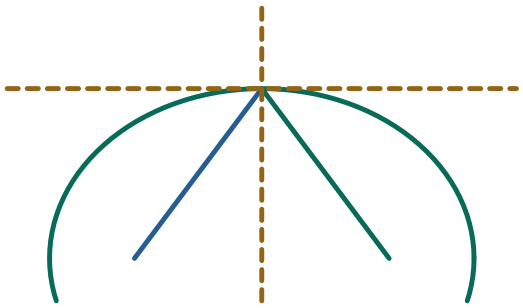
If one angle of intersecting lines is θ, the adjacent one is $180° - \theta$. Their bisectors make an angle $\theta/2 + (180° - \theta)/2 = 90°$. Opposite angles share the same bisector line.



The interior and exterior bisectors have perpendicular directions.

2. Use the focal directions

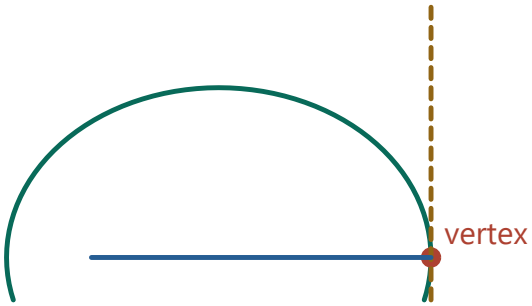
At a point P of an ellipse, mark rays toward F_1 and F_2. The outside angle bisector is tangent by the focal reflection property. The inside angle bisector is therefore its perpendicular, the normal.



The exterior bisector is tangent; the other is the normal.

3. Construct the tangent

One can bisect the angle between the focal rays, then draw the perpendicular through P. At an axis vertex symmetry gives the tangent perpendicular to the corresponding axis.



At an axial vertex the axis itself supplies the normal.

Check. The external bisector makes equal acute angles with the two focal lines, satisfying reflection.

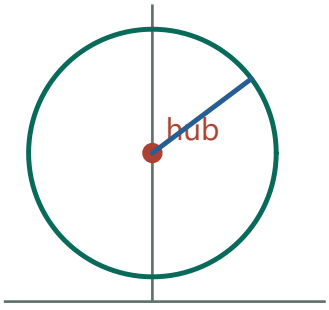
Exercise 618 · Vertical speed on a Ferris wheel

Printed page 65 · Circular motion / Rates of change

Jamie is riding a Ferris wheel that takes fifteen seconds for each complete revolution. The diameter of the wheel is 10 meters and its center is 6 meters above the ground. **(a)** When Jamie is 9 meters above the ground and rising, at what rate (in meters per second) is Jamie gaining altitude? **(b)** When is Jamie rising most rapidly? At what rate?

1. Convert the wheel to a height model

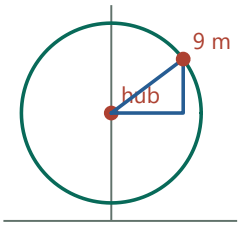
The radius is 5 m. Let $y=6+5\sin\theta$ and take θ increasing at $\omega=2\pi/15$ rad/s. Differentiation gives vertical velocity $5\omega\cos\theta$.



The hub stays 6 m above ground as a 5 m radius rotates.

2. Use the indicated height

At y=9 the vertical displacement from the hub is 3. On the radius-5 circle the horizontal displacement has magnitude √(25−9)=4. On the rising side the velocity is positive, so 5cos θ=4.



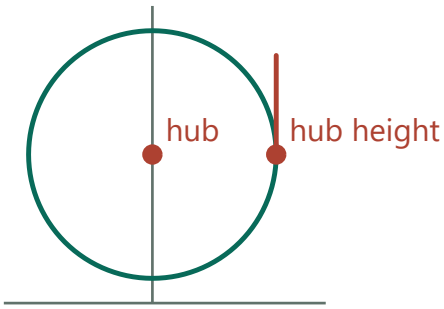
On the rising side a 3 m rise leaves a 4 m horizontal leg.

1. 9=6+5sin θ ⇒ sin θ=3/5
Isolate the vertical fraction of the radius.

2. $\cos^2\theta=1-(3/5)^2=16/25 \Rightarrow \cos\theta=4/5$
Use the positive root because height is increasing with θ here.

3. Read the two rates

At that instant the gain is 4ω=8π/15 m/s. The largest possible rising value uses cos θ=1. This occurs at hub height 6 m on the rising side, with upward speed 5ω=2π/3 m/s.



The upward speed is largest when the rim tangent is vertical.

Check. Using y=6+5sinθ gives dy/dt=5cosθ·2π/15, with cosθ=4/5 when rising at 9 m.

Exercise 619 · Three ways to isolate y

Printed page 65 · Exponential functions / Systems of equations / Quadratic equations

Solve for y: **(a)** $(1.5)^y = 3.6(1.25)^y$ **(b)** $\begin{bmatrix}6 & 5\\ 7 & 6\end{bmatrix}\begin{bmatrix}x\\ y\end{bmatrix}=\begin{bmatrix}19\\ 96\end{bmatrix}$ **(c)** $6x^2+9y^2=144$

1. (a) Gather the factors that contain y

Both exponential bases are positive. In particular, $(1.25)^y$ is positive for every real y, so dividing by it never discards a solution.

1 **$(1.5)^y = 3.6(1.25)^y$**
Start with the given equation.

2 **$(1.5)^y/(1.25)^y = 3.6$**
Divide both sides by the same positive factor.

3 **$(1.5/1.25)^y = 3.6$**
Combine the powers with equal exponents.

4 **$1.2^y = 3.6$**
Simplify the quotient of the bases.

2. (a) Bring down the exponent

The positive sides admit natural logarithms. Since 1.2 exceeds 1, ln(1.2) is positive and division by it is valid.

1 **$\ln(1.2^y) = \ln(3.6)$**
Take the natural log of both sides.

2 **$y \ln(1.2) = \ln(3.6)$**
Apply the power law.

3 **$y = \ln(3.6)/\ln(1.2) \approx 7.026$**
Divide by the coefficient of y.

3. (b) Read and scale the two matrix rows

Multiplying a matrix by a column vector forms one scalar equation per row. Give the x-terms the same coefficient so subtraction can eliminate x.

1 **$6x + 5y = 19$**
The first row gives 6 times x plus 5 times y.

2 **$7x + 6y = 96$**
The second row gives 7 times x plus 6 times y.

3 **$42x + 35y = 133$**
Multiply both sides of the first equation by 7.

4 **$42x + 36y = 576$**
Multiply both sides of the second equation by 6.

4. (b) Eliminate x and check the pair

Subtract the scaled first equation from the scaled second. The resulting coefficient of y is 1, so y is determined without division by any unknown.

1 **$(42x + 36y) - (42x + 35y) = 576 - 133$**
Subtract the two equalities in the same order on each side.

2 **$y = 443$**
The x-terms cancel and $36y - 35y$ leaves y.

3 **$6x = 19 - 5(443) = -2196$**
Use the first row to recover x as a check.

4 **$x = -366, \quad 7(-366) + 6(443) = 96$**
Divide by 6 and check the second row.

5. (c) Take both real square roots

Removing a square generally produces two branches. They meet where the radicand is zero, and there is no real y when the radicand is negative.

1 **$9y^2 = 144 - 6x^2$**
Subtract $6x^2$ from both sides.

2 **$y^2 = 16 - (2/3)x^2$**
Divide every term by the nonzero coefficient 9.

3 **$y = \pm\sqrt{(16 - (2/3)x^2)}$**
Both square-root signs give the same square.

4 **$16 - (2/3)x^2 \geq 0 \iff -\sqrt{24} \leq x \leq \sqrt{24}$**
A real square is nonnegative, so x^2 cannot exceed 24.

Check. For (b), x=−366 gives both right-hand sides.

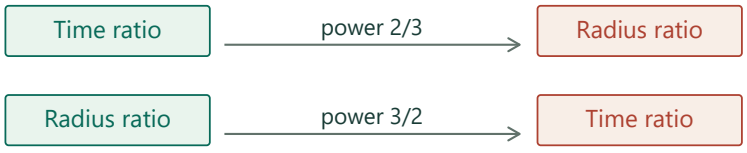
Exercise 620 · Scale Jupiter moon orbits

Printed page 65 · Kepler's third law / Power functions

The moons of Jupiter are still being discovered (28 known as of 2000). Galileo found the first four of them in 1610. Europa is 671000 km from the center of Jupiter and completes one orbit every 3.551 days. What is the distance from the center of Jupiter to Io, whose period is 1.769 days? Ganymede and Callisto are 1070000 km and 1880000 km from the center of Jupiter, respectively. What are their periods?

1. Calibrate with Europa

For all moons around Jupiter, r^3/T^2 has the same value. Thus r/r_E=(T/T_E)^(2/3), equivalently T/T_E=(r/r_E)^(3/2). Subscript E denotes Europa.



1. $r^3/T^2 = r_E^3/T_E^2 \Rightarrow (r/r_E)^3 = (T/T_E)^2$
Cross-multiply and divide by the positive reference values.

2. r/r_E=(T/T_E)^(2/3); T/T_E=(r/r_E)^(3/2)
Take the positive cube or square root to isolate the wanted ratio.

Use Europa as the common reference orbit.

2. Use a known period for Io

Its period is shorter than Europa's, so the orbital distance is smaller. Substituting Io's period gives 671000(1.769/3.551)^(2/3)≈421671 km from Jupiter's center.



The shorter period corresponds to a smaller orbital radius.

3. Use known distances for the other moons

For Ganymede the period is 3.551(1070/671)^(3/2)≈7.151 days. For Callisto it is 3.551(1880/671)^(3/2)≈16.653 days; the farthest moon has the longest period.



A larger orbit gives a longer period under the same law.

Check. Each result has the same T^2/r^3 as Europa.

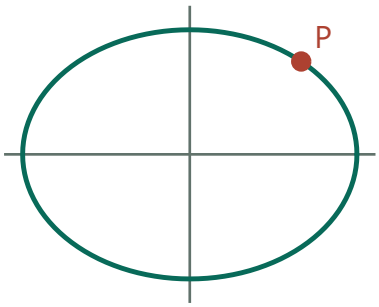
Exercise 621 · Two ellipse tangent methods

Printed page 65 · Ellipses / Tangent lines / Slopes and tangent lines

The point $P = (6, 5)$ is on the ellipse $5x^2+9y^2 = 405$. Verify this and make a sketch. Using two different methods, find an equation for the line that intersects the ellipse tangentially at P.

1. Set the scale of the drawing

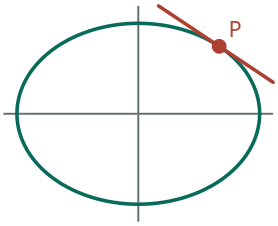
Write $x^2/81+y^2/45=1$. The intercepts are $(\pm 9,0),(0,\pm\sqrt{45})$. In quadrant I mark $P=(6,5)$, which satisfies $5\cdot 36+9\cdot 25=405$.



The first-quadrant point is (6, 5).

2. Differentiate along the curve

Implicit differentiation gives $10x+18y\,y'=0$. At P the slope is $-60/90=-2/3$. The point-slope equation $y-5=-(2/3)(x-6)$ becomes $2x+3y=27$.



The tangent falls two units per three units to the right.

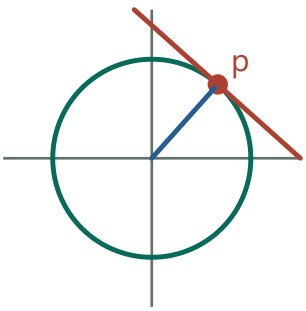
1. $10x+18yy'=0 \Rightarrow y'=-10x/(18y)$
Isolate the derivative. At P, $y=5$ is nonzero.

2. $3(y-5)=-2(x-6) \Rightarrow 3y-15=-2x+12$
Clear the denominator in the point-slope equation.

3. $2x+3y=27$
Move the x-term left and the constants right.

3. Start a second construction on a circle

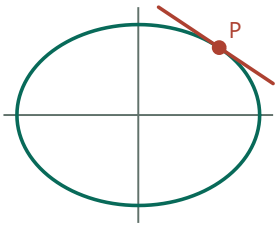
Scale $u=x/9$ and $v=y/\sqrt{45}$. The ellipse becomes the unit circle, with P mapped to $p=(6/9,5/\sqrt{45})$. The radius p is the normal to the circle tangent, so $p\cdot(u,v)=1$.



Scale to a unit circle, whose radius is normal to its tangent.

4. Return that tangent to the ellipse

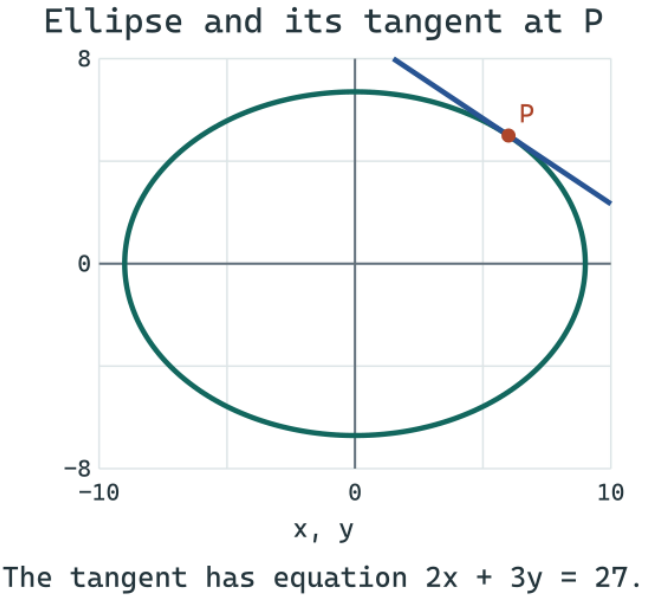
Substitute u and v to obtain 6x/81+5y/45=1, again 2x+3y=27. Scaling is invertible, so the single point of contact is preserved.



1. 6x/81+5y/45=1 ⇒ 2x/27+y/9=1
Reduce the two coefficients of the transformed circle tangent.

2. 2x+3y=27
Multiply every term by the common denominator 27.

Undoing both scale factors returns the same tangent line.



Check. Substitute y=9−2x/3 in the ellipse to obtain $9(x-6)^2=0$.

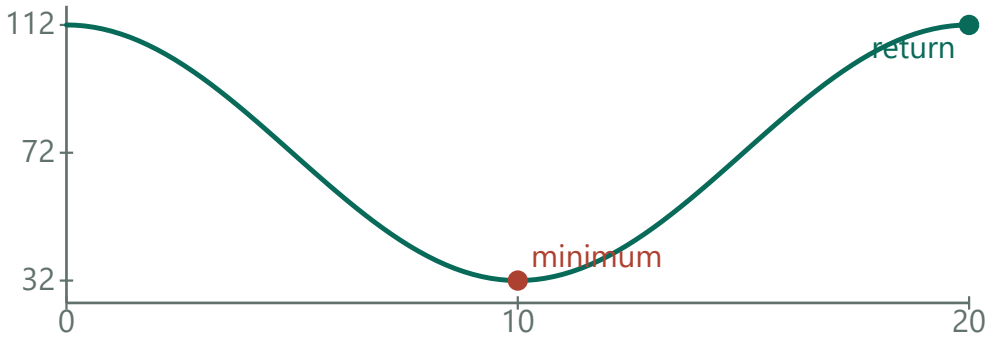
Exercise 622 · Mean and peak harmonic speed

Printed page 65 · Simple harmonic motion / Rates of change

The height of an object moving up and down is described by $y = 72 + 40\cos 18t$ (in degree mode). This is an example of *simple harmonic motion*. To at least three decimal places, confirm that the *average speed* of the object is $2/\pi$ times its *greatest speed*.

1. Measure distance over one repetition

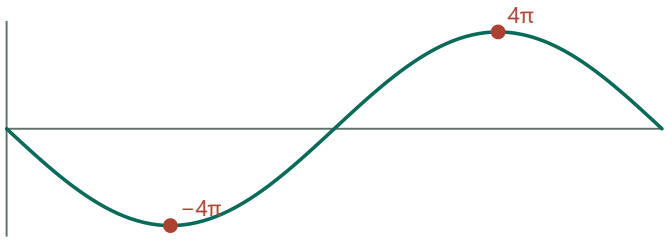
The angular rate is 18° per time unit, so the period is 360/18=20. The object travels from maximum to minimum and back: two trips of 2·40, for 160 units total. Its average speed is 160/20=8.



Two 80-unit journeys occur in one 20-unit cycle.

2. Calculate its largest instantaneous speed

Convert 18°/unit to π/10 radians/unit. For y=72+40cos(πt/10), velocity is −4πsin(πt/10). Its magnitude is largest when the sine has magnitude 1, giving 4π.



The velocity magnitude peaks at the midline crossings.

1. 18°/unit = 18π/180 = π/10 rad/unit
Convert before differentiating sine or cosine.

2. v=40[−sin(πt/10)](π/10)=−4πsin(πt/10)
The rate of phase change multiplies the derivative by the chain rule.

3. Compare speeds

The ratio of average speed to largest speed is 8/(4π)=2/π≈0.636620, or 0.637 to three decimals. Total distance accounts for reversal of direction.



The mean is about 63.7% of the peak.

Check. The mean of the absolute sine over a full revolution is 4/(2π).

Exercise 623 · Expected rental returns

Printed page 66 · Transition matrices / Probability

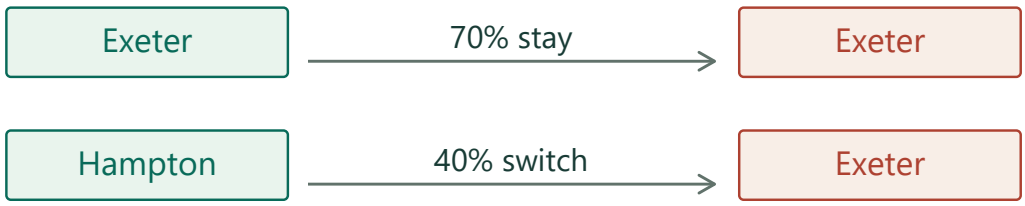
The Hoyts company has a fleet of 1000 cars for one-day local rentals, which must be leased from either the Exeter office or the Hampton office. The company expects that 70% of the cars rented in Exeter return to the Exeter office, 60% of the cars rented in Hampton return to the Hampton office, and all cars return to one office or the other. Assume that all 1000 cars are rented each day, and that 200 of the rental cars are in Exeter and 800 are in Hampton on Monday morning.

(a) What distribution of cars is expected on Tuesday morning?

(b) What distribution of cars is expected on Wednesday morning?

1. Separate cars by their starting offices

Of Exeter's cars, 70% return there. Of Hampton's cars, 40% go to Exeter, since 60% remain in Hampton. Add both contributions to get Exeter's next expected count.



Two contributions enter the Exeter office.

2. Update Monday to Tuesday

Exeter receives $0.70 \cdot 200 + 0.40 \cdot 800 = 140 + 320 = 460$. The remaining $1000 - 460 = 540$ are expected at Hampton.



Tuesday: 140 + 320 expected cars return to Exeter.

3. Use Tuesday as the new input

Wednesday Exeter receives 0.70·460+0.40·540=322+216=538. Hampton has the remaining 462. These are expected morning distributions.



Wednesday: 322 + 216 expected cars return to Exeter.

Check. Both days total 1000 cars; these are expected counts.

Exercise 624 · Rental recurrence

Printed page 66 · Transition matrices / Sequences and recursion

(Continuation) Verify that the equations

$$E_0 = 200 \qquad E_n = 0.70E_{n-1} + 0.40H_{n-1}$$
$$H_0 = 800 \qquad H_n = 0.30E_{n-1} + 0.60H_{n-1}$$

provide a recursive description of the rental-car distribution.

(a) What information do E_n and H_n convey when $n = 1$?

(b) Verify that E_n and H_n could be written in matrix form as

$$\begin{bmatrix} E_n \\ H_n \end{bmatrix} = \begin{bmatrix} 0.7 & 0.4 \\ 0.3 & 0.6 \end{bmatrix} \begin{bmatrix} E_{n-1} \\ H_{n-1} \end{bmatrix}$$

1. Follow arrivals rather than departures

On day n Exeter receives 70% of the previous Exeter count and 40% of the previous Hampton count. Hampton receives the complementary shares. Thus E_n=.70E_{n-1}+.40H_{n-1} and H_n=.30E_{n-1}+.60H_{n-1}.

from E	from H
.70	.40
.30	.60

Read the destinations by row: Exeter first, Hampton second.

2. Apply the starting values

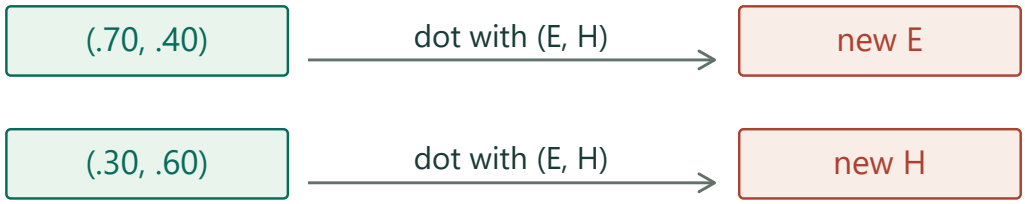
$E_0=200$ and $H_0=800$ describe Monday morning. At $n=1$ the two sums give $E_1=460, H_1=540$, the expected counts on Tuesday.



Apply the transition once to Monday's (200, 800).

3. Read the matrix multiplication

Put E above H in a column. The Exeter row (.70,.40) dotted with yesterday's column gives E_n; the Hampton row (.30,.60) gives H_n. Each column sums to 1, preserving all 1000 cars.



Each column accounts for all cars at one starting office.

Check. Column sums are 1, so adding both equations preserves E+H.

Exercise 625 · Matrix powers count days

Printed page 66 · Transition matrices / Matrix multiplication

(Continuation) Let $\mathbf{v} = \begin{bmatrix} 200 \\ 800 \end{bmatrix}$ be the initial distribution of cars and $\mathbf{M} = \begin{bmatrix} 0.7 & 0.4 \\ 0.3 & 0.6 \end{bmatrix}$. Thus $\mathbf{Mv}$ is the distribution of cars on Tuesday morning.

(a) What does the matrix product $(\mathbf{M})\mathbf{Mv} = \mathbf{M}^2\mathbf{v}$ represent?

(b) What does the matrix product $\mathbf{M}^{10}\mathbf{v}$ represent?

(c) Matrix $\mathbf{M}$ is an example of a *transition matrix*. After you label its rows and columns, interpret the entries of $\mathbf{M}$ as probabilities.

(d) The sum of the entries in each column of $\mathbf{M}$ is 1; how is this significant?

1. Interpret another multiplication

v describes Monday, and applying M once gives Tuesday. $M(Mv)=M^2v$ applies the same one-day rule again, yielding Wednesday's expected counts. $M^{10}v$ applies it ten rental days after Monday.



Consecutive vector heads represent Mon, Tue and Wed.

1. $Mv=(.70\cdot200+.40\cdot800,$ $.30\cdot200+.60\cdot800)=(460,540)$
Use each destination row against Monday's starting counts, preserving the Exeter–Hampton order.

2. $M^2v=(.70\cdot460+.40\cdot540,$ $.30\cdot460+.60\cdot540)=(538,462)$
Wednesday's multiplication uses Tuesday's result as its input. The updated entries still total 1000.

2. Label a column by origin

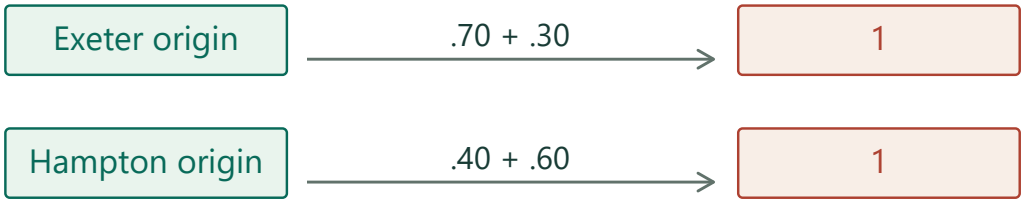
With locations ordered Exeter, Hampton, the first column describes a car starting in Exeter: .70 returns there and .30 ends at Hampton. The second describes Hampton: .40 to Exeter and .60 to Hampton.

from E	from H
.70	.40
.30	.60

A single column follows all possible destinations from one office.

3. Label a row by destination

The first row gathers all arrivals to Exeter, the second all arrivals to Hampton. Because ending in exactly one office is certain, the two destination probabilities in each starting column add to 1.



Every car ends the day at one of the two offices.

Check. Associativity gives $M^2v=M(Mv)$, applying one-day transitions twice.

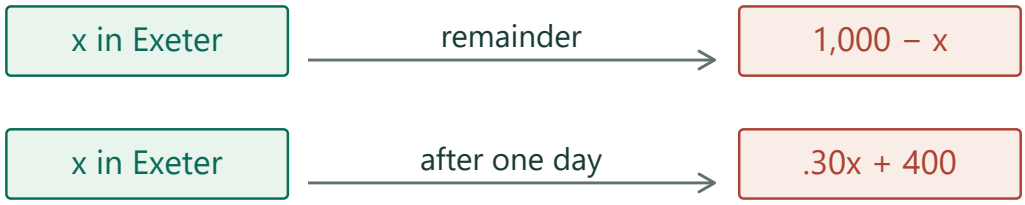
Exercise 626 · Rental equilibrium

Printed page 66 · Transition matrices / Steady-state distributions

(Continuation) Solve the equation $\mathbf{M}\begin{bmatrix} x \\ 1000-x \end{bmatrix} = \begin{bmatrix} x \\ 1000-x \end{bmatrix}$. Interpret.

1. Encode the fixed distribution

If x cars are at Exeter, the remaining 1000−x are at Hampton. Equilibrium means the next expected Exeter count equals x again.



Equilibrium makes the incoming count equal the existing count.

2. Solve the single equation

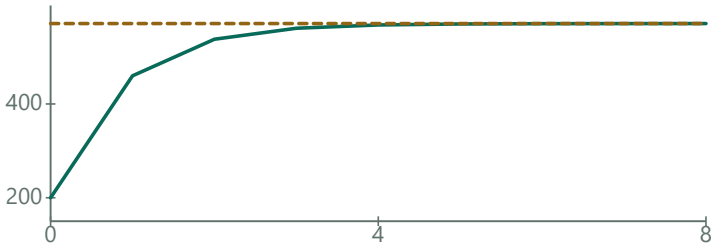
Use x=.70x+.40(1000−x)=.30x+400. Hence .70x=400 and x=4000/7. The Hampton expectation is 1000−4000/7=3000/7.



The fixed expected split is 4000/7 versus 3000/7.

3. Explain the long-run meaning

Any Exeter count updates by x′=.30x+400. Subtract the fixed value x*=4000/7 to get x′−x*=.30(x−x*). Each day retains just 30% of the previous gap, so expectations approach this split.



Starting at 200, each gap from equilibrium shrinks by 70%.

1. x* = .30x* +400; x′=.30x+400
Write the fixed-point equation next to the update.

2. x′−x*=(.30x+400)−(.30x*+400)=.30(x−x*)
Subtract to remove the constant. Since .30<1, repeated gaps shrink.

Check. Expected cross-office flows agree: .3(4000/7)=.4(3000/7).

Exercise 627 · Why parabolic reflectors focus

Printed page 66 · Parabolas / Conic reflection properties

The reflection property of parabolas. Let F be the focus of a parabola, and let P be an arbitrary point on the parabola. Let μ be the line through P that is parallel to the axis of symmetry of the parabola; this means that μ intersects the directrix perpendicularly at a point N. Let λ be the perpendicular bisector of FN.

(a) Explain why P is on λ.

(b) Explain why λ is tangent to the parabola.

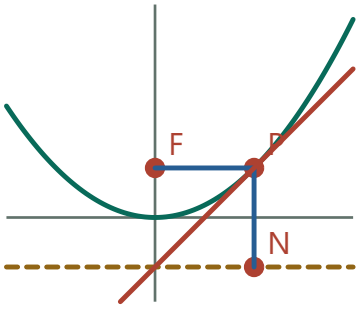
(c) Explain why λ bisects angle FPN.

(d) Explain why it makes sense for a car headlight or a spotlight to have a bulb at the focus of a parabolic reflector.

(e) Explain why it makes sense for a solar oven, a satellite dish, or a parabolic microphone to have a parabolic reflector.

1. Connect the focus to the nearest directrix point

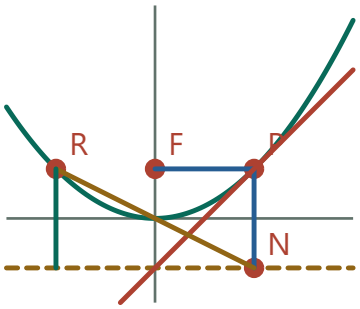
The line μ through P is perpendicular to the directrix and meets it at N. Thus PN is the shortest distance from P to the directrix. Since P is on the parabola PF=PN, placing P on λ, the perpendicular bisector of FN.



Example: PF and the perpendicular drop PN both equal 2.

2. Prove the line supports the parabola

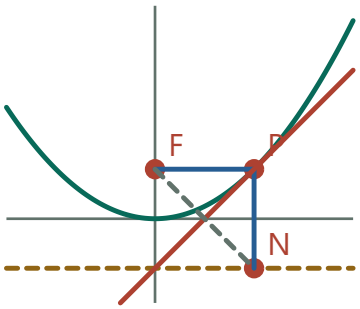
For every point R on the parabola, RF equals its distance to the directrix and so RF≤RN. All points of the curve lie on the F side of λ or on λ itself. Equality requires N to be the perpendicular foot, so R must lie both on μ and λ. Their intersection is P.



For another R, the distance to the line is at most RN.

3. Read the equal angles

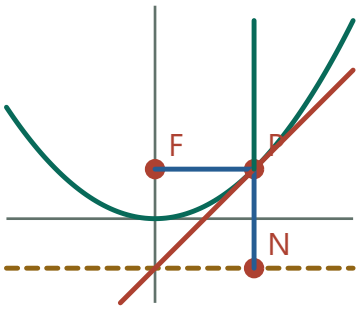
Triangle FPN has equal sides PF=PN. Reflection across λ exchanges F and N and fixes P; λ therefore bisects angle FPN. At the vertex F,P,N are collinear; λ is their perpendicular bisector and the tangent there.



The tangent is the perpendicular bisector of FN.

4. Use the reflection in both directions

An incoming ray F→P reflects along the axial line parallel to μ, outward from the reflector. A bulb at F therefore makes a parallel beam. Running the paths in reverse collects parallel incoming rays at F, concentrating sunlight, signals or sound there.



The focal ray meets the tangent and leaves parallel to the axis.

Check. At the vertex, F, P and N are collinear. P is the midpoint of FN; the perpendicular bisector is the vertex tangent. An axial ray reflects along the axis.

Exercise 628 · Bounds with a variable log base

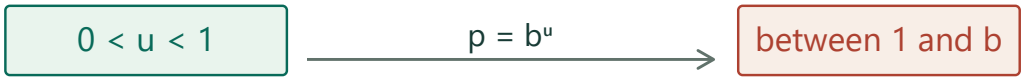
Printed page 67 · Logarithms / Inequalities

What else can be said about the positive number p, given that

(a) $0 < \log_7 p < 1$? **(b)** $0 < \log_{1/2} p < 1$? **(c)** $0 < \log_b p < 1$?

1. Turn the logarithm into a variable

Let $u=\log_b p$. Then $0<u<1$ and $p=b^u$, so compare p with the endpoint values $b^0=1$ and $b^1=b$.



One exponent interval has two possible orientations.

2. Use the order for the two specified bases

Powers of 7 increase with u, giving $1<p<7$. Powers of ½ decrease with u, giving $½<p<1$.



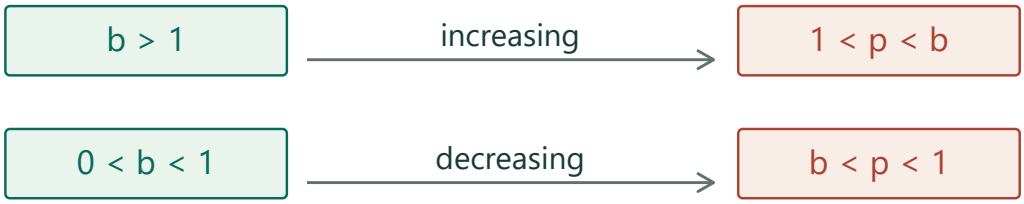
For base ½ the powers decrease; for 7 they increase.

1. $7^0<7^u<7^1 \Rightarrow 1<p<7$
Raising 7 to larger exponents increases the output, and $p=7^u$.

2. $(1/2)^0>(1/2)^u>(1/2)^1 \Rightarrow 1>p>1/2$
Increasing the exponent decreases a positive power of a base between zero and one. Reverse the order to write $1/2<p<1$.

3. State both general cases

For $b>1$, $1<p<b$. For $0<b<1$, $b<p<1$. In either case p lies strictly between b and 1. Other real bases do not define the usual logarithm.



In both cases the logarithm bounds exclude the endpoints.

Check. If $r=\log_b p$ lies between 0 and 1, $p=b^r$ lies strictly between b^0 and b^1.

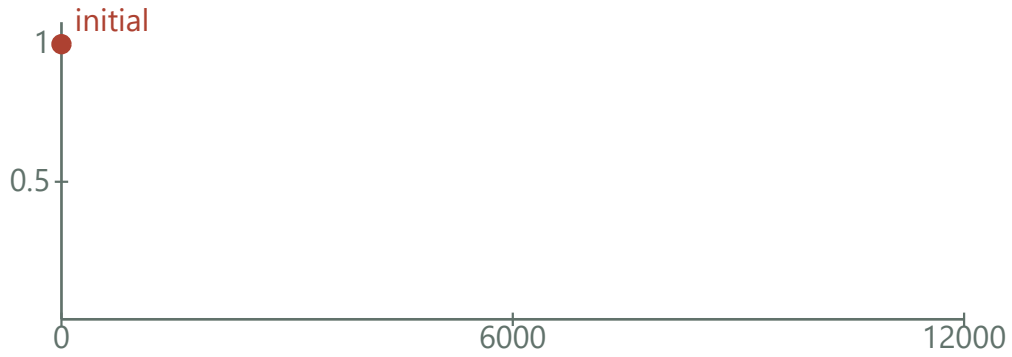
Exercise 629 · Radioactive survival

Printed page 67 · Half-life / Exponential growth and decay / Probability

An atom of carbon-14 is unstable, meaning that it can spontaneously transform itself (by radioactive decay) into nitrogen at any instant. The probability that this will actually happen to a specific atom of carbon-14 in the course of a year is only 0.0121 percent, however. In other words, there is a 99.9879 percent chance that any specific atom of carbon-14 will survive another year. Suppose that one million carbon-14 atoms are placed in a container. How many of these atoms are expected to be carbon-14 atoms one year later? The half-life question: How much time will it take for half the atoms to decay?

1. Convert percent to probability

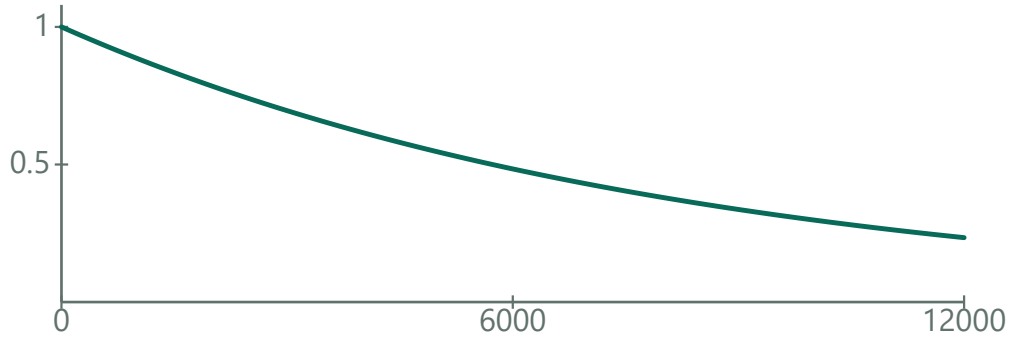
The annual decay probability is 0.0121/100=0.000121. An atom's annual survival probability is 1−0.000121=.999879.



Measure surviving carbon as a fraction of the original amount.

2. Track the expected number surviving

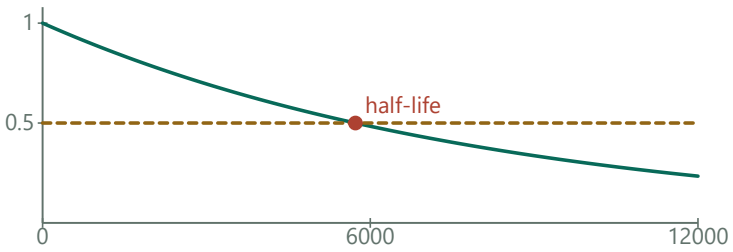
Multiplying one million by .999879 gives 999879 expected after one year. Repeating the survival factor gives $1{,}000{,}000(.999879)^t$ after t years in the exponential model.



Every year multiplies the fraction by 0.999879.

3. Find the half-life

Half the original expected count corresponds to $(.999879)^t=.5$. Taking logs yields $t=\ln(.5)/\ln(.999879)\approx 5728.14$ years. In a particular finite sample the actual time for half its atoms to decay varies.



Read when the height reaches 50% of the initial value.

1. $.999879^t=1/2 \Rightarrow t\ln(.999879)=\ln(1/2)$
Both fractions are positive, so take logarithms.

2. $t=\ln(1/2)/\ln(.999879)$
Divide by the negative, nonzero log. This is an equality, so no inequality changes.

Check. After two such intervals the survival fraction is one quarter, since $.5^2=.25$.

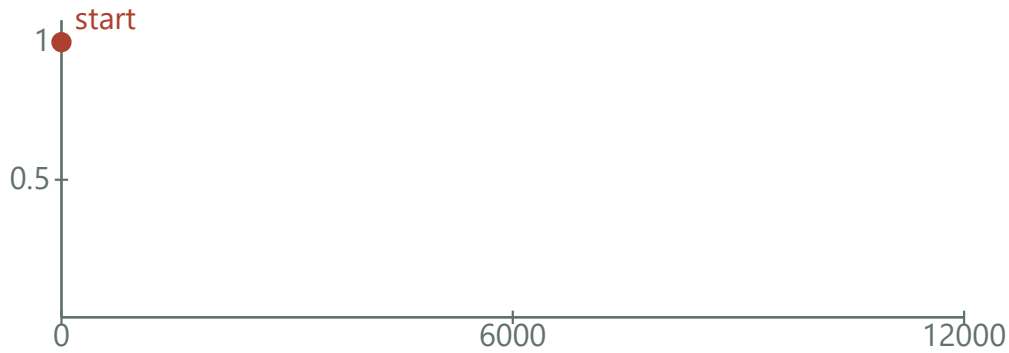
Exercise 630 · Date a charcoal sample

Printed page 67 · Half-life / Logarithms

(Continuation) Carbon-14, which is produced from nitrogen when solar radiation bombards the upper atmosphere, makes up about 1 trillionth of all the carbon found in things that rely on air to live. When an organism dies, it no longer takes in air, and its supply of carbon-14 diminishes exponentially as described above. Apply this principle to estimate the age of a lump of charcoal (found in a cavern at an ancient campsite) that has only 32 percent as much carbon-14 as it had when the charcoal was still firewood.

1. Choose the comparison quantity

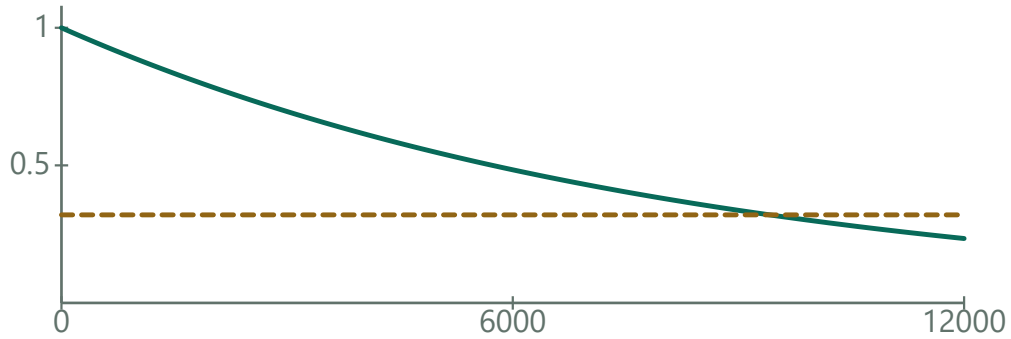
Let N_0 be the carbon-14 content when atmospheric replenishment stopped. The measured charcoal has $N(t)/N_0=.32$, so the absolute starting amount cancels.



Measure surviving carbon as a fraction of the original amount.

2. Apply the survival factor

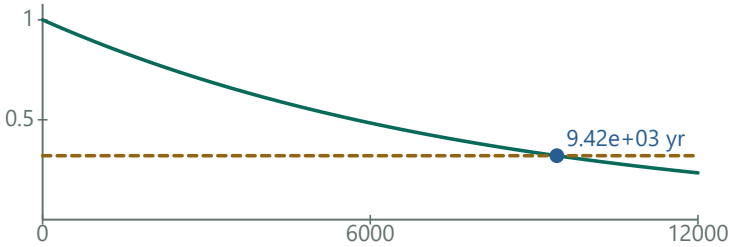
Using the previous exercise, the retained fraction after t years is $(.999879)^t$. Set this equal to .32 and take natural logarithms.



Every year multiplies the fraction by 0.999879.

3. Estimate the age

t=ln(.32)/ln(.999879)≈9416 years. The result lies between one and two half-lives, consistent with retaining between 50% and 25%.



Read when the height reaches 32% of the initial value.

1. $N_0(.999879)^t=.32N_0 \Rightarrow (.999879)^t=.32$
Divide by the positive original sample amount.

2. $t\ \ln(.999879)=\ln(.32) \Rightarrow t=\ln(.32)/\ln(.999879)$
Take logs and isolate the exponent.

Check. The fraction is between one half and one quarter, so the age lies between one and two half-lives.

Exercise 631 · A runner who may stop

Printed page 67 · Probability / Geometric probability distributions

Courtney is supposed to run laps around the outdoor track. At the start of each lap, however (even the first), there is always an 8 percent chance that Courtney will call it quits for the day. What is the probability that today Courtney will run **(a)** no laps? **(b)** at least four laps? **(c)** exactly four laps?

1. Locate the decisions

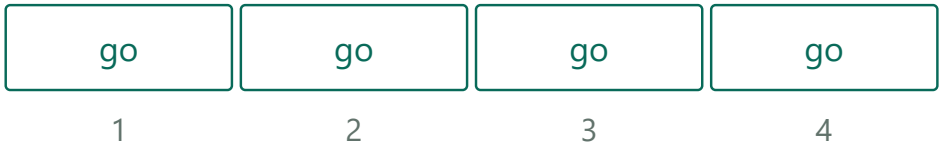
There is one decision before each lap, including lap 1. Stopping at a decision has probability .08, and conditional on reaching it, continuing has probability .92.



Decide before each lap, including the first.

2. Separate zero from at least four

No laps means stopping at the first decision, probability .08. To run at least four laps, Courtney must pass the first four decisions: $.92^4$. What happens afterward is unrestricted.



At least four laps requires four successive continuations.

3. Require exactly four

Pass the first four decisions and then stop before lap 5. The probability is $.92^4 \cdot .08$, equivalently $.92^4 - .92^5$.



Exactly four laps also requires stopping before lap 5.

1. P(exactly 4) = P(at least 4) − P(at least 5)
Every path reaching lap 5 also reached lap 4.

2. $.92^4 - .92^5 = .92^4(1 - .92) = .92^4 \cdot .08$
Factor the common four-lap survival probability.

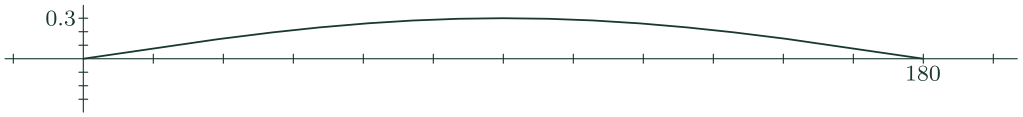
Check. Subtract the chance of at least five laps from that of at least four to obtain the same exact-four probability.

Exercise 632 · Snapshots of a vibrating string

Printed page 67 · Simple harmonic motion / Amplitude and period / Rates of change

Interpreted in degree mode, the equation $y = 0.3\cos(36000t)\sin(x)$ models the motion of a stretched, vibrating string that is 180 centimeters long ($x = 0$ is one end of the string and $x = 180$ is the other). The amplitude of the vibration is 0.3 cm.

(a) Verify that the period of the vibration is 0.01 second. This means that the string vibrates 100 times per second.

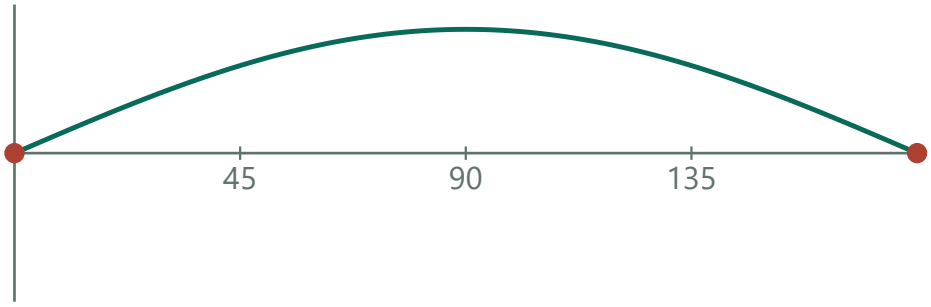


(b) Taking a photo of the string corresponds to choosing a t-value. For example, the diagram shows the string at the instant $t = 0$. Notice that the amplitude has been exaggerated to make it more visible. Sketch photos of the string for $t = 0.001$, $t = 0.0025$, $t = 0.003$, and $t = 0.005$ second.

(c) How far does the center of the string move during one complete vibration? Calculate an approximate value for the speed of the center of the string when $t = 0.0025$ second. Does the center of the string move with constant speed?

1. Separate time from position

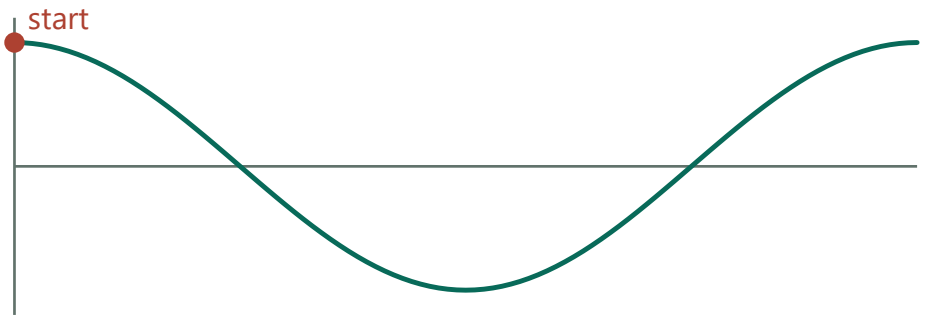
At fixed t the shape is sin(x°) multiplied by the signed height .3cos(36000t°). The sine vanishes at both ends and reaches 1 halfway along the string at x=90 cm.



Fixed ends stay at zero throughout the motion.

2. Read the period and frame heights

The time factor advances 360° in 360/36000=.01 s, or 100 cycles/s. At t=.001,.0025,.003,.005 s its angles are 36°,90°,108°,180°. The corresponding signed central heights are about .243,0,−.093,−.300 cm.



At an antinode, one cosine cycle supplies the time factor.

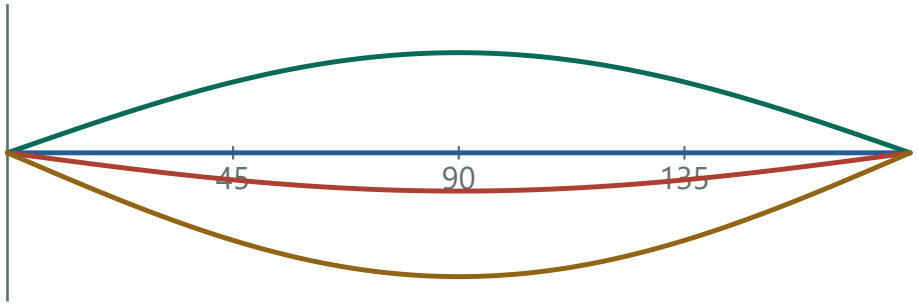
Time, phase and signed height at the string midpoint

Time (s)	Time phase (°)	Midpoint height (cm)
0.001	36	0.243
0.0025	90	0.000
0.003	108	−0.093
0.005	180	−0.300

3. Sketch successive photographs

Draw a half-sine arch through the fixed ends for each height. It bows up for the positive height, lies flat at zero, and bows down for the two negative heights. Its amplitude changes while the horizontal positions stay fixed.

Snapshot time · All four times



Snapshots at 0.001, 0.0025, 0.003 and 0.005 s (green, blue, red, gold).

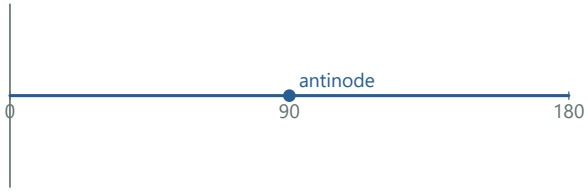


Fixed ends

Look at the ends of the drawn string. Their horizontal separation stays fixed and their vertical displacement is zero at every time.

Compare settings · Snapshot time

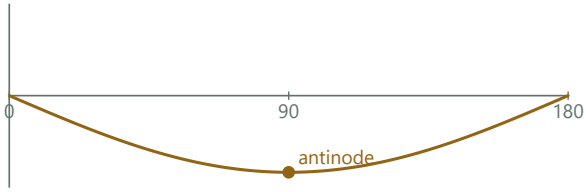
0.0025 s



The signed multiplier is 0.0000 cm at t = 0.0025 s.

The string lies flat at its equilibrium position.

0.005 s

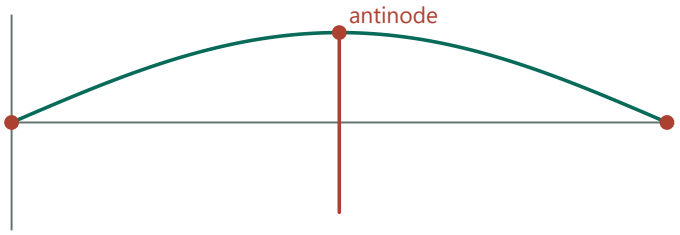


The signed multiplier is -0.3000 cm at t = 0.005 s.

At x = 90 cm the height is -0.3000 cm. The two endpoints remain at height zero.

4. Measure the midpoint's motion

Convert angular speed to 200π rad/s. The midpoint starts at +.3 cm, goes to −.3 and back, traveling 1.2 cm per cycle. Its velocity is −60πsin(200πt) cm/s; at .0025 s speed is 60π≈188.50 cm/s. At the turning points speed is zero.

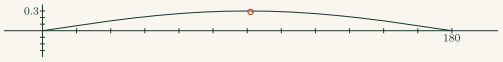


A moving antinode passes from top to bottom and back each cycle.

1. 36000°/s · π/180° = 200π rad/s
Convert angular speed before taking the derivative.

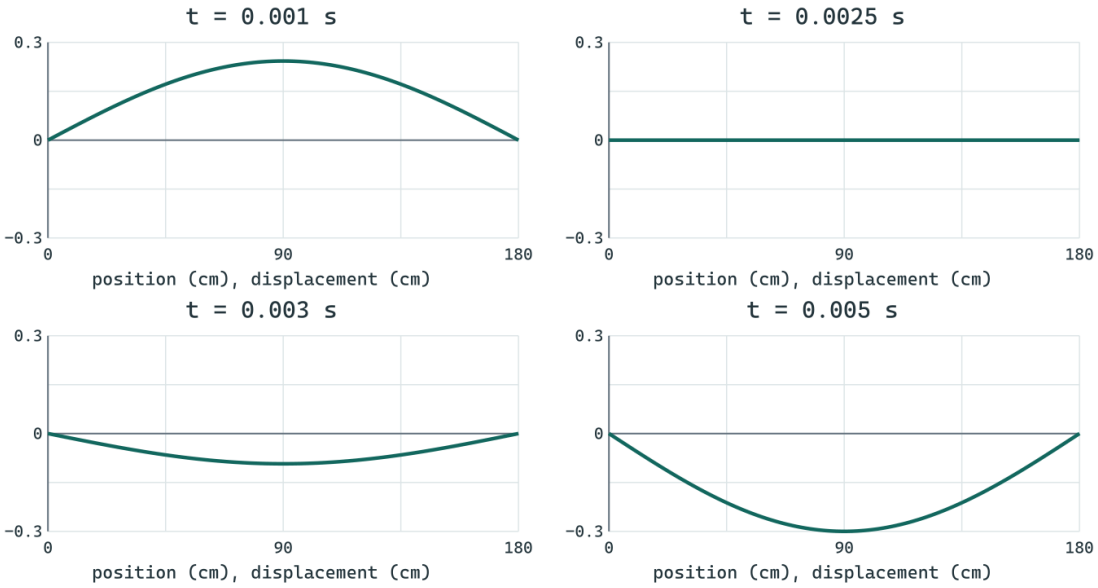
2. y_mid=.3cos(200πt) ⇒ v_mid=−.3(200π)sin(200πt)
At the center, sin 90°=1. Apply the chain rule.

3. 200π(.0025)=π/2 ⇒ v_mid=−60π cm/s
The negative value describes downward motion; speed is its magnitude.



Midpoint of the string

This central point has the largest possible displacement in this mode. Track its up-and-down movement while keeping its horizontal coordinate fixed.



The vertical scale is exaggerated. All four snapshots use identical axes.

Check. Both endpoints stay fixed since sin0°=sin180°=0.

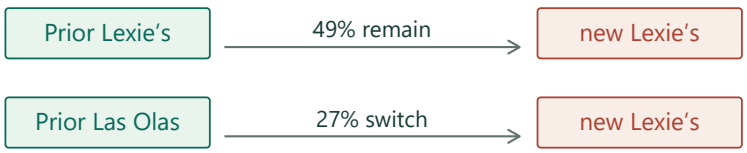
Exercise 633 · Changing restaurant choices

Printed page 68 · Transition matrices / Steady-state distributions

Let's assume students at PEA go to Las Olas or Lexie's once a week. Each week, 51% of the students who went to Lexie's the previous week decide to go to Las Olas and 27% of the students who went to Las Olas the previous week decide to go to Lexie's. Assume that this trend continues indefinitely and that the student population of PEA numbers 1000. If 420 students went to Las Olas last week, how many students should Lexie's expect this week? In two weeks? In three weeks? In the long run?

1. Write the headcount recursion

Let X denote this week's expected Lexie's count. Among its customers 49% stay, and 27% of Las Olas customers switch there. Thus X′=.49X+.27(1000−X)=.22X+270.



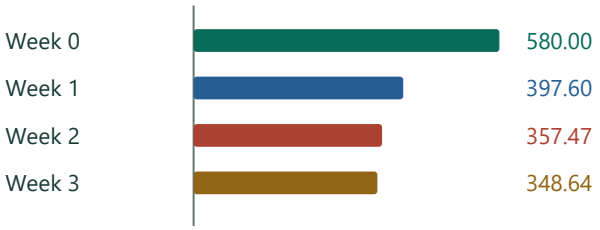
Add the retained diners and incoming diners.

1. X′=.49X+.27·1000−.27X
Distribute .27 over the other restaurant's count.

2. X′=(.49−.27)X+270=.22X+270
Collect the two contributions that depend on the current count.

2. Follow the calendar

Last week 1000−420=580 went to Lexie's. Successive weekly counts are 397.6 this week, 357.472 next week, 348.64384 two weeks from this week, and 346.7016448 three weeks from this week.



The measured starting week precedes this week.

1. .22(580)+270=397.6; .22(397.6)+270=357.472
Feed each week's expected count back into the same rule.

2. .22(357.472)+270=348.64384
This is two updates after the present-week value of 397.6.

3. .22(348.64384)+270=346.7016448
Apply one more transition for three weeks from the present.

3. Interpret the requested horizons

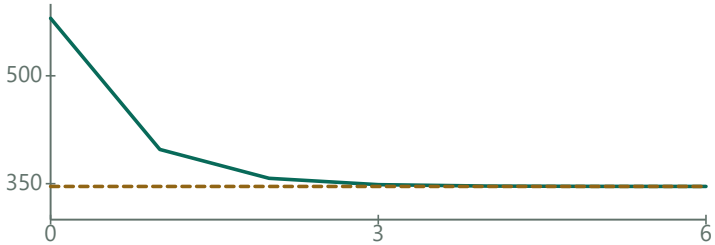
Counting from today gives about 398 now, 349 in two weeks and 347 in three. Counting transitions from the initial observation instead gives about 357 and 349 at the second and third transitions.



Count the time horizon from the specified reference week.

4. Find the long-run count

Set X=.22X+270, giving X*=270/.78≈346.154. The difference X′−X*=.22(X−X*) shrinks each week, proving convergence.



Each weekly update preserves just 22% of the previous gap.

1. X−.22X=270 ⇒ X*=270/.78
At equilibrium subtract .22X from both sides.

2. X′−X*=(.22X+270)−(.22X*+270)=.22(X−X*)
The constant cancels and the gap is multiplied by a factor less than 1.

Check. At equilibrium expected switching flows .51X and .27(1000−X) balance.

Exercise 634 · Order the transition matrix

Printed page 68 · Transition matrices / Matrix multiplication

(Continuation) In setting up #633, Sam considered the following possible matrix equations, using X for Lexie's and O for Las Olas. Which of these represent the scenario presented in #633?

(a) $\begin{bmatrix} X_n \\ O_n \end{bmatrix} = \begin{bmatrix} 0.49 & 0.51 \\ 0.27 & 0.73 \end{bmatrix} \begin{bmatrix} X_{n-1} \\ O_{n-1} \end{bmatrix}$

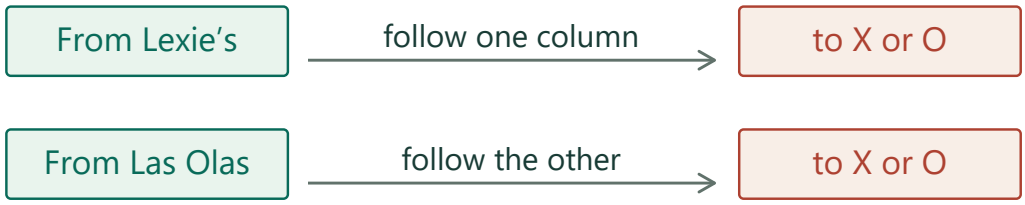
(b) $\begin{bmatrix} X_n \\ O_n \end{bmatrix} = \begin{bmatrix} 0.49 & 0.27 \\ 0.51 & 0.73 \end{bmatrix} \begin{bmatrix} X_{n-1} \\ O_{n-1} \end{bmatrix}$

(c) $\begin{bmatrix} O_n \\ X_n \end{bmatrix} = \begin{bmatrix} 0.73 & 0.51 \\ 0.27 & 0.49 \end{bmatrix} \begin{bmatrix} O_{n-1} \\ X_{n-1} \end{bmatrix}$

(d) $\begin{bmatrix} O_n \\ X_n \end{bmatrix} = \begin{bmatrix} 0.51 & 0.49 \\ 0.27 & 0.73 \end{bmatrix} \begin{bmatrix} O_{n-1} \\ X_{n-1} \end{bmatrix}$

1. Choose a test for each column

With column vectors, column j lists destinations for a student starting at restaurant j. Therefore the order on the input vector decides the meaning of each column.



The input order determines the meaning of each column.

2. Use the order (X,O)

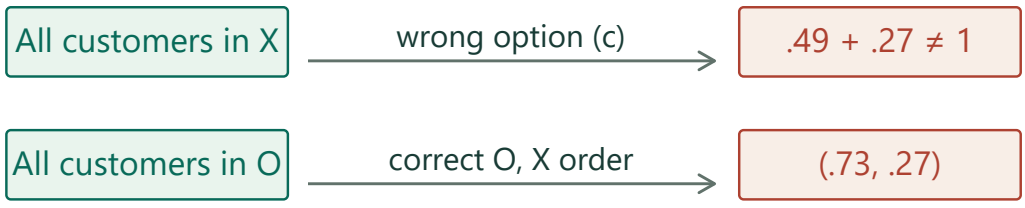
A former Lexie's customer goes to X,O with probabilities .49,.51. A Las Olas customer goes to X,O with .27,.73. Placing these columns in order gives rows (.49,.27),(.51,.73), option (b).

from X	from O
.49	.27
.51	.73

Rows give new X and O in that order.

3. Reverse both labels

For order (O,X) the rows are (.73,.51),(.27,.49), option (c). The other choices substitute probabilities for the wrong origins or destinations. You can reject (a) with an all-X test input (1,0): its first column gives (.49,.27), totaling only .76. For (d) an all-O test input should give (.73,.27) in (O,X) order, but it gives (.51,.27).



Test a proposed matrix on an entire group at one restaurant.

Check. Each correct column represents a complete distribution of the destination given one starting restaurant.

Exercise 635 · Mismatched vector orders

Printed page 68 · Transition matrices / Vectors and components

(Continuation) Explain why the equation below is an accurate representation of the scenario in #633, yet not as useful for calculations as either of the correct equations in #634.

$$\begin{bmatrix} X_n \\ O_n \end{bmatrix} = \begin{bmatrix} 0.27 & 0.49 \\ 0.73 & 0.51 \end{bmatrix} \begin{bmatrix} O_{n-1} \\ X_{n-1} \end{bmatrix}$$

1. Read the input carefully

The input column is ordered (O,X), but the output is ordered (X,O). For the first output row, .27O+.49X counts next week's Lexie's customers.



The first output entry is a Lexie's count.

2. Check the other row

The second row gives .73O+.51X, the next Las Olas count. Both equations correctly reproduce the switching assumptions.



The second output entry is a Las Olas count.

3. Try another week

The output X,O arrives in reverse order from what the matrix expects. Swap the two entries before multiplying again. The consistently ordered matrices in 634 allow ordinary matrix powers directly.



Restore the input order before repeating this matrix.

Check. Test an all-O input (1000,0): (270,730) is correct in the stated output order.

Exercise 636 · One lump sum

Printed page 69 · Compounding / Present and future value

On 1 July 2012, you deposit 1000 dollars into an account that pays 6 percent interest annually. How much is this investment worth on 1 July 2032?

1. Count anniversaries

From July 1, 2012 to July 1, 2032 there are 20 complete annual interest periods.



Twenty complete 1-year periods separate the two dates.

2. Form the annual multiplier

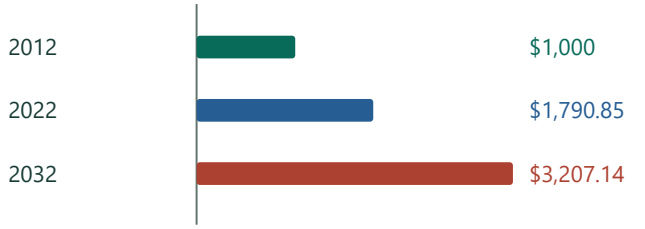
A balance B earns .06B over a year, so the next balance is 1.06B. In the following year interest applies to the new balance.



Add 6% of the balance present at the start of the year.

3. Compound and round

After 20 multiplications the investment is $1000(1.06)^{20} \approx \$3{,}207.14$, rounded to cents.



The balance compounds through all twenty years.

1. $B_0 = 1000$; $B_1 = 1000(1.06) = 1060$
The subscript counts completed annual credits.

2. $B_2 = 1060(1.06) = 1123.60 = 1000(1.06)^2$
The first year's interest earns interest as part of the updated balance.

3. $B_{20} = 1000(1.06)^{20}$
Repeating twenty times supplies twenty factors of 1.06.

Check. Dividing the unrounded future value by 1.06 twenty times recovers \$1000.

Exercise 637 · Staggered deposits

Printed page 69 · Compounding / Present and future value

(Continuation) On 1 July 2013, you deposit 1000 dollars into an account that pays 6 percent interest annually. How much is this investment worth on 1 July 2032? Answer the same question for deposits made on 1 July 2014, 1 July 2015, and so forth, until you see a pattern developing.

1. Count periods for each deposit

Each July 1 deposit earns one interest payment per complete year before July 1, 2032. Deposit years 2013, 2014, 2015 give 19,18,17 periods respectively.



Each deposit compounds only after its own deposit date.

2. Apply the same growth factor

At 6% annual compounding, their separate values are $1000(1.06)^{19} \approx \$3{,}025.60$, $1000(1.06)^{18} \approx \$2{,}854.34$, and $1000(1.06)^{17} \approx \$2{,}692.77$.



Compare the future values at the same date in 2032.

3. Describe the pattern

A single \$1000 deposit in year j has future value 1000(1.06)^(2032−j). Each year of delay divides the future value by 1.06; a deposit on the final date earns zero interest periods.



A deposit in year j earns 2032 − j annual factors.

Check. The 2032 deposit earns zero periods and is worth its face value.

Exercise 638 · Sum a savings stream

Printed page 69 · Annuities / Geometric sequences and series

(Continuation) Suppose that you deposit 1000 dollars into the same account on 1 July *every* year. The problem is now to calculate the combined value of *all* these deposits on 1 July 2032, including the deposit made on that final day. Rather than getting the answer by tediously adding the results of *twenty-one* separate (but similar) calculations, we can find a shorter way. Let V stand for the number we seek, and observe that

$$V = 1000(1.06)^0 + 1000(1.06)^1 + \cdots + 1000(1.06)^{19} + 1000(1.06)^{20}$$

is the very calculation that we wish to avoid. Obtain a second equation by multiplying both sides of this equation by 1.06, then find a way of combining the two equations to obtain a compact, easy-to-calculate formula for V.

1. Count terms by years of growth

The final deposit has no interest yet. The first deposit has accrued 20 yearly factors. Together the 21 deposits give $V=1000(1+1.06+\cdots+1.06^{20})$.



Each term is $1,000 times 1.06 raised to the indicated exponent.

2. Shift the exponents

Multiplying by 1.06 gives $1.06V=1000(1.06+1.06^2+\cdots+1.06^{21})$. Line it up against the original sum: terms with exponents 1 through 20 occur in both.



Multiplying by 1.06 aligns each term with its successor.

3. Subtract and isolate V

Only the last shifted term and the first original term survive: $.06V = 1000(1.06^{21} - 1)$. Thus $V = 1000(1.06^{21} - 1)/.06 \approx \$39{,}992.73$.



Only the end terms survive cancellation when the ratio differs from 1.

1. $1.06V - V = (1.06 - 1)V = .06V$
Combine the two multiples of the same total.

2. $1000 \cdot 1.06^{21} - 1000 = 1000(1.06^{21} - 1)$
At the ends of the aligned sums only the new highest power and the original constant survive.

3. $V = (1000/.06)(1.06^{21} - 1)$
Divide both sides by the nonzero rate before evaluating.

Check. A yearly ledger $B_1 = 1000$, $B_{n+1} = 1.06B_n + 1000$ gives the same B_{21}.

Exercise 639 · General geometric sum

Printed page 69 · Geometric sequences and series

(Continuation) Any list $first$, $first \cdot multiplier$, $first \cdot multiplier^2$, …, in which each term is obtained by multiplying its predecessor by a fixed number, is called a geometric *sequence*. A geometric *series*, on the other hand, is an addition problem formed by taking consecutive terms from some geometric sequence. Two examples: $16 + 24 + 36 + 54$ is a four-term geometric series whose sum is 130, and $32 - 16 + 8 - 4 + \cdots + 0.125$ is a nine-term geometric series whose sum is 21.375. Consider now the typical geometric series, which looks like $first + first \cdot multiplier + first \cdot multiplier^2 + \cdots + last$. Find a compact, easy-to-calculate formula for the sum of all these terms.

1. Name the first and last terms

Write S=a+ar+⋯+ar^(n−1), where the last term is ar^(n−1). The first exponent is zero, so n terms end with exponent n−1.



Each term is a times r raised to the indicated exponent.

2. Move each term to its successor

Then $rS=ar+ar^2+\cdots+ar^n$. Subtracting it from S cancels each intermediate power and gives $(1-r)S=a-ar^n$.

0	1	2	…	n−1	
	1	2	…	n−1	next

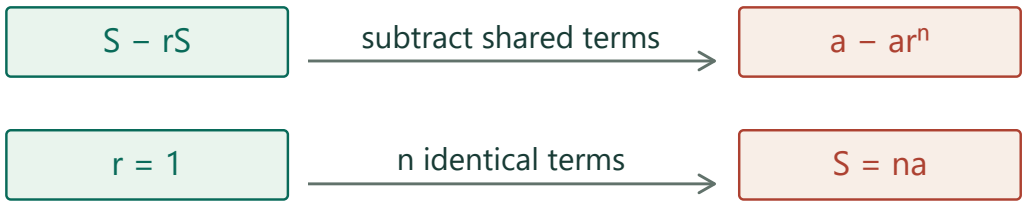
Multiplying by r aligns each term with its successor.

1. $S-rS=(a+ar+\cdots+ar^{n-1})-(ar+\cdots+ar^{n-1}+ar^n)$
Write the original and shifted terms in the same order.

2. $S-rS=a-ar^n \Rightarrow S(1-r)=a(1-r^n)$
The overlapping terms cancel. Factor each side.

3. Use the last term directly

Since $ar^n=r\cdot\text{last}$, $S=(\text{first}-r\cdot\text{last})/(1-r)$ when $r\neq1$. If $r=1$ all n terms equal a, giving $S=na$ without division by zero.



Only the end terms survive cancellation when the ratio differs from 1.

Check. For just one term the formula returns a, which checks the exponent convention.

Exercise 640 · Count powers in a sum

Printed page 69 · Geometric sequences and series / Sigma notation

$$\text{Write the sum of } 3383(1.04)^0+3383(1.04)^1+3383(1.04)^2+\cdots+3383(1.04)^n \text{ compactly.}$$

1. Recognize the series parameters

The first term is 3383 and each successive term multiplies by 1.04. The exponents 0 through n give n+1 terms.



Each term is 3383 times 1.04 raised to the indicated exponent.

2. Align the two sums

Multiplying the sum S by 1.04 shifts its exponent range to 1 through n+1. In 1.04S−S all shared terms cancel.

0	1	2	…	n	
	1	2	…	n	next

Multiplying by 1.04 aligns each term with its successor.

3. Write the compact value

The leftover equation is .04S=3383(1.04^(n+1)−1). Dividing by .04 gives S=3383(1.04^(n+1)−1)/.04, which returns 3383 for n=0.

1.04S − S → subtract shared terms → $3383(1.04^{n+1}-1)$

Only the end terms survive cancellation when the ratio differs from 1.

1. 1.04S−S=(1.04−1)S=.04S
Factor the common S on the left.

2. 3383·1.04^(n+1)−3383=3383[1.04^(n+1)−1]
The highest shifted term survives, and the constant came from subtracting the original sum.

3. S=84575[1.04^(n+1)−1]
Divide by .04, using 3383/.04=84575.

Check. At n=0 the result is exactly 3383.

Exercise 641 · Remove common logarithms

Printed page 69 · Logarithm rules / Exponential functions

Convert the following equations into non-logarithmic forms:

(a) $2.3\log x + \log y = 1.845$ **(b)** $\log y = 3.204 - 0.510x$

1. (a) Record the inputs and combine the logs

This problem uses common logarithms. The original log x and log y require x>0 and y>0. The power and product rules below therefore apply to positive arguments.

1 **2.3 log x + log y = 1.845**
Both logs are in base 10.

2 **log(x^2.3) + log y = 1.845**
Move the coefficient onto x as an exponent.

3 **log(x^2.3 · y) = 1.845**
Adding the two logs multiplies their arguments.

2. (a) Undo the common logarithm

Exponentiating in base 10 reverses a common log. The original positivity of x makes x^2.3 nonzero, and the solved expression then automatically gives y>0.

1 **10^(log(x^2.3 · y)) = 10^1.845**
Raise 10 to each side.

2 **x^2.3 · y = 10^1.845**
Cancel the inverse functions on the positive argument.

3 **y = 10^1.845 / x^2.3, x > 0**
Divide by x^2.3 to give an explicit equivalent form.

3. (b) Convert the linear exponent

Only y appears inside a logarithm in this part, so the given equation requires y>0 and allows every real x. The exponential form always gives such a y.

1 **log y = 3.204 − 0.510x**
Read the second common-log equation.

2 **y = 10^(3.204 − 0.510x)**
Exponentiate both sides in base 10.

3 **y = 10^3.204 · 10^(−0.510x)**
A sum in an exponent becomes a product.

4 **y = 10^3.204 · (10^(−0.510))ˣ**
Group −0.510 and x using the power-of-a-power rule.

Check. Taking common logarithms of the positive right sides reproduces the initial equations.

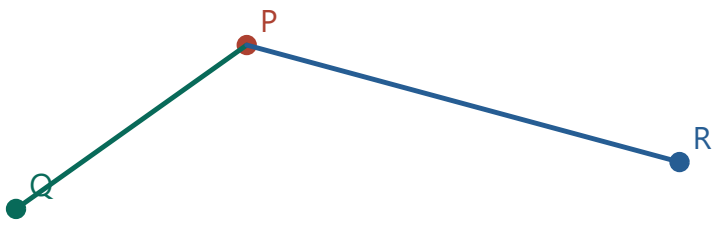
Exercise 642 · Plane through three points

Printed page 69 · Planes in space / Cross products

Find an equation for the plane that contains the points $P = (3, 1, 4)$, $Q = (1, 4, -2)$, and $R = (9, -1, 1)$. There is more than one possible form for your answer, and more than one workable approach. You will have to make some choices.

1. Construct two independent planar directions

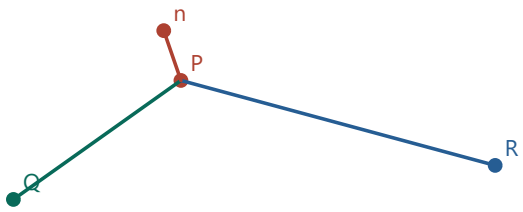
Subtract coordinates: Q−P=(−2,3,−6) and R−P=(6,−2,−3). These vectors are not multiples, so they span the plane.



Two displacement vectors begin at the same point.

2. Choose a perpendicular vector

Their cross product is $(-21,-42,-14)$. Divide by -7 to get the simpler normal $n=(3,6,2)$. Checking its dot product with each direction gives zero.



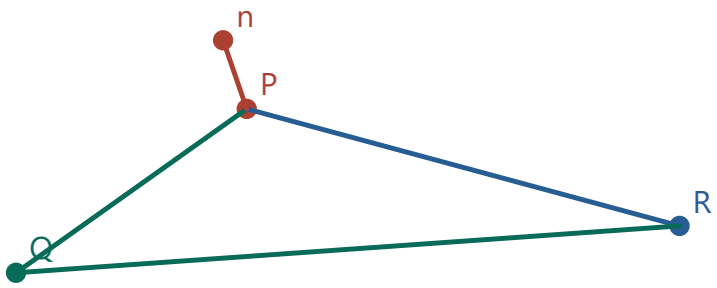
The cross product gives a direction normal to the plane.

1. $n_1=3(-3)-(-6)(-2)=-21$; $n_2=(-6)6-(-2)(-3)=-42$
Use the differences of products from the y,z and z,x coordinates.

2. $n_3=(-2)(-2)-3\cdot6=-14$; $n/(-7)=(3,6,2)$
Form the remaining component and reduce the common factor.

3. Pass the plane through P

A point X belongs when $n\cdot X=n\cdot P$. At P the value is $3\cdot3+6\cdot1+2\cdot4=23$, so $3x+6y+2z=23$. Both Q and R give the same 23.



Every point in this triangular patch has $n \cdot X = 23$.

Check. The other points give $3+24-4=23$ and $27-6+2=23$.

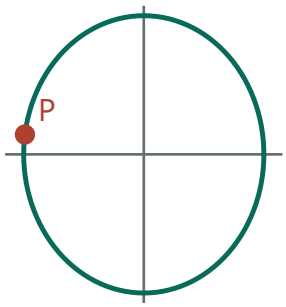
Exercise 643 · Reflect an ellipse tangent

Printed page 69 · Ellipses / Tangent lines / Reflections

The point $P=(-12,2)$ is on the ellipse $4x^2+3y^2=588$. Find an equation for the line through P that is tangent to the ellipse.

1. Identify what a tangent slope measures

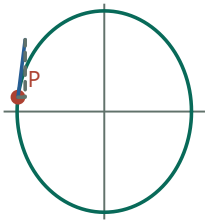
Along the ellipse nearby x and y change together. At $P=(-12,2)$ both vary smoothly, so differentiate the implicit equation with respect to x.



The point lies on the upper-left portion of the ellipse.

2. Solve for the slope

From $8x+6y\,y'=0$, $y'=-4x/(3y)$. Substitute $x=-12$ and $y=2$: the slope is $48/6=8$.



The tangent direction rises 8 for one unit of run.

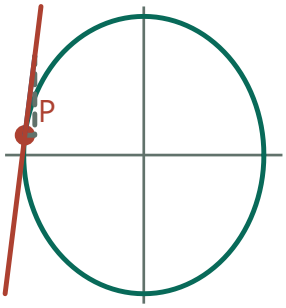
1. $8x+6yy'=0 \Rightarrow 6yy'=-8x$
Subtract the derivative of $4x^2$ from both sides.

2. $y'=-8x/(6y)=-4x/(3y)$
Divide by 6y, which is nonzero at P.

3. $-4(-12)/(3\cdot 2)=48/6=8$
The negative x-coordinate makes the slope positive.

3. Anchor the line at P

Point-slope form is $y-2=8(x+12)$, giving $y=8x+98$. Its positive slope agrees with the upward direction of the upper-left part of the ellipse.



Extend that direction through P to get $y = 8x + 98$.

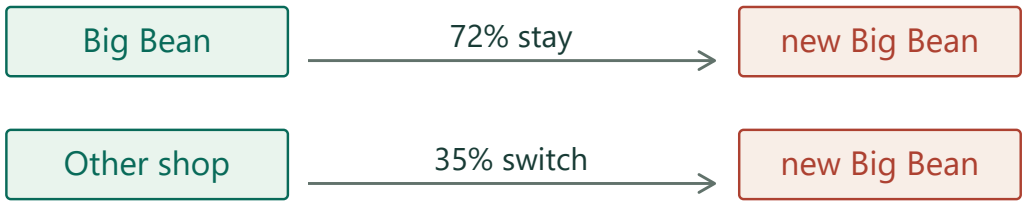
Check. Reflecting the tangent at (12,2), $y=-8x+98$, across the y-axis gives the same equation.

Exercise 644 · Competing lunch spots

The lunchtime crowd in Exeter frequents either The Big Bean or Me & Ollie's. Competition between the two restaurants is high, and customer surveys disclose that 35% of those who eat at the Me & Ollie's will switch to The Big Bean the next day, while 28% of those who eat at The Big Bean will switch to Me & Ollie's the next day. Given that 50% of the lunchtime crowd ate at each of the two restaurants today, predict the distribution of customers **(a)** tomorrow **(b)** the day after tomorrow **(c)** three weeks from now. This sequence of lunchtime distributions is an example of a *Markov chain*.

1. Follow tomorrow's Big Bean customers

Let b_n be the Big Bean fraction n days from today. It retains .72 of its own customers and gains .35 of the other restaurant's. Then $b_{n+1}=.72b_n+.35(1-b_n)=.37b_n+.35$.



Add arrivals that start in each of the two shops.

2. Advance the first two days

Starting from $b_0=.5$, compute $b_1=.535$ and $b_2=.54795$. The complementary Me & Ollie's shares are .465 and .45205.



Start with half the customers in each shop.

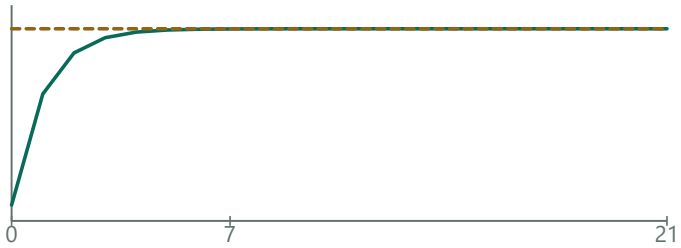
1. $b_1=.37(.5)+.35=.535$
Use today's half share as the first input.

2. $b_2=.37(.535)+.35=.54795$
The second update uses tomorrow's share, without rounding.

3. $1-b_2=.45205$
The two restaurants account for the entire customer population.

3. Extend to three weeks

A fixed fraction satisfies $b^*=.37b^*+.35$, so $b^*=5/9$. Subtract this fixed equation from the recursion: deviations shrink by .37 daily. Thus $b_{21}=5/9+(.5-5/9)(.37)^{21}\approx 55.5556\%$, leaving 44.4444% at Me & Ollie's.



The deviation from 5/9 multiplies by 0.37 daily.

1. $b^*=.37b^*+.35 \Rightarrow .63b^*=.35 \Rightarrow b^*=35/63=5/9$
Combine the present and next shares at equilibrium.

2. $b_{n+1}-b^*=.37(b_n-b^*) \Rightarrow b_n-b^*=(.37)^n(b_0-b^*)$
Subtract the fixed equation, then iterate the multiplier n times.

3. $b_{21}=5/9+(.37)^{21}(1/2-5/9)$
Add the equilibrium share back after 21 updates.

Check. The fixed share .35/.63=5/9 equates the two switching flows.

Exercise 645 · Equivalent products and logs

Printed page 70 · Exponent rules / Change of base

Find equivalent expressions:

(a) $\dfrac{(2.86)^x}{(1.43)^x}$ (b) $(1.4)^x+(1.4)^{x+1}+(1.4)^{x+2}$ (c) $\dfrac{\log k}{\log m}$

1. (a) Combine powers with the same exponent

Both given bases are positive, and the denominator never vanishes. Consequently the equal-exponent quotient rule holds for every real x.

1 **$(2.86)^x/(1.43)^x = (2.86/1.43)^x$**
Divide the bases under the shared exponent.

2 **$(2.86/1.43)^x = 2^x$**
Twice 1.43 is 2.86.

2. (b) Factor out the smallest displayed power

The positive base 1.4 makes these exponent laws valid for every real x. Isolate the common power before adding the coefficients.

1 **$1.4^{(x+1)} = 1.4^x \cdot 1.4$**
Separate the last factor using the sum-of-exponents law.

2 **$1.4^{(x+2)} = 1.4^x \cdot 1.4^2$**
The third term has two extra factors of 1.4.

3 **$1.4^x + 1.4^{(x+1)} + 1.4^{(x+2)} = 1.4^x(1+1.4+1.4^2)$**
Distribute in reverse to factor out 1.4^x.

4 **$1.4^x(1+1.4+1.4^2) = 4.36 \cdot 1.4^x$**
Add the coefficients 1, 1.4, and 1.96.

3. (c) Recognize the change of base

The numerator needs k>0 and the denominator needs m>0. The denominator log m must also be nonzero, excluding m=1. These are precisely the conditions needed for log base m of k.

1 **$u = \log_m k \iff m^u = k$**
Name the exponent on m that produces k.

2 **$\log(m^u) = \log k$**
Apply the same fixed logarithm to the positive sides.

3 **$u \log m = \log k$**
Move the exponent u into the coefficient.

4 **$(\log k)/(\log m) = u = \log_m k$**
Divide by the nonzero log m.

Check. Raising m to the answer in (c) gives k.

Exercise 646 · Find the seventh term

Printed page 70 · Geometric sequences and series

The third term of a geometric sequence is 40. The sixth term is 135. What is the seventh term of this sequence?

1. Count the ratio steps

Let r be the geometric multiplier. Moving from term 3 to term 6 applies it three times, so $40r^3=135$.



Count three gaps, from term 3 through term 6.

2. Recover the multiplier

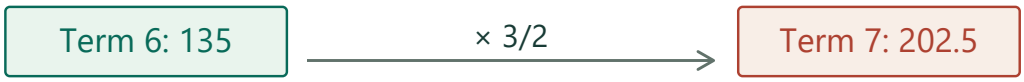
Divide by 40: $r^3=27/8=(3/2)^3$. Thus $r=3/2$.



A constant ratio raised to the third power gives 27/8.

3. Advance one more term

The seventh term is the sixth times r, or $135\cdot 3/2=202.5$.

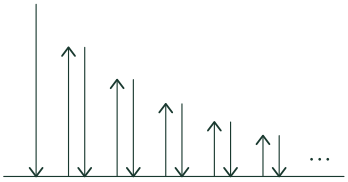


Take one more geometric step.

Check. $40(3/2)^4=202.5$.

Exercise 647 · Repeated bounces

Printed page 70 · Geometric sequences and series / Sequences and recursion



A speckled green superball has a 75% rebound ratio. When you drop it from a height of 16 feet, it bounces and bounces and bounces . . .

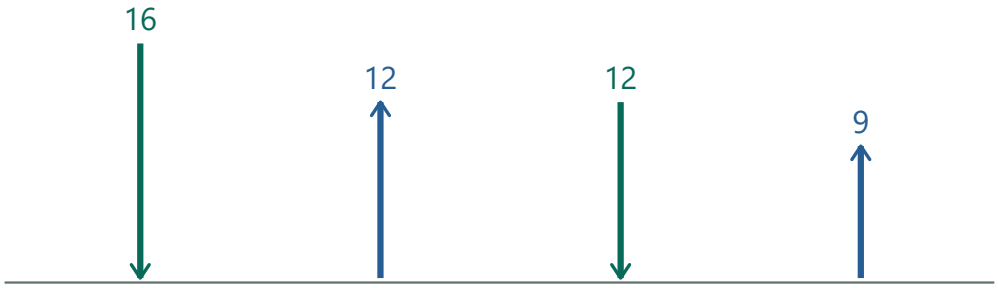
(a) How high does the ball bounce after it strikes the ground for the third time?

(b) How high does the ball bounce after it strikes the ground for the seventeenth time?

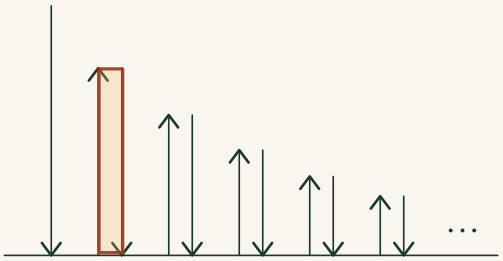
(c) When it strikes the ground for the second time, the ball has traveled a total of 28 feet in a *downward* direction. Verify this. How far downward has the ball traveled when it strikes the ground for the seventeenth time?

1. Distinguish a rebound height from a path length

Each bounce reaches 75% of the preceding height. With 16 ft as the initial height, rebound j reaches $16(.75)^j$ ft.



In feet, each new rebound is three quarters of the preceding height.



First rebound and return

The first upward arrow ends at the height from which the next downward trip begins. Heights and total distances refer to different quantities.

2. Evaluate the requested rebounds

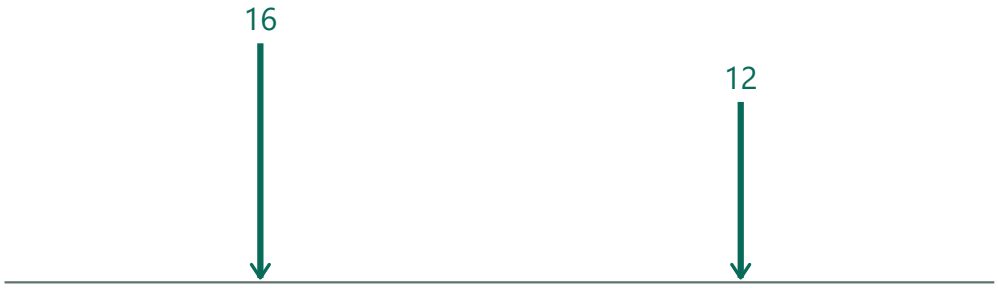
For $j=3$, $16(.75)^3=6.75$ ft. For $j=17$, $16(.75)^{17}\approx.12027$ ft.



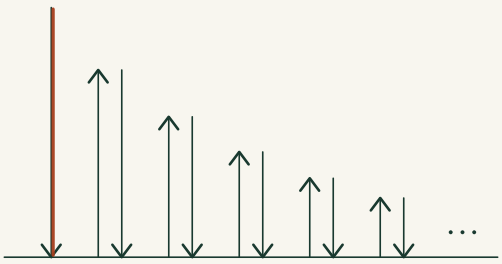
The exponent counts rebounds after the original drop.

3. Count only the downward legs

By impact 2, the ball has descended 16 ft, then 12 ft: total 28 ft. At impact 17 the initial drop and 16 subsequent descents give 17 terms with exponents 0 through 16.



At the second impact, count the two completed downward legs.

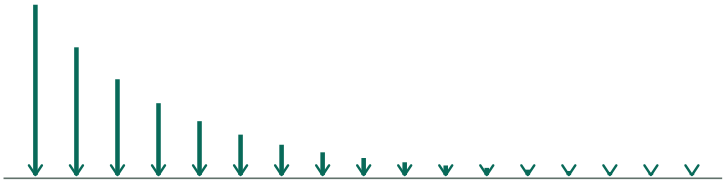


Initial descent

This entire downward arrow occurs before the first rebound. Count it when measuring downward distance through successive impacts.

4. Sum the descents

The resulting sum is $16[1-(.75)^{17}]/(1-.75)=64[1-(.75)^{17}]$ ft. The upward leg after impact 17 has not started at the specified instant.



Impact 17 completes seventeen progressively shorter descents.

1. $D=16(1+r+\cdots+r^{16})$, $r=.75$
Each exponent counts completed rebounds before that descent.

2. $D-rD=16(1-r^{17}) \Rightarrow D=16(1-r^{17})/(1-r)$
Subtract the shifted sum; the 16 interior powers cancel.

Check. There have been 17 descents but only 16 completed rebounds at ground impact 17.

Exercise 648 · Rebound path length

Printed page 70 · Geometric sequences and series

(Continuation) At the top of its second rebound, the ball has traveled 21 feet *upward*.
(a) At the top of its seventeenth rebound, how far upward has the ball traveled?
(b) At the top of its seventeenth rebound, how far has the ball traveled in total?

1. Identify the first upward legs

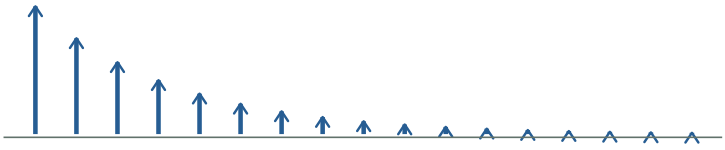
The initial 16-ft drop is downward. The first two rises are 12 and 9 ft, accounting for the given 21 ft. At the top of rebound 17, all 17 upward legs are complete.



The first two complete rises add 12 + 9 feet.

2. Sum the upward distances

They form a geometric series beginning with 12 and ratio .75. Its sum is $U=12[1-(.75)^{17}]/(1-.75)=48[1-(.75)^{17}]$ ft.



Include all seventeen rises through the top of rebound 17.

1. $U=16r+16r^2+\cdots+16r^{17}=12(1+r+\cdots+r^{16})$
Factor out the first rise $16r=12$ and count 17 completed rises.

2. $(1-r)U=12(1-r^{17}) \Rightarrow U=48(1-r^{17})$ ft
Subtract the shifted sum and divide by .25.

3. Add all completed descents

The latest downward leg ended at impact 17. From exercise 647 its cumulative length is $D=64[1-(.75)^{17}]$ ft. Total travel is $U+D=112[1-(.75)^{17}]$ ft.



The distance traveled includes both totals, regardless of direction.

Check. $D-U=16[1-(.75)^{17}]$, the net descent from release height.

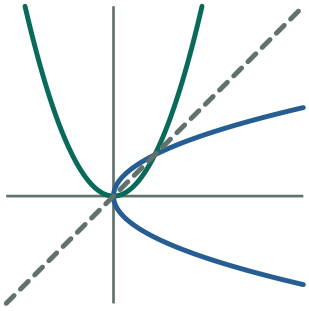
Exercise 649 · When squaring has an inverse

Printed page 70 · Inverse functions / Domain and range

Consider the function $f(x) = x^2$. Does it have an inverse? If so, write a formula for it; if not, why not? In considering this question, please draw a graph of $y = f(x)$ as well as its reflection over the line $y = x$.

1. Reflect the graph one point at a time

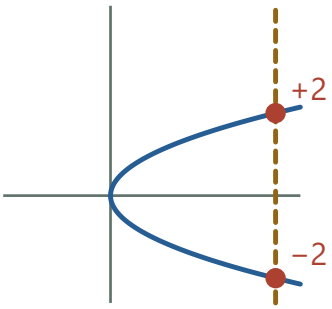
The upright parabola $y=x^2$ has points $(2,4),(-2,4),(0,0)$. After reflection in $y=x$ these become $(4,2),(4,-2),(0,0)$, lying on the sideways parabola $x=y^2$.



Reflecting the parabola exchanges x and y.

2. Test the inverse requirement

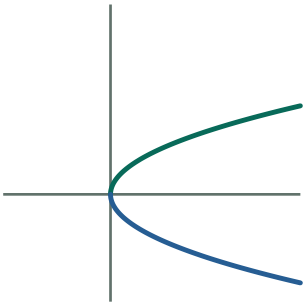
At input 4 the reflected relation has two outputs, 2 and −2. Thus the original squaring function on all real x fails to have an inverse function.



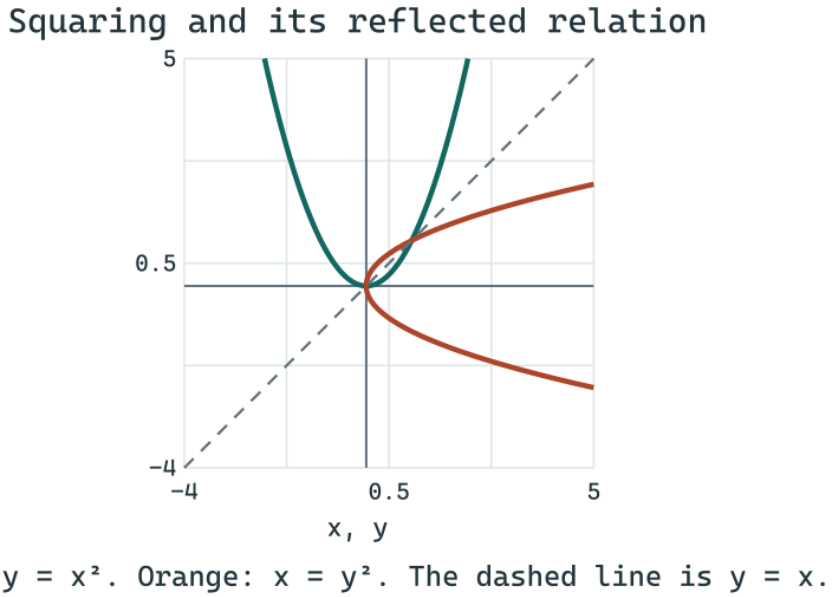
The vertical line x = 4 meets the inverse relation twice.

3. Choose one branch if the domain changes

Restricting the original domain to $x \geq 0$ keeps the inverse $y = \sqrt{x}$; restricting it to $x \leq 0$ gives $y = -\sqrt{x}$. Each has one output for each nonnegative input.



Selecting either half of the original produces one inverse branch.



Check. Both 2 and −2 map to 4, violating one-to-one correspondence.

Exercise 650 · Present value of a college fund

Printed page 70 · Compounding / Present and future value

Sometimes it is necessary to invest a certain amount of money at a fixed interest rate for a fixed number of years so that a financial goal is met. The initial amount invested is called the *present value* and the goal is called the *future value*. The parents of a child born today decide that \$350 000 will be needed for college expenses. They find a certificate of deposit that pays 0.5 percent interest each month. How much (present value) should they invest so that there is \$350 000 on the child's 18th birthday?

1. Match the time unit to the rate

Interest posts monthly. Eighteen years contain 18·12=216 compounding periods, and a 0.5% monthly rate multiplies a balance by 1.005 each period.



Count monthly compounding periods before applying any rate.

2. Write the forward equation

An initial deposit P grows to $P(1.005)^{216}$. Set this equal to the target \$350,000.



Grow the unknown deposit forward to the target date.

1. $B_1 = P + .005P = 1.005P$
After one month the account contains the original deposit plus interest on it.

2. $B_2 = 1.005B_1 = P(1.005)^2$
The next interest calculation uses the entire updated balance.

3. $B_{216} = P(1.005)^{216} = 350000$
Each month adds one multiplier. The required future balance supplies the right side.

3. Discount the target to today

Divide by that growth factor: $P=350000/(1.005)^{216}\approx\$119{,}178.72$. Compounding the unrounded present value forward gives the target.



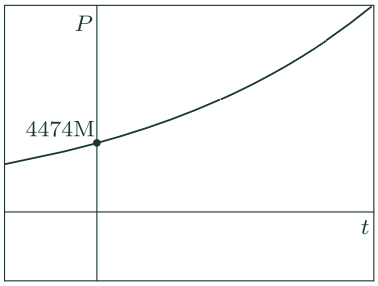
Discount that same growth to recover the initial amount.

Check. Compounding the unrounded deposit forward reproduces the target.

Exercise 651 · Two meanings of population growth

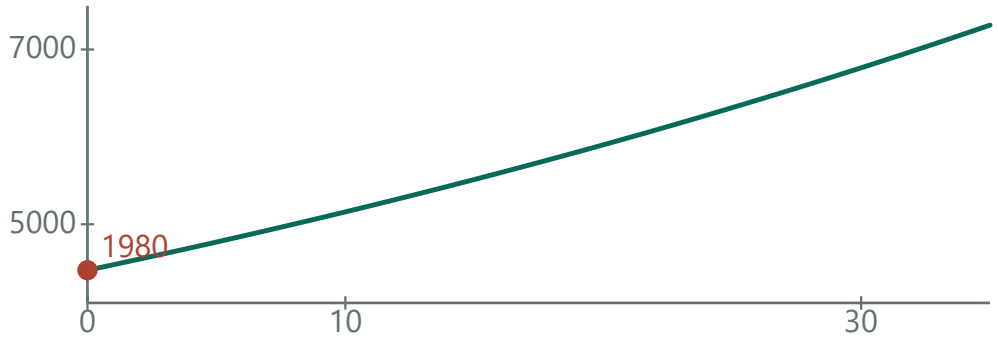
Printed page 71 · Exponential growth and decay / Rates of change

The size of the earth's human population at the beginning of the year $1980 + t$ can be approximated by exponential functions. One such function is $P(t) = 4\,474\,469\,000(1.0140)^t$. Its graph is shown at right. At what rate is $P(t)$ increasing when **(a)** $t = 10$ and when **(b)** $t = 30$? There are at least two ways to interpret each of these questions. Provide an answer for each of your interpretations.

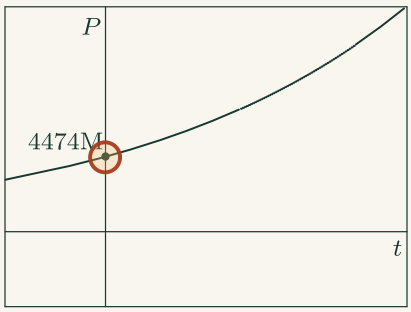


1. Choose a time scale for “rate”

t counts years since 1980. The annual multiplier 1.014 gives an exact one-year change. The slope of the tangent to the smooth exponential gives a second, instantaneous interpretation.



P(t) is population in millions; horizontal distance is years.



Population at the starting year

The black point fixes the scale of the population model at time zero. Time in this graph is measured from that year.

2. Calculate the one-year change

P(t+1)−P(t)=(1.014−1)P(t)=.014P(t). Dividing by the one-year interval, this is about 71.986 million people/year at t=10 and 95.062 million/year at t=30.



The one-year change is the rise of a secant of width one.

3. Calculate the instantaneous slope

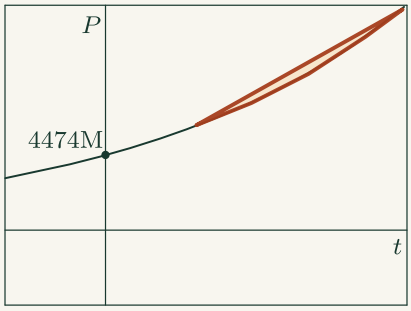
Writing P(t)=P(0)e^(t ln1.014) gives P′(t)=P(t)ln1.014. Substitution gives about 71.487 and 94.403 million/year, respectively. The upward-curving graph has a larger next-year average than its starting tangent.



The tangent uses an interval shrinking to zero.

1. P(t)=Ce^(t ln 1.014)
Write the constant multiplier as e raised to its natural log.

2. P′(t)=Ce^(t ln 1.014)·ln 1.014=P(t)ln 1.014
Differentiate the exponential and multiply by its inner rate.



Growing slope

Trace the upper-right part of the curve. To estimate its slope compare the rise over a chosen horizontal interval, or shrink that interval toward a tangent.

Check. The instantaneous rates are slightly smaller than the next-year averages since this growth curve steepens.

Exercise 652 · What stays constant in growth?

Printed page 71 · Exponential growth and decay / Rates of change / Percentage change

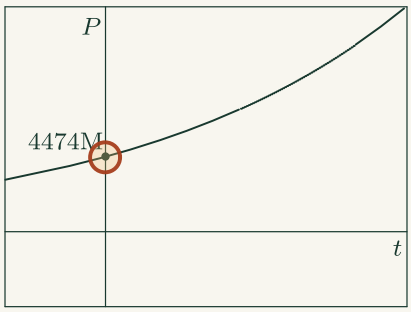
(Continuation) Does it make sense to say that the earth's human population is growing at a *constant rate*? Discuss.

1. Ask what is being held constant

The model uses P(t+1)=1.014P(t). The ratio P(t+1)/P(t) stays fixed, so the annual percentage increase stays at 1.4%.



The annual increase is always 1.4% of the starting amount.



Population at the starting year · Figure from exercise 651

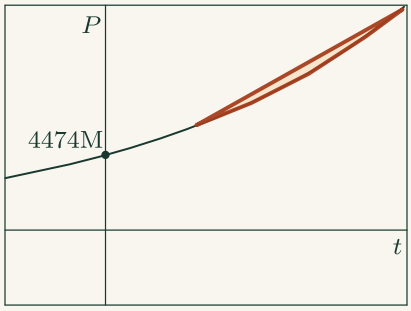
The black point fixes the scale of the population model at time zero. Time in this graph is measured from that year.

2. Compare absolute changes

The yearly number added is .014P(t). A larger starting population therefore produces a greater numerical gain. The instantaneous relative rate P′(t)/P(t)=ln1.014 is constant, while P′(t) rises.



The same multiplier adds more people to the larger base.

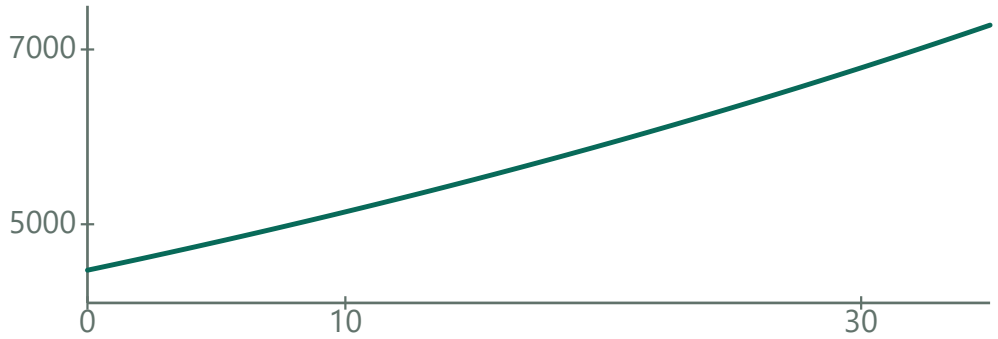


Growing slope · Figure from exercise 651

Trace the upper-right part of the curve. To estimate its slope compare the rise over a chosen horizontal interval, or shrink that interval toward a tangent.

3. Separate the approximation from the population

The property follows from the chosen mathematical model. The given formula approximates demographic history and does not establish a fixed real-world growth percentage for all years.



A smooth exponential represents the model, not individual censuses.

Check. Dividing next year's modeled gain by today's population always gives .014.

Exercise 653 · A string's second mode

Printed page 71 · Simple harmonic motion / Amplitude and period / Rates of change

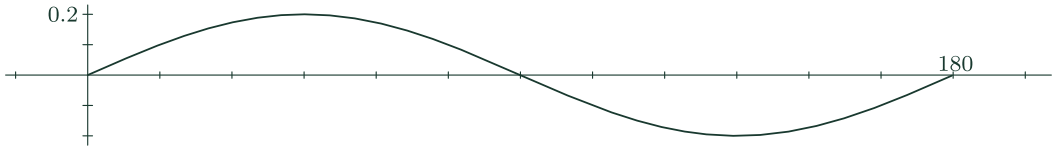
Interpreted in degree mode, the equation $y = 0.2\cos(72000t)\sin(2x)$ models the motion of a stretched, vibrating string that is 180 centimeters long ($x = 0$ is one end of the string and $x = 180$ is the other). The amplitude of the vibration is 0.2 cm.

(a) Verify that the period of the vibration is 0.005 second. This means that the string vibrates 200 times per second.

(b) Taking a photo of the string corresponds to choosing a t-value. For example, the diagram below shows the string at the instant $t = 0$. Notice that the amplitude has been exaggerated to make it more visible. Sketch photos of the string for $t = 0.001$, $t = 0.0025$, $t = 0.003$, and $t = 0.005$ second.

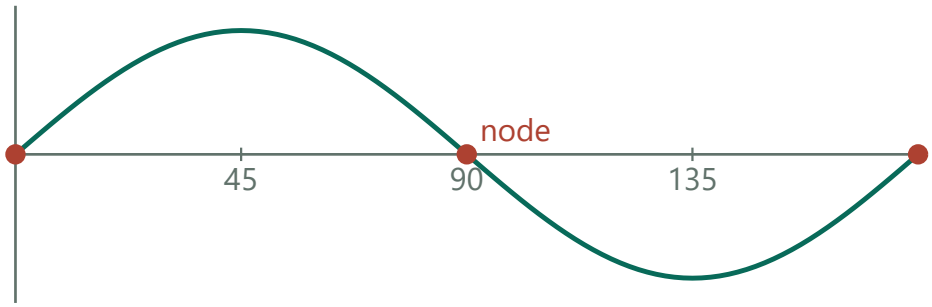
(c) How far does the center of the string move during one complete vibration?

(d) What two points on the string move the most during one complete vibration? For either one of them, calculate its speed at the instant when it is crossing the x-axis.

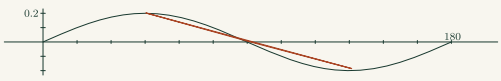


1. Identify the fixed points and moving lobes

The position factor sin(2x°) is zero at x=0,90,180 cm. It reaches +1 at x=45 and −1 at x=135, so the two lobes move in opposite directions.



Fixed ends stay at zero throughout the motion.

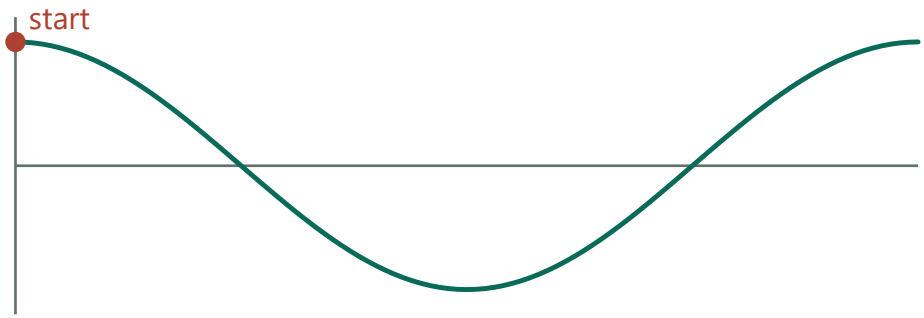


Oppositely moving lobes

Track the top of the left lobe and the bottom of the right one. Their signed heights have equal size and opposite signs at the same instant.

2. Set the time scale

A full 360° time cycle takes 360/72000=.005 s, or 200 cycles per second. Each snapshot multiplies the same two-lobe profile by .2cos(72000t°).

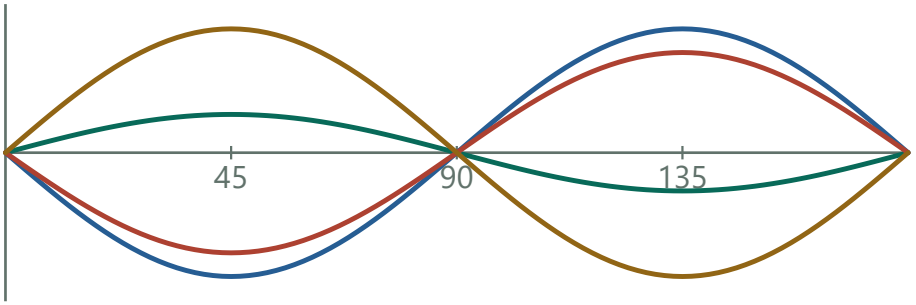


At an antinode, one cosine cycle supplies the time factor.

3. Draw four snapshots

At .001,.0025,.003,.005 s the height multipliers are respectively .06180,−.2000,−.16180,.2000 cm. Use that signed height at x=45 and its negative at x=135, joining smoothly through the three fixed points.

Snapshot time · All four times



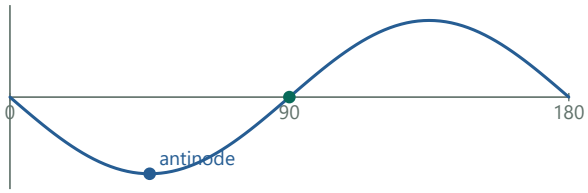
Snapshots at 0.001, 0.0025, 0.003 and 0.005 s (green, blue, red, gold).

Signed heights of the two vibrating lobes

Time (s)	Height at x = 45 cm (cm)	Height at x = 135 cm (cm)
0.001	0.06180	−0.06180
0.0025	−0.20000	0.20000
0.003	−0.16180	0.16180
0.005	0.20000	−0.20000

Compare settings · Snapshot time

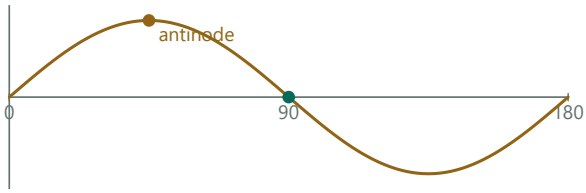
0.0025 s



The signed multiplier is -0.2000 cm at t = 0.0025 s.

At x = 45 cm the height is -0.2000 cm. The other antinode has the opposite height.

0.005 s

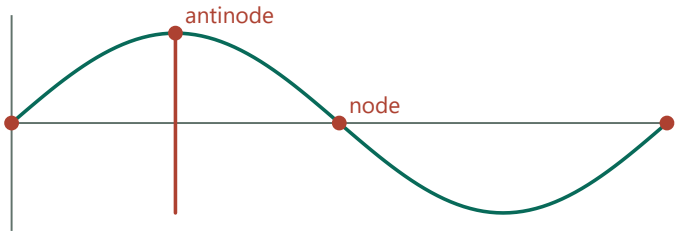


The signed multiplier is 0.2000 cm at t = 0.005 s.

At x = 45 cm the height is 0.2000 cm. The other antinode has the opposite height.

4. Measure center and maximum motion

The center x=90 is fixed and travels zero. Each antinode travels 4(.2)=.8 cm per cycle. Its angular speed is 400π rad/s, so its speed at a zero crossing is .2·400π=80π cm/s.



A moving antinode passes from top to bottom and back each cycle.

1. 72000°/s · π/180° = 400π rad/s
Use radians for the derivative of the oscillation.

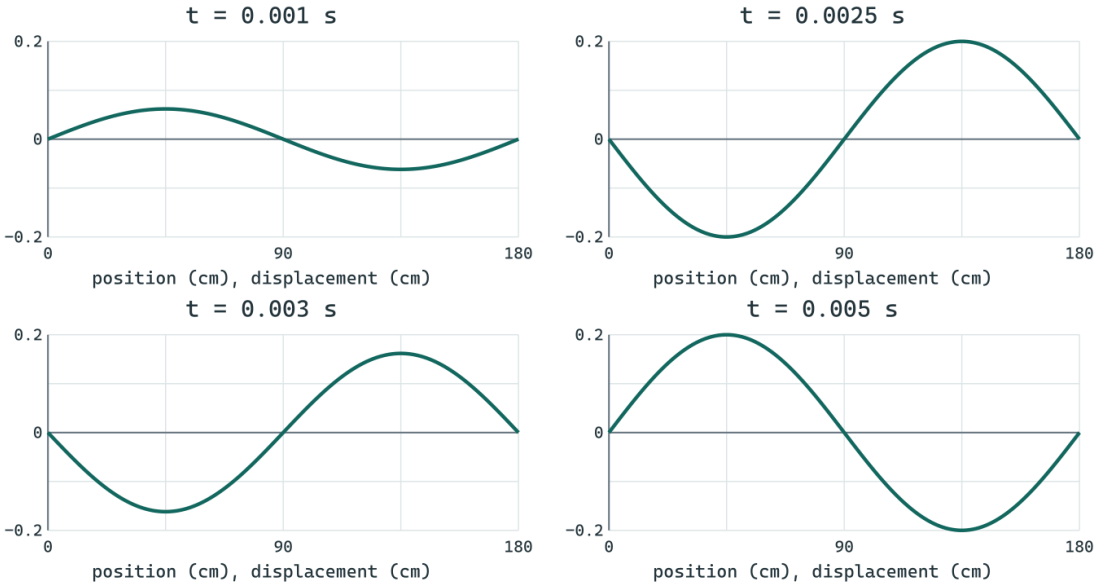
2. v=−80π sin(400πt) sin(2x°)
Differentiate the cosine in time; the spatial factor stays fixed.

3. abs(v)=80π at a zero crossing of an antinode
There both sine factors have magnitude one. At the center the spatial factor is zero.



Fixed middle node

The two lobes meet the axis in the middle. Watch this point through every snapshot while the neighboring points move.



The vertical scale is exaggerated. All four snapshots use identical axes.

Check. The two lobes move in opposite directions because sin(2x°) changes sign at 90°.

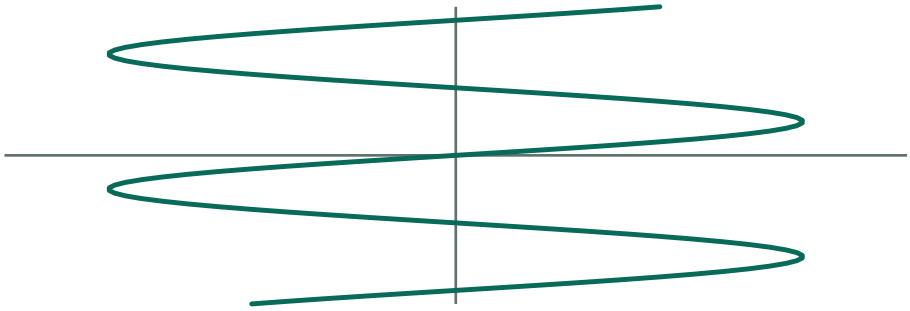
Exercise 654 · All sine inverses versus one branch

Printed page 71 · Inverse trigonometric functions / Inverse functions

Use a graphing tool to draw the graph of $x = \sin y$ in radians. Explain why it is not the same as the graph of $y = \sin^{-1} x$.

1. Graph the swapped relation

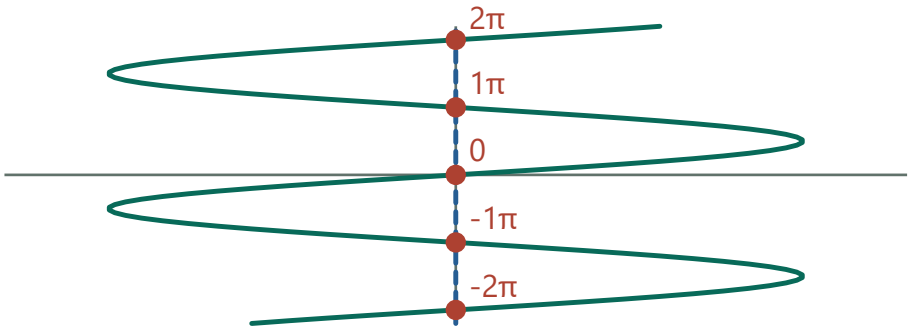
Start with the full sine wave and swap its coordinates. The result $x=\sin y$ winds up and down a strip $-1 \le x \le 1$ as y ranges over every real angle.



Swapping $y = \sin x$ gives $x = \sin y$.

2. Test the vertical line x=0

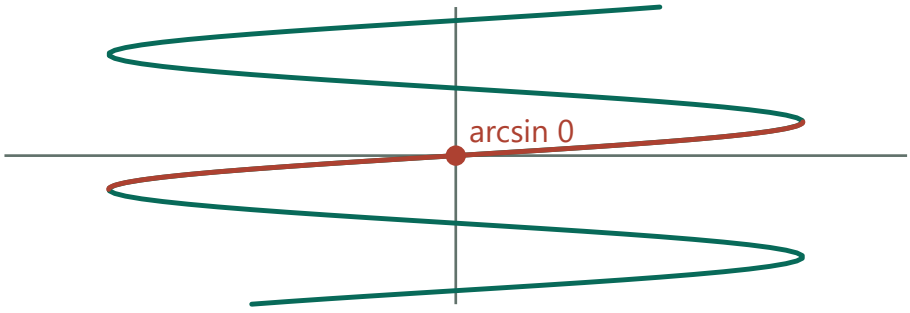
The relation accepts $y=0, \pi, -\pi, 2\pi, \ldots$ because each sine equals zero. Thus one x can have many y-values.



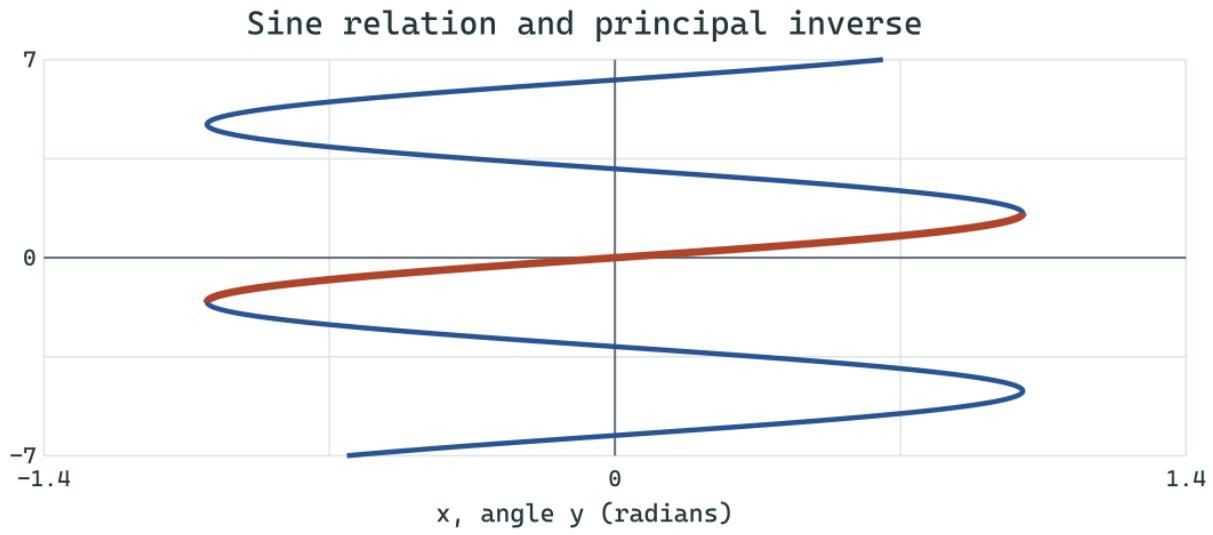
The same input 0 has many possible angles.

3. Choose the inverse-function branch

arcsin returns the unique angle in $[-\pi/2, \pi/2]$ having the prescribed sine. Its graph is only the central increasing arc of the swapped relation. In particular $\arcsin(0)=0$.



Keep the central increasing arc to make an inverse function.



Orange is the principal arcsine branch. Blue includes the additional solutions.

Check. At x=0 the relation permits y=kπ; arcsin returns 0.

Exercise 655 · The inverse-sine correspondence

Printed page 71 · Inverse trigonometric functions / Domain and range

Many times you have used the function $\sin^{-1}$ to find angles, trusting (for example) that calculators will always return an angle whose sine is 0.3 when you request $\sin^{-1}(0.3)$. Now it is time to consider the rule that makes $\sin^{-1}$ a function.

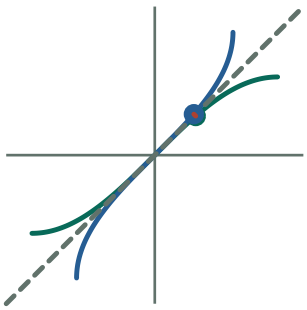
(a) The point $(\frac{\pi}{6}, 0.5)$ is on the graph of $y = \sin x$ (working in radian mode). There is a corresponding point that is on the graph of $y = \sin^{-1} x$. What is it?

(b) Write four other sets of corresponding points — two with negative x-coordinates and two with positive x-coordinates.

(c) Notice that the point $(\frac{5\pi}{6}, 0.5)$ is on the graph of $y = \sin x$ but the point $(0.5, \frac{5\pi}{6})$ is not on the graph of $y = \sin^{-1} x$. Explain.

1. Read the inverse point (a)

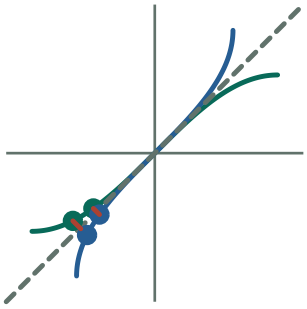
A sine point (angle,value) reverses to (value,angle) when the angle is in [−π/2,π/2]. Thus (π/6,½) corresponds to (½,π/6).



The angle π/6 and height 1/2 exchange places.

2. Use two negative angles (b)

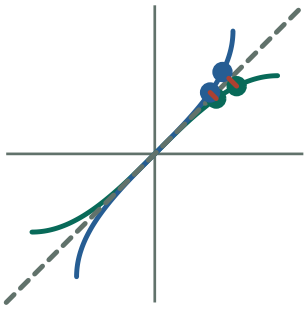
Sine pairs $(-\pi/3,-\sqrt{3}/2)$ and $(-\pi/4,-\sqrt{2}/2)$ reverse to $(-\sqrt{3}/2,-\pi/3)$ and $(-\sqrt{2}/2,-\pi/4)$.



The two negative angle–height pairs exchange coordinates.

3. Use two positive angles (b)

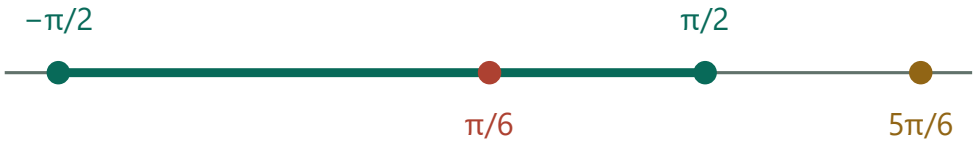
The pairs $(\pi/4,\sqrt{2}/2)$ and $(\pi/3,\sqrt{3}/2)$ reverse to $(\sqrt{2}/2,\pi/4)$ and $(\sqrt{3}/2,\pi/3)$. Each original angle lies on the increasing branch.



The two positive pairs reflect across the same diagonal.

4. Read the principal range (c)

Although $\sin(5\pi/6)=\tfrac{1}{2}$, that angle is outside the range of arcsin. The angle in the chosen interval with sine $\tfrac{1}{2}$ is $\pi/6$, so $\arcsin(\tfrac{1}{2})=\pi/6$.



Only $\pi/6$ lies in the arcsine output interval at input 1/2.

Check. Sine is strictly increasing on the chosen interval, so the returned angle is unique.

Exercise 656 · Difference of logarithms

Printed page 71 · Logarithm rules

Explain why the expression $\log(a) - \log(b)$ should not be confused with $\log(a - b)$. Rewrite $\log(a) - \log(b)$ in an equivalent logarithmic form.

1. Express each input as a power

Fix a base B>0 with B≠1 so the logarithm is defined as the inverse of an exponential. The original log a and log b each require a positive input.

1 **u = log_B a ⇔ Bᵘ = a**
Write a as a power of the common base.

2 **v = log_B b ⇔ Bᵛ = b**
Write b as a power of that same base.

3 **a/b = Bᵘ/Bᵛ = B^(u−v)**
Divide by b>0. The quotient of like bases subtracts exponents.

2. Take the log of that quotient

The quotient a/b is positive whenever a and b are positive, so its logarithm exists on the entire original domain.

1 **log_B(a/b) = log_B(B^(u−v))**
Take log base B of the positive quotient.

2 **log_B(a/b) = u − v**
Undo the exponential.

3 **log_B a − log_B b = log_B(a/b)**
Replace u and v by their original names.

3. Test subtraction inside the argument

Putting a difference inside the logarithm additionally requires a>b. Even when both expressions exist they can have different values.

1 **a = 8, b = 2**
Choose positive inputs for which 8−2 is also positive.

2 **log_B 8 − log_B 2 = log_B(8/2) = log_B 4**
The proved rule uses division.

3 **log_B(8−2) = log_B 6 ≠ log_B 4**
Distinct positive inputs have distinct logarithms.

4 **a = b > 0 ⇒ log_B a − log_B b = 0**
Here a−b=0, so the proposed log of the difference is undefined.

Check. For a=b the difference equals 0 while log(a−b) is undefined.

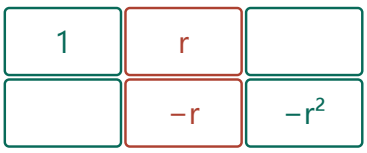
Exercise 657 · Telescoping geometric products

Printed page 71 · Geometric sequences and series / Algebraic manipulation

Multiply each of the following by $1 - r$:
(a) $1 + r$ (b) $1 + r + r^2$ (c) $1 + r + r^2 + \cdots + r^{1995}$

1. Distribute in the small example (a)

Multiplying 1+r by 1−r means subtracting r(1+r) from 1+r. The middle terms cancel: $1+r-r-r^2=1-r^2$.



When like powers align, the r terms cancel.

2. Watch the same cancellation in (b)
Subtract $r+r^2+r^3$ from $1+r+r^2$. The result is $1-r^3$.

1	r	r^2	
	$-r$	$-r^2$	$-r^3$

A longer product has two canceling interior columns.

3. Track just the endpoints in (c)
The unshifted powers run from 0 to 1995; the shifted powers from 1 to 1996. Every shared power cancels, leaving $1-r^{1996}$.

1	r	…	r^{1995}	
	$-r$	…	$-r^{1995}$	$-r^{1996}$

Only the two end columns survive in the large product.

1. $S=1+r+\cdots+r^{1995}$; $rS=r+r^2+\cdots+r^{1996}$
Name the long sum and multiply each term by r.

2. $(1-r)S=S-rS=1-r^{1996}$
Every power from r through r^{1995} appears once with each sign. No division is involved, so $r=1$ is allowed.

Check. At $r=1$ the original product and each simplified answer are zero.

Exercise 658 · Evaluate finite geometric sums

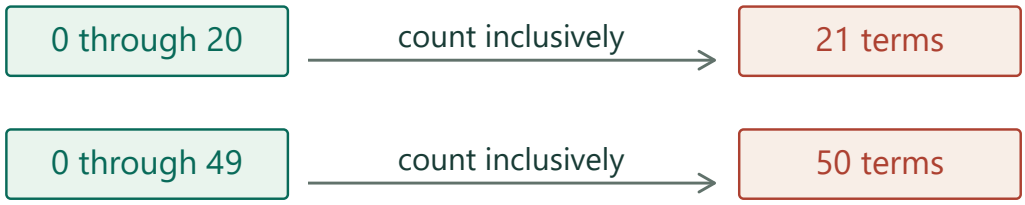
Printed page 72 · Geometric sequences and series

Find sums for the following geometric series:

(a) $3000 + 3150 + \cdots + 3000(1.05)^{20}$ **(b)** $18 - 12 + 8 - \cdots + 18(-2/3)^{49}$

1. Identify first terms, ratios and counts

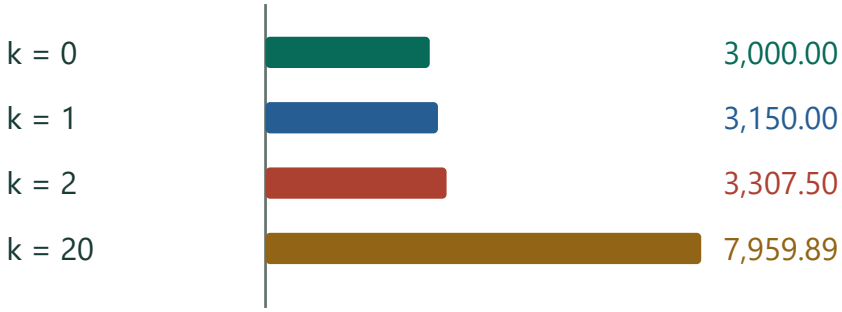
For a geometric sum starting at a with ratio r, n terms sum to $a(1-r^n)/(1-r)$. Exponents 0 through 20 give 21 terms in (a), and 0 through 49 give 50 in (b).



The exponent in the geometric-sum formula is the term count.

2. Use the positive ratio (a)

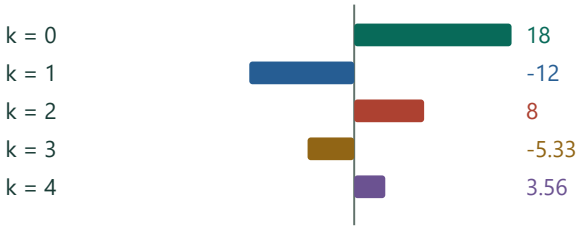
The first term is 3000 and 3150/3000=1.05. Substitution gives $3000(1.05^{21}-1)/.05$.



A positive ratio above 1 makes the terms grow.

3. Use the negative ratio (b)

Now a=18 and r=−2/3. The exponent 50 is even, so the result is $18[1-(2/3)^{50}]/(1+2/3)=(54/5)[1-(2/3)^{50}]$.



The negative ratio alternates signs and shrinks magnitudes.

1. $n=50$, $r^{50}=(-2/3)^{50}=(2/3)^{50}$
An even number of negative factors gives a positive power.

2. $18/(1-r)=18/(1+2/3)=18\cdot3/5=54/5$
Simplify the constant multiplying the finite-sum numerator.

Check. The negative-ratio sum lies slightly below its infinite limit 54/5 because the number of terms is even.

Exercise 659 · Sigma notation practice

Printed page 72 · Geometric sequences and series / Sigma notation

(Continuation) It is often convenient to use what is called *sigma notation* to describe a series. For example, #658(a) and #658(b) can be described by

$$\sum_{k=0}^{20} 3000(1.05)^k, \text{ and either } \sum_{n=1}^{50} 18\left(-\frac{2}{3}\right)^{n-1} \text{ or } \sum_{p=0}^{49} 18\left(-\frac{2}{3}\right)^{p},$$

respectively. The symbol Σ is the Greek letter *sigma*, which corresponds to the English S (for *sum*). Evaluate the series (a) and (b), then express (c) and (d) using sigma notation:

(a) $\sum_{k=0}^{9} 12\left(\frac{3}{5}\right)^k$ **(b)** $\sum_{n=4}^{7} 5n$ **(c)** $1+4+9+\cdots+361$ **(d)** $8+4+2+\cdots+\frac{1}{4}+\frac{1}{8}$

Notice that some of these series are *not* geometric. By the way, calculators have the built-in capacity to evaluate sigma notation.

1. Read the bounds inclusively (a)

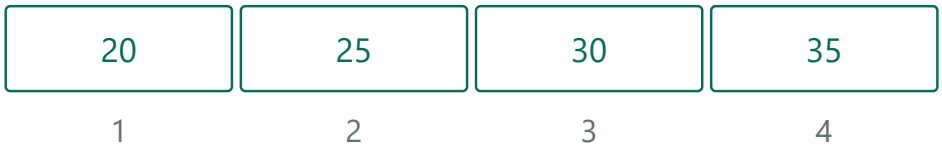
k=0 to 9 gives ten terms. With a=12 and r=3/5 the sum is $12[1-(.6)^{10}]/.4=30[1-(.6)^{10}]$. The notation instructs you to substitute k=0,1,...,9 and add. Writing the first terms as 12+7.2+4.32 shows that each is 3/5 of the previous term. In the finite-sum formula the exponent 10 counts terms, including the k=0 term.



The first index is zero, so there are ten terms.

2. Evaluate the ordinary sum (b)

Insert n=4,5,6,7 into 5n. The result is 5(4+5+6+7)=110; the terms have a common difference. Writing them out gives 20+25+30+35. Their differences are 5 but their ratios differ, so the geometric-series formula from (a) would not apply.



Evaluate 5n at the four integers from 4 through 7.

3. Describe the squares (c)

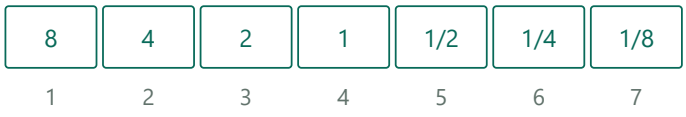
The pattern is $1^2,2^2,3^2,\ldots,19^2$, since $361=19^2$. Its sigma form is $\Sigma\, k^2$ for $k=1,\ldots,19$.



The final square 361 fixes the upper index.

4. Describe the halvings (d)

Start at 8 and multiply by ½ each time. Because $8(\tfrac{1}{2})^6=\tfrac{1}{8}$, write $\Sigma\, 8(\tfrac{1}{2})^k$ for $k=0,\ldots,6$. Checking the last two substitutions gives $8/32=1/4$ and $8/64=1/8$, exactly the two endpoint terms shown in the question.



1. $8(1/2)^k=1/8 \Rightarrow (1/2)^k=1/64$
Use the stated final term to locate its exponent.

2. $1/64=(1/2)^6 \Rightarrow k=6$
The exponents run from 0 through 6, giving seven terms.

The final halving uses exponent 6.

Check. The last terms are $19^2=361$ and $8/64=1/8$.

Exercise 660 · Fifty bounces and beyond

Printed page 72 · Geometric sequences and series / Limits and convergence

The rebound ratio of a speckled green superball is 75%. It is dropped from a height of 16 feet. Consider the instant when the ball strikes the ground for the fiftieth time.

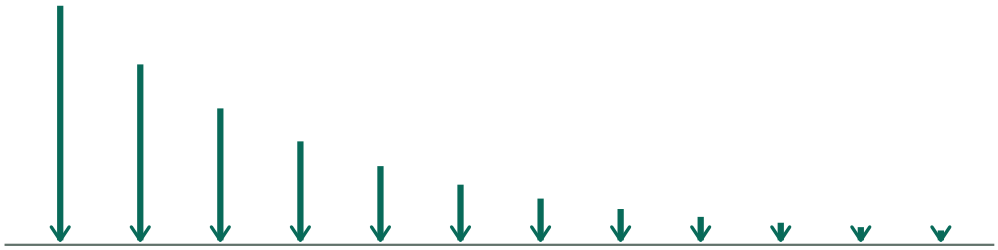
(a) How far downward has the ball traveled at this instant?

(b) How far (upward and downward) has the ball traveled at this instant?

(c) How far would the ball travel if you just let it bounce and bounce and bounce . . . ?

1. Count the down legs

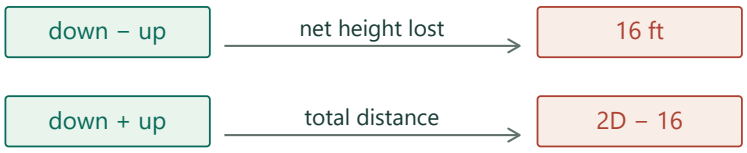
At impact 50 the first fall of 16 ft has been followed by 49 more descents. They form a series with exponents 0 through 49. Sum to $D=16[1-(.75)^{50}]/.25=64[1-(.75)^{50}]$ ft.



The first twelve of fifty downward legs show the geometric pattern.

2. Recover the upward distance from position

The net drop from release to the ground is 16 ft. Therefore $D-U=16$ and the total traveled is $D+U=2D-16$ ft.



1. $D-U=16 \Rightarrow U=D-16$
The total downward movement exceeds the upward movement by the net drop.

2. $D+U=D+(D-16)=2D-16$
Add lengths of all legs without cancelling opposite directions.

Displacement and total distance combine legs differently.

3. Pass to indefinitely many bounces

For impact n the same formula uses $(.75)^n$. This tends to zero, so all downward travel tends to 64 ft, upward travel to 48 ft, and total travel to 112 ft.



The infinite sums account for the initial 16-ft drop.

Check. The infinite upward sum is $12/(1-.75)=48$ ft and downward sum is 64 ft.

Exercise 661 · Four names for the same sum

Printed page 72 · Sigma notation / Geometric sequences and series

(Continuation) Explain why the following are equivalent.

$$\textbf{(a)}\ \sum_{n=0}^{49} 16(0.75)^n \quad \textbf{(b)}\ \sum_{n=1}^{50} 16(0.75)^{n-1} \quad \textbf{(c)}\ 16+\sum_{n=1}^{49} 12(0.75)^{n-1} \quad \textbf{(d)}\ 16+\sum_{n=0}^{48} 12(0.75)^n$$

1. Compare exponent lists

In (a) the exponent n runs from 0 to 49. In (b) n runs from 1 to 50 but the exponent n−1 still runs from 0 to 49. They list exactly the same 50 terms.

n = 0	1	2	…	49
n = 1	2	3	…	50

Shifting the starting index also shifts the exponent by one.

2. Separate the initial 16 ft

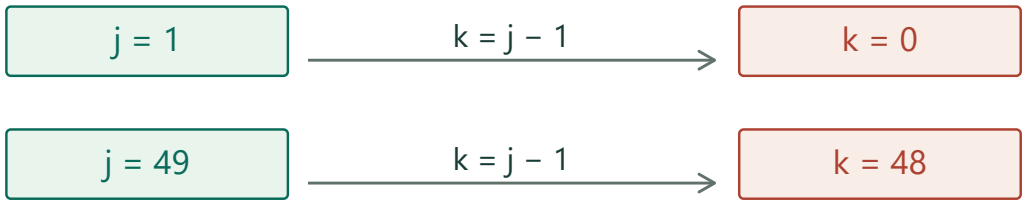
The remaining terms start at 16·.75=12. Rewriting $16(.75)^j$ as 12(.75)^(j−1) for j=1,…,49 gives (c). One way to see the accounting is to write the list in both forms: 16, 16·.75, $16\cdot.75^2$,… is also 16, 12, 12·.75,… . The separate 16 is a single term, so there are 49 more terms after it.

16	12	9	…	last
1	2	3	4	5

Separate the first leg before reindexing the remaining 49.

3. Shift the index once again

Let k=j−1; the range becomes 0 through 48 and the last part becomes $\Sigma 12(.75)^k$ as in (d). The order and value of every summand remain the same.



The new limits enumerate the same tail.

Check. Each list begins 16,12,9 and contains fifty entries.

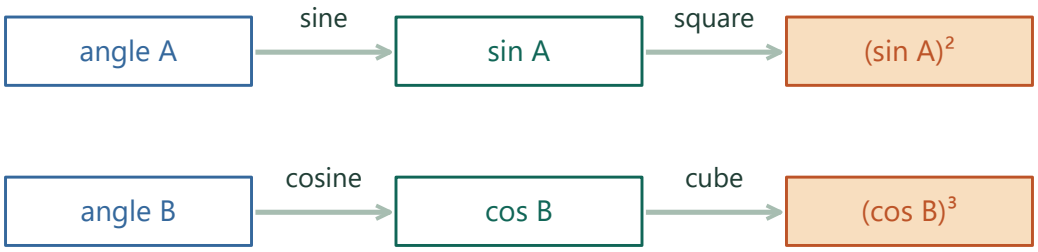
Exercise 662 · Trig powers and inverse notation

Printed page 72 · Inverse trigonometric functions / Exponent rules

In most mathematics books, the notation $\sin^2 A$ is often used in place of the clearer $(\sin A)^2$, or $\cos^3 B$ in place of $(\cos B)^3$. Why do you think that writers of mathematics fell into this strange habit? It is unfortunate that this notation is inconsistent with notation commonly used for inverse functions. Explain.

1. Decide what is squared

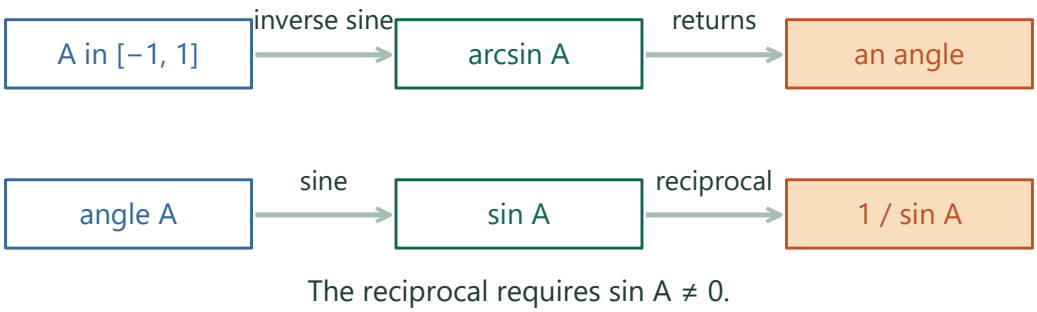
$\sin^2 A$ groups the operation as $(\sin A)^2$. The exponent sits on the function name to avoid the interpretation $\sin(A^2)$ and dispense with parentheses. The same convention gives $\cos^3 B=(\cos B)^3$.



The power acts on the trigonometric output; follow the arrows to see the grouping.

2. Distinguish the inverse convention

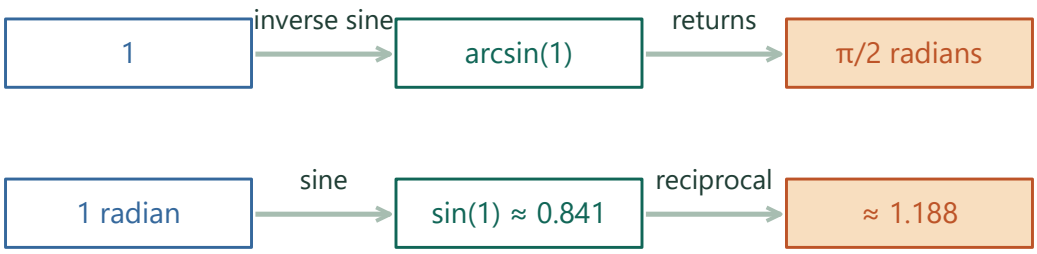
By convention $\sin^{-1}A$ names an inverse function: the angle arcsin A in its principal range. To take the reciprocal of the sine value write $(\sin A)^{-1}=1/\sin A=\csc A$.



Inverse sine returns an angle. The reciprocal instead requires a nonzero sine and returns a number.

3. Try a shared input

At A=1, arcsin(1)=π/2 whereas 1/sin(1 radian) is about 1.188. The two placements call for different operations.



The displayed inputs share the numeral 1. One is a sine value and the other is a radian measure, which leads to different results.

Check. At A=1 (radians), arcsin(1)=π/2 and 1/sin(1) differs.

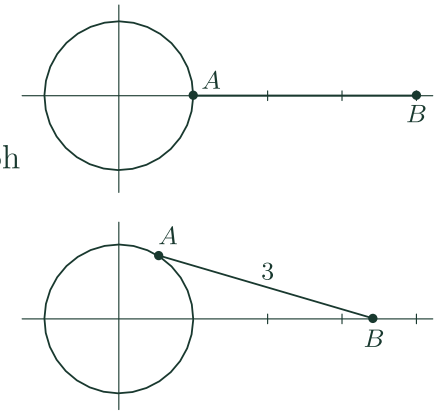
Exercise 663 · A crank drives a slider

Printed page 73 · Circular motion / Parametric equations / Periodic functions

A wheel of radius 1 meter is centered at the origin, and a rod AB of length 3 meters is attached at A to the rim of the wheel. The wheel is turning counterclockwise, one rotation every 4 seconds, and, as it turns, the other end $B = (b, 0)$ of the rod is constrained to slide back and forth along a segment of the x-axis. Given any time t seconds, the position of B is determined. This functional relationship is expressed by writing $b = f(t)$. The top figure shows this apparatus when $t = 0$ and $b = 4$, and the bottom figure corresponds to $t = 0.64$.

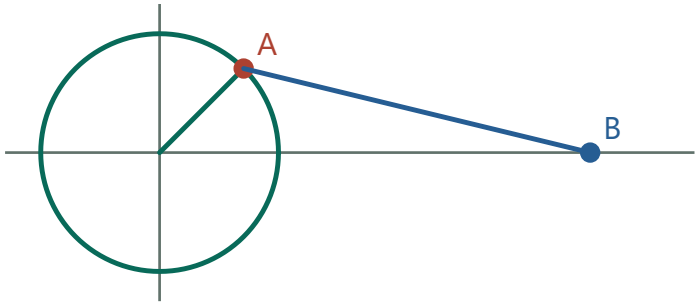
(a) Calculate $f(0)$, $f(1)$, $f(2)$, and $f(11)$. Calculate $f(t)$ for a t-value that you and *only* you choose.

(b) Explain why f is a *periodic* function of t, and sketch a graph of $b = f(t)$ for $0 \le t \le 8$. What is the range of values of f?

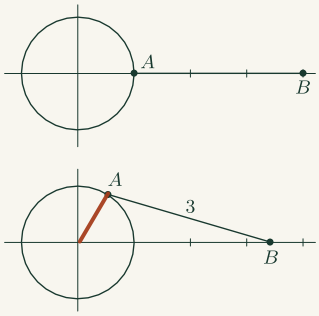


1. Locate the rotating end A

Each four seconds adds 2π radians, so θ=πt/2. On the unit wheel A=(cos θ,sin θ). B=(b,0) stays on the horizontal axis and to the right of A.



The rotating pin A drives B along the horizontal guide.



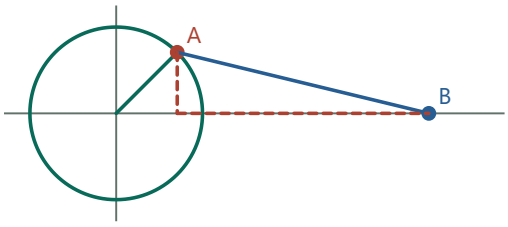
Rotating radius

In the lower drawing the wheel radius goes from its hub to the rod pin A. The pin's vertical coordinate changes as the rim turns.

2. Use the rigid length

The horizontal rod component is $b-\cos\theta$ and the vertical component has magnitude $|\sin\theta|$. Thus $(b-\cos\theta)^2+\sin^2\theta=9$. Take the positive horizontal root to obtain $b=\cos\theta+\sqrt{(9-\sin^2\theta)}$.

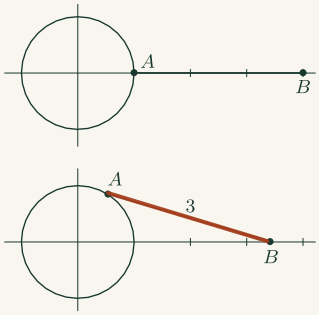
Crank time · 0.5 s



The rod is the 3 m hypotenuse of the component triangle.

1. $(b-\cos\theta)^2=9-\sin^2\theta$
Subtract the squared vertical leg from the squared rod length.

2. $b-\cos\theta=+\sqrt{(9-\sin^2\theta)} \Rightarrow b=\cos\theta+\sqrt{(9-\sin^2\theta)}$
B is right of A, so the horizontal displacement is positive.

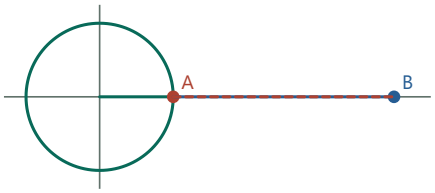


Connecting rod

The lower sloping member joins A to the slider B. Its length is fixed while its horizontal projection changes.

Compare settings · Crank time

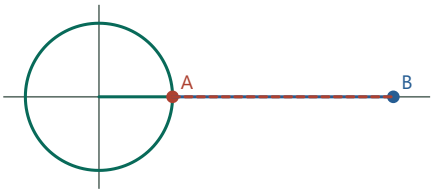
0 s



At t = 0 s the rod's horizontal component is 3.000 m.

The vertical component of the rod has magnitude 0.000 m. Its horizontal projection is 3.000 m, putting B at x = 4.000 m.

4 s



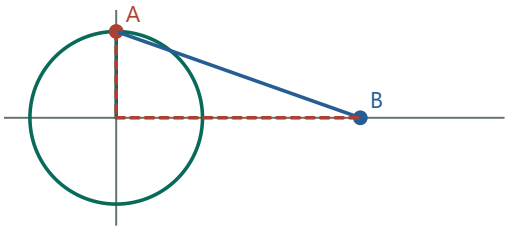
At t = 4 s the rod's horizontal component is 3.000 m.

The vertical component of the rod has magnitude 0.000 m. Its horizontal projection is 3.000 m, putting B at x = 4.000 m.

3. Evaluate specified times

At $t=0,1,2,11$, the positions are $4,\sqrt{8},2,\sqrt{8}$ m. For an additional choice $t=.5$ gives $\theta=\pi/4$ and $b=(1+\sqrt{17})/\sqrt{2}$ m.

Crank time · 1 s



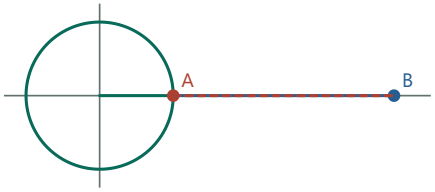
At t = 1, A is above the hub and B is $\sqrt{8}$ m from it.

1. $\theta=\pi/4 \Rightarrow b=1/\sqrt{2}+\sqrt{(9-1/2)}$
At half a second, sine and cosine both equal $1/\sqrt{2}$.

2. $\sqrt{(17/2)}=\sqrt{17}/\sqrt{2} \Rightarrow b=(1+\sqrt{17})/\sqrt{2}$
Combine terms over their common denominator.

Compare settings · Crank time

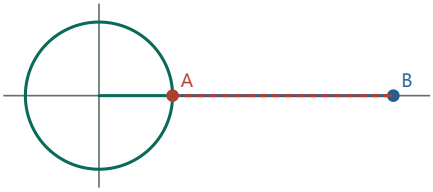
0 s



At t = 0 s the rod's horizontal component is 3.000 m.

The vertical component of the rod has magnitude 0.000 m. Its horizontal projection is 3.000 m, putting B at x = 4.000 m.

4 s

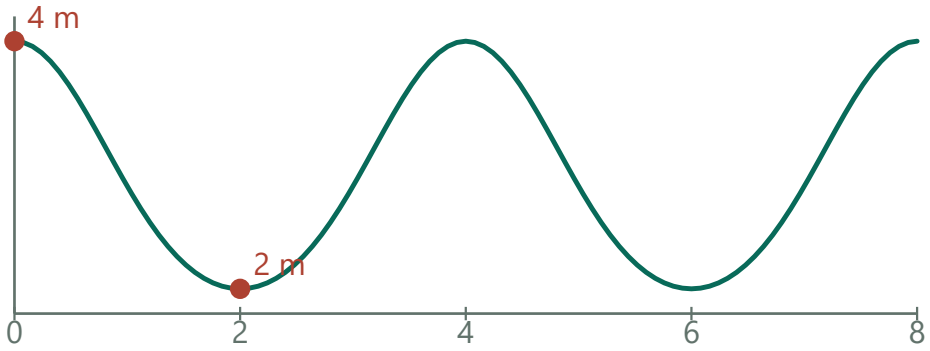


At t = 4 s the rod's horizontal component is 3.000 m.

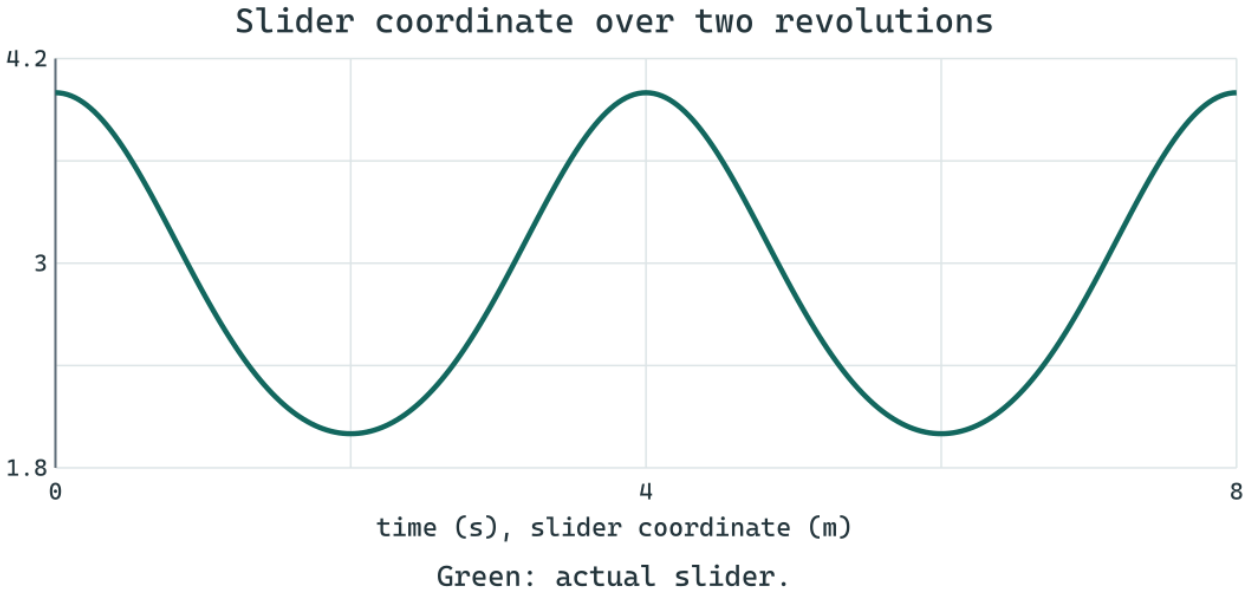
The vertical component of the rod has magnitude 0.000 m. Its horizontal projection is 3.000 m, putting B at x = 4.000 m.

4. Draw the 8-second record

The mechanism repeats after each full 4-second turn. The distance from hub to B lies between $3-1$ and $3+1$ by the triangle inequalities, both attained for horizontal wheel radii. Draw two smooth cycles from 4 down to 2 and back on $0 \le t \le 8$.



Each four-second cycle repeats the same horizontal motion.



Check. At any listed time $(b-\cos\theta)^2+\sin^2\theta=9$ verifies the rod length.

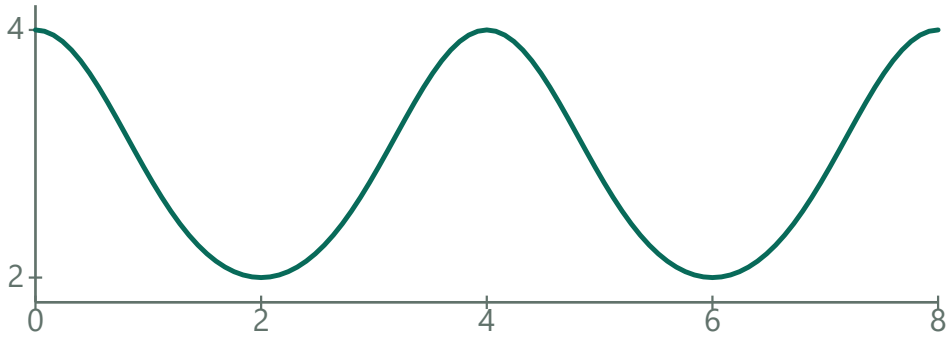
Exercise 664 · Slider versus a sinusoid

Printed page 73 · Periodic functions / Sinusoidal functions / Circular motion

(Continuation) Calculate a general formula for $f(t)$, and use it to refine your sketch.
(a) Solve the equation $f(t) = 3$.
(b) In degree mode, compare the graph of $g(t) = 3 + \cos(90t)$ with the graph of f.
(c) Do you think that the graph of the periodic function $f(t)$ is *sinusoidal*? Explain.
(d) Why was this particular $g(t)$ chosen?

1. Write the slider as a function

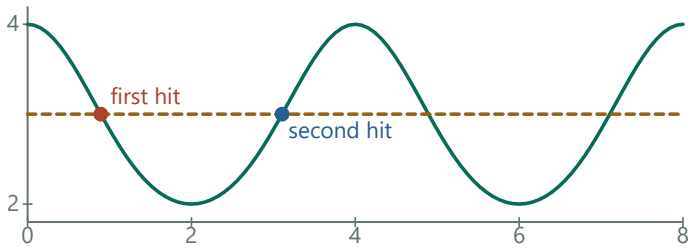
At θ=πt/2 the rod triangle has squared horizontal length 9−sin²θ, so f(t)=cos θ+√(9−sin²θ). This refines intermediate heights between the extrema.



The fixed rod length determines the green slider curve.

2. Find all center-coordinate crossings (a)

Set b=3 in the original length constraint: (3−cos θ)²+sin²θ=9. Hence 10−6cos θ=9, or cos θ=1/6. The solutions are t=4k±(2/π)arccos(1/6), k any integer, restricted to the time interval of interest.



The two crossings in each turn have cos θ = 1/6.

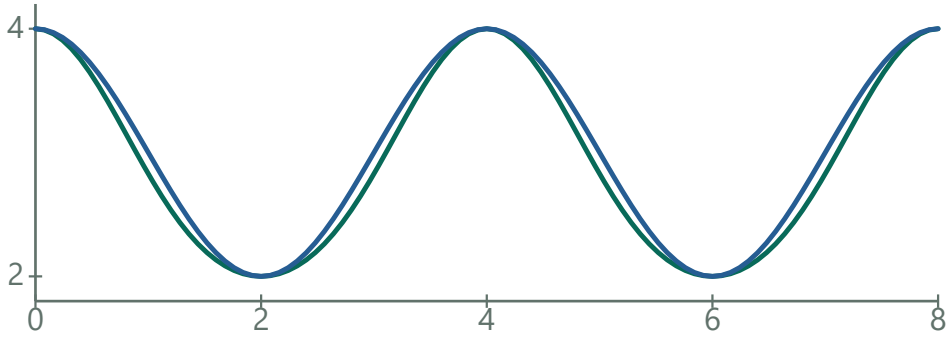
1. 9−6cos θ+cos²θ+sin²θ=9 ⇒ 10−6cos θ=9
Expand the rod constraint and use sin²θ+cos²θ=1.

2. cos θ=1/6 ⇒ θ=2πk±arccos(1/6)
Cosine repeats after 2π and is even.

3. t=2θ/π=4k±(2/π)arccos(1/6)
Undo θ=πt/2 to express both branches in seconds.

3. Compare with the suggested cosine (b,d)

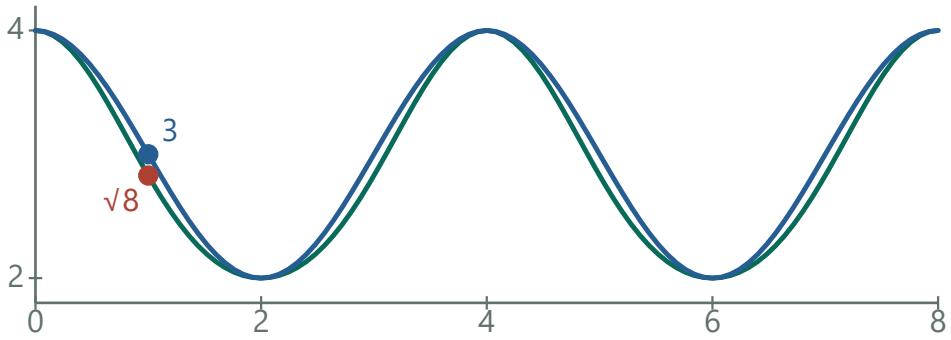
In degrees g(t)=3+cos(90t). Its baseline 3, amplitude 1 and period 4 place its extrema at the same heights and times as f. The horizontal rod reach $\sqrt{(9-\sin^2\theta)}$ is at most 3, so the actual curve lies at or below g, touching it at sin θ=0.



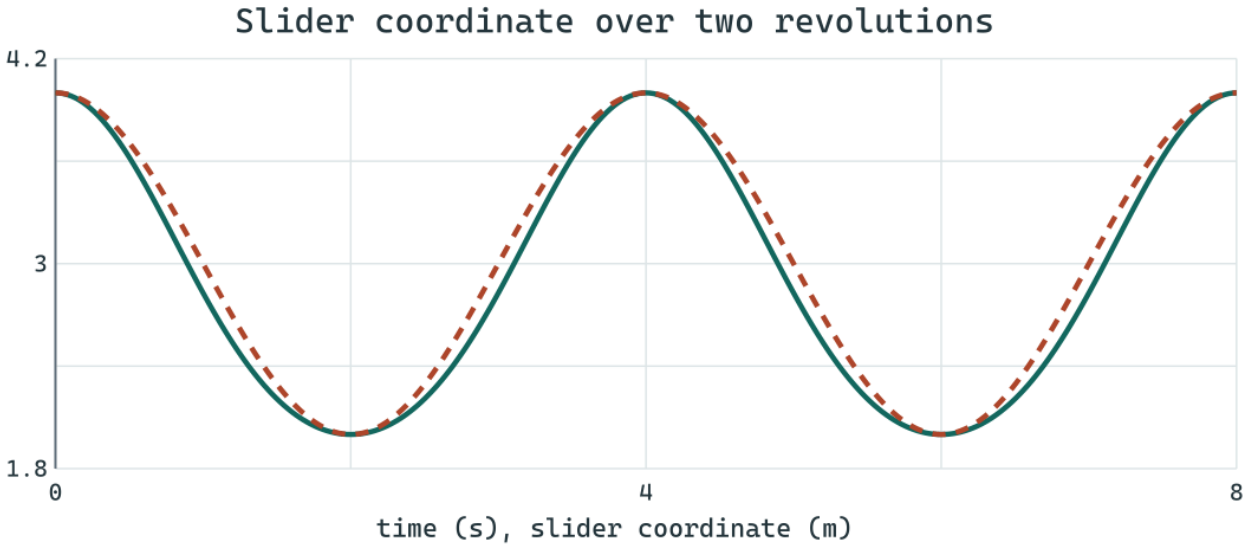
The cosine model meets the actual motion at its extrema.

4. Decide whether f is sinusoidal (c)

Any sinusoid with these extrema and times must reach midline 3 a quarter cycle after its peak. At t=1 the slider reads √8, so its full graph cannot be that sinusoid.



At the quarter cycle the actual position lies below 3.



Green: actual slider. Dashed orange: comparison cosine.

Check. Substitute either family from (a) into the original unsquared length equation.

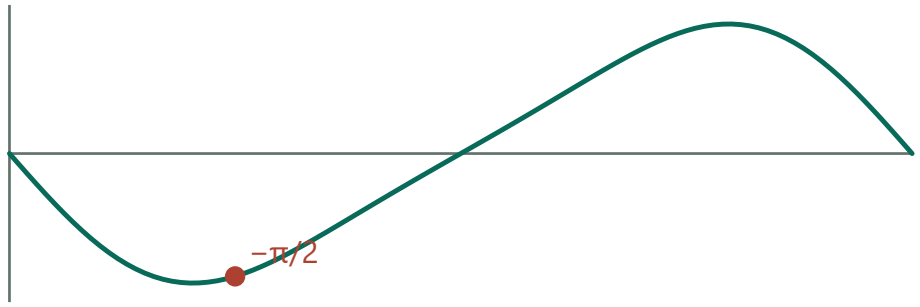
Exercise 665 · When is the slider fastest?

Printed page 73 · Circular motion / Rates of change / Optimization

(Continuation) What is the *velocity* of B when $t = 1$? Find an instant t of time when you think B is moving most rapidly. Explain your choice.

1. Differentiate the horizontal position

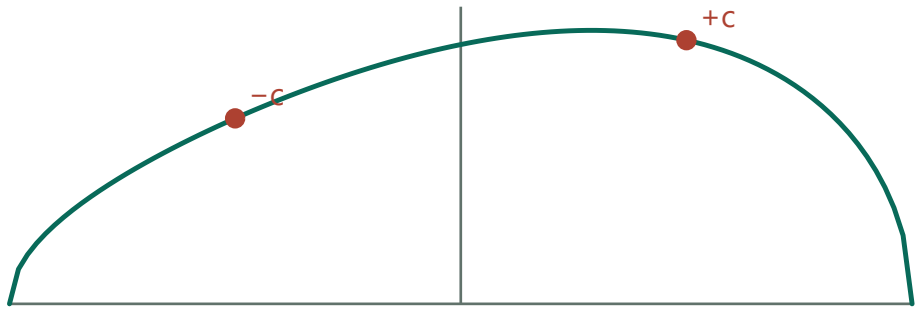
Put $\theta=\pi t/2$ and $c=\cos\theta$. Then $b=c+\sqrt{(8+c^2)}$. By the chain rule $v=(1+c/\sqrt{(8+c^2)})(-\sin\theta)(\pi/2)$. At $t=1$, $c=0$ and $\sin\theta=1$, giving $v=-\pi/2$ m/s, toward the left.



Velocity is negative when the slider moves left.

2. Compare speeds irrespective of direction

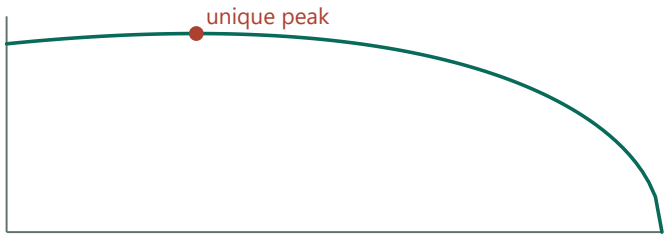
Let $q=\sqrt{(8+c^2)}$. Speed is $(\pi/2)\sqrt{(1-c^2)}(1+c/q)$. For $\pm c$ of the same magnitude, the positive c gives greater speed except at zero-speed endpoints. It suffices to maximize on $0\le c\le 1$.



At matched magnitudes, positive cosine gives the greater speed.

3. Locate the unique peak

The derivative of log(speed) on $0 \le c < 1$ is $-c/(1-c^2)+(q-c)/q^2$. Its first term strictly decreases, and its second is $8/[q^2(q+c)]$, also decreasing. The sum starts positive and tends to $-\infty$, so it crosses zero only once and locates the global maximum.



Maximizing speed over $0 \le c \le 1$ isolates the global peak.

1. $\ln(\text{speed}) = \ln(\pi/2) + (1/2)\ln(1-c^2) + \ln(q+c) - \ln q$
Use $q = \sqrt{(8+c^2)} > 0$ and restrict to $c < 1$ so speed is positive.

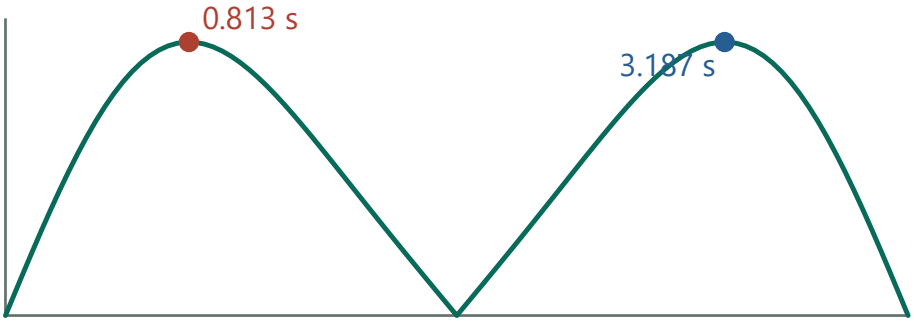
2. $q' = c/q \Rightarrow (q'+1)/(q+c) = (c+q)/[q(q+c)] = 1/q$
Differentiate $\ln(q+c)$ and simplify.

3. $(\ln \text{speed})' = -c/(1-c^2) + 1/q - c/q^2$
Differentiate the other two varying terms and combine.

4. $1/q - c/q^2 = (q-c)/q^2 = 8/[q^2(q+c)]$
Multiply by $(q+c)/(q+c)$ and use $q^2 - c^2 = 8$.

4. Convert back to times

Bisection brackets c between .28944 and .28945, with $c \approx .289445$. In one revolution the two angles with that cosine give $t \approx .813059$ s and 3.186941 s. Both have speed about 1.656624 m/s, with opposite velocity signs, repeated every four seconds.



Two positions in the turn give the same maximum speed.

Check. The sign test brackets the root between c=.28944 and .28945. Direct substitution in the velocity gives opposite signs but the same speed at the two reported times.

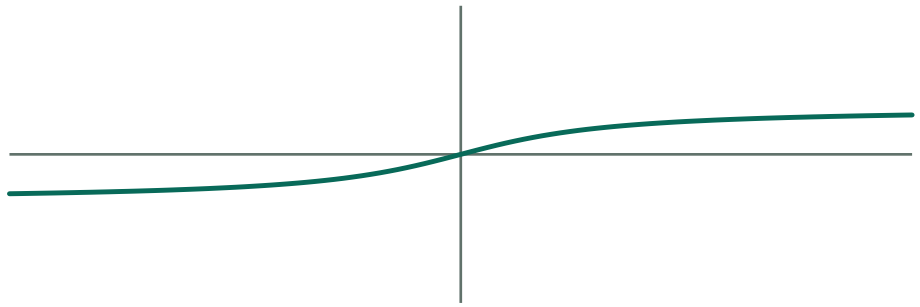
Exercise 666 · Inverse tangent branches

Printed page 73 · Inverse trigonometric functions / The tangent function

With a graphing calculator, draw the graph of $x = \tan y$ in radians, then explain why it is not the same as the graph of $y = \tan^{-1} x$.

1. Draw one branch of the relation

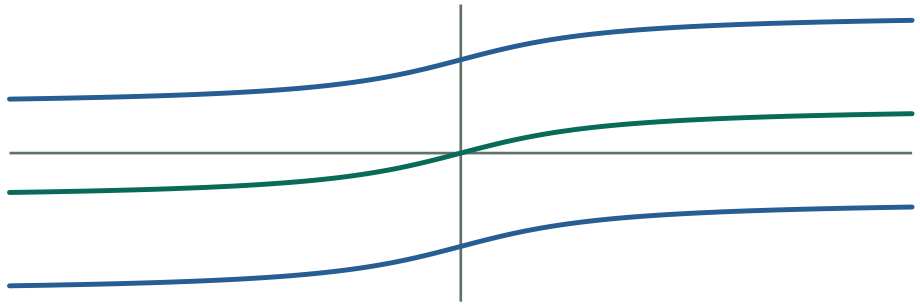
Take y between $-\pi/2$ and $\pi/2$ and compute $x=\tan y$. As y increases, x runs from arbitrarily negative to arbitrarily positive through zero. This branch is $y=\arctan x$.



Swap the central tangent branch to obtain arctangent.

2. Recover the other branches

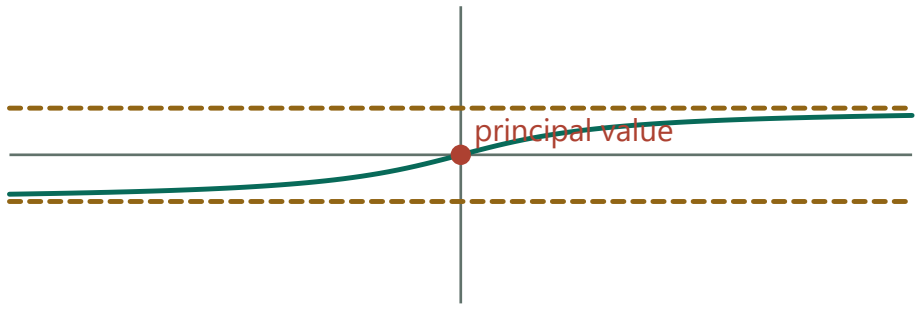
Tangent repeats after π radians. Moving any point of the first branch vertically by $k\pi$ leaves its x-coordinate unchanged, so the relation consists of $y=\arctan x+k\pi$, k any integer.



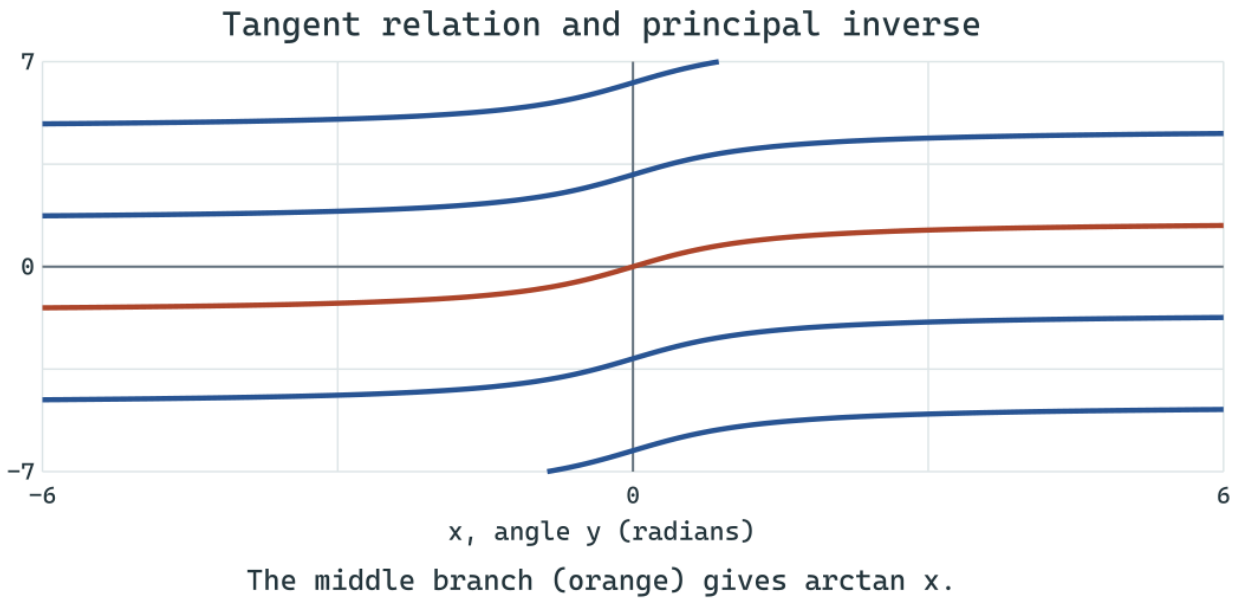
Adding any multiple of π reproduces a branch of $x = \tan y$.

3. Distinguish the inverse function

A vertical line meets every branch. For example $x=0$ permits $y=0,\pi,-\pi,\ldots$. The arctangent function returns just the value in $(-\pi/2,\pi/2)$, giving one angle per x.



Only outputs strictly between $-\pi/2$ and $\pi/2$ are principal.



Check. At x=0 the relation admits every kπ; the function takes just 0.

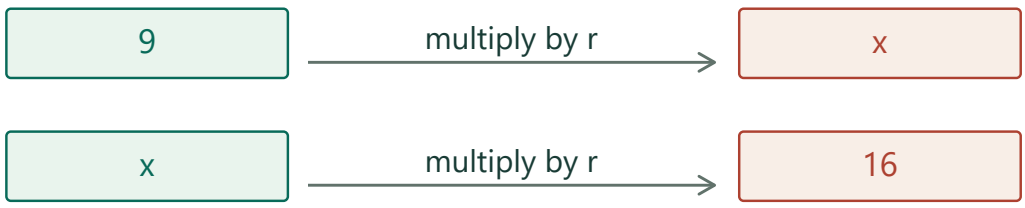
Exercise 667 · Positive geometric mean

Printed page 73 · Geometric sequences and series / Radicals and rational powers

In order that the sequence 9 , x , 16 , $\ldots$ of positive numbers be geometric, what must x be? This value of x is called the *geometric mean of 9 and 16*.

1. Express the same ratio twice

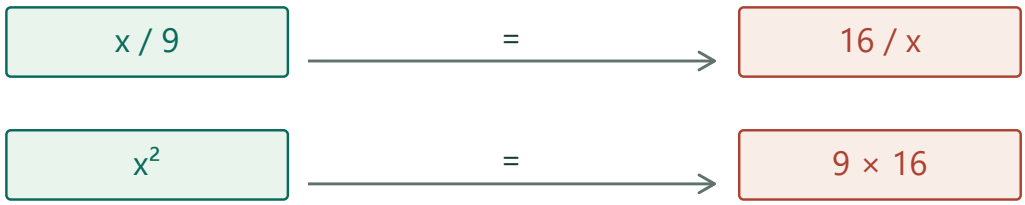
A geometric sequence multiplies each entry by the same number. The first transition has ratio x/9; the second has ratio 16/x.



The middle term links two equal ratios.

2. Solve the condition

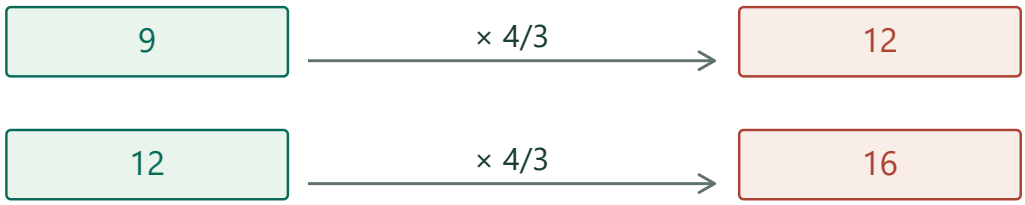
Set x/9=16/x. Positivity permits multiplication by 9x, giving $x^2=144$.



Cross multiplication determines its square.

3. Choose the admissible root

The positive root is x=12. Indeed 12/9=16/12=4/3. This explains why the geometric mean is the square root of the product 9·16.



The positive requirement selects the geometric mean.

Check. Both ratios equal 4/3.

Exercise 668 · Association of matrix products

Printed page 73 · Matrix multiplication / Associativity

Matrix multiplication is a complicated process. You might therefore be surprised by the results of the following two calculations:

(a) $\begin{bmatrix}3 & -2\\ 1 & 5\end{bmatrix}\left(\begin{bmatrix}4 & 1\\ -3 & 2\end{bmatrix}\begin{bmatrix}-2 & 6\\ 0 & 3\end{bmatrix}\right)$ (b) $\left(\begin{bmatrix}3 & -2\\ 1 & 5\end{bmatrix}\begin{bmatrix}4 & 1\\ -3 & 2\end{bmatrix}\right)\begin{bmatrix}-2 & 6\\ 0 & 3\end{bmatrix}$

Test this apparent coincidence on three random square matrices of your own choosing. Is this phenomenon familiar to you?

1. Respect order while grouping

Call the three printed matrices A,B,C in order. Each product entry comes from a row dotted with a column. Parentheses only decide which adjacent pair is computed first.



Keep the factor order fixed and choose only the grouping.

2. Multiply the right pair first (a)

BC has rows $(-8,27),(6,-12)$. For example its upper right entry is $4\cdot6+1\cdot3=27$. Multiplying A(BC) gives rows $(-36,105),(22,-33)$.

−8	27
6	−12

First multiply B and C to get this intermediate matrix.

1. $\text{row}_1(BC)=(4\cdot(-2)+1\cdot0,4\cdot6+1\cdot3)=(-8,27)$
Dot the first row of B with each column of C.

2. $\text{row}_2(BC)=((-3)(-2)+2\cdot0,(-3)\cdot6+2\cdot3)=(6,-12)$
Repeat with the second row of B.

3. $\text{row}_1(A(BC))=(3(-8)-2\cdot6,3\cdot27-2(-12))=(-36,105)$
Keep the new matrix on the right of A.

4. $\text{row}_2(A(BC))=(-8+5\cdot6,27+5(-12))=(22,-33)$
Multiply the second row of A by those same columns.

3. Multiply the left pair first (b)

AB has rows $(18,-1),(-11,11)$. Now (AB)C also gives rows $(-36,105),(22,-33)$, although the intermediate matrices differ.

18	−1
−11	11

Multiplying A and B first gives a different intermediate matrix.

1. $\text{row}_1(AB)=(3\cdot4-2(-3),3\cdot1-2\cdot2)=(18,-1)$
First multiply A and B, keeping their order.

2. $\text{row}_2(AB)=(4+5(-3),1+5\cdot2)=(-11,11)$
Obtain the other row of the intermediate matrix.

3. $\text{row}_1((AB)C)=(18(-2)-1\cdot0,18\cdot6-1\cdot3)=(-36,105)$
Dot the first new row with the original columns of C.

4. $\text{row}_2((AB)C)=((-11)(-2)+11\cdot0,(-11)\cdot6+11\cdot3)=(22,-33)$
Both entries match the second row from the other grouping.

4. Test and explain the rule

For another choice take A=diag(2,3), B with rows (1,1),(0,1), C=diag(4,5). Both orders of grouping give rows (8,10),(0,15). This is associativity: expanding entry (i,j) in either grouping sums the same products $A_{ik}B_{kl}C_{lj}$ over k,l.

8	10
0	15

For the second test both groupings give this matrix.

Check. Both groupings have entry (i,j) equal to the same double sum $\Sigma_k\Sigma_l\, A_{ik}B_{kl}C_{lj}$. This is associativity.

Exercise 669 · Quarter-cycle sine shift

Printed page 73 · Sinusoidal functions / Phase shifts

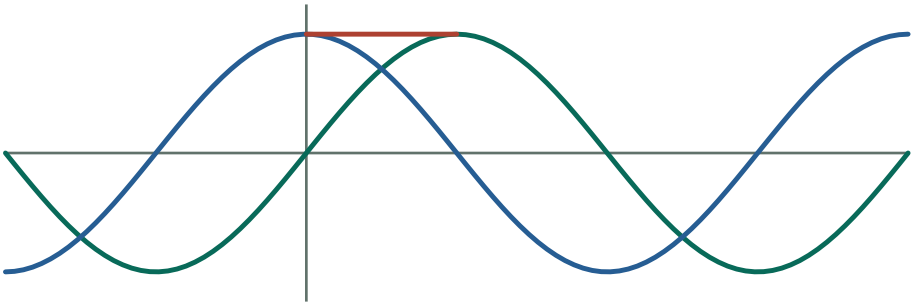
Often trigonometric functions can be simplified.

(a) Working in degree-mode, graph $y = \sin(x + 90)$. On the basis of your graph, suggest a simpler expression that is equivalent to $\sin(x + 90)$.

(b) Write the radian-mode version of (a) and answer it.

1. Track the quarter-cycle shift

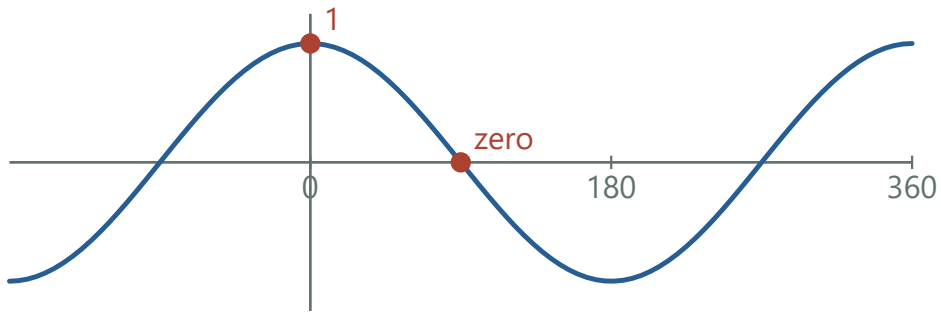
Adding 90° inside sine shifts its graph left by a quarter-turn. The new curve begins at height 1 when x=0°, then crosses zero at x=90°.



The sine peak moves left by 90° to the cosine peak.

2. Confirm the identity in degrees

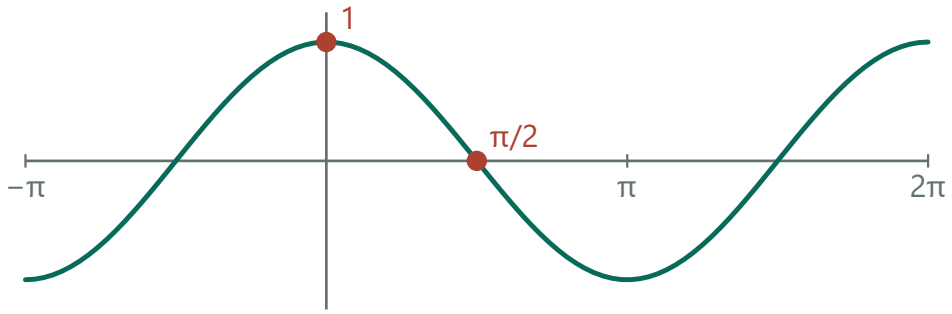
The sine addition formula gives sin(x+90°)=sin x cos90°+cos x sin90°=cos x. Thus the graph ceases to need the phase shift in its formula.



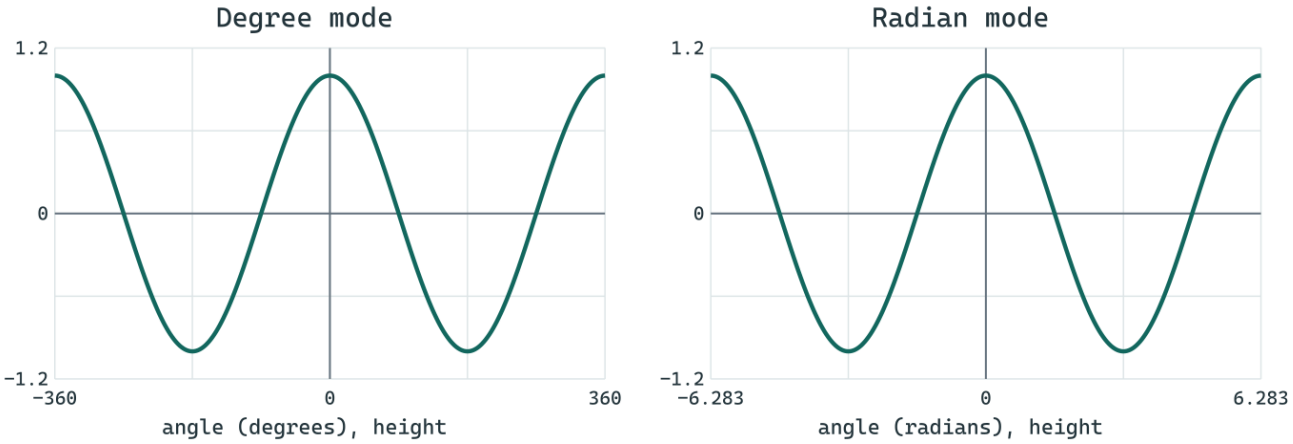
In degrees sin(x + 90°) and cos x coincide.

3. Change units consistently

A quarter-turn is π/2 radians. With radians for both input and shift the statement is sin(x+π/2)=cos x.



In radians the identical shift is π/2.



Each shifted sine coincides with the cosine in its respective units.

Check. The addition identity reduces to sin x·0+cos x·1.

Exercise 670 · Discount lottery payments

Assume that N.H. Lottery commissioners invest their revenue at 8% annual interest.
(a) To make a \$50000 payment in one year, how much money should they invest now?
(b) To make a \$50000 payment in four years, how much money should they invest now?
(c) What is the present value of a \$500000 prize in ten equal annual payments, the first of which is due immediately?

1. Discount one payment by its waiting time

With 8% annual compounding, today's P becomes $P(1.08)^j$ after j years. To fund a later \$50,000 payment, use $P=50000/(1.08)^j$.



Discount by undoing the annual 8% multiplier.

2. Compute the two single-payment amounts

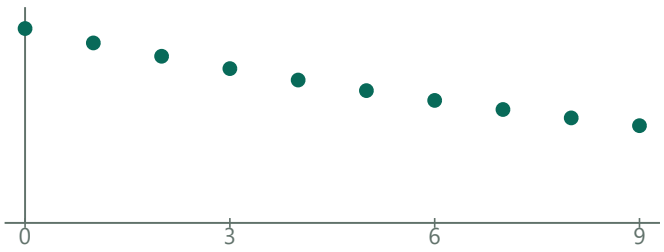
For one year the amount is \$50,000/1.08≈\$46,296.30. For four years it is $\$50,000/1.08^4 \approx \$36,751.49$.



The same future payment has lower value when it is farther away.

3. Place the installment plan on a timeline

A $500,000 prize in ten equal installments pays $50,000 each at years j=0 through 9. Add their present values: 50000Σ(1.08)^(−j). With r=1/1.08, this is $50000(1-r^{10})/(1-r)\approx\$362{,}344.40$.



The immediate installment is at year 0; the last is at year 9.

1. $P=50000(1+r+\cdots+r^9)$, $r=1/1.08$
The immediate installment has exponent 0; the other nine are discounted.

2. $(1-r)P=50000(1-r^{10})$
Subtract rP to cancel the nine common intermediate powers.

3. $P=50000(1-r^{10})/(1-r)$
Divide by 1−r, which is positive since the discount factor is below one.

Check. The initial payment has discount factor 1; each later payment receives one additional year's discount.

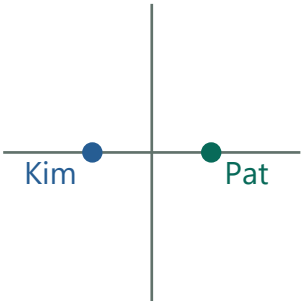
Exercise 671 · Thunder delay as a locus

Printed page 74 · Hyperbolas / Distance and loci

Pat and Kim are talking on the telephone during a thunderstorm. After one of the lightning flashes, Pat hears the rumble of thunder twice — the first sound coming through the open window, and the second sound coming through the telephone ten seconds later. Given that Pat lives two miles east of the center of town, Kim lives two miles west of the center of town, both on the same east-west road, and that sound takes five seconds to travel a mile through air, draw a map that shows some of the places where the lightning could have struck. For example, could the lightning have struck the road on which Pat and Kim live? Assume that light and electricity take no time to reach their destinations.

1. Place the houses on a map

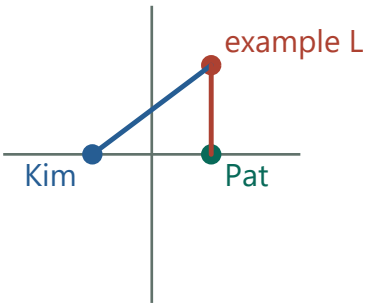
Use town center as origin, east as positive x. Pat is at (2,0) and Kim at (−2,0). The telephone adds negligible delay; the second sound traveled through air to Kim.



The houses sit 4 miles apart on one east–west axis.

2. Convert a time difference to a distance difference

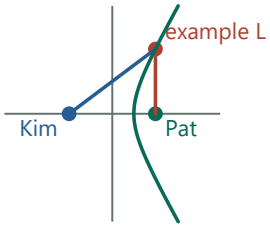
Sound needs 5 seconds per mile. The 10-second delay makes LK−LP=2 miles, with L the strike point. A constant difference of distances from two fixed points defines a hyperbola.



A 10-second delay means one path is 2 miles longer.

3. Draw only the correct branch

The foci are ±2, so c=2. The difference equals 2a=2, hence a=1 and $b^2=c^2-a^2=3$. Draw the right branch of $x^2-y^2/3=1$ with $x\ge1$, closer to Pat.



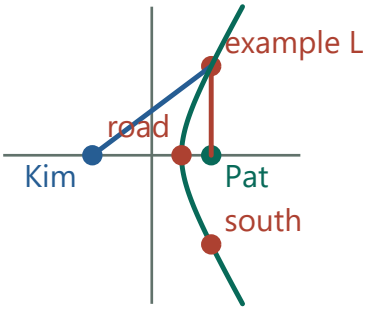
The signed difference selects the Pat-side branch.

1. $2a=2 \Rightarrow a=1$; $c^2=a^2+b^2 \Rightarrow b^2=2^2-1^2=3$
The distance difference is twice the transverse semiaxis; the foci set c.

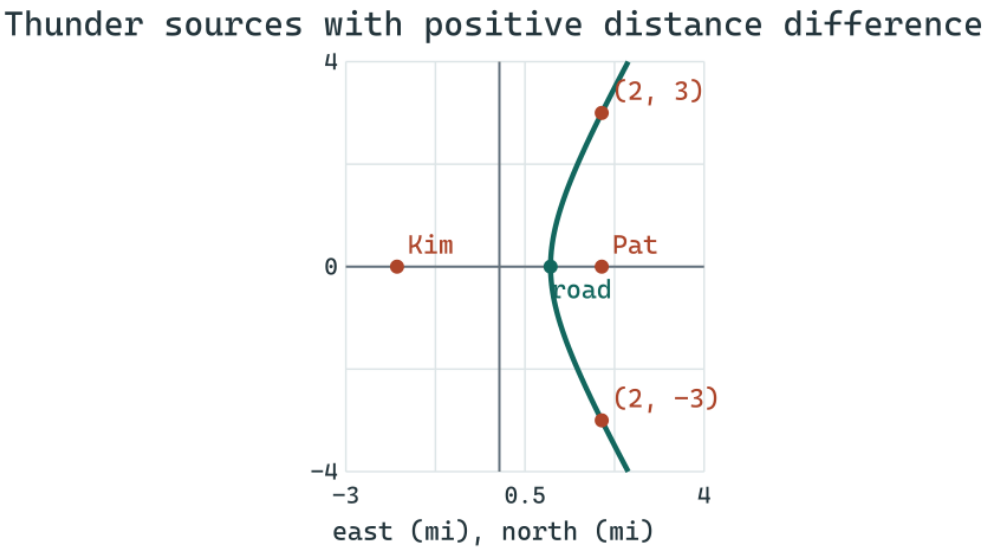
2. $x^2/a^2-y^2/b^2=1 \Rightarrow x^2-y^2/3=1$, $x\ge1$
Use the right branch because Pat, the right focus, is nearer.

4. Put several possible strikes on it

On the road L=(1,0) is 3 miles from Kim and 1 from Pat. Above or below the road, (2,3) and (2,−3) have distances 5 and 3, so each fits the delay.



Road, northern and southern sites all satisfy the same delay.



Check. Swapping west and east would reverse the signed delay, selecting the wrong branch.

Exercise 672 · Domains of inverse and elementary functions

Printed page 74 · Domain and range / Inverse functions

The domain of the function $\sin^{-1}$ consists of all numbers between -1 and 1, inclusive. What is the domain of the function $\cos^{-1}$? What is the domain of the square-root function? What is the domain of log? What is the domain of an exponential function?

1. Start with the meaning of domain

The domain is the set of allowable inputs. For arccos the input is the horizontal coordinate of a point on the unit circle. It can be any number from −1 to 1. The two endpoints occur at angles 180° and 0°, so the domain includes both ends: [−1,1].



The real arccos input stays in the closed interval [−1, 1].

2. Test the boundary of a square root

A real square is never negative. Therefore a real square root of x exists exactly when $x \geq 0$. At the boundary $\sqrt{0}=0$, so the domain is $[0,\infty)$.



The real square root admits zero and all positive inputs.

3. Ask which inputs can be logarithms' arguments

For a fixed log base $b>0$, $b\neq 1$, $\log_b x$ asks for an exponent t with $b^t=x$. Every such power is positive, and every positive x occurs. Thus the domain is $(0,\infty)$; zero is excluded.



A real logarithm requires a strictly positive argument.

4. Switch from the output to the exponent

For b^x the input is the exponent. Negative exponents give reciprocals, zero gives 1, and positive exponents and real powers are defined for every real x when $b>0$. Its domain is all real numbers, including for the constant case $b=1$.



The exponent of a positive base may be any real number.

Check. The square, restricted to nonnegative inputs, has range $[0,\infty)$. Exponentials with standard base $b>0$, $b\neq 1$ have positive outputs, explaining the log domain. The exceptional constant function 1^x still admits every real x.

Exercise 673 · Ranges of standard functions

Printed page 74 · Domain and range / Trigonometric ratios

When working in degree mode, the range of the function $\sin^{-1}$ consists of all numbers between -90 and 90, inclusive. What is the range of the function $\cos^{-1}$? What is the range of the square-root function? What is the range of log? What is the range of $f(x) = 2-7\sin 45x$? What is the range of the exponential function $E(x) = b^x$?

1. Choose the principal angle

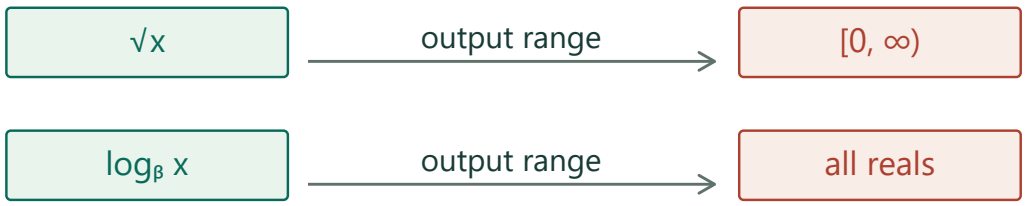
A range lists possible outputs. An arccos output is an angle; in degree mode choose the unique angle between 0° and 180° having the supplied cosine. Inputs 1 and −1 give those endpoints, so the full range is [0°,180°].



A principal arccos output includes its two endpoint angles.

2. Identify root and logarithm outputs

The principal square root returns a nonnegative number. Every $t \geq 0$ occurs as $\sqrt{(t^2)}$, proving the range is $[0,\infty)$. A logarithm can return any real t because $\log_\beta(b^t)=t$ for a fixed $b>0$, $b\neq 1$. Its range is all real numbers.



The square root is nonnegative; the logarithm can return any exponent.

3. Transform the sine interval

As sin(45x) varies from −1 to 1, multiplication by −7 interchanges the ends and gives values from −7 to 7. Adding 2 shifts them to [−5,9]. At x=2 in degree mode, sine is 1 and the output is −5; at x=6, sine is −1 and the output is 9.



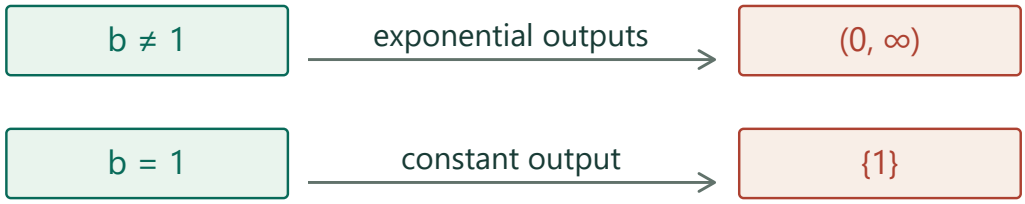
1. $-1 \le \sin(45x^\circ) \le 1 \Rightarrow 7 \ge -7\sin(45x^\circ) \ge -7$
Multiplication by the negative coefficient reverses both inequalities.

2. $-7 \le -7\sin(45x^\circ) \le 7 \Rightarrow -5 \le 2-7\sin(45x^\circ) \le 9$
Put the smaller endpoint first and add 2 to all three parts.

Reflect and scale the sine interval, then add the midline height.

4. Read the exponential outputs

For b>0, b≠1, the graph stays above zero. Every positive target y is attained at $x=\log_b y$, so the range is (0,∞). If the base 1 is included, then 1^x always equals 1, and its range is the single value {1}.



A positive base never produces zero.

Check. Sine's two extremes attain both endpoints of the stated range.

Exercise 674 · Log reciprocals and a finite sum

Printed page 74 · Logarithm rules / Geometric sequences and series

Simplify the following expressions: **(a)** $\log_c A + \log_c \frac{1}{A}$ **(b)** $p + pm + pm^2 + \cdots + pm^w$

1. (a) Combine the reciprocal pair

A real base c must satisfy $c>0$ and $c\neq1$. The first argument requires $A>0$. That same condition makes $1/A$ defined and positive.

1. **$\log_c A + \log_c(1/A) = \log_c(A \cdot 1/A)$**
   Use the product law on the two positive arguments.

2. **$\log_c(A \cdot 1/A) = \log_c 1$**
   A nonzero number times its reciprocal is 1.

3. **$\log_c 1 = 0$**
   The zero exponent gives $c^0=1$.

2. (b) Align two copies of the finite sum

Let w be a nonnegative integer. There are $w+1$ terms, starting with p. Multiplying the sum by m shifts each power up one place.

1. **$S = p + pm + \cdots + pm^w$**
   Name the finite sum. At $w=0$ it is simply p.

2. **$mS = pm + pm^2 + \cdots + pm^{(w+1)}$**
   Distribute m over every term of S.

3. **$S - mS = p - pm^{(w+1)}$**
   Subtract. Every power from m^1 through m^w cancels.

3. (b) Separate the exceptional ratio

Factoring makes it possible to divide out the common coefficient of S, provided that coefficient is nonzero. The excluded ratio belongs to the original problem and can be evaluated directly.

1 **(1−m)S = p(1−m^(w+1))**
Factor S on the left and p on the right.

2 **m ≠ 1 ⇒ S = p(1−m^(w+1))/(1−m)**
Only this branch permits division by 1−m.

3 **m = 1 ⇒ S = (w+1)p**
All w+1 original terms are p, including the first.

Check. Multiply S by 1−m to cancel all interior terms.

Exercise 675 · Principal tangent inverse

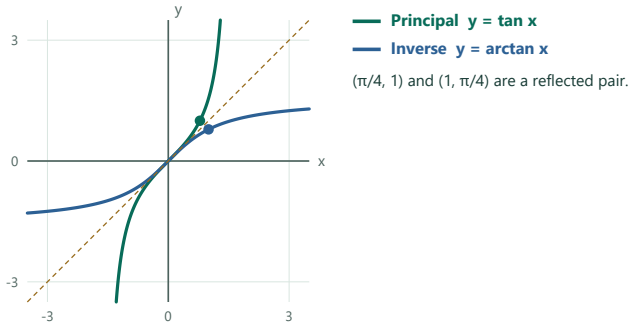
Printed page 74 · Inverse trigonometric functions / Domain and range

You have used the function $\tan^{-1}$ many times to find angles. Now it is time to consider the rule that makes $\tan^{-1}$ a function.
(a) The point $(\frac{\pi}{4}, 1)$ is on the graph of $y = \tan x$ (working in radian mode). There is a corresponding point that is on the graph of $y = \tan^{-1} x$. What is it?
(b) Write four other sets of corresponding points — two with negative x-coordinates and two with positive x-coordinates.
(c) Notice that the point $(\frac{3\pi}{4}, -1)$ is on the graph of $y = \tan x$ but the point $(-1, \frac{3\pi}{4})$ is not on the graph of $y = \tan^{-1} x$. Explain.

1. Swap the given tangent point (a)

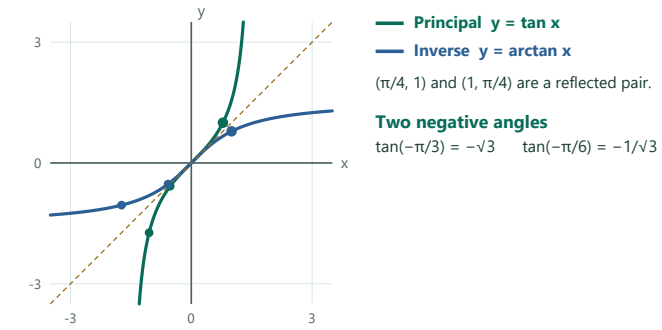
On the principal interval (−π/2,π/2), tangent is one-to-one. The point (π/4,1) therefore corresponds to (1,π/4) on the inverse.

Only the principal tangent branch has arctangent as its inverse



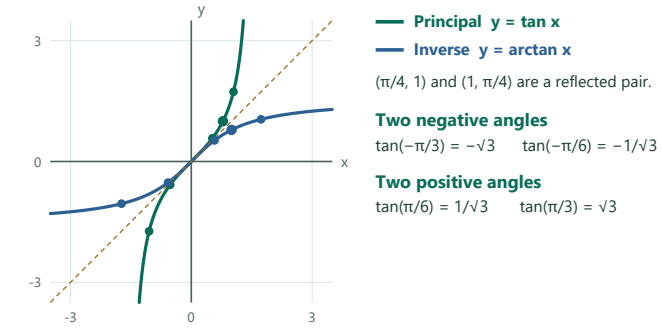
2. Choose two negative examples (b)

Angles $-\pi/3$ and $-\pi/6$ have tangents $-\sqrt{3}$ and $-1/\sqrt{3}$. The ordinary points pair each angle with its tangent; the inverse points are $(-\sqrt{3},-\pi/3),(-1/\sqrt{3},-\pi/6)$.



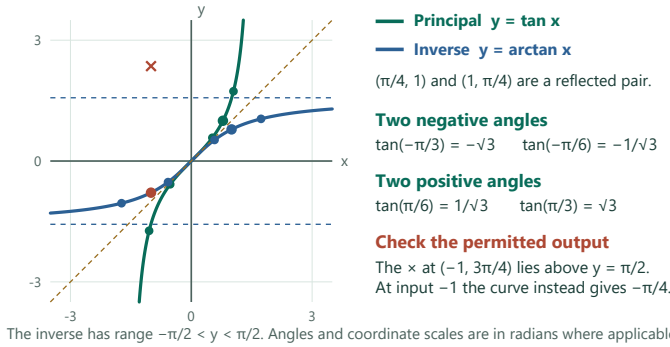
3. Choose two positive examples (b)

Likewise $(\pi/6,1/\sqrt{3})$ and $(\pi/3,\sqrt{3})$ give the inverse points $(1/\sqrt{3},\pi/6)$ and $(\sqrt{3},\pi/3)$.



4. Choose the principal angle (c)

The tangent at $3\pi/4$ is -1. Subtract π to reach $-\pi/4$ inside the principal range. Consequently $\arctan(-1)=-\pi/4$.



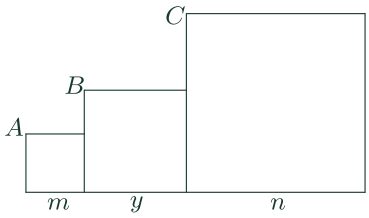
The inverse has range $-\pi/2 < y < \pi/2$. Angles and coordinate scales are in radians where applicable.

Check. Tangent repeats every π; only one representative falls in the principal interval.

Exercise 676 · Three aligned square corners

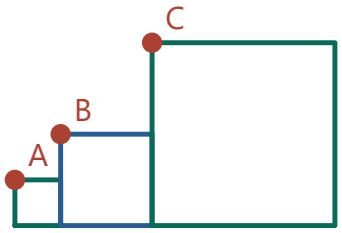
Printed page 74 · Similarity and scaling / Collinearity / Radicals and rational powers

Three squares are placed next to each other as shown. The vertices A, B, and C are collinear. In terms of m and n, find the dimension y.

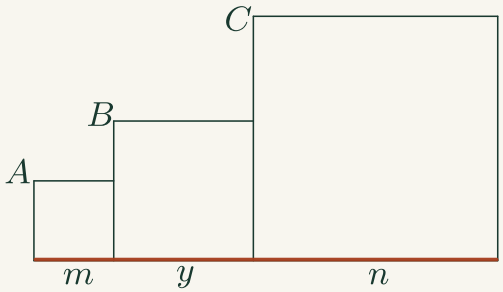


1. Assign coordinates using the square sides

Take the shared baseline as y-coordinate zero. With outside square sides m and n and middle side y, the indicated collinear corners may be written (0,m),(m,y),(m+y,n).



Example with square sides 1, 2 and 4 on a common baseline.

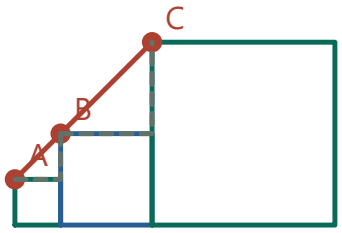


Common baseline

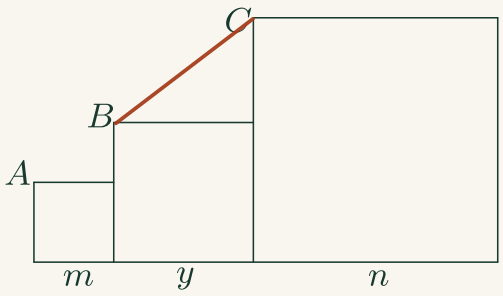
Measure all three top heights from the same baseline; each height equals its square's own width.

2. Compare consecutive slopes

From the first corner to the middle, rise/run=(y−m)/m. From the middle to the last, it is (n−y)/y. Collinearity equates these slopes.



The rise divided by run is identical for the two segments.

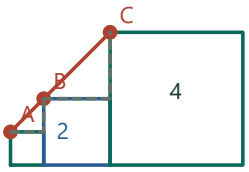


Second pair of square corners

Continue from B to C. Compare this segment's vertical and horizontal changes with those of the first segment.

3. Solve for the side length

Cross multiplication gives $y^2-my=mn-my$. Cancel $-my$ to get $y^2=mn$. A square's side is positive, so $y=\sqrt{(mn)}$.



The middle side here is $\sqrt{(1 \times 4)} = 2$, as $y^2 = mn$ predicts.

1. $(y-m)/m=(n-y)/y \Rightarrow y(y-m)=m(n-y)$
Multiply both sides by the product my. The two side lengths in the denominators are positive.

2. $y^2-my=mn-my \Rightarrow y^2=mn$
Distribute on both sides and add my.

3. $y=\sqrt{(mn)}$
The negative square root cannot represent a square's side.

Check. For equal outside squares the middle side is the same, as the horizontal-line case requires.

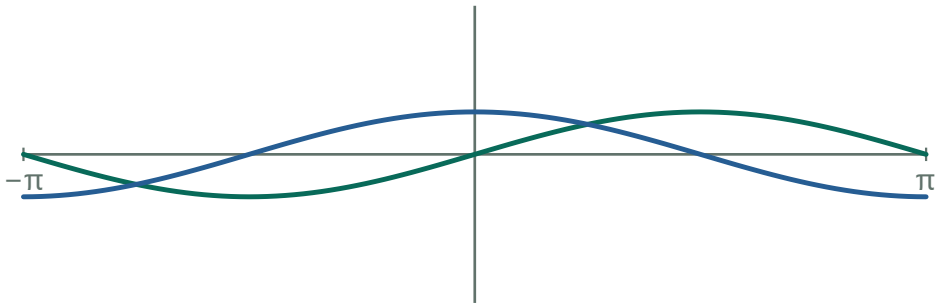
Exercise 677 · Rewrite a cosine as a sine

Printed page 74 · Sinusoidal functions / Phase shifts

Show that $y = 3\cos 2x$ can be rewritten in the equivalent form $y = a\sin(mx + b)$, thereby confirming that this cosine curve is sinusoidal.

1. Recall the phase relation

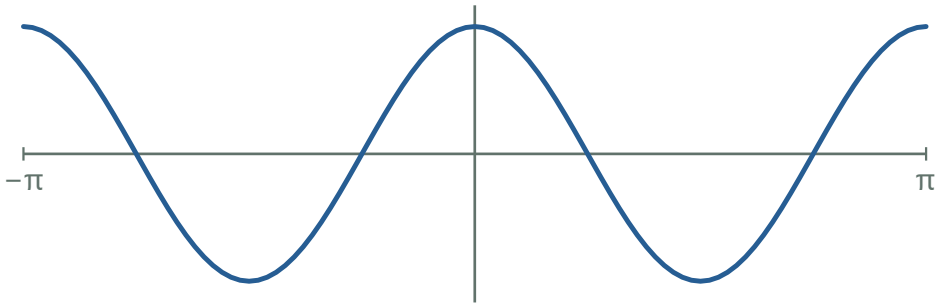
Sine reaches its peak a quarter-turn after angle zero, whereas cosine starts there. Moving the sine wave left by a quarter-turn aligns it with cosine.



An ordinary cosine is a sine shifted by one quarter turn.

2. Insert the given angle

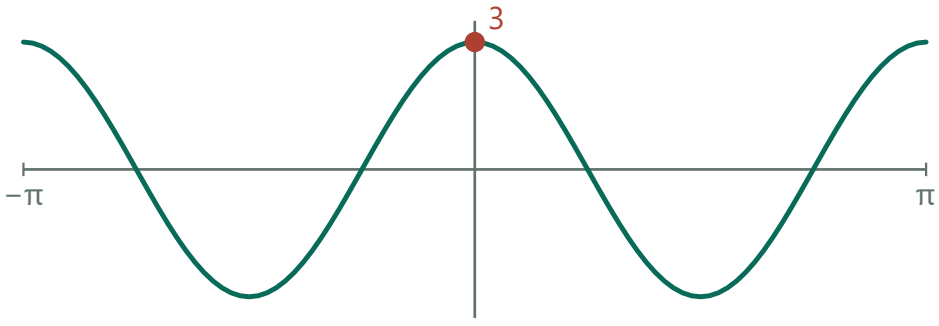
Putting u=2x in cos u=sin(u+π/2) gives 3cos(2x)=3sin(2x+π/2). In degree mode use b=90° instead.



Doubling the input doubles the angular rate.

3. Read the requested parameters

In radian measure a=3, m=2 and b=π/2. At x=0 both expressions give 3, verifying the shift direction.



The scale 3 and inside shift π/2 match the starting height.

Check. At x=0 both formulas attain the peak 3.

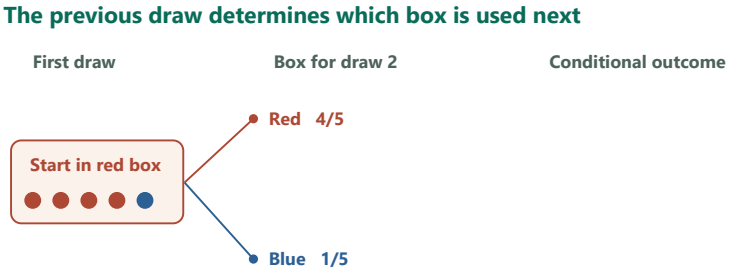
Exercise 678 · Two boxes choose the next draw

Printed page 75 · Transition matrices / Probability

There are two boxes. The red box contains four red marbles and one blue marble, and the blue box contains five blue marbles and three red marbles. The following experiment is done a hundred times: A marble is randomly drawn from one of the boxes, its color recorded, then replaced in the same box. The first drawing is from the red box, but each subsequent drawing is determined by the color of the marble most recently drawn: if it is red, the next drawing comes from the red box; if it is blue, the next drawing comes from the blue box. What is the probability that the first marble is red? the first marble is blue? the second marble is red? the second marble is blue? the hundredth marble is red?

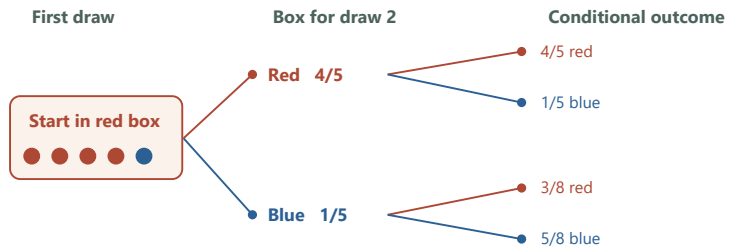
1. Read the starting draw

The first draw uses the red box, with four reds and one blue. Thus $p_1 = P(\text{red}) = 4/5$, and $P(\text{blue}) = 1/5$. Replacement keeps both boxes' compositions fixed.



2. Condition on the previous color

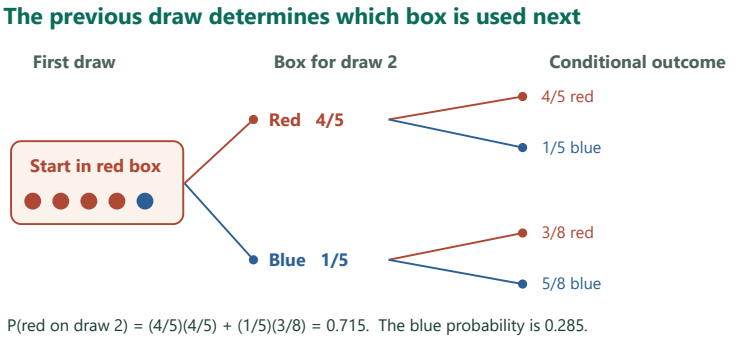
A red result brings the next draw to the red box, where red chance is 4/5. A blue result sends it to the blue box, where red chance is 3/8. Thus $p_{n+1} = (4/5)p_n + (3/8)(1 - p_n) = 3/8 + (17/40)p_n$.



1. $p_{n+1} = (4/5)p_n + (3/8) - (3/8)p_n$	Distribute 3/8 over the previous blue probability $1 - p_n$.
2. $4/5 - 3/8 = (32 - 15)/40 = 17/40$	Combine the two coefficients of p_n.

3. Compute the second draw

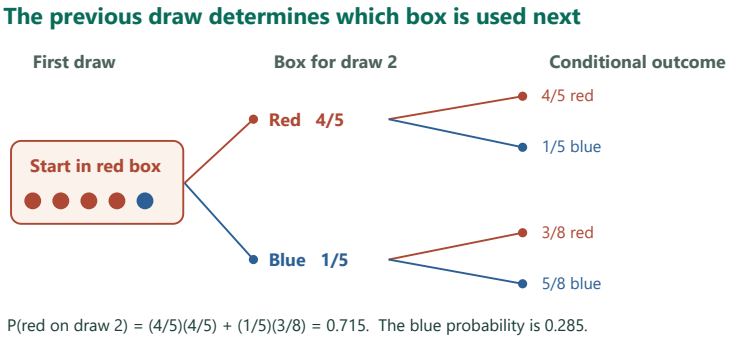
Insert $p_1=.8$, obtaining $.375+.425(.8)=.715$ red and $1-.715=.285$ blue.



P(red on draw 2) = (4/5)(4/5) + (1/5)(3/8) = 0.715. The blue probability is 0.285.

4. Fast-forward to draw 100

A fixed probability obeys $p^*=3/8+(17/40)p^*$, giving $p^*=15/23$. Deviations from it multiply by 17/40 per draw. Therefore $p_{100}=15/23+(4/5-15/23)(17/40)^{99}\approx.652174$.



P(red on draw 2) = (4/5)(4/5) + (1/5)(3/8) = 0.715. The blue probability is 0.285.

At each update $p_{n+1} = 3/8 + (17/40)p_n$. The draw-100 chance of red is about 0.652174.

1. $(1-17/40)p^*=3/8 \Rightarrow (23/40)p^*=15/40$	Bring the fixed-probability terms together using denominator 40.
2. $p^*=15/23$; $p_{n+1}-p^*=(17/40)(p_n-p^*)$	Subtract the fixed equation from the update.
3. $p_{100}=p^*+(17/40)^{99}(p_1-p^*)$	From index 1 to 100 there are 99 applications of the gap multiplier.

Check. Red-to-blue and blue-to-red probabilities balance at the fixed value.

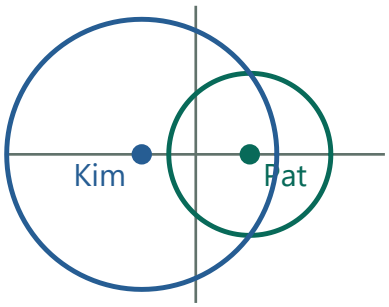
Exercise 679 · Triangulate a lightning flash

Printed page 75 · Circle geometry / Systems of equations / Distance and loci

Pat lives two miles east of the center of town, Kim lives two miles west of the center of town, both on the same east-west road. They are talking on the telephone during a thunderstorm. Fifteen seconds after a lightning flash, Pat hears the rumble of thunder. Ten seconds after that, Pat hears the *same* peal of thunder through the telephone. Because sound takes five seconds to travel a mile through air, and light and electricity take no time to reach their destinations, Pat reasons that there are only two places where the lightning could have struck. What is the reasoning? Could Kim have reached the same conclusion?

1. Convert each arrival time separately

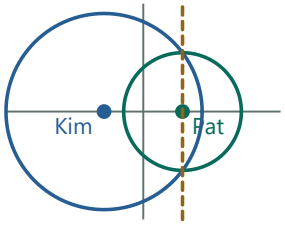
Pat hears thunder 15 s after the flash, so L is 15/5=3 miles from Pat. The telephone sound arrives at 25 s, giving 25/5=5 miles from Kim.



Two arrival times give circles of radii 3 and 5 miles.

2. Intersect the two circles

At P=(2,0), K=(−2,0), impose $(x-2)^2+y^2=9$ and $(x+2)^2+y^2=25$. Subtract the first from the second: 8x=16, so x=2.



Subtracting their equations fixes the common x-coordinate.

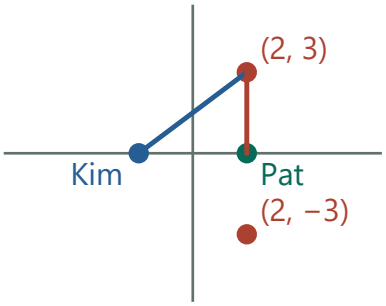
1. $(x+2)^2-(x-2)^2=25-9$
Subtract circle equations to cancel the same y^2.

2. $(x+2-x+2)(x+2+x-2)=16 \Rightarrow 4\cdot 2x=16$
Factor the difference of squares.

3. x=2
Divide by 8; both original circles still have to hold.

3. Recover the two locations

Substitute into the first circle to get $y^2=9$ and $y=\pm 3$. Both are the same distances from the houses, on opposite sides of the road. Kim can infer the same two positions if she registers both arrival times, including Pat's local sound over the phone.



The mirrored intersections give two possible strike locations.

Check. Both candidates have 3–4–5 geometry and mirror symmetry across the road.

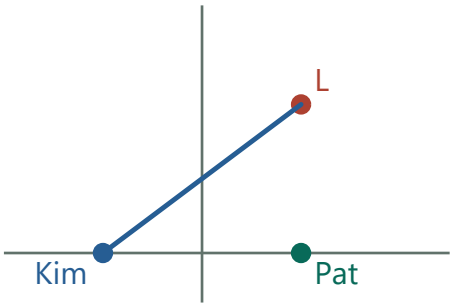
Exercise 680 · Choose the signed distance delay

Printed page 75 · Hyperbolas / Signed distances

(Continuation) Which equation, $LK - LP = 2$ or $LP - LK = 2$, best fits the situation that you have been investigating?

1. Identify the later path

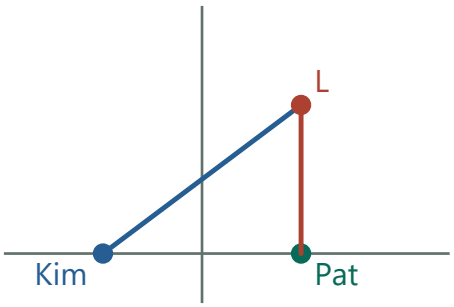
The first sound came through the air to Pat. The telephone relayed the sound that traveled through air to Kim, which arrived later. Thus LK is the longer distance.



The later arrival traveled through the air to Kim.

2. Convert the delay

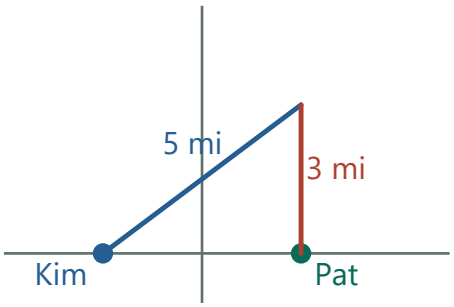
At 5 seconds per mile, an extra 10 seconds corresponds to an extra 10/5=2 miles of air travel.



Compare the two paths from the same strike location.

3. Choose the sign

Longer minus shorter is positive, so LK−LP=2. For the candidate (2,3) the two distances are 5 and 3 miles, verifying the sign.



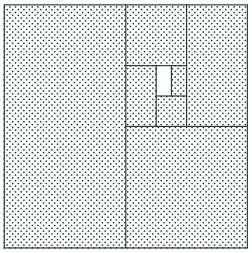
The example has LK − LP = 5 − 3 = 2 miles.

Check. At (2,3), LK−LP=5−3=2.

Exercise 681 · An infinite square filling

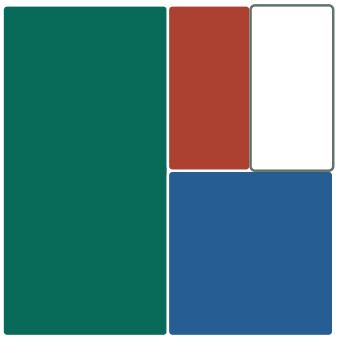
Printed page 75 · Geometric sequences and series / Limits and convergence

What does the figure at right suggest to you about the geometric series $\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\cdots$?

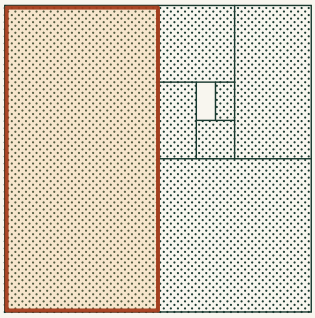


1. Read the successive pieces

Start with total area 1. The first piece takes 1/2, the next takes half the unused area (1/4), the next takes half again (1/8).



The first three pieces use 1/2, 1/4 and 1/8 of the whole area.

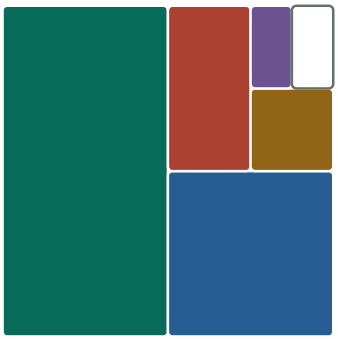


First half

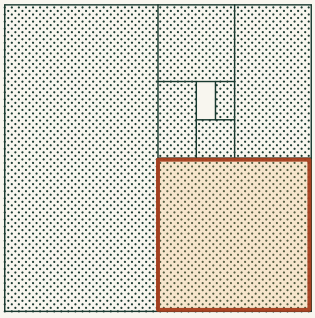
This full-height strip occupies half the original unit square. Start the running sum with its area.

2. Track the omitted region

Each step halves the still-unused region. After n steps its remaining area is $(1/2)^n$. The pieces used so far cover the complementary area $1-(1/2)^n$.



After n halvings the white remainder has area $(1/2)^n$.

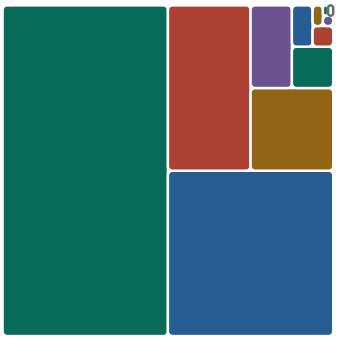


Quarter of the original

The lower-right region takes half of what remained after the initial piece. Keep track of the reference area when calling a piece a half.

3. Pass to the limit

The remainder tends to zero as n increases. Thus 1/2+1/4+1/8+⋯=1 even though any finite collection still leaves some area unused.



The white remainder tends to zero as more pieces are taken.

Check. The geometric formula gives (1/2)/(1−1/2)=1.

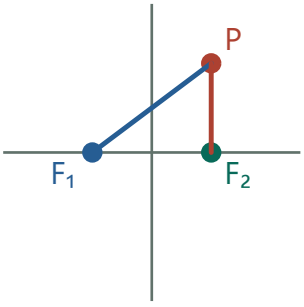
Exercise 682 · Derive the thunder hyperbola

Printed page 75 · Hyperbolas / Distance and loci / Algebraic manipulation

Let $F_1 = (-2, 0)$, $F_2 = (2, 0)$, and $P = (x, y)$. Write an equation that says that $PF_1 - PF_2 = 2$ and make a sketch. What does this have to do with Pat and Kim? Starting with the step $(PF_1)^2 = (2 + PF_2)^2$, simplify your equation until it reaches the form $hx^2 - ky^2 = m$.

1. Write the signed distance rule

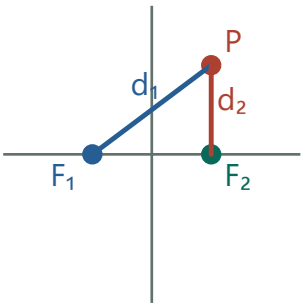
The left and right focal distances at P are $d_1=\sqrt{((x+2)^2+y^2)}$ and $d_2=\sqrt{((x-2)^2+y^2)}$. The thunder delay requires $d_1-d_2=2$, making the right focus closer.



Subtract PF_2 from PF_1. The right focus lies closer to this P.

2. Remove the first radical

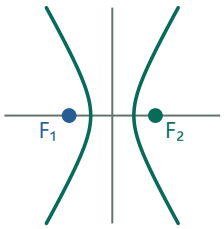
Square $d_1=2+d_2$ to obtain $d_1^2-d_2^2=4+4d_2$. The difference of squared coordinate distances is $(x+2)^2-(x-2)^2=8x$, so $d_2=2x-1$.



The squared distances differ by 8x, leaving one distance to isolate.

3. Remove the second radical

Square again: $(x-2)^2+y^2=(2x-1)^2$. Expanding and collecting gives $3x^2-y^2=3$.



The twice-squared equation contains two symmetric branches.

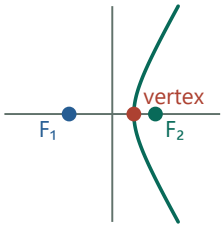
1. $x^2-4x+4+y^2=4x^2-4x+1$
Expand the two squares after isolating the nonnegative distance.

2. $4+y^2=3x^2+1 \Rightarrow 3x^2-y^2=3$
Cancel $-4x$ and collect like terms.

3. $x^2-y^2/3=1$, $2x-1\geq 0$
Divide by 3 and retain the sign condition from the step before squaring.

4. Recover the original branch

The quadratic implies $|x|\geq 1$. The equation $d_2=2x-1$ also demands $2x-1\geq 0$, selecting $x\geq 1$. Draw that right branch with vertex (1,0) and asymptotes $y=\pm\sqrt{3}x$; the extra left branch would reverse the delay.

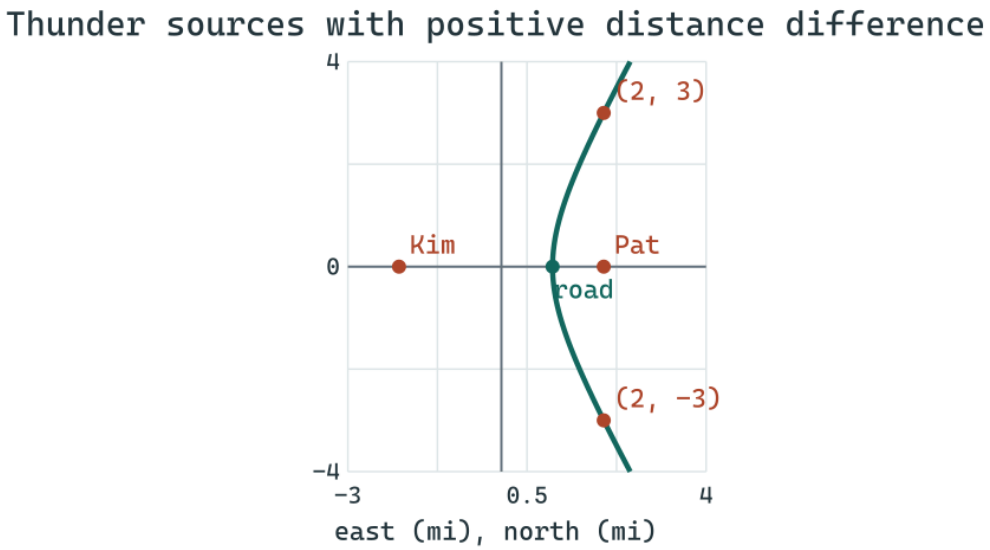


Restore the sign condition $PF_1 - PF_2 = 2$.

1. $x\geq 1 \Rightarrow 2x-1\geq 0 \Rightarrow d_2=2x-1$
Reverse the second squaring using the nonnegative root.

2. $d_1^2-d_2^2=8x=4+4d_2 \Rightarrow d_1^2=(2+d_2)^2$
The coordinate-distance difference restores the first squared equation.

3. $d_1=2+d_2 \Rightarrow d_1-d_2=2$
Both distances are nonnegative, completing the converse on this branch.



Check. On the left branch the distances instead differ by −2.

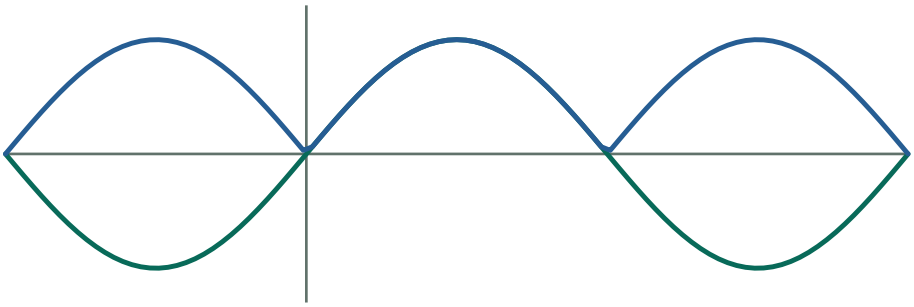
Exercise 683 · Fold a sine wave

Printed page 75 · Absolute value / Periodic functions / Graph transformations

Consider the function f defined by $f(x) = |\sin x|$. Draw its graph, and find the period. Explain why this graph is not sinusoidal.

1. Fold the sine graph

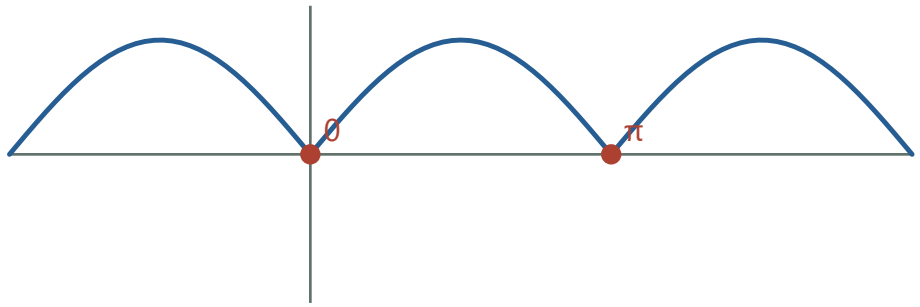
Where sin x≥0 keep the sine curve. Where sin x<0 reflect its height across the x-axis. All heights are now between 0 and 1.



Reflect the negative sine arches above the axis.

2. Find the least period

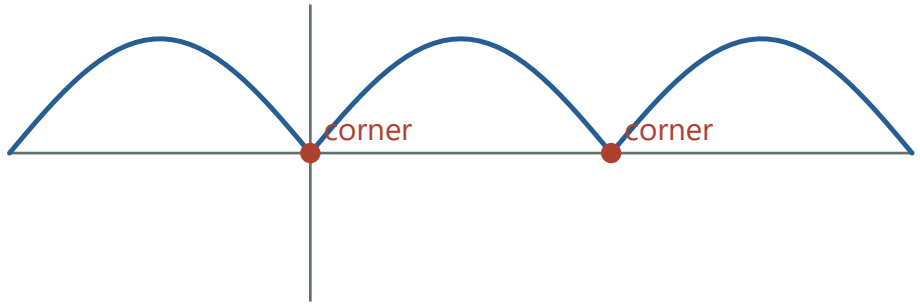
Because $\sin(x+\pi) = -\sin x$, taking absolute values reproduces the whole curve after π radians (180°). Its zeros occur precisely at $k\pi$, so no smaller positive shift can reproduce the zero at 0.



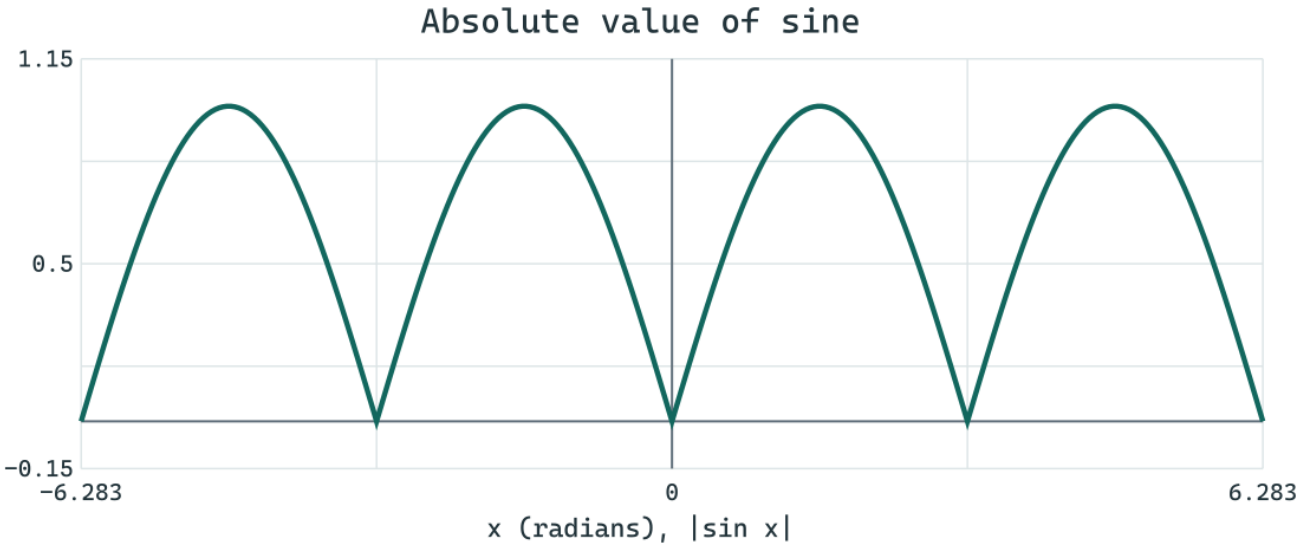
Adjacent zeros delimit one complete repeat of the folded graph.

3. Examine the joins

At consecutive zeros the reflected arcs meet with sharp corners. An ordinary nonconstant sinusoid has a smooth tangent everywhere, so this periodic function is not sinusoidal.



At each zero the descending and ascending arcs form a corner.



Check. Sharp corners at $k\pi$ distinguish it from any nonconstant sinusoid.

Exercise 684 · Constant half-life

Printed page 75 · Half-life / Exponential growth and decay

If $p(t)$ is an exponentially decreasing function, what is the usual name for the number t that solves the equation $p(t) = \frac{1}{2}p(0)$? What is the usual name for the number u that solves the equation $p(37 + u) = \frac{1}{2}p(37)$?

1. Use a fraction of the starting amount

For p(s)=Ab^s with 0<b<1, a wait of h changes a starting value by p(s+h)/p(s)=b^h. The starting time s cancels.



1. p(s+h)/p(s)=Ab^(s+h)/(Ab^s)
Compare the later and earlier positive amounts.

2. Ab^{s}b^h/(Ab^s)=b^h
Split the power and cancel the nonzero starting value; s disappears.

The fractional change depends on the waiting time alone.

2. Find a 50% reduction

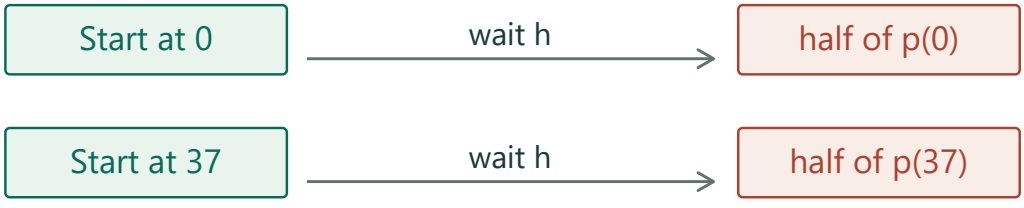
Set b^h=½. Taking logarithms gives h=ln(½)/ln b, a positive duration because both logs are negative.



Select h so that b^h = 1/2.

3. Name both intervals

Starting at s=0 yields t=h; starting at s=37 yields u=h. Both durations are the half-life of the same exponential decrease.



An equal waiting time halves either starting amount.

Check. Taking s=0 and s=37 yields the same equation for h.

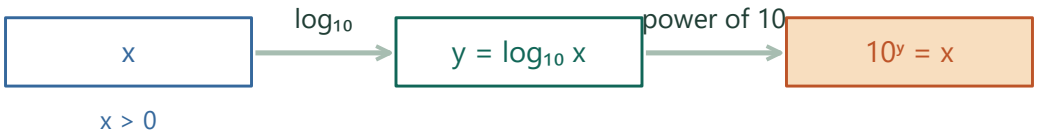
Exercise 685 · The inverse of log

Printed page 75 · Logarithms / Inverse functions

A calculator has a sin function and a $\sin^{-1}$ function, a cos function and a $\cos^{-1}$ function, and a tan function and a $\tan^{-1}$ function. Does it have a $\log^{-1}$ function?

1. Read the operation to be undone

A base-ten log reports the exponent y with $10^y=x$. Its inverse must take that exponent and produce x again.



Start with a positive input x and follow the arrows to recover it.

2. Identify the appropriate calculator function

For common log the inverse is 10^x, often a shifted key. For natural log the inverse is e^x or exp. More generally the inverse of log_b is the exponential b^x.

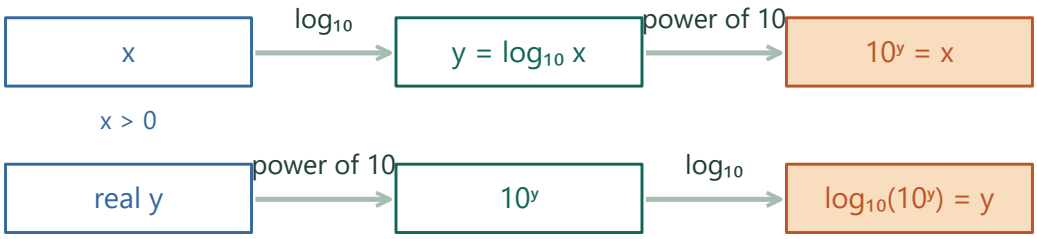


For base e the corresponding pair is exp and ln.

A base determines its inverse key. Each logarithm is undone by exponentiation with that same base.

3. Check the two directions

$10^{(\log_{10} x)}=x$ for $x>0$, and $\log_{10}(10^y)=y$ for every real y. An inverse-function key performs this undoing.



Read both compositions and notice where positivity is required.

Check. $10^{(\log_{10} x)}=x$ for $x>0$ and $\log_{10}(10^y)=y$ for real y.

Exercise 686 · Repeating two-digit block

Printed page 75 · Geometric sequences and series / Limits and convergence

The repeating decimal $0.13131313\ldots$ can be thought of as an infinite geometric series. Write it in the form $a + ar + ar^2 + \cdots$. Express the series using sigma notation, and then find its sum. What is the rational number equivalent to $0.13131313\ldots$?

1. Split the repeating blocks

Write .131313...=.13+.0013+.000013+⋯. Each new block shifts two decimal places, multiplying the preceding term by .01.

13	00	00
00	13	00
00	00	13

Two-digit blocks occupy successive pairs of decimal places.

2. Write the series

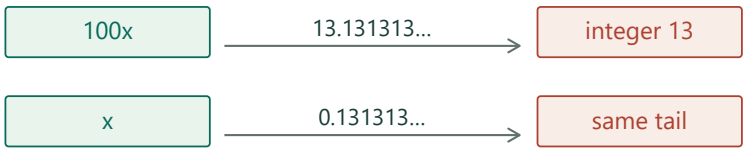
With a=.13 and r=.01 the sigma notation is Σ .13(.01)^k for k=0,1,2,… . The ratio has magnitude below 1.



Each new place block is one hundredth as large.

3. Sum to a fraction

The infinite geometric sum is a/(1−r)=.13/.99=13/99. Independently, shifting the original number by two places gives 100x−x=13, hence the same fraction.



1. 100x=13.131313…, x=0.131313…
A shift by the block length aligns the repeating tails.

2. 99x=13 ⇒ x=13/99
Subtract and divide by the nonzero integer 99.

Subtract to cancel the identical infinite decimal tails.

Check. If x=.131313…, then 100x−x=13.

Exercise 687 · An exponential inside tangent

Printed page 75 · Logarithms / Trigonometric equations

Find the solution to $\log_3(T) = \frac{1}{2}$, then find all the solutions to $\log_3(\tan x) = \frac{1}{2}$.

1. Undo the logarithm

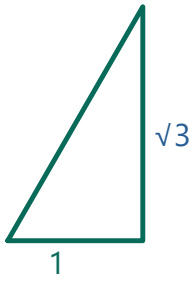
$\log_3 T = ½$ means T=3^(1/2)=√3, positive as required by the logarithm.



Undo the logarithm first.

2. Use that value for tangent

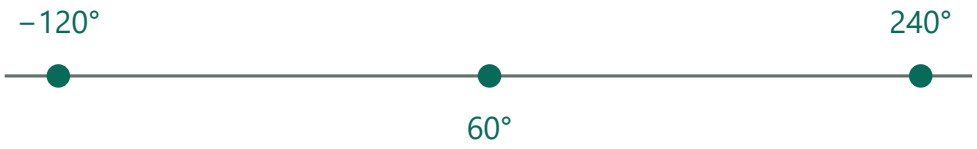
Set tan x=√3. The 30°–60°–90° triangle supplies x=60°=π/3 as one solution.



The tangent of 60° is √3 / 1.

3. Include every repeat

Tangent repeats after π, so x=π/3+kπ radians, or 60°+180°k in degrees, for every integer k. All these angles have the same positive tangent, so they meet the log domain.

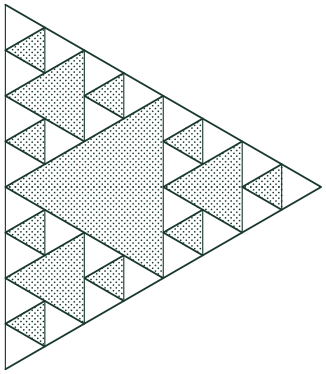


Adjacent solutions are separated by 180°.

Check. Substitution gives $\log_3(\sqrt{3})=1/2$ for every solution.

Exercise 688 · Painting midpoint triangles

Printed page 76 · Fractals and iteration / Geometric sequences and series / Sigma notation



An equilateral triangle with an area of one square unit is painted step-by-step as follows: Step 1 consists of painting the triangle formed by joining the midpoints of the sides. Step 2 then consists of applying the same midpoint-triangle process to each of the three small unpainted triangles. Step 3 then consists of applying the midpoint-triangle process to each of the nine very small unpainted triangles. The result is shown at right. In general, each step consists of applying the midpoint-triangle process to each of the (many) remaining unpainted triangles left by the preceding step. Let P_n be the area that was painted *during* step n, and let U_n be the total unpainted area left after n steps have been completed.

(a) Find U_1, U_2, U_3, P_1, P_2, and P_3.

(b) Write a recursive description of U_n in terms of U_{n-1}. Find an explicit formula for U_n.

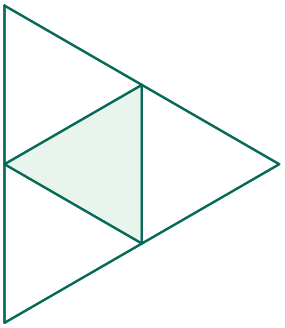
(c) Write a recursive description of P_n in terms of P_{n-1}. Find an explicit formula for P_n.

(d) Use your work to evaluate the sum $\frac{1}{4} + \frac{3}{16} + \frac{9}{64} + \cdots + \frac{3^{99}}{4^{100}} + \frac{3^{100}}{4^{101}}$.

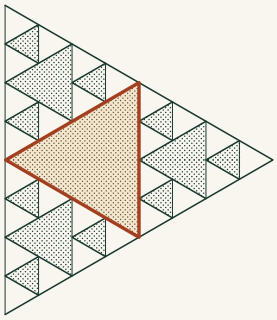
(e) Express the series of part (d) using sigma notation.

1. Study one midpoint subdivision

Joining midpoints splits an equilateral triangle into four congruent quarter-area triangles. Paint the middle one, leaving three quarters of each previously unpainted piece.



Join the midpoints and paint the central quarter.



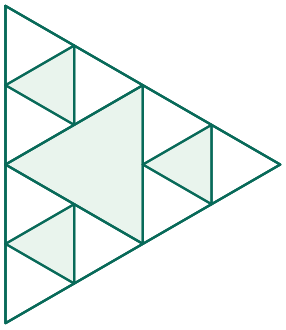
First painted triangle

This central triangle joins the three midpoints. Compare its base and altitude with those of the original whole triangle.

2. Track remaining area (a,b)

Starting $U_0=1$, take $U_n=(3/4)U_{n-1}=(3/4)^n$. Thus $U_1,U_2,U_3=3/4,9/16,27/64$. For instance, after step 1 the three remaining corner triangles each occupy 1/4 of the original; at step 2 each retains 3/4 of its area. Multiplying 3/4 by 3/4 gives the 9/16 total still unpainted.

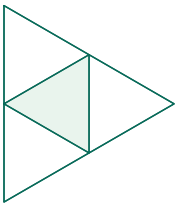
Midpoint subdivisions · Stage 2



Three surviving triangles each retain three quarters of their area.

Compare settings · Midpoint subdivisions

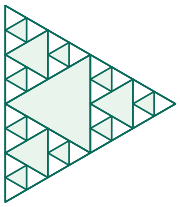
Stage 1



3/4 of the original triangle remains unpainted.

3/4 of the original triangle remains unpainted.

Stage 3

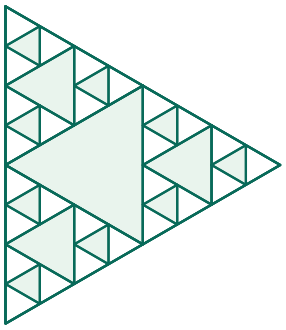


27/64 of the original triangle remains unpainted.

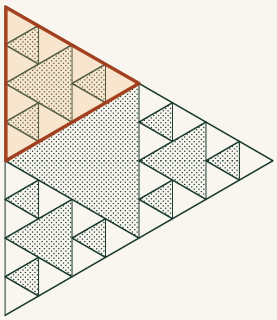
27/64 of the original triangle remains unpainted.

3. Track newly painted area (a,c)

At stage n paint one quarter of the previous unpainted area, so $P_n=(1/4)(3/4)^{(n-1)}$. This gives $P_1,P_2,P_3=1/4,3/16,9/64$ and recursion $P_n=(3/4)P_{n-1}$ for $n\ge2$.



Only the central quarters of the previously unpainted pieces are new.



One surviving corner triangle

The upper-left region has the same shape as the original at half the side length. Subsequent subdivisions are already visible inside it.

4. Sum all stages listed (d,e)

The last term $3^{100}/4^{101}$ is the 101st increment. Thus the sum is $\Sigma\ 3^j/4^{(j+1)}$, $j=0,\dots,100$. The painted area equals all area minus the remainder: $1-(3/4)^{101}$. As a check, the sum after the first two steps is $1/4+3/16=7/16$. The complement $1-9/16$ gives the same painted area.



1. $P_n=(1/4)(3/4)^{(n-1)}=3^{(n-1)}/4^n$
Collect powers of the denominator.

2. $4^n=4^{101}\Rightarrow n=101;\ \Sigma P_j=1-U_{101}$
The matching increment is stage 101; the covered area is its complement.

At stage two the painted amount is the complement of the remainder.

Check. Subtract the final unpainted area from the original unit area.

Exercise 689 · Present value of a sports contract

Printed page 76 · Annuities / Present and future value

A basketball player signed a seven-year contract for \$121 million. What is the present value of this contract, given a 9% interest rate assuming seven equal payments, the first one due immediately?

1. Place the installments in time

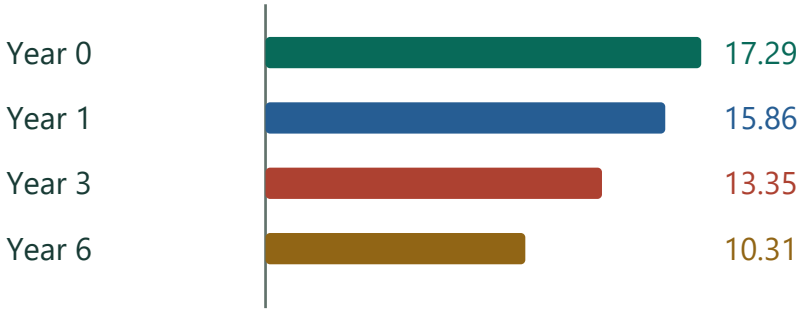
Each annual payment is 121/7 million dollars. Since the first is paid immediately, the seven payments arrive at years 0,1,…,6.



The first of seven equal installments is paid now.

2. Discount each separately

At 9% annual interest an amount payable j years later has present value equal to that amount divided by 1.09^{j}. Add (121/7)Σ(1.09)^(−j) for j=0,…,6.



Discount each payment back to year 0 (millions of dollars).

3. Evaluate the geometric sum

With r=1/1.09 the multiplier is $(1-r^{7})/(1-r)$. Multiplying by 121/7 gives about $94.828 million, below the sum of the stated future payments.



Earlier payments lose less value under discounting.

1. $S=1+r+\cdots+r^{6} \Rightarrow S-rS=1-r^{7}$
The six intermediate discount powers cancel.

2. $PV=(121/7)(1-r^{7})/(1-r)$ million, r=1/1.09
Solve for S and multiply by one installment. The discount ratio is less than one.

Check. It lies between the face value and the value obtained by discounting every installment for all six years.

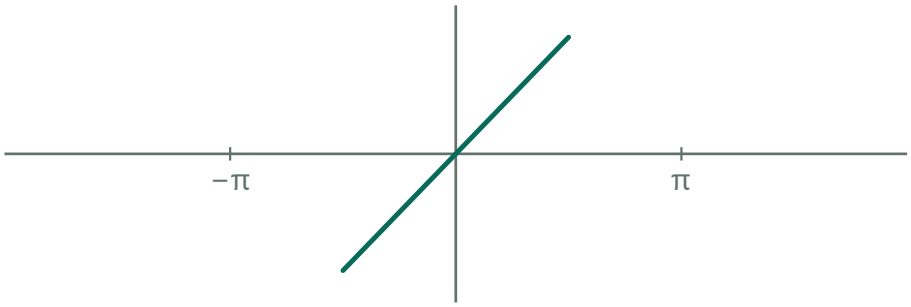
Exercise 690 · A folded angle graph

Printed page 76 · Inverse trigonometric functions / Periodic functions

Function f is defined by $f(x) = \sin^{-1}(\sin x)$. Draw its graph, and find its period.

1. Start on the principal interval

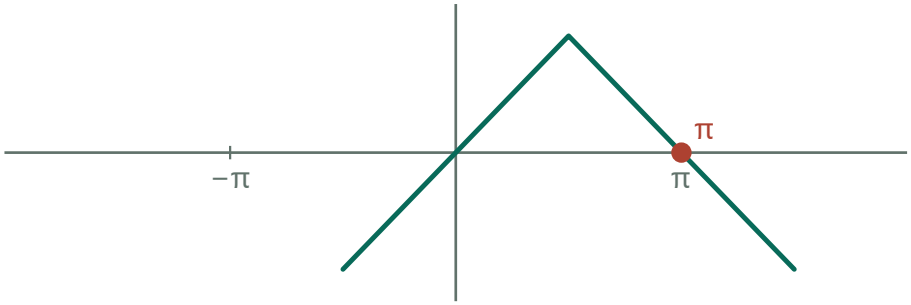
On $-\pi/2 \le x \le \pi/2$ the angle x already lies in the range of arcsin. Thus arcsin(sin x)=x. Draw the rising segment between the interval's endpoints.



On the principal interval the output equals the input.

2. Fold the next interval

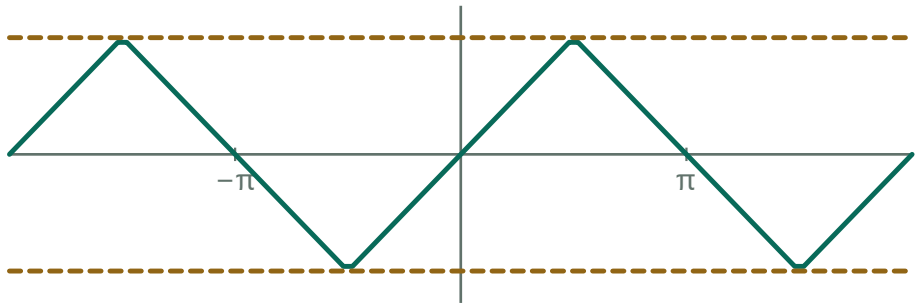
For $\pi/2 \le x \le 3\pi/2$, the angle $\pi - x$ lies in the allowed inverse range and has the same sine as x. The graph there is $y = \pi - x$, a falling line segment.



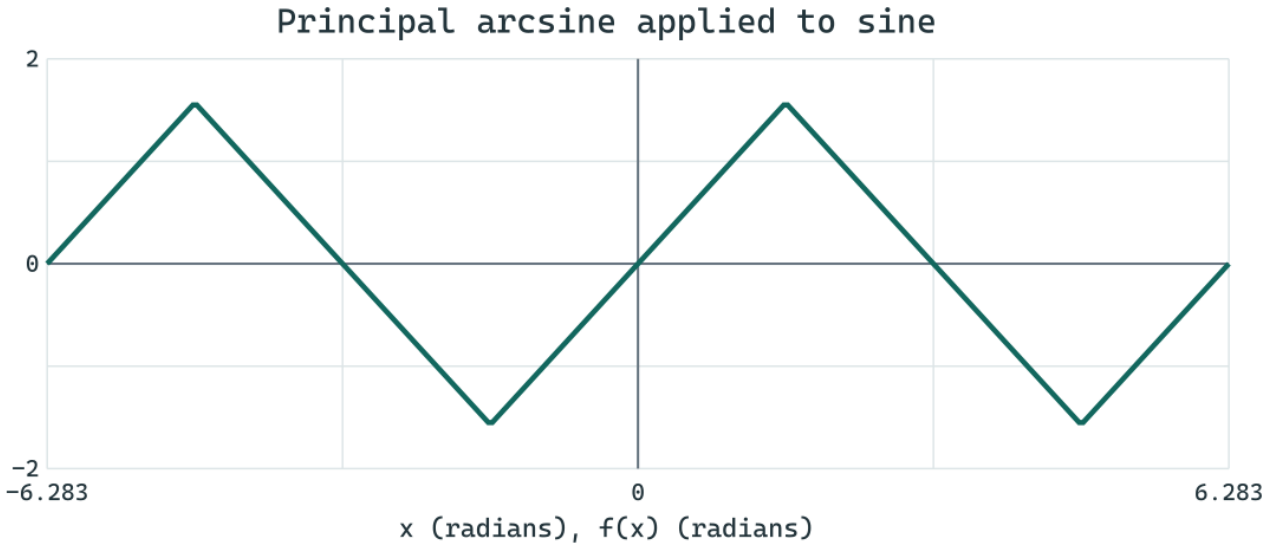
Past π/2 the principal angle folds back toward zero.

3. Repeat the triangular pattern

Sine repeats after 2π, so copy both segments horizontally with period 2π. The peaks π/2 occur at π/2+2kπ and the troughs −π/2 at −π/2+2kπ. Their spacing makes 2π the least period.



The triangular graph remains between −π/2 and π/2.



Check. The range lies entirely inside the principal inverse-sine interval.

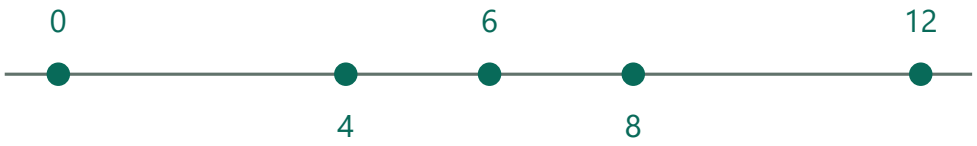
Exercise 691 · A common period

Printed page 76 · Periodic functions

Given that the period of function f is 6, and the period of function g is 4, what can be said about the periodicity of the function h defined by $h(x) = f(x) + g(x)$?

1. Find a shared shift

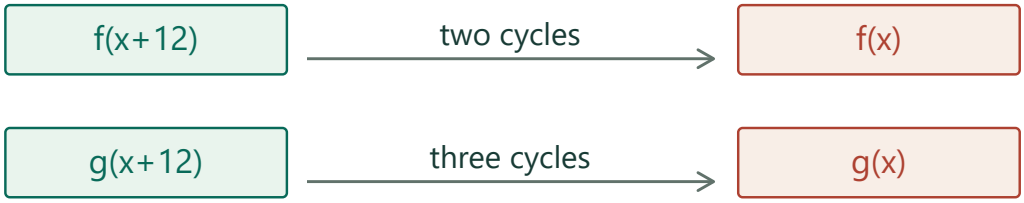
Increasing x by 12 advances f by two periods of 6 and g by three periods of 4. Each returns to its original value.



A shift of 12 includes two 6-unit and three 4-unit periods.

2. Apply it to the sum

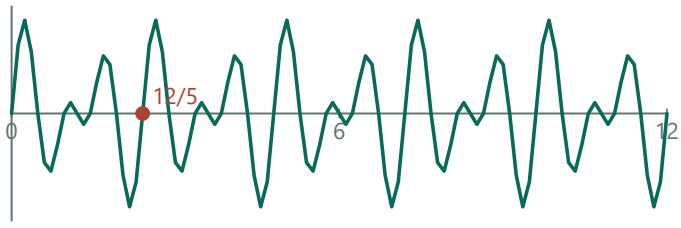
$h(x+12)=f(x+12)+g(x+12)=f(x)+g(x)=h(x)$. Therefore 12 is always a period. Showing a least period would require excluding every shorter positive repeat.



Their sum must also repeat after 12.

3. Test why least periods can change

Let $\theta=2\pi x/12$, $f=\sin(10\theta)+\sin(6\theta)$ and $g=\sin(15\theta)-\sin(6\theta)$. The integer frequencies in f and g have greatest common divisors 2 and 3, giving least periods 6 and 4. In h the common terms cancel, leaving frequencies 10 and 15; their greatest common divisor 5 gives least period 12/5.



In the cancellation example the combined wave repeats sooner.

1. For a period T, each surviving frequency m has mT/12 an integer.
Distinct sine and cosine harmonics are independent, so each phase must complete whole turns.

2. 10T/12, 6T/12 integers ⇒ 2T/12 is an integer
Use $2=2\cdot6-10$. The smallest positive T satisfying both is 6.

3. 15T/12, 6T/12 integers ⇒ 3T/12 is an integer
Use $3=15-2\cdot6$. The smallest possible T for g is 4.

4. $h=\sin(10\theta)+\sin(15\theta)$ ⇒ 5T/12 is an integer
Subtract the surviving frequency conditions; $T=12/5$ works in both.

Check. Even fundamental periods 6 and 4 need not give least period 12 for the sum. Put θ=2πx/12, f=sin(10θ)+sin(6θ), and g=sin(15θ)−sin(6θ). The integer frequencies have greatest common divisors 2 and 3, giving respective least periods 6 and 4. The common terms cancel, leaving frequencies 10 and 15 with greatest common divisor 5, so the sum has least period 12/5.

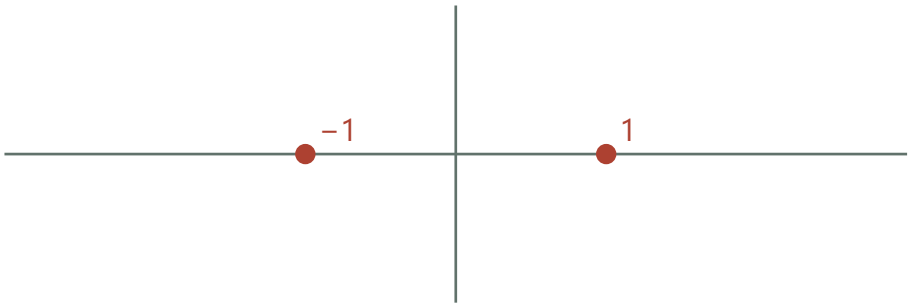
Exercise 692 · A hyperbola at two scales

Printed page 76 · Hyperbolas / Asymptotes

Graph $\frac{x^2}{1} - \frac{y^2}{3} = 1$ in windows of various sizes. Include the small window determined by $-5 \le x \le 5$, $-5 \le y \le 5$ and the large window with $-1000 \le x \le 1000$, $-1000 \le y \le 1000$. Describe the significant features of this graph, which is an example of a *hyperbola*.

1. Locate the vertices before zooming

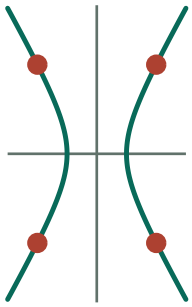
Rewrite as $y^2=3x^2-3$, so real y requires $|x|\ge1$. At $x=\pm1$ both branches meet the horizontal axis; switching the sign of either coordinate preserves the equation.



Real heights occur only for |x| ≥ 1.

2. Sketch within the small window

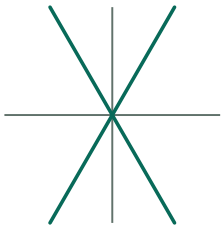
Mark the vertices (±1,0) and the four points (±2,±3). Join the points on each side into a separate smooth branch opening left or right, symmetric in both axes.



A close view separates the two branches and their vertices.

3. Compare with the large window

Far from the origin, the constant -3 in y^2 is small compared with $3x^2$. The branch directions approach $y=\pm\sqrt{3}x$. At the ±1000 scale the gap between the vertices is tiny and the curves look nearly straight.



In a ±1000 window the central gap is visually tiny.

1. $\sqrt{3}\,\mathrm{abs}(x)-\mathrm{abs}(y)=\sqrt{(3x^2)}-\sqrt{(3x^2-3)}$
Compare the asymptote and curve at the same horizontal position, using nonnegative heights.

2. $=3/[\sqrt{(3x^2)}+\sqrt{(3x^2-3)}]$
Multiply by the conjugate and use $3x^2-(3x^2-3)=3$.

3. gap→0 as abs(x)→∞
The fixed numerator is divided by a denominator that grows without bound.



Check. Solving gives $y=\pm\sqrt{(3x^2-3)}$, so real points require $x\le-1$ or $x\ge1$.

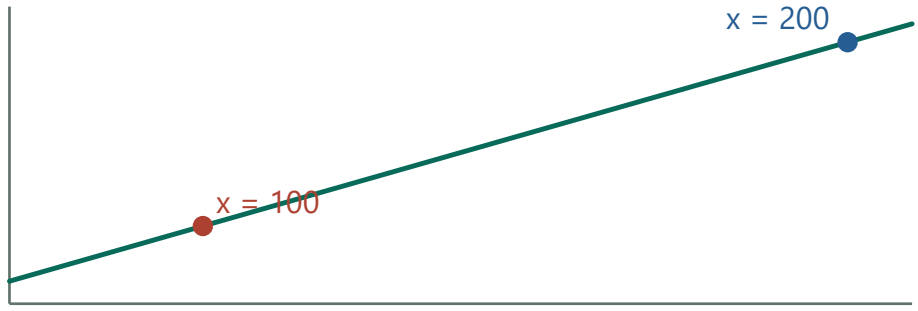
Exercise 693 · A limiting chord slope

Printed page 76 · Hyperbolas / Asymptotes / Limits and convergence

The part of the hyperbola, $\frac{x^2}{1}-\frac{y^2}{3}=1$, above the x-axis is described by $y=\sqrt{3x^2-3}$. Choose two first-quadrant points on the graph of $y=\sqrt{3x^2-3}$ that have large x-values, and calculate the slope of the line they determine. You should find that the slope is close to 1.732. Now notice that $\sqrt{3x^2-3}$ is virtually the same as $x\sqrt{3}$ when x is large and positive. Give an explanation. Because the line $y=x\sqrt{3}$ becomes indistinguishable from the hyperbola, it is called an *asymptote* for the hyperbola, and the two graphs are said to be *asymptotic*.

1. Choose points on the actual curve

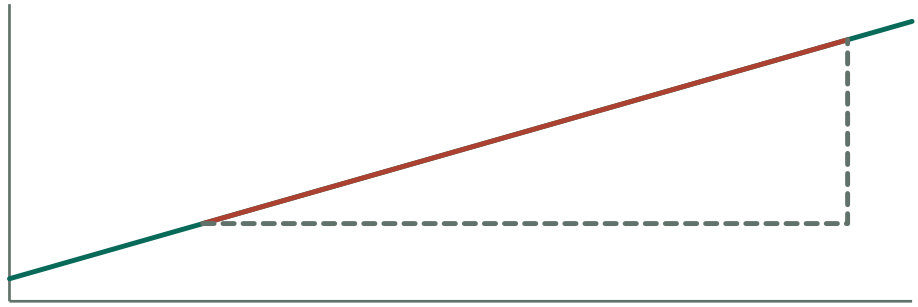
In quadrant I take $x=100$ and $x=200$. Their ordinates are $\sqrt{29997}$ and $\sqrt{119997}$, both from $y=\sqrt{(3x^2-3)}$.



Choose two genuine upper-branch coordinates.

2. Measure the secant slope

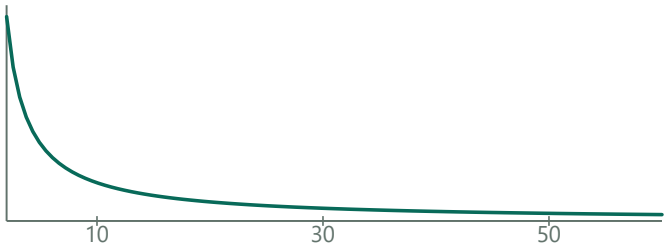
Rise divided by run is $(\sqrt{119997}-\sqrt{29997})/(200-100)\approx 1.73209$, close to $\sqrt{3}\approx 1.73205$.



Divide the rise of this secant by its horizontal run of 100.

3. Prove that the gap vanishes

Multiply $\sqrt{3}\cdot x-\sqrt{(3x^2-3)}$ by its conjugate over itself. The result is $3/[\sqrt{3}\cdot x+\sqrt{(3x^2-3)}]$, which tends to zero as x increases. Thus the curve approaches the actual line $y=\sqrt{3}\cdot x$ in vertical distance.



Plot the vertical gap $\sqrt{3}x-\sqrt{(3x^2-3)}$; it tends to zero.

1. Let $a=\sqrt{3}\cdot x$, $b=\sqrt{(3x^2-3)}$. Then $a^2-b^2=3$.
The squared leading terms cancel, leaving the constant gap of squares.

2. $a-b=(a^2-b^2)/(a+b)=3/(a+b)$
The denominator is positive on the first-quadrant branch.

3. $3/(a+b)\to 0$ as $x\to\infty$
Both terms in the denominator grow; the vertical separation therefore vanishes.

Check. Dividing the ordinate by x gives $\sqrt{(3-3/x^2)}\to\sqrt{3}$.

Exercise 694 · Logs turn multiplication into addition

Consider a geometric sequence of positive terms a, ar, ar^2, $\cdots$. Make a new list by writing down the logarithms of these terms. This new list is an example of what is called an *arithmetic sequence*. What special property does it have?

1. Translate the positivity assumption

A real logarithm needs a positive argument. The given positive sequence guarantees this at every index. Its first two terms also identify the signs of the initial value and the ratio.

1 **$a_1 = a > 0, \quad a_2 = ar > 0$**
Read the first two positive terms of the sequence.

2 **$r = a_2/a_1 > 0$**
A quotient of positive numbers is positive.

3 **$a_n = ar^{(n-1)}, \quad n = 1, 2, \ldots$**
Term n has n−1 copies of the ratio after the initial a.

2. Rewrite a typical log term

Choose a fixed real logarithm base B with B>0 and B≠1. The original assumptions a>0 and r>0 justify taking logs of both factors separately.

1 **$L_n = \log_B(ar^{(n-1)})$**
Apply the same logarithm at every index.

2 **$L_n = \log_B a + \log_B(r^{(n-1)})$**
The input is a product of positive factors.

3 **$L_n = \log_B a + (n-1)\log_B r$**
Bring the exponent outside the second logarithm.

3. Identify the defining arithmetic property

The difference of any two consecutive transformed terms is independent of n. This includes a constant original sequence, where r=1 and the difference is zero.

1 **$L_{n+1} = \log_B a + n \log_B r$**
Replace n by n+1 in the formula.

2 **$L_{n+1} - L_n = n \log_B r - (n-1) \log_B r$**
Subtract the earlier term and cancel log_B a.

3 **$L_{n+1} - L_n = \log_B r$**
Exactly one increment remains at every index.

Check. Exponentiating consecutive log terms multiplies the original terms by r.

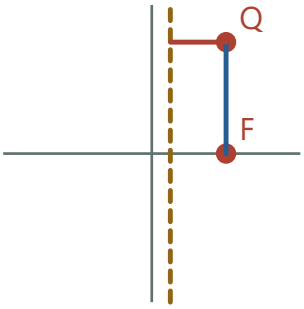
Exercise 695 · A focus and a line

Printed page 77 · Hyperbolas / Foci and directrices / Eccentricity

A new look at a familiar hyperbola: Draw the vertical line $x = 1/2$, and let F be $(2, 0)$. Verify that the distance from the point $Q = (2, 3)$ to F is twice the distance from Q to the line $x = 1/2$. Find other points that have the same property. Write an equation that describes all such points $P = (x, y)$. By analogy with ellipses, this hyperbola is said to have *eccentricity* 2. Explain.

1. Measure to the line perpendicularly

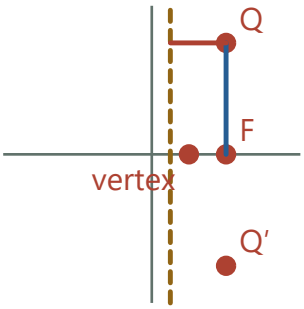
For Q=(2,3) and F=(2,0), QF=3. The shortest path to x=.5 is horizontal with length 2−.5=1.5. Thus QF is twice that distance.



The focal segment is 3; the horizontal directrix gap is 1.5.

2. Find other examples

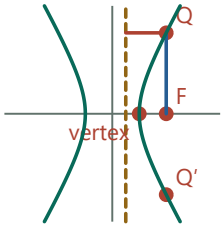
At (2,−3) the same distances follow by symmetry. At (1,0) the distances to F and the directrix are 1 and .5, again in ratio 2.



Reflection and the near vertex give two more examples.

3. Write the full distance condition

For P=(x,y), impose $\sqrt{((x-2)^2+y^2)}=2|x-.5|$. Both sides are nonnegative, so squaring is equivalent. Expansion yields $(x-2)^2+y^2=4(x-.5)^2$, or $x^2-y^2/3=1$.



The resulting squared equation describes a hyperbola.

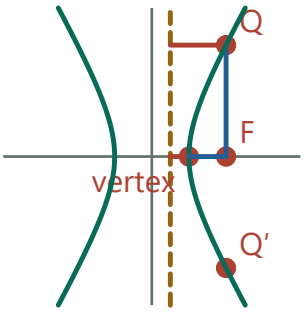
1. $x^2-4x+4+y^2=4(x^2-x+1/4)$
Expand both sides of the squared distance equation.

2. $x^2-4x+4+y^2=4x^2-4x+1 \Rightarrow 3+y^2=3x^2$
Distribute 4, then cancel the common linear terms.

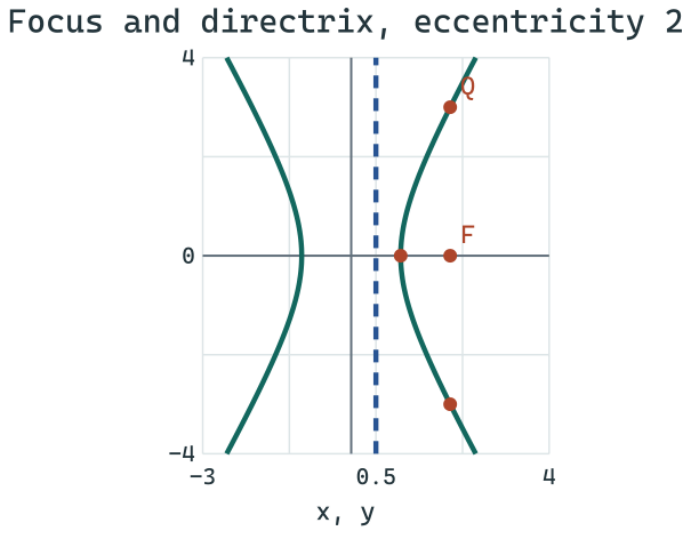
3. $3x^2-y^2=3 \Rightarrow x^2-y^2/3=1$
Collect terms and divide by 3. The squaring introduced no sign ambiguity.

4. Read the eccentricity

The focal distance divided by directrix distance equals 2 throughout the locus. This constant ratio is the eccentricity; the standard hyperbola also gives $c/a=2/1$.



The ratio of paired distances is always 2.



Dashed line: x = 1/2. For Q, the focal and directrix distances are 3 and 1.5.

Check. Here a=1,c=2, independently giving e=c/a=2.

Exercise 696 · Coffee cooling from measurements

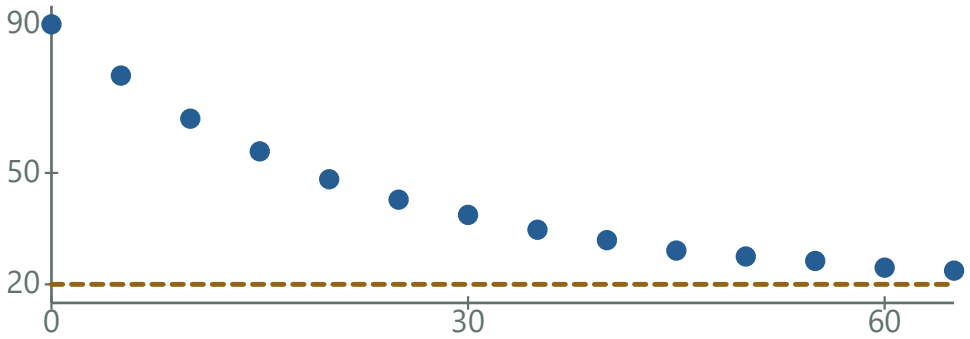
Printed page 77 · Exponential growth and decay / Mathematical modeling

The data at right displays how the temperature (in degrees Celsius) of a cup of coffee diminishes as time (in minutes) increases. *Newton's Law of Cooling* says that the rate at which an object's temperature changes is proportional to the difference between the temperature of the object and the ambient (surrounding) temperature, which is 20 degrees in this example. Create a third column of data by subtracting 20 from each entry in the second column. Create a fourth column of data by dividing each entry in the third column by the entry immediately above it. What conclusion do you draw from the entries in the fourth column? Use your findings to write a formula that describes the temperature of the coffee as a function of time.

time	*temp*
0.0	90.0
5.0	76.2
10.0	64.6
15.0	55.8
20.0	48.3
25.0	42.8
30.0	38.7
35.0	34.7
40.0	31.9
45.0	29.1
50.0	27.5
55.0	26.3
60.0	24.5
65.0	23.7

1. Measure excess above room temperature

The 20°C background remains when cooling ends. Subtract it from each measured temperature. The excesses are 70,56.2,44.6,35.8,28.3,22.8,18.7,14.7,11.9,9.1,7.5,6.3,4.5,3.7°C.



The vertical distance above 20°C is excess temperature.

2. Compare equal five-minute intervals

Starting at the second row, divide each excess by the preceding one. For instance 56.2/70≈.803. The ratios are .803,.794,.803,.791,.806,.820,.786,.810,.765,.824,.840,.714,.822. There is no previous reading for the first row.



Compare consecutive excesses over equal five-minute intervals.

time	temp
0.0	90.0
5.0	76.2
10.0	64.6
15.0	55.8
20.0	48.3
25.0	42.8
30.0	38.7
35.0	34.7
40.0	31.9
45.0	29.1
50.0	27.5
55.0	26.3
60.0	24.5
65.0	23.7

Time intervals in the left column

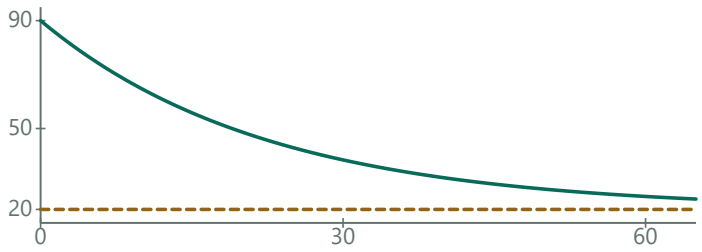
Follow the time entries downward. Equal differences between successive times let us compare what fraction survives in each interval.

Cooling measurements relative to room temperature (20 °C)

Time (min)	Reading (°C)	Excess (°C)	Excess / prior excess
0	90.0	70.0	No prior reading
5	76.2	56.2	0.803
10	64.6	44.6	0.794
15	55.8	35.8	0.803
20	48.3	28.3	0.791
25	42.8	22.8	0.806
30	38.7	18.7	0.820
35	34.7	14.7	0.786
40	31.9	11.9	0.810
45	29.1	9.1	0.765
50	27.5	7.5	0.824
55	26.3	6.3	0.840
60	24.5	4.5	0.714
65	23.7	3.7	0.822

3. Use the recurring fraction

The ratios cluster around .8: about 80% of the excess remains after each five minutes. For t minutes there are t/5 such periods. Starting excess is 70, giving T(t)≈20+70(.8)^(t/5) °C.



The fitted excess retains about 80% every five minutes.

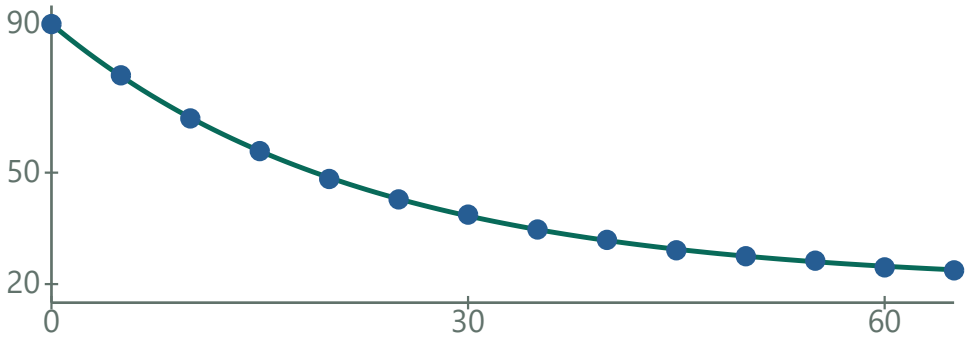
1. $E_0=90-20=70$; $E_1\approx.8\cdot70=56$
E counts excess degrees above the room temperature. One five-minute interval retains about four fifths of that excess.

2. $E_2\approx.8\cdot56=70(.8)^2$
Use the same multiplier on the updated excess in the second interval.

3. k=t/5 ⇒ T(t)≈20+70(.8)^(t/5)
There are t/5 five-minute intervals. Add 20 back to turn an excess into a temperature.

4. Test against the measurements

The model starts at 90°C, approaches 20°C, and at 65 minutes predicts about 23.85°C, close to the recorded 23.7°C. Measurement variation explains why one fitted multiplier does not reproduce every row exactly.



Overlay actual readings and the smooth model.

time	*temp*
0.0	90.0
5.0	76.2
10.0	64.6
15.0	55.8
20.0	48.3
25.0	42.8
30.0	38.7
35.0	34.7
40.0	31.9
45.0	29.1
50.0	27.5
55.0	26.3
60.0	24.5
65.0	23.7

Recorded temperature column

These are measured temperatures. Subtract the stable room temperature from each reading before comparing cooling factors.

Check. At 65 minutes the model gives about 23.85°C, close to the observed 23.7°C; the readings are approximate.

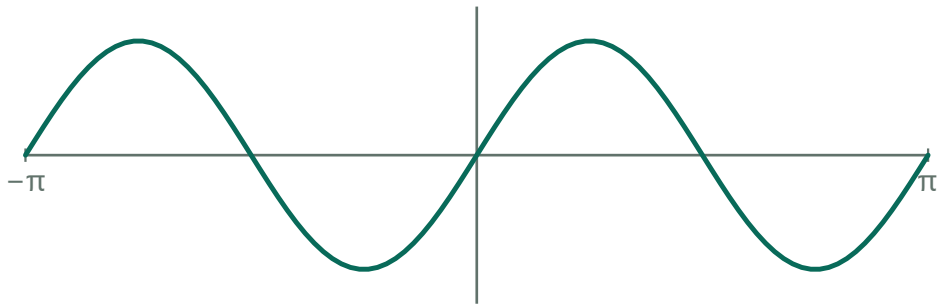
Exercise 697 · A double-angle wave

Printed page 77 · Sinusoidal functions / Trigonometric identities

Given that $f(x) = \sin(x)\cos(x)$, find the period of function f, and the range of its values. Do you think that f is sinusoidal? Explain.

1. Recognize a product of two trig functions

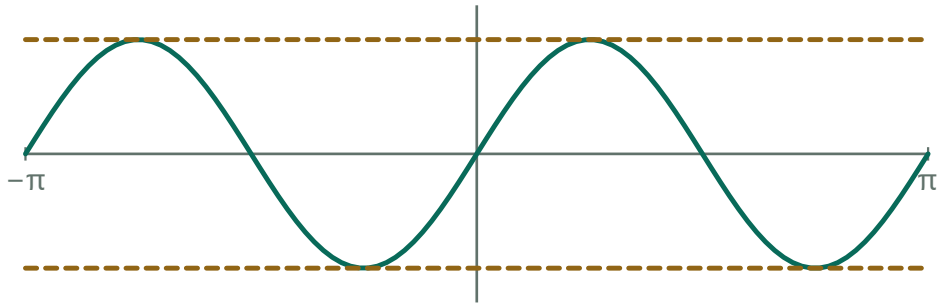
The double-angle identity reads sin(2x)=2sin x cos x. Divide it by 2 to isolate the product in f.



Multiplying sine and cosine produces a double-angle wave.

2. Identify amplitude and period

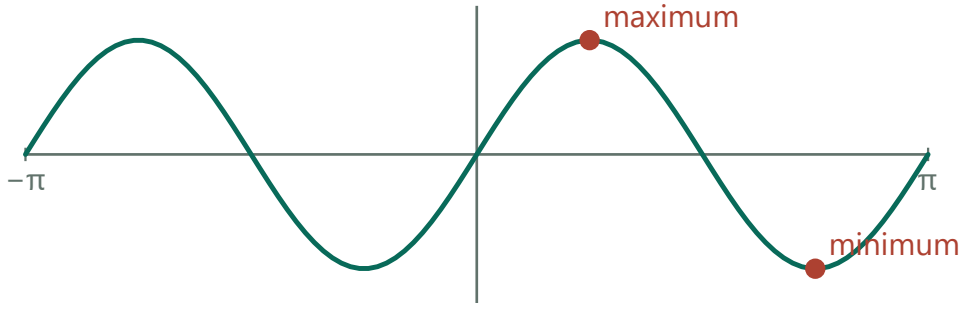
Now f=(1/2)sin(2x). The multiplier outside gives amplitude 1/2; the doubled angle completes a whole turn in π radians, or 180°.



Half the double-angle sine ranges from −1/2 to 1/2.

3. Read its shape and extremes

This is exactly a sinusoid. Its range is [−1/2,1/2]; for example x=π/4 gives sin(2x)=1 and realizes the maximum.



At π/4 the doubled angle reaches π/2.

Check. At x=π/4 both factors are √2/2 and the maximum .5 is attained.

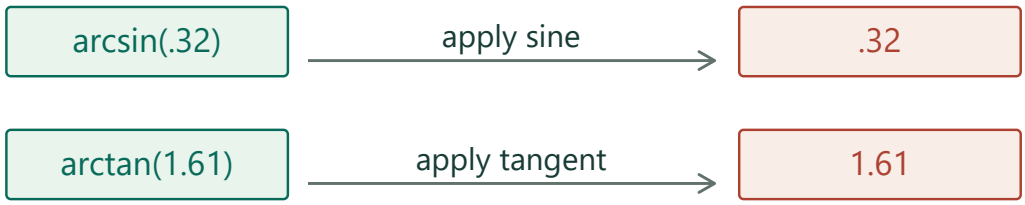
Exercise 698 · Principal values by hand

Without using a calculator, simplify the following (in degrees):

(a) $\sin\left(\sin^{-1} 0.32\right)$ **(b)** $\tan\left(\tan^{-1} 1.61\right)$ **(c)** $\cos^{-1}(\cos 123)$ **(d)** $\sin^{-1}(\sin 137)$

1. Undo an inverse with its original function (a,b)

arcsin(.32) is defined as an angle with sine .32, so taking its sine gives .32. Likewise tangent applied to arctan(1.61) returns 1.61. Read each expression from the inside outward. The inner result is an angle; the outer operation takes a trigonometric ratio of that angle. The outputs in (a) and (b) are numbers, not angles in degrees.



Apply the original trig function after taking its principal inverse.

2. Check the cosine principal interval (c)

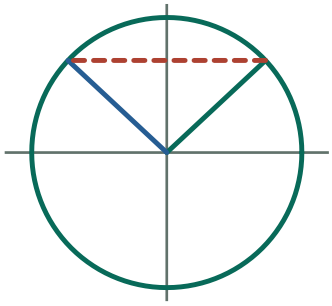
arccos(cos θ) returns θ itself when $0° \le \theta \le 180°$. Since 123° lies in this interval, the result is 123°.



123° already belongs to the arccos output interval.

3. Reflect a sine angle into range (d)

The principal arcsine must lie in $[-90°, 90°]$. Angle $137°$ has the same sine as $180° - 137° = 43°$, which lies in range. The result is $43°$. On the unit circle the points at $137°$ and $43°$ are mirror images across the vertical axis: they have the same height. This is why sine cannot distinguish them and its inverse has to choose a principal angle.



43° and 137° share a height; only 43° is a principal arcsine.

Check. 137° and 43° have the same unit-circle height.

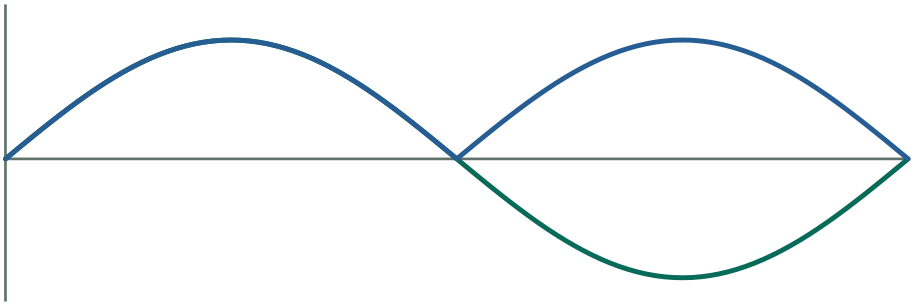
Exercise 699 · Construct a nonsinusoidal wave

Printed page 77 · Periodic functions / Absolute value

By means of a formula, invent a periodic but *non-sinusoidal* function f, whose period is 10, and whose values $f(x)$ oscillate between the extremes 5 and 9.

1. Start with a folded wave

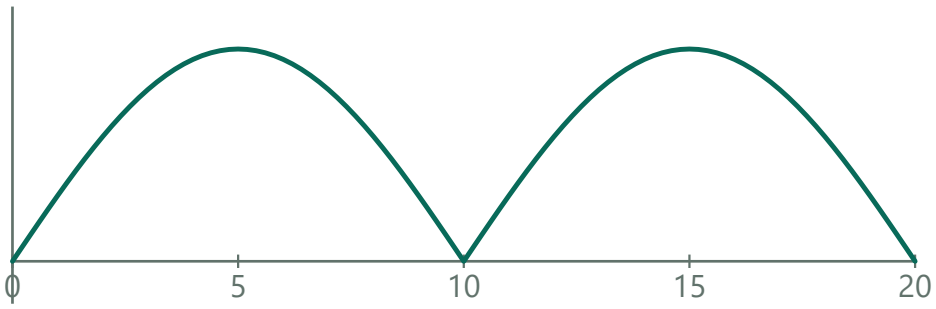
For a non-sinusoidal example use the absolute value of sine, which reflects its negative arches upward and leaves sharp corners at its zeros.



Absolute value folds each negative arc upward.

2. Choose the repeat distance

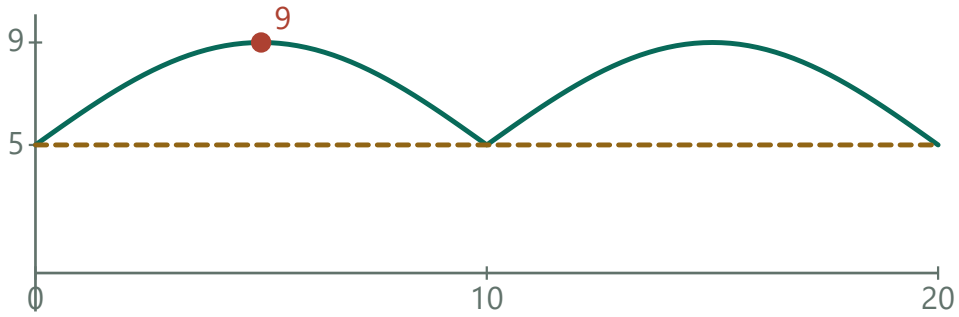
In radians |sin u| repeats every π. Taking u=πx/10 makes the fundamental period in x equal to 10.



The same folded shape now repeats every ten input units.

3. Scale its heights

The folded sine ranges from 0 to 1. Multiply by 4 and add 5: f(x)=5+4|sin(πx/10)|. It reaches 5 at 0 and 9 at 5; its corners distinguish it from a sinusoid.



Scale by 4 and raise by 5, preserving the sharp joins.

Check. Corners at multiples of 10 prevent the graph from being a sinusoid.

Exercise 700 · All possible stopping times

Printed page 77 · Probability / Geometric sequences and series

Courtney is about to start running laps on the track. At the beginning of each lap, there is an 8% chance that Courtney will decide to stop for the day, and a 92% chance that another lap will be run. Thus the probability that Courtney runs *exactly* two laps is $(0.92)(0.92)(0.08)$. Give two justifications for the equation

$$1 = (0.08) + (0.92)(0.08) + (0.92)^2(0.08) + (0.92)^3(0.08) + \cdots$$

1. Name the event in each term

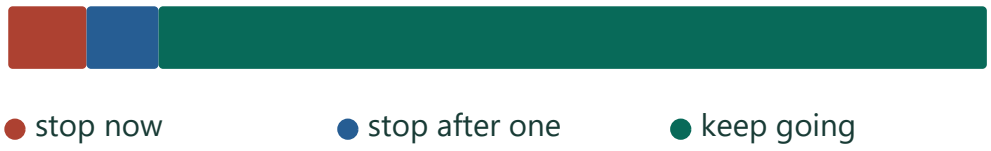
Exactly n laps means continuing n times and stopping on the next decision. Its probability is $(.92)^n(.08)$, including n=0 for an immediate stop.



The two possible decisions account for the full probability.

2. Use disjoint events

Two different stopping counts cannot both occur. The only omitted possibility is never stopping, whose probability is bounded by $.92^n$ for every n and hence is zero. Adding all stopping probabilities gives 1.



Each branch separates a new stopping event from the remainder.

3. Use a geometric sum independently

The first term is .08 and the ratio is .92. The sum .08/(1−.92) is independently 1. Through n laps the finite total is 1−.92^(n+1), leaving precisely the chance of continuing n+1 times.



After ten decisions the remaining continuing probability is 0.92^{10}.

Check. The partial sum through n laps is 1−.92^(n+1), with the remaining probability exactly that of surviving n+1 decisions.

Exercise 701 · General stopping probability

(Continuation) Given that p and q are positive numbers whose sum is 1, find the sum of the infinite series $q + pq + p^2q + p^3q + \cdots$.

1. Locate the geometric ratio

Successive terms $q, pq, p^2q, \ldots$ are multiplied by p. Since p, q are positive and $p+q=1$, p lies strictly between zero and one.



The two possible decisions account for the full probability.

2. Write a finite partial sum

The first N terms sum to $q(1-p^N)/(1-p)$. The relation $q=1-p$ reduces this to $1-p^N$.



Each branch separates a new stopping event from the remainder.

3. Take the limit

As N grows, p^N tends to zero. The infinite series therefore sums to 1.



Example p = 0.7 and N = 10. The continuing chance is 0.7^{10}.

Check. The first N terms sum to $1-p^N$, approaching 1.

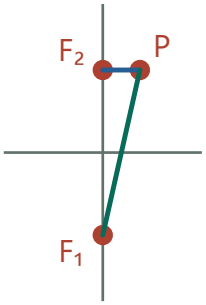
Exercise 702 · Vertical focal-difference hyperbola

Printed page 77 · Hyperbolas / Foci and directrices

Let $F_1 = (0, -5)$ and $F_2 = (0, 5)$. The equation $|PF_1 - PF_2| = 8$ is satisfied by many points $P = (x, y)$. Identify and plot the two points P that are on the y-axis, and the four points P whose y-coordinate is ± 5. Write and simplify a Cartesian equation that describes this *hyperbola*. (Hint: It may help to start with the equation $PF_1 = PF_2 \pm 8$.) For large values of x and y, the hyperbola is indistinguishable from a pair of lines that intersect at the origin. Find equations for these *asymptotes*. Sketch a graph of this *hyperbola*.

1. Resolve the sign on the upper branch

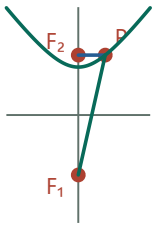
With foci (0,−5),(0,5), let d be the distance to the upper focus for a point above the center. The lower focus is farther by 8, so its distance is d+8.



At an upper-branch point the lower focus is farther away.

2. Eliminate the distances

Subtract the squares from the coordinate formulas: $(y+5)^2-(y-5)^2=20y$. From $(d+8)^2-d^2=16d+64$ get $d=5y/4-4$. Substituting into $d^2=x^2+(y-5)^2$ yields $y^2/16-x^2/9=1$.



The upper branch satisfies $y^2/16 - x^2/9 = 1$.

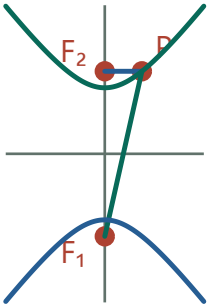
1. $20y=16d+64 \Rightarrow d=(20y-64)/16=5y/4-4$
Subtract the squared distances to isolate the upper focal distance.

2. $25y^2/16-10y+16=x^2+y^2-10y+25$
Expand $d^2=(5y/4-4)^2$ and $x^2+(y-5)^2$.

3. $9y^2/16-x^2=9 \Rightarrow y^2/16-x^2/9=1$
Cancel $-10y$, move terms to opposite sides and divide by 9.

3. Check squaring in reverse

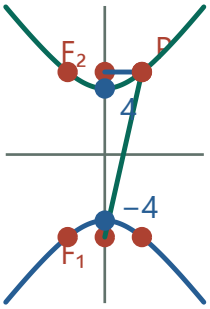
On the upper branch $y\ge4$. Substitution into the original squared distances gives $(5y/4-4)^2$ and $(5y/4+4)^2$. Their bases are nonnegative, so the distances differ by exactly 8. Reflect in the x-axis for the lower branch.



Reflecting swaps the distance roles and adds the lower branch.

4. Mark six points

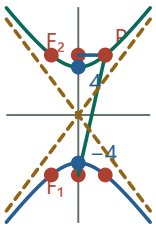
With $x=0$ obtain $(0,\pm4)$. At $y=\pm5$, $x^2/9=25/16-1=9/16$, so $x=\pm9/4$. All four sign combinations give the remaining points.



Mark the two vertices and four points with $|y| = 5$.

5. Draw the limiting directions

The equation gives |y|=(4/3)√(x²+9). For large |x| this approaches (4/3)|x| in vertical distance, so the asymptotes are y=±4x/3. Draw branches opening upward and downward through their vertices.

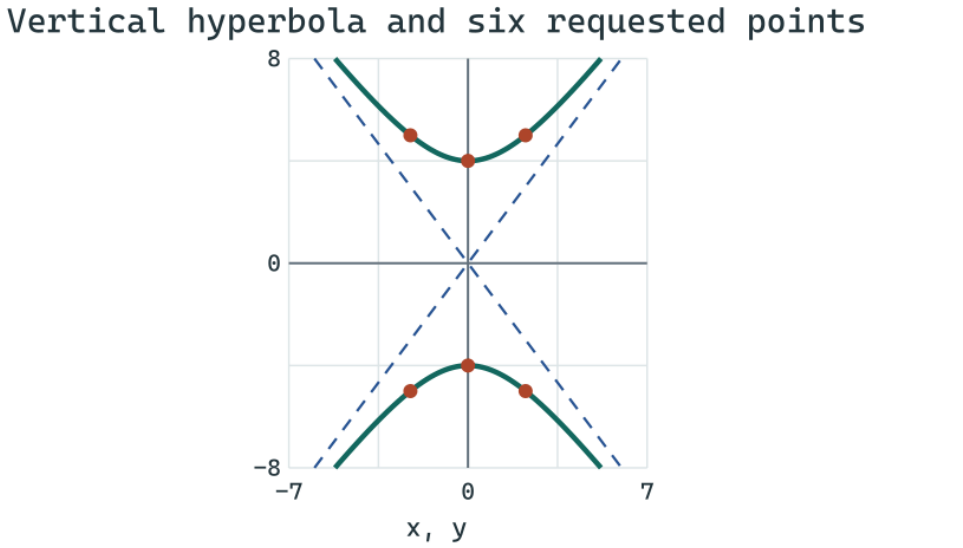


Far from the center the branches approach their diagonal guides.

1. abs(y)−4abs(x)/3=(4/3)[√(x²+9)−abs(x)]
Subtract the asymptote height from the branch height.

2. √(x²+9)−abs(x)=9/[√(x²+9)+abs(x)]
Multiply by the conjugate and cancel x².

3. abs(y)−4abs(x)/3→0 as abs(x)→∞
The numerator remains fixed and the denominator grows.



Check. At (9/4,5), distances are 9/4 and 41/4, differing by 8.

Exercise 703 · Angle addition via rotation

Printed page 78 · Angle addition formulas / Rotations / Matrix multiplication

You have used different methods to establish the identities

$$\cos(\alpha+\beta)=\cos\alpha\cos\beta-\sin\alpha\sin\beta \quad \text{and} \quad \sin(\alpha+\beta)=\sin\alpha\cos\beta+\cos\alpha\sin\beta.$$

(a) Justify the identities using rotation matrices.

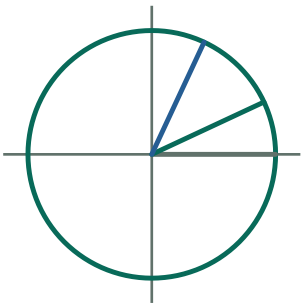
(b) Show how to use the identities above to derive

$$\cos(\alpha-\beta)=\cos\alpha\cos\beta+\sin\alpha\sin\beta \quad \text{and} \quad \sin(\alpha-\beta)=\sin\alpha\cos\beta-\cos\alpha\sin\beta.$$

(c) Compare the graphs $y=\sin 2x$ and $y=2\sin x\cos x$ and explain any observations.

1. Apply rotations to the horizontal unit vector

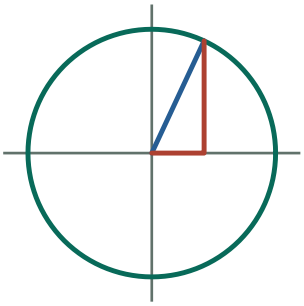
Rotate (1,0) first by β to (cosβ,sinβ). Rotating again by α applies the matrix with rows (cosα,−sinα) and (sinα,cosα). The combined rotation angle is α+β.



Example: turn by β = 25° and then α = 40°.

2. Compare the two coordinates

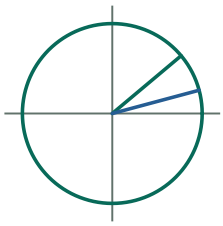
The product's first coordinate gives cos(α+β)=cosα cosβ−sinα sinβ. Its second gives sin(α+β)=sinα cosβ+cosα sinβ.



The final horizontal and vertical coordinates give cosine and sine.

3. Reverse the second angle

Replace β by −β. Cosine is even and sine is odd, so cos(α−β)=cosα cosβ+sinα sinβ and sin(α−β)=sinα cosβ−cosα sinβ.



Using −β instead turns from 40° back to 15°.

1. cos(α−β)=cos α cos(−β)−sin α sin(−β)
Use the sum identity with the second rotation reversed.

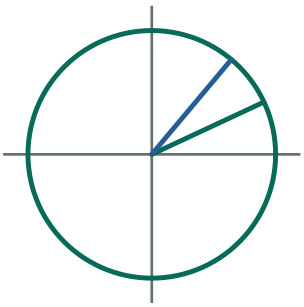
2. cos(α−β)=cos α cos β+sin α sin β
The cosine stays unchanged while the sine changes sign.

3. sin(α−β)=sin α cos(−β)+cos α sin(−β)
Make the same substitution in the second coordinate.

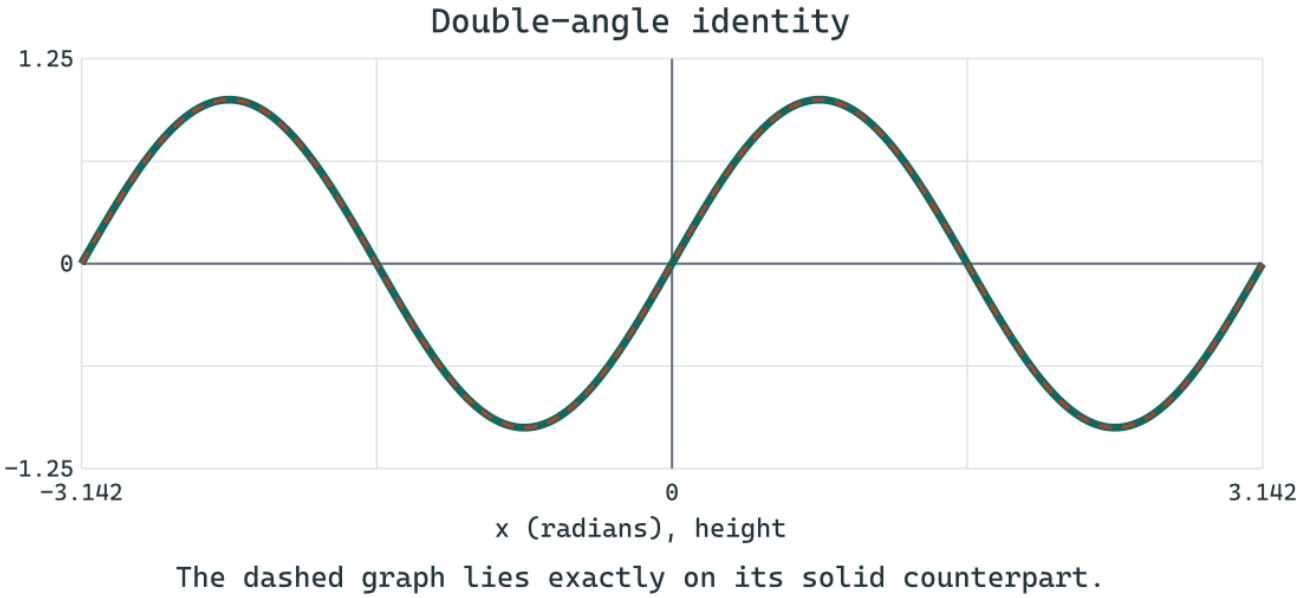
4. sin(α−β)=sin α cos β−cos α sin β
Only the term with the odd sine changes sign.

4. Double one angle

Set α=β=x in the sine sum identity. Both terms equal sin x cos x, proving sin(2x)=2sin x cos x for every x. Their plotted graphs coincide everywhere.



Taking both angles equal makes the final angle twice the first.



Check. The two coordinates of the product column have squared sum 1, as a rotation requires.

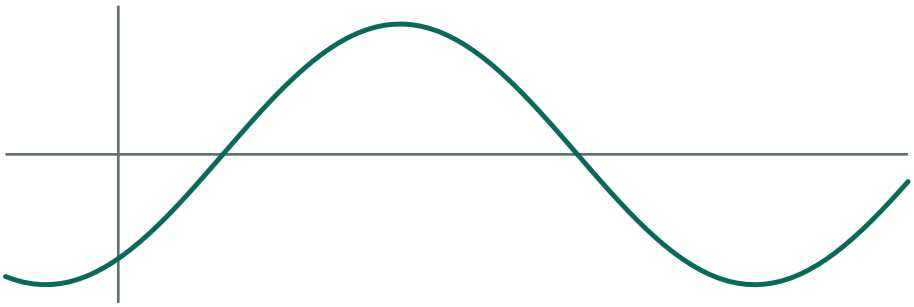
Exercise 704 · Read amplitude and phase

Printed page 78 · Sinusoidal functions / Phase shifts / Amplitude and period

Graph the curve $y = 3\sin x - 4\cos x$, which should look sinusoidal. In other words, it resembles $y = a\sin(x - p)$. Use a calculator to find approximate values for the *amplitude* a and the *phase shift* p.

1. Read a sine model from the graph

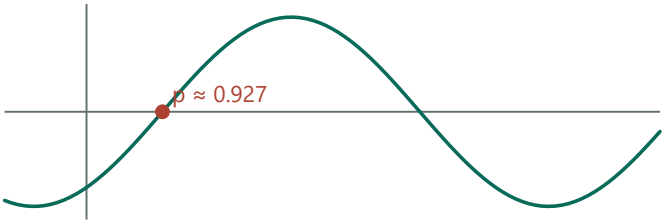
The given wave rises through its midline shortly after x=0. For the form a sin(x−p), p is that rising zero crossing and a is the peak height above the midline.



Locate a rising crossing of the horizontal midline.

2. Locate the phase

At a zero, 3sin x=4cos x, so tan x=4/3. At the quadrant-I solution the curve rises; its positive phase shift is p≈.92730 rad or 53.130°.



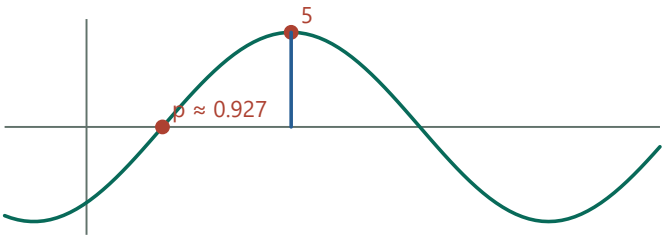
The first positive rising zero satisfies tan p = 4/3.

1. 3sin x=4cos x ⇒ 3tan x=4
At a zero, cos x cannot vanish: then sin x=±1 and the left side is nonzero.

2. x=arctan(4/3) in quadrant I
Use the rising crossing. There the slope 3cos x+4sin x is positive.

3. Determine the scale

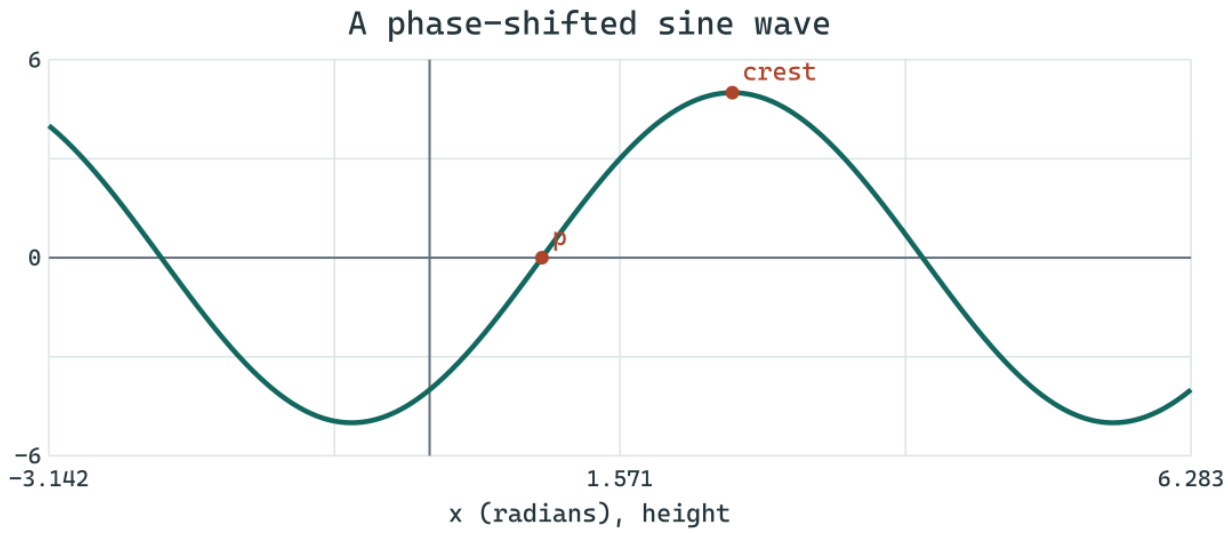
The expansion gives coefficients a cos p=3 and a sin p=4. Squaring and adding gives $a^2=9+16=25$. Thus a=5, consistent with the graphed amplitude.



A quarter turn after that crossing the height is 5.

1. $a^2\cos^2 p+a^2\sin^2 p=3^2+4^2$
Square both matched coefficient equations and add them.

2. $a^2(\cos^2 p+\sin^2 p)=25 \Rightarrow a^2=25 \Rightarrow a=5$
The factor in parentheses is 1 and amplitude is positive.



Check. At x=0 the fitted function gives −4 and rises, agreeing with the graph.

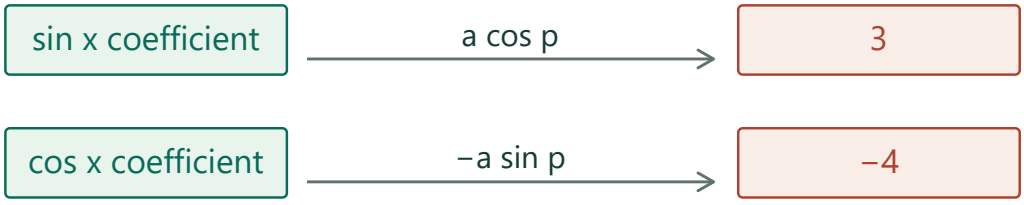
Exercise 705 · Prove the phase fit

Printed page 78 · Sinusoidal functions / Phase shifts / Angle addition formulas

(Continuation) Use the $\sin(\alpha-\beta)$ identity to show that the curve $y = 3\sin x - 4\cos x$ can be rewritten equivalently as $y = a\sin(x-p)$.

1. Expand the proposed translated sine

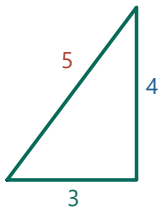
Using the difference identity gives a sin(x−p)=(a cos p)sin x−(a sin p)cos x. To match the original choose a cos p=3, a sin p=4.



Match like trig terms in the expanded expression.

2. Use the 3–4–5 triangle

The matching coefficients give $a=\sqrt{(3^2+4^2)}=5$. Both sine and cosine must be positive, so take $p=\arctan(4/3)$ in quadrant I; $\cos p=3/5$, $\sin p=4/5$.



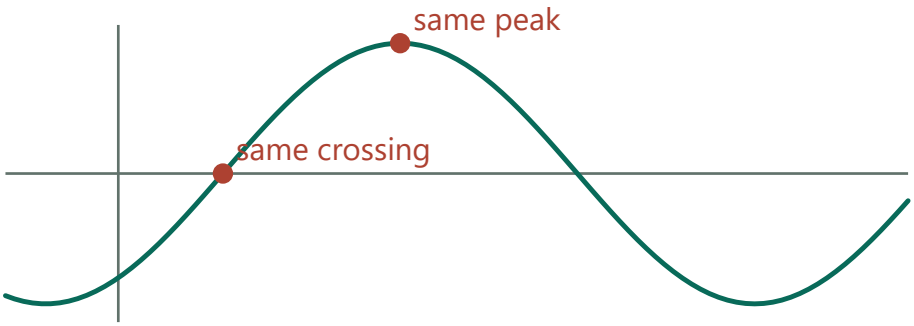
The coefficient triangle supplies $\cos p = 3/5$ and $\sin p = 4/5$.

1. $(a \sin p)/(a \cos p)=4/3 \Rightarrow \tan p=4/3$
Divide the two coefficient equations; $a \cos p=3$ is nonzero.

2. $\cos p=3/5$, $\sin p=4/5$
Divide the original coefficients by the positive amplitude 5.

3. Verify the entire function

Substitution yields $5\sin x(3/5)-5\cos x(4/5)=3\sin x-4\cos x$. Matching the coefficients establishes the identity for every x.



Matching both coefficients proves equality at every input.

Check. The 3–4–5 triangle provides both coefficients and determines the quadrant.

Exercise 706 · Audit a logarithm derivation

Printed page 78 · Logarithm rules / Domain and range

Asked to simplify the expression $\log_c k + \log_c \frac{1}{k}$, Kirby wrote the following:

$$\log_c k + \log_c \frac{1}{k} = \frac{\log k}{\log c} + \frac{\log c}{\log k} = (\log k - \log c) + (\log c - \log k) = 0$$

What do you think of Kirby's analysis?

1. Check Kirby's first change of base

Use common logs in the change-of-base fractions. The original problem requires $c>0$, $c\neq 1$ and $k>0$. In particular, $\log c$ can be a denominator. Nothing in the problem requires $k\neq 1$.

1. **$\log_c k = (\log k)/(\log c)$**
   Kirby correctly changes the base in the first term.
2. **$\log_c(1/k) = \log(1/k)/(\log c)$**
   The denominator must still be $\log c$ in the second term.
3. **$\log(1/k) = \log(k^{-1}) = -\log k$**
   The reciprocal of the argument negates its logarithm.
4. **$\log_c(1/k) = -(\log k)/(\log c)$**
   Kirby instead swapped numerator and denominator.

2. Test the next replacement

A quotient of two logarithms changes the base. Subtracting those two logarithms instead takes the logarithm of a quotient. A single legal choice of inputs exposes the error.

1. **$c = 10, \quad k = 100$**
   Both arguments and the base obey the original restrictions.
2. **$(\log k)/(\log c) = 2/1 = 2$**
   Evaluate Kirby's first change-of-base fraction.
3. **$\log k - \log c = 2 - 1 = 1$**
   It differs from 2, so the proposed replacement fails.
4. **$k = 1 \Rightarrow \log k = 0$**
   Kirby's extra denominator would even reject a permitted input.

3. Recover the correct value

Kirby's last number agrees with the original sum. The valid route combines the positive arguments before the common logarithm is evaluated.

1 **log_c k + log_c(1/k) = log_c(k · 1/k)**
Apply the product rule on k>0.

2 **log_c(k · 1/k) = log_c 1 = 0**
The factors cancel, and $c^0=1$.

Check. The original expression works at k=1, where dividing by log k would fail.

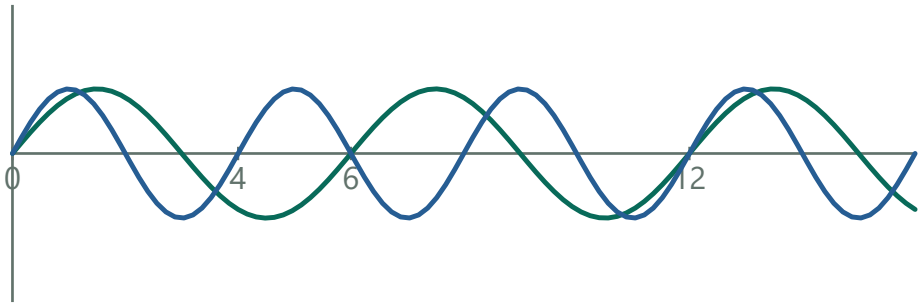
Exercise 707 · Periods of two sine waves

Printed page 78 · Periodic functions / Amplitude and period

Working in degree mode, find the periods for the three functions $f(x) = \sin(60x)$, $g(x) = \sin(90x)$, and $h(x) = f(x) + g(x)$.

1. Compute both individual periods

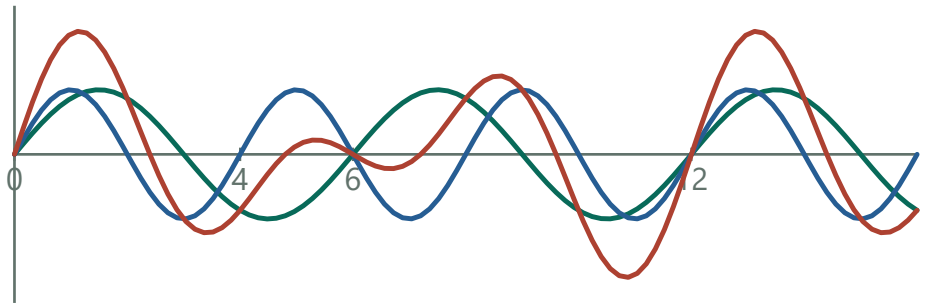
One turn is 360°. At 60° per input unit the sine period is 6 units. At 90° per unit it is 4. After 12 units both waves have completed whole cycles, so their sum repeats.



Individual waves repeat over 6 units and 4 units.

2. Test whether a shorter repeat is possible

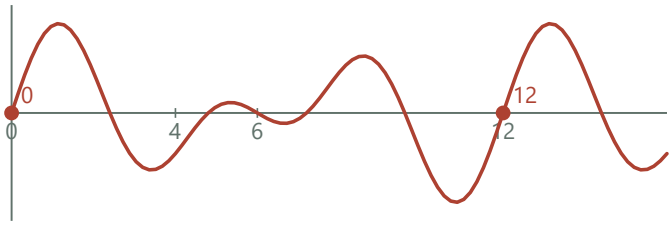
Write $z=\pi x/6$, so $h=\sin(2z)+\sin(3z)$. A shift T in x adds $\delta=\pi T/6$ to z. In the shifted wave, angle addition gives coefficients $\cos(2\delta),\sin(2\delta),\cos(3\delta),\sin(3\delta)$ for the two sine/cosine pairs.



Both independent frequency components must return to phase.

3. Determine the least common shift

The four different harmonic functions are independent: multiplying by each and integrating through a full 2π cycle isolates its coefficient. Equality requires 2δ and 3δ both be multiples of 2π. Their difference δ is then one too. Thus T must be a multiple of 12, and the least positive period is 12.



The earliest common repeat occupies 12 units.

1. $\sin(m(z+\delta))=\sin(mz)\cos(m\delta)+\cos(mz)\sin(m\delta)$
Expand the shifted terms for m=2 and m=3 separately.

2. $\cos(m\delta)=1$ and $\sin(m\delta)=0 \Rightarrow m\delta=2\pi k_m$
Each independent harmonic must regain its original sine coefficient and zero cosine coefficient.

3. $\delta=3\delta-2\delta=2\pi(k_3-k_2) \Rightarrow T=6\delta/\pi$
Subtract the two whole-turn conditions and convert back to horizontal units.

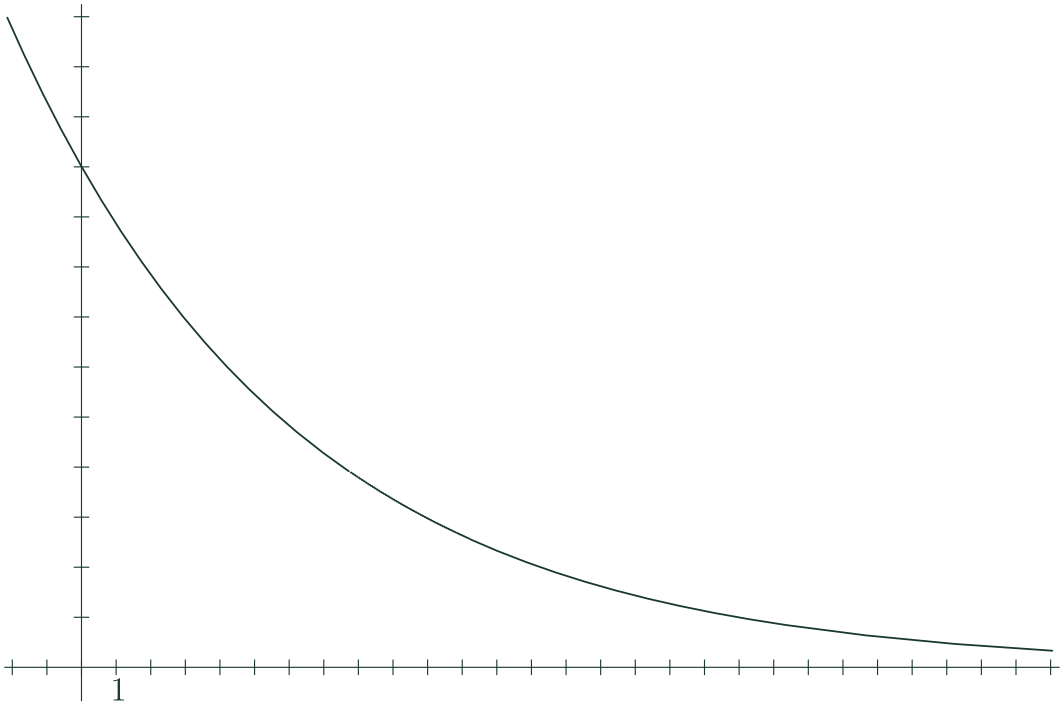
4. $T=12(k_3-k_2)$
The least positive candidate 12 gives full turns for both frequencies.

Check. Over 12 units the two terms complete exactly two and three cycles.

Exercise 708 · Measure a graph's half-life

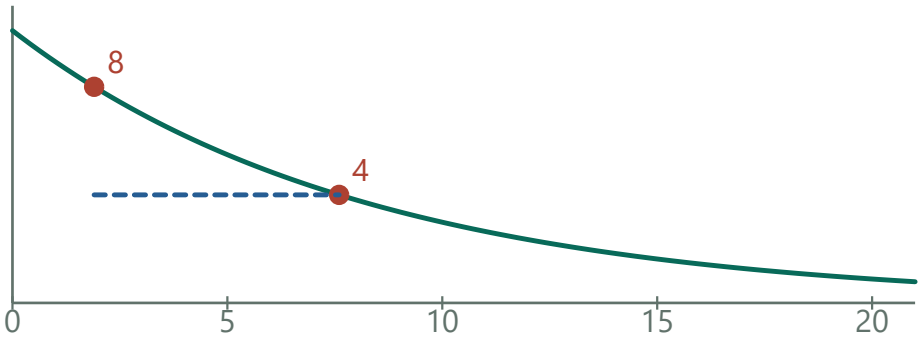
Printed page 78 · Half-life / Exponential growth and decay / Graph interpretation

The graph $y = ab^t$ of an exponential function is shown below. Using only a pencil and a ruler, calculate the half-life of this function by marking two first-quadrant points on the graph. Repeat the process using a different pair of points. Explain why you and your classmates should arrive at the same answer, no matter what choices you made.

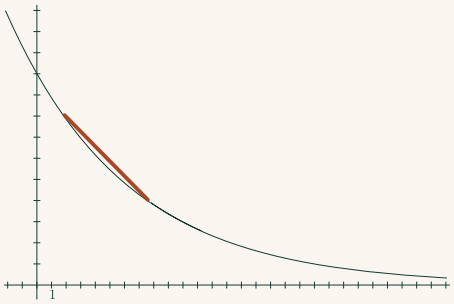


1. Mark two heights on the curve

Read positive heights using the vertical grid. Mark where the curve reaches 8 and then 4 divisions. A ruler gives horizontal coordinates of about 1.9 and 7.6 in printed tick units.



Approximate reconstruction from the printed ruler readings.

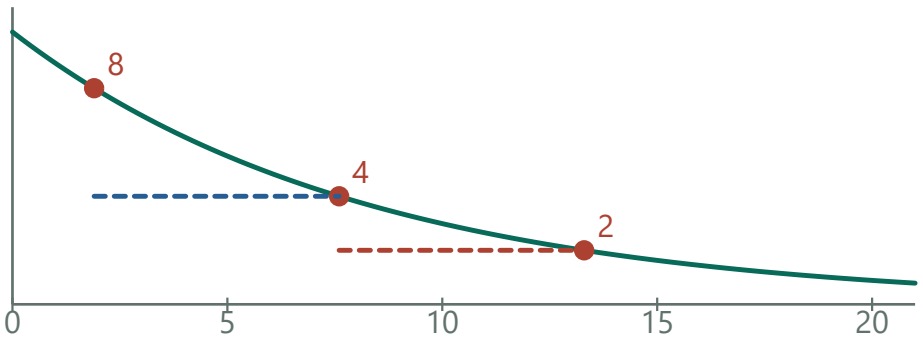


First halving interval

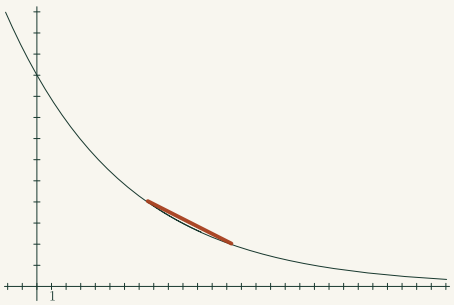
The endpoints represent heights eight and four on the vertical scale. Read their horizontal positions by dropping a vertical to the axis from each end.

2. Measure a second pair

Mark 4 and 2 divisions. Their coordinates are about 7.6 and 13.3. The two horizontal gaps are both approximately 5.7 units, within drawing and ruler precision.



Halving the next height measures the same horizontal gap.

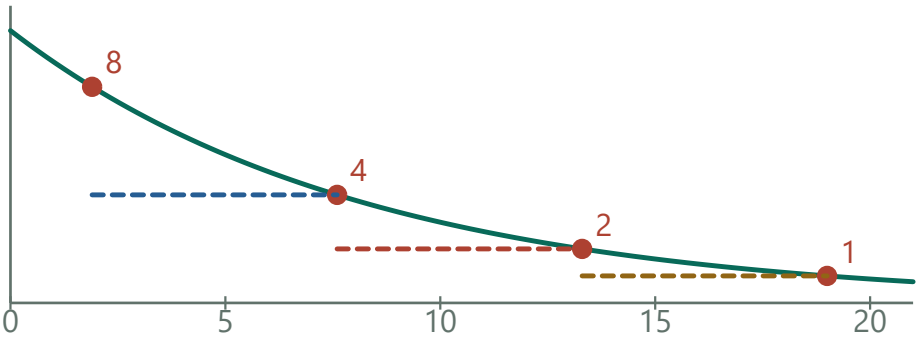


Next halving interval

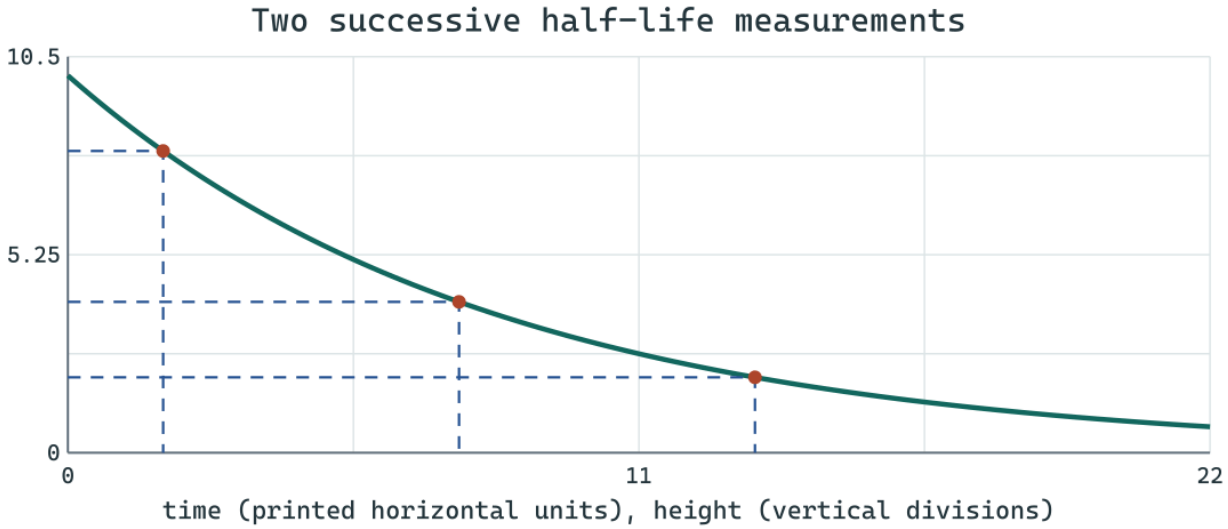
The height falls from four divisions to two. Read the difference of the two horizontal coordinates from the same scale.

3. Explain the agreement

For $y=ab^t$ the ratio of the heights at $t+h$ and t is b^h. Halving requires $b^h=\tfrac{1}{2}$, independent of t. Any pair of starting and halved heights on the curve therefore gives the same horizontal interval.



Any fixed multiplicative decrease uses an equal time interval.



Curve reconstructed from the printed scale. Adjacent marked times differ by about 5.7 units.

Check. Measured in printed tick units, approximate crossings are (1.9,8), (7.6,4) and (13.3,2). Both successive horizontal differences are about 5.7.

Exercise 709 · A particular birthday

Printed page 79 · Probability / Complementary events

Assuming that there are 365 birthdates in a year, consider the following probability questions.
(a) If you were to ask for the birthday of a random Exonian, what is the probability that the response will be the 18th of August?
(b) If you were to ask for the birthday of a random Exonian, what is the probability that it will *not* be 18 August?
(c) If you ask two random Exonians to state their birthdays, what is the probability that *neither* will say 18 August?
(d) If you ask all 1017 Exonians this birthday question, what is the probability that *no one* will say 18 August?
(e) If you ask all 1017 Exonians this birthday question, what is the probability that *someone* (that means at least one person) will say 18 August?

1. Model one birthday (a)

Use the book's 365-day model and assume all dates are equally likely and birthdays are independent. August 18 accounts for one of the 365 days, so one random person has probability 1/365 of naming it.



Out of 365 equally likely dates, exactly one matches.

2. Count all the other dates (b)

A birthday either is August 18 or is one of the other 364 dates. Thus the chance of missing the date is 364/365, also equal to $1-1/365$.



The complement counts the 364 dates that miss.

3. Require two misses (c)

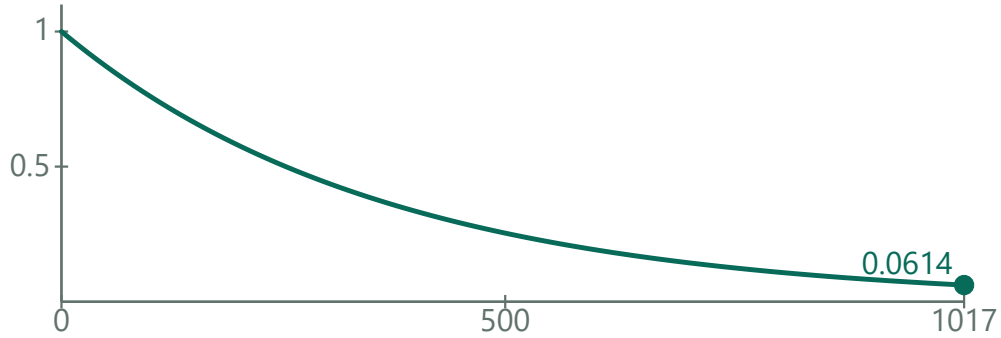
Both people must miss. The first misses with probability 364/365. Independence leaves the same probability for the second, even after learning the first result. Multiplication gives $(364/365)^2$.



Two independent people must both miss.

4. Extend the multiplication to the whole school (d)

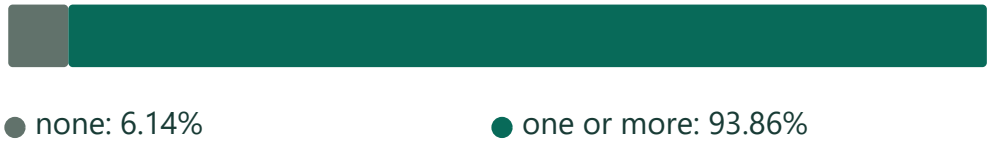
For no one among 1,017 students to give August 18, the miss condition must hold for every student. Multiply 1,017 copies of the same factor. The answer is $(364/365)^{1017}$, about 0.0614.



The chance that everyone misses falls with the number asked.

5. Turn zero hits into at least one hit (e)

The possibilities 0, 1, 2,... birthday matches exhaust all outcomes. All counts from 1 upward together have probability 1 minus the zero-count probability, or $1-(364/365)^{1017}$, about 0.9386.



Subtract the no-match probability from the whole.

August 18 in the independent, equally likely birthday model

August 18 event	People	Exact probability	Chance (%)
One birthday matches	1	1/365	0.2740
One birthday misses	1	364/365	99.7260
Neither matches	2	$(364/365)^{2}$	99.4528
Nobody matches	1,017	$(364/365)^{1017}$	6.1414
At least one matches	1,017	$1 - (364/365)^{1017}$	93.8586

Check. The probabilities of none and at least one add to one.

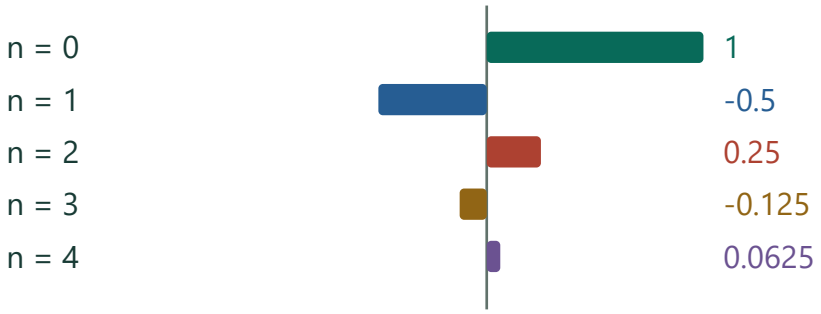
Exercise 710 · Limits of geometric sums

Printed page 79 · Geometric sequences and series / Limits and convergence

Suppose that r is a number strictly between -1 and 1. What can be said about the value of r^n when n is a large positive integer? What can be said about $\dfrac{a-ar^n}{1-r}$ when n is a large positive integer? Give an example of an infinite geometric series whose sum is $\dfrac{12}{1-(1/3)}$.

1. Track the magnitude of r^n

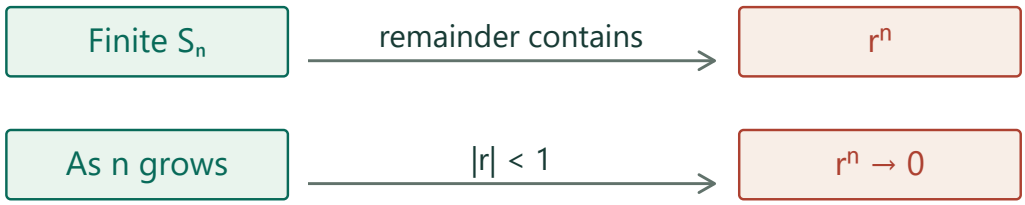
For $-1<r<1$, repeated multiplication by $|r|$ shrinks the magnitude toward zero. Negative r alternates signs, but the limit remains zero.



Example $r = -1/2$: signs alternate while magnitudes shrink.

2. Apply the limit to the finite sum

With $1-r$ fixed and nonzero, $(a-ar^n)/(1-r)$ approaches $a/(1-r)$. The limiting value is the infinite geometric sum.



Only a diminishing power can disappear in the limit.

3. Construct the requested example

Take $a=12$ and $r=1/3$. Then $12+4+4/3+\cdots$ has sum $12/[1-(1/3)]=18$.



With $a = 12$ and $r = 1/3$ the partial sums approach 18.

Check. Multiply its partial sums by 2/3 to get 12 minus a vanishing remainder.

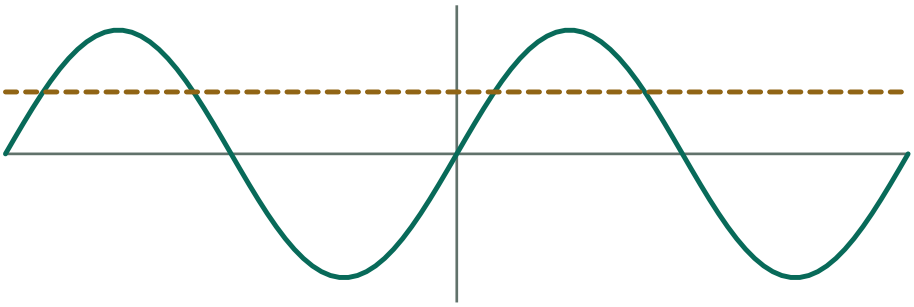
Exercise 711 · Inverse-trig ranges in radians

Printed page 79 · Inverse trigonometric functions / Domain and range

In radian mode, what are the ranges of $\sin^{-1}$ and $\cos^{-1}$?

1. Recall why an inverse needs a range convention

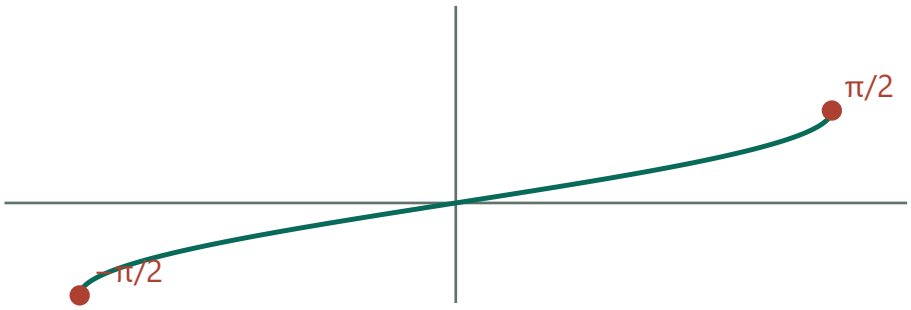
Sine and cosine repeat their values through infinitely many angles. Each inverse chooses a single monotonic portion of the original curve.



A horizontal level meets the full sine graph repeatedly.

2. Trace the arcsine branch

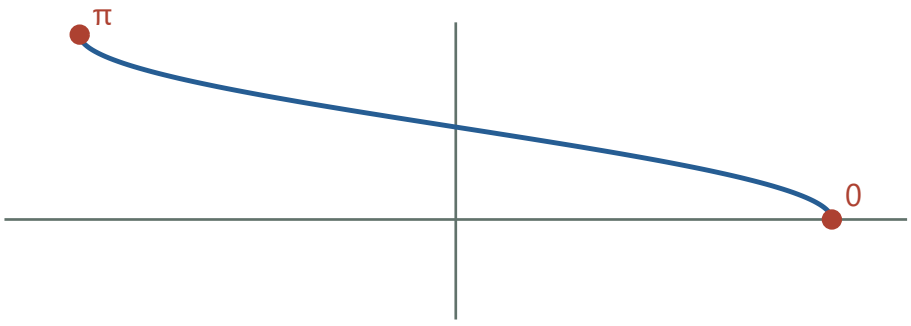
Sine increases from −1 to 1 on [−π/2,π/2]. Its inverse therefore returns exactly the angles in that closed interval.



Arcsine returns angles in its closed increasing branch.

3. Trace the arccosine branch

Cosine decreases from 1 to −1 on [0,π], reaching both endpoints. Its inverse has range [0,π].



Arccosine returns angles from 0 to π.

Check. Sine and cosine respectively cover $[-1,1]$ exactly once on these intervals.

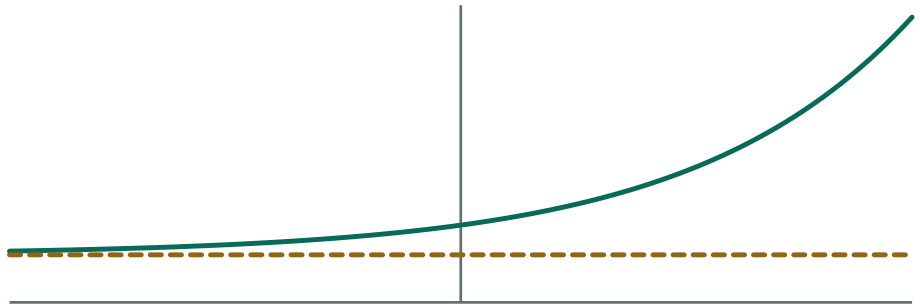
Exercise 712 · Linearize a shifted exponential

Printed page 79 · Exponential growth and decay / Data linearization / Logarithm rules

If you attempt to "linearize" the equation $y = 5 \cdot 2^x + 8$ by applying logarithms, you will run into difficulty. Why? Explain how the difficulty can be avoided.

1. Identify the obstruction

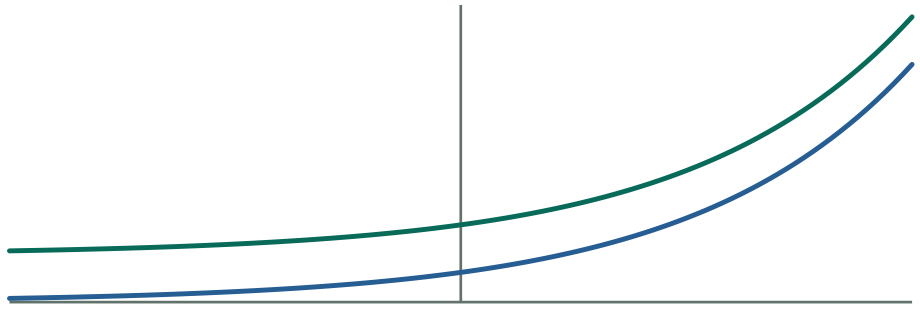
In $y=5\cdot2^x+8$, applying log directly produces $\log(5\cdot2^x+8)$. No logarithm rule separates the two terms across the plus sign.



The measured height includes an additive baseline of 8.

2. Subtract the vertical offset

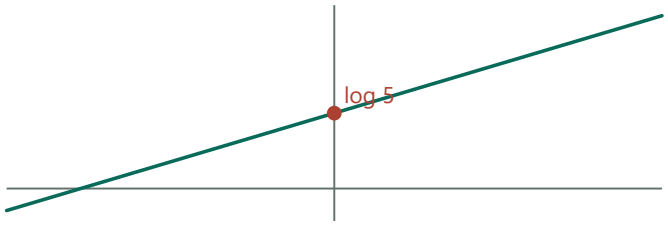
Subtract 8, the curve's horizontal asymptote: $y-8=5\cdot2^x$. This positive quantity is now a product of powers, to which log rules do apply.



Remove the offset so only the exponential excess remains.

3. Read the straight-line coordinates

Taking logs gives log(y−8)=log5+x log2. A plot of x against vertical coordinate log(y−8) is a straight line of slope log2 and intercept log5.



Taking the common logarithm of the excess straightens the data.

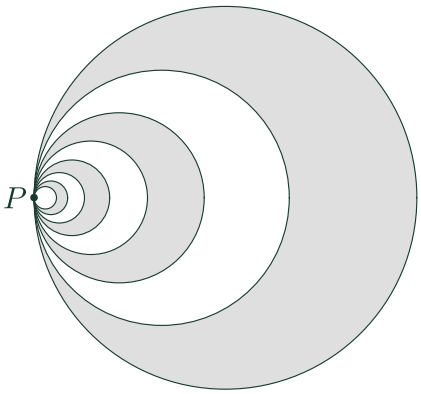
1. log(y−8)=log(5·2ˣ)=log 5+log(2ˣ)
After subtracting the offset the argument is a positive product.

2. log(y−8)=log 5+x log 2
The power rule puts x in the slope-times-input position.

Check. The log is defined because 5·2ˣ is positive.

Exercise 713 · Alternating nested disks

Printed page 79 · Geometric sequences and series / Circle geometry / Limits and convergence



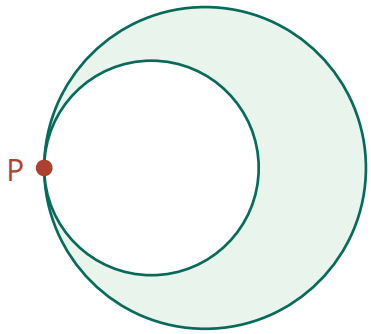
Consider the following recursive procedure: Step 1 consists of drawing a circle of radius 1 and painting it gray. Step 2 consists of drawing a circle with radius two-thirds of the radius from Step 1, which is internally tangent to the circle in Step 1 at a point P, and painting it white. Step 3 consists of drawing a circle with radius four-ninths (or two-thirds of the radius from Step 2), which is internally tangent to the circle in Step 2 at point P and painting it gray. And so on, as shown in the diagram. Let $A(n)$ represent the area of the gray regions at the end of Step n.

(a) Determine $A(1)$, $A(2)$, and $A(3)$.

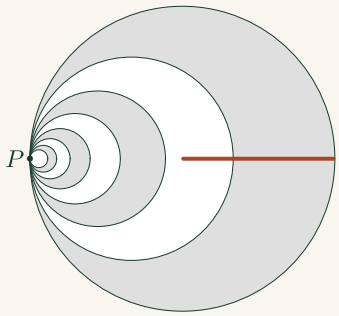
(b) If this process continues indefinitely, what portion of the original circle of radius 1 will be painted gray?

1. Track radii and areas separately

Each circle drawn tangent at P has radius 2/3 of its predecessor. Squaring that ratio gives area factor 4/9. A white disk removes gray from the total; the next gray disk restores some.



Both circles share P; their radius ratio is 2/3.



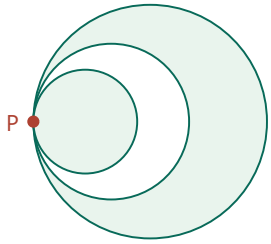
Outer radius

The segment joins the largest disk's center to its right edge. Compare radii before converting a scale factor to an area factor.

2. Calculate the first three gray areas

$A_1=\pi$. Then $A_2=\pi-(4/9)\pi=5\pi/9$. The third adds back $(4/9)^2\pi$, giving $A_3=61\pi/81$.

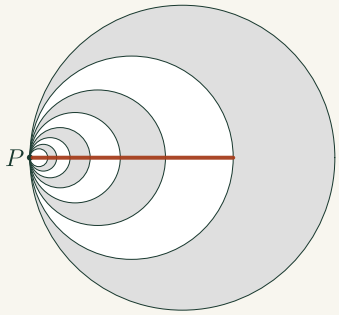
Number of nested disks · 3 disks



The third disk restores a smaller amount of gray area.

1. $A_3/\pi=1-4/9+16/81=(81-36+16)/81$
Express the alternating disk contributions with denominator 81.

2. $A_3=61\pi/81$
The third disk restores its gray area.

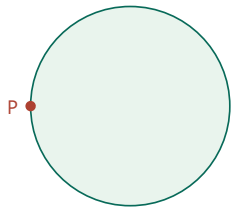


Next disk

The first white disk runs from the common point P to this new right edge. It covers a portion of the first gray disk.

Compare settings · Number of nested disks

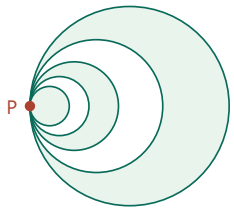
1 disk



1 disk leave a shaded fraction of 1.

1 disk leave a shaded fraction of 1.

5 disks

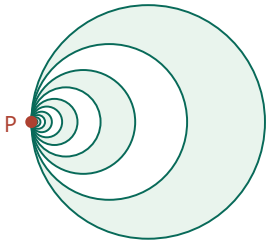


5 disks leave a shaded fraction of 4621/6561.

5 disks leave a shaded fraction of 4621/6561.

3. Sum the alternating changes

The net gray area tends to $\pi[1-4/9+(4/9)^2-\cdots]=\pi/(1+4/9)=9\pi/13$. Divide by the original area π: the gray portion is 9/13.



Alternating removals and additions lead to gray fraction 9/13.

1. Let $S=1+r+r^2+\cdots$ with $r=-4/9$. Then $S-rS=1$
The ratio has magnitude below one, so the infinite sum converges and its shift cancels term by term.

2. $S=1/(1+4/9)=1/(13/9)=9/13$
Divide by $1-r$, then invert the fractional denominator. Multiplying by π restores area units.

Check. The gray and white fractions are 9/13 and 4/13, summing to 1.

Exercise 714 · Thirteen cooling steps

Printed page 79 · Exponential growth and decay / Sequences and recursion

A sequence of temperature differences is defined recursively by the initial value $D_0 = 70$ and the equation $D_n - D_{n-1} = -0.2D_{n-1}$. Calculate D_{13}.

1. Interpret the difference equation

D_n-D_{n-1} is the change from the previous reading. A change of $-.2D_{n-1}$ means 80% of that previous difference remains.



The 20% change is taken from the previous excess.

2. Iterate the multiplier

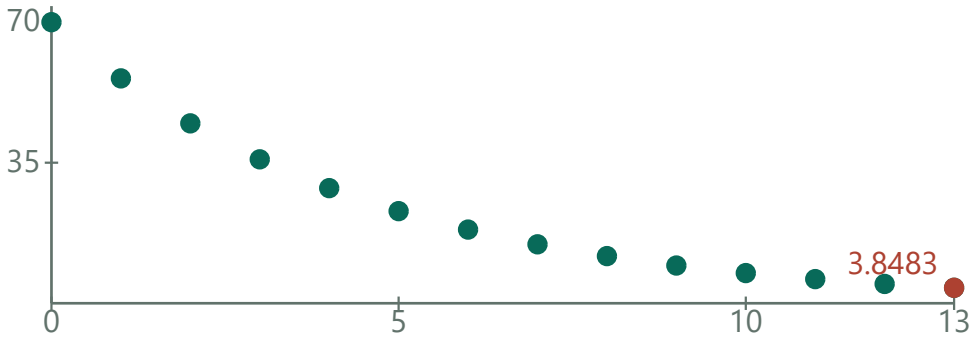
Add D_{n-1} to both sides to get $D_n = .8D_{n-1}$. Starting from index zero, thirteen updates contribute thirteen factors.



Thirteen transitions carry index 0 to 13.

3. Evaluate

$D_{13} = 70(.8)^{13} \approx 3.8483$ temperature units. In the five-minute cooling model this corresponds to 65 minutes.



Only whole five-minute steps appear in this sequence.

Check. The result agrees with the excess-temperature model at 65 minutes in 696.

Exercise 715 · Meaning of an infinite sum

Printed page 79 · Limits and convergence / Geometric sequences and series

When $t = 0.2$, the infinite series $1 + t + t^2 + t^3 + \cdots$ *equals* 1.25. What does this mean? For what t-values is it correct to say that the series $1 + t + t^2 + t^3 + \cdots$ has a *sum*? What is the sum?

1. Define what an infinite sum means

Take the limit of the partial sums 1+t+⋯+t^(N−1). A finite limit as N grows defines the infinite series' sum.



A partial sum stops after a finite number of terms.

2. Use the finite geometric formula

For t≠1 the partial sum is (1−t^N)/(1−t). If |t|<1, then t^N→0, leaving 1/(1−t). At t=.2 this is 1/.8=1.25.



For t = 0.2 the sequence of partial sums approaches 1.25.

3. Exclude all other real values

A convergent series must have individual terms tending to zero. For |t|≥1 the terms t^n fail that test, including t=1 and t=−1, so there is no ordinary finite sum.



Exactly −1 < t < 1 gives a convergent geometric sum.

1. $t^n = S_{n+1} - S_n$
Subtract consecutive partial sums to leave only the newest term.

2. $S_n \to S$ implies $S_{n+1} - S_n \to S - S = 0$
A finite series limit forces its terms to vanish; this fails for abs(t)≥1.

Check. At t=.2 the limiting expression gives 1/.8=1.25.

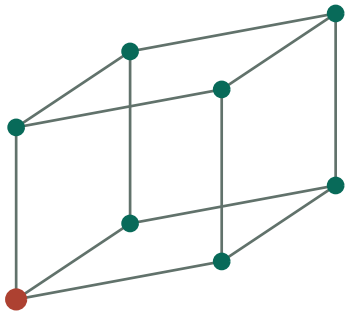
Exercise 716 · Distances between cube vertices

Printed page 80 · Probability / Coordinate geometry

Two distinct vertices of a cube are to be randomly chosen. Find the probability that the chosen vertices will be the endpoints of **(a)** an edge of the cube; **(b)** a face diagonal of the cube; **(c)** an interior diagonal of the cube.

1. Fix one vertex of the cube

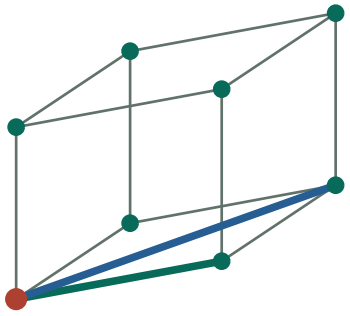
By symmetry any first vertex has the same seven possible partners. Mark its coordinates as (0,0,0) and the others with entries 0 or 1.



Fix one of eight vertices as the start; seven partners remain.

2. Count changes of one or two coordinates

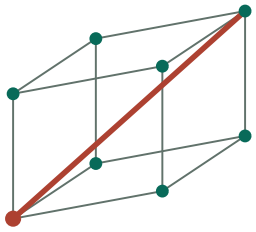
An edge partner differs in one coordinate, giving 3 choices. A face-diagonal partner differs in two coordinates, also 3 choices (choose the coordinate left unchanged).



Green changes one coordinate; blue changes two.

3. Count the space diagonal

Only the opposite vertex differs in all three coordinates. The resulting probabilities are 3/7 for an edge, 3/7 for a face diagonal, and 1/7 for an interior diagonal. They exhaust the seven partners.



Only the opposite vertex changes all three coordinates.

1. $d^2=(\Delta x)^2+(\Delta y)^2+(\Delta z)^2$
Apply Pythagoras first in a face and then perpendicular to that face.

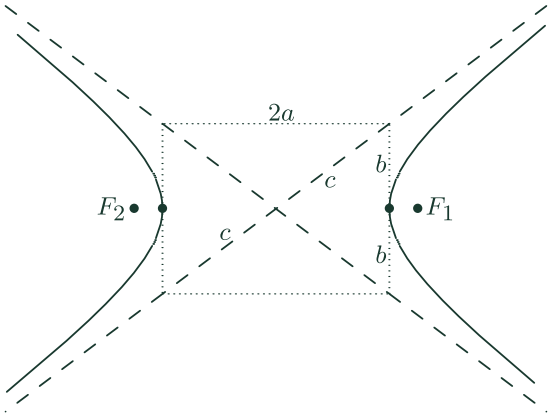
2. $d^2=1,2,3 \Rightarrow d=1,\sqrt{2},\sqrt{3}$
Each changed coordinate contributes one square unit, distinguishing the three distance classes.

Check. Unordered counts 12,12,4 total 28=8 choose 2.

Exercise 717 · Why focal difference is twice a

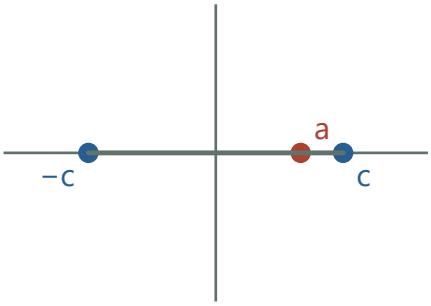
Printed page 80 · Hyperbolas / Foci and directrices

Some hyperbola terminology. Every hyperbola has two focal points, F_1 and F_2 (which is what Pat and Kim were). The focal points are located on the *major* symmetry axis, with the hyperbola *center* midway between them. The *vertices* are the points where the hyperbola intersects the major axis. As shown, it is customary to let $2a$ be the distance between the vertices and $2c$ be the distance between the focal points. Notice that $a < c$ and that a hyperbola does not intersect its minor axis. According to one definition of hyperbola, the difference between the two focal radii $|PF_1 - PF_2|$ is constant for any point P on the curve. Explain why this constant equals $2a$. (Hint: Try a special position for P.)



1. Use the location where the lengths are easiest

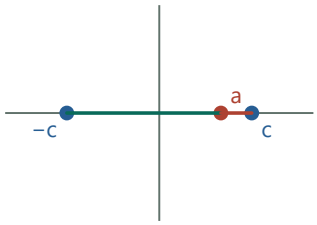
The focal-distance difference is constant by definition, so evaluate it at a vertex. Place the center at the origin, foci (±c,0) and right vertex (a,0), with c>a.



Example c = 3, a = 2: choose the right-hand vertex.

2. Read distances on the axis

The distance to the left focus is a+c. The distance to the right focus is c−a. These lie along the same axis, so ordinary subtraction measures their difference.

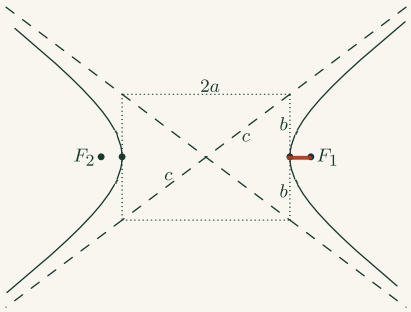


The two distances at this vertex are a + c and c − a.

1. d(left)=abs(a−(−c))=a+c
A horizontal segment has length equal to its absolute change in x.

2. d(right)=abs(a−c)=c−a
The right focus lies beyond the vertex since c>a.

3. d(left)−d(right)=a+c−c+a=2a
Subtracting c−a changes both its signs.

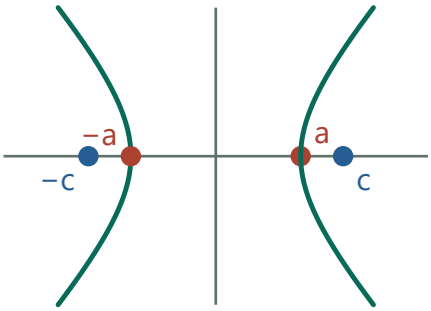


Vertex to the near focus

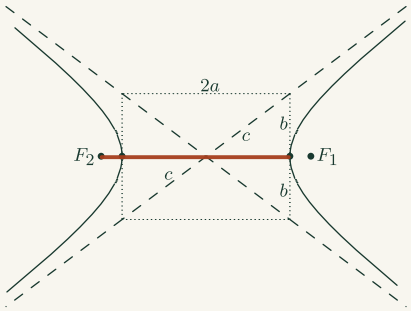
The vertex on the right branch is between the center and the near focus. Both lie on the horizontal axis, so subtract their coordinates to measure the gap.

3. Apply the constant to the whole curve

The difference is (a+c)−(c−a)=2a. At the other vertex the two distances exchange roles, leaving the same absolute difference 2a throughout the hyperbola.



At the opposite vertex their roles switch; the absolute difference persists.



Vertex to the far focus

To reach the other focus from the same vertex, pass through the center. Both coordinate distances now contribute to the length.

Check. At the opposite vertex their roles reverse but the absolute difference is unchanged.

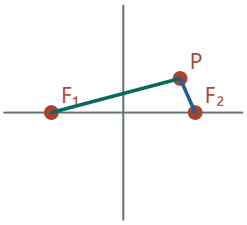
Exercise 718 · From focal distances to a hyperbola

Printed page 80 · Hyperbolas / Foci and directrices / Algebraic manipulation

(Continuation) Let $F_1 = (c, 0)$, $F_2 = (-c, 0)$, and $P = (x, y)$. Use the distance formula and some algebra to convert the equation $|PF_1 - PF_2| = 2a$ into the Cartesian form $\dfrac{x^2}{a^2} - \dfrac{y^2}{c^2 - a^2} = 1$. Now *define* $b^2 = c^2 - a^2$, find the *asymptotes* of $\dfrac{x^2}{a^2} - \dfrac{y^2}{b^2} = 1$, and confirm that the above diagram is correctly labeled. Notice that the asymptotes are the extended diagonals of the $2a$-by-$2b$ rectangle (shown dotted) that is tangent to the hyperbola at its vertices. It is interesting that the length of a diagonal of this rectangle is exactly $2c$.

1. Choose the nearer distance on one branch

On the right branch let $d=\sqrt{((x-c)^2+y^2)}$ be the distance to (c,0). The farther distance toward (−c,0) is d+2a. Comparing their squares gives $4cx=4ad+4a^2$, so $d=cx/a-a$.



Example a = 2, c = 3: the right focus is nearer.

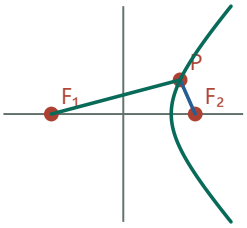
1. $(d+2a)^2-d^2=4ad+4a^2$
Subtract the near distance squared from the far distance squared.

2. $(x+c)^2-(x-c)^2=4cx$
The vertical components cancel in that same difference.

3. $4ad=4cx-4a^2 \Rightarrow d=cx/a-a$
Equate the differences and divide by 4a>0.

2. Eliminate d

Set $(cx/a-a)^2=(x-c)^2+y^2$. The terms $-2cx$ cancel. Rearrangement gives $(c^2-a^2)x^2/a^2-y^2=c^2-a^2$. Divide by $c^2-a^2>0$ and name it b^2 to obtain $x^2/a^2-y^2/b^2=1$.



Eliminating the distances produces the right-hand branch.

1. $c^2x^2/a^2-2cx+a^2=x^2-2cx+c^2+y^2$
Expand the two squared expressions.

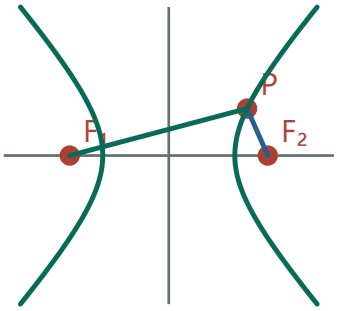
2. $c^2x^2/a^2-x^2-y^2=c^2-a^2$
Cancel −2cx and move the remaining terms.

3. $(c^2-a^2)x^2/a^2-y^2=c^2-a^2$
Factor the x^2 terms using the common denominator a^2.

4. $x^2/a^2-y^2/b^2=1$, $b^2=c^2-a^2>0$
Divide by the positive squared conjugate semiaxis.

3. Verify the reverse implication

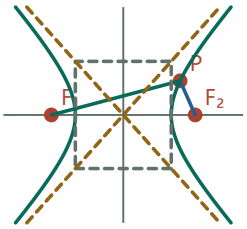
This equation forces $|x|\ge a$. On $x\ge a$, substitute $y^2=(c^2-a^2)(x^2/a^2-1)$ into both distance formulas. Their squares become $(cx/a-a)^2$ and $(cx/a+a)^2$. Both bases are positive, so the distances differ by 2a. Reflect across the y-axis for the left branch.



Symmetry adds the left branch with reversed distance roles.

4. Read the rectangle and asymptotes

The rectangle corners are $(\pm a,\pm b)$, giving diagonal slopes $\pm b/a$. From $|y|=(b/a)\sqrt{(x^2-a^2)}$, its gap to $(b/a)|x|$ is $ab/(|x|+\sqrt{(x^2-a^2)})$, tending to zero. Thus the extended diagonals are asymptotes.



The rectangle diagonals give the asymptote directions.

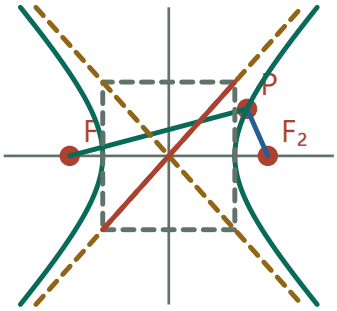
1. gap=(b/a)(abs(x)−√(x²−a²))
Compare the positive branch height with the matching diagonal height.

2. gap=(b/a)·a²/[abs(x)+√(x²−a²)]
The conjugate changes the numerator to $x^2-(x^2-a^2)$.

3. gap=ab/[abs(x)+√(x²−a²)]→0
The fixed numerator is divided by an unbounded positive quantity.

5. Check the marked lengths

At $x=\pm a$ the hyperbola has $y=0$, so the vertical rectangle sides touch at the vertices. Each full diagonal has length $\sqrt{[(2a)^2+(2b)^2]}=2\sqrt{(a^2+b^2)}=2c$, the focal separation.



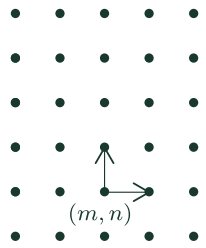
A full rectangle diagonal equals the distance between the foci.

Check. Its full diagonal is $\sqrt{[(2a)^2+(2b)^2]}=2c$.

Exercise 719 · Two lattice jumps

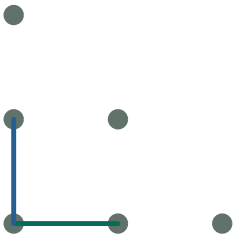
Printed page 80 · Lattice paths / Probability / Counting and permutations

A bug jumps from lattice point to lattice point on a piece of graph paper, one jump per second, according to the following pattern: From (m, n), the bug jumps only to $(m+1, n)$ or $(m, n+1)$, each equally likely. Find all the places the bug could be, two seconds after it leaves the origin $(0, 0)$. Are all these places equally likely?

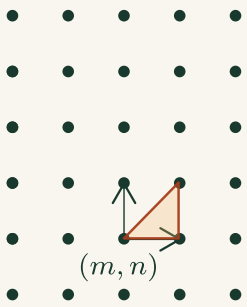


1. List the two decisions in order

Use R for right and U for up. Two equally likely independent decisions make four ordered paths RR,RU,UR,UU, each of probability 1/4.



Each move chooses the next horizontal or vertical neighbor.

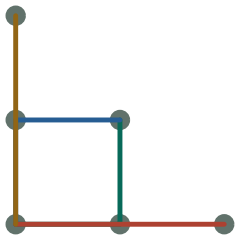


Right then up

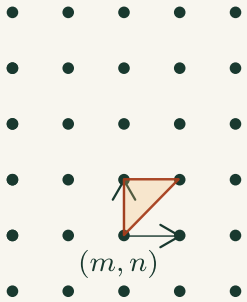
Follow the horizontal jump first and continue up from its new endpoint. Both moves use the same grid spacing.

2. Translate paths to coordinates

RR ends at (2,0), UU at (0,2), and both RU and UR end at (1,1). Different routes can reach the same final point.



Two routes meet at (1,1); the straight routes end elsewhere.

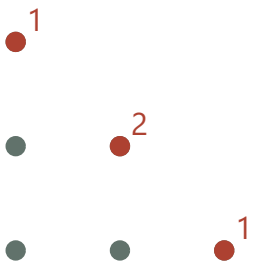


Up then right

Follow the vertical jump first, then add the horizontal move. Compare this final dot with the other mixed path.

3. Collect endpoint probabilities

The probabilities are 1/4 at (2,0), 1/2 at (1,1), and 1/4 at (0,2). Thus the possible endpoints are not equally likely.



Labels count the four equally likely paths by endpoint.

Check. Four equally likely ordered move strings account for all cases.

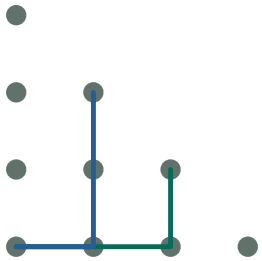
Exercise 720 · Three lattice jumps

Printed page 80 · Lattice paths / Probability / Binomial coefficients

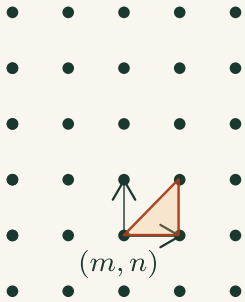
(Continuation) Where could the bug be, three seconds after it leaves the origin? Are all these places equally likely?

1. Describe an endpoint after three jumps

If k jumps go right and 3−k go up, the endpoint is (k,3−k). There are $2^3=8$ equally likely ordered paths.



Every three-jump endpoint lies on x + y = 3.

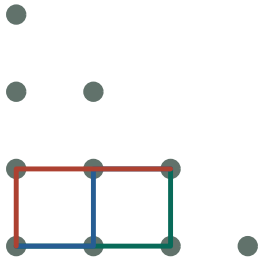


Right then up · Figure from exercise 719

Follow the horizontal jump first and continue up from its new endpoint. Both moves use the same grid spacing.

2. Count orders for each k

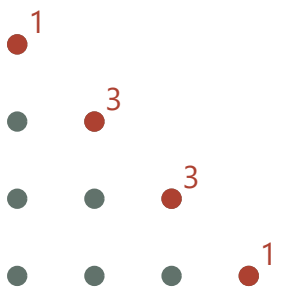
For k=0 or 3 there is one order. For k=1 choose which of three positions is right; for k=2 choose which one is up. The counts are 1,3,3,1.



These are the three orders that end at (2,1).

3. Convert counts to probabilities

In order (0,3),(1,2),(2,1),(3,0), the probabilities are 1/8,3/8,3/8,1/8. Endpoints with several possible routes are more likely.



Divide each endpoint count by the eight equally likely paths.

Check. For one right move choose one of three positions for it; similarly for one up move.

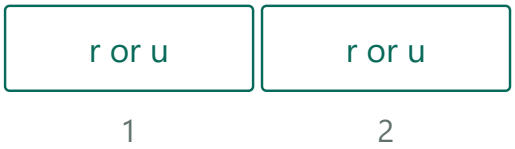
Exercise 721 · Binomials and paths

Printed page 80 · Binomial coefficients / Lattice paths

(Continuation) Expand $(r+u)^2$ and $(r+u)^3$, looking for connections with the jumping bug. In algebra, it is customary to collect terms like rru, rur, and urr into a single term $3r^2u$. Is there any reason to distinguish these terms in this example, however?

1. Give each factor a jump meaning

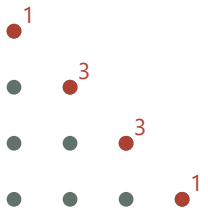
From each factor (r+u) choose one summand. Ordered choices are ordered right and up moves. Two factors produce rr,ru,ur,uu.



Choose one term in each factor, in order.

2. Expand and group like endpoints

Algebra gives $(r+u)^2=r^2+2ru+u^2$ and $(r+u)^3=r^3+3r^2u+3ru^2+u^3$. The exponent of each variable counts one kind of jump.



The coefficient of r^ku^{3-k} counts paths with that endpoint.

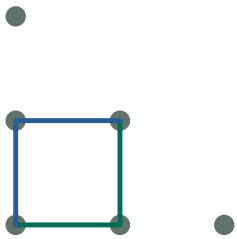
1. $(r+u)^3=(r^2+2ru+u^2)(r+u)$
Use the two-step path count first, then choose the last step.

2. $= r^3+2r^2u+ru^2+r^2u+2ru^2+u^3$
Each old term produces one new term for each last-step choice.

3. $= r^3+3r^2u+3ru^2+u^3$
Group the paths with equal endpoints.

3. Preserve paths when needed

rru,rur,urr all end at (2,1), giving coefficient 3. They visit different intermediate vertices, so keep their order when asking about a route even though their endpoints agree.



Grouping monomials forgets the intermediate vertex.

Check. The coefficients sum to 4 or 8, counting all possible paths for two or three jumps.

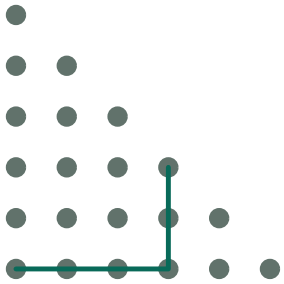
Exercise 722 · Five jumps to a specified point

Printed page 80 · Lattice paths / Binomial coefficients

(Continuation) It would take the bug five seconds to reach $(3, 2)$ from the origin. Given the bug's random behavior, how likely is it that this will happen?

1. Determine the necessary moves

To move from (0,0) to (3,2) the bug must make exactly three right moves and two up moves, totaling five seconds.



One path must have three moves right and two up.

2. Count positions within the sequence

Choose the three right-jump positions in $C(5,3)=5\cdot4/(2\cdot1)=10$ ways. The remaining two positions are forced to be up.



Choose the three positions for R. The other two must be U.

1. $C(5,3)=5!/(3!2!)=(5\cdot4\cdot3!)/(3!\cdot2\cdot1)$
Three R positions and two U positions have interchangeable labels within each group.

2. $C(5,3)=20/2=10$
Cancel the positive 3! and evaluate.

3. Compare with all paths

Five binary decisions produce $2^5=32$ equally likely paths. The probability is $10/32=5/16$.



● 10 favorable ● 22 other paths

All 32 five-move strings have equal probability.

Check. Selecting the two up-jump positions gives $C(5,2)=10$ again.

Exercise 723 · An arithmetic term formula

An *arithmetic sequence* is a list in which each term is obtained by adding a constant amount to its predecessor. For example, the list $4.0\ ,\ 5.2\ ,\ 6.4\ ,\ 7.6\ ,\ \ldots$ is arithmetic. The first term is 4.0; what is the fiftieth? What is the millionth term? What is the nth term?

1. Identify the increment

Each difference is 1.2: for instance 5.2−4.0=1.2. The first term already has value 4.0 before any increments.



Equal differences drive an arithmetic sequence.

2. Move to term n

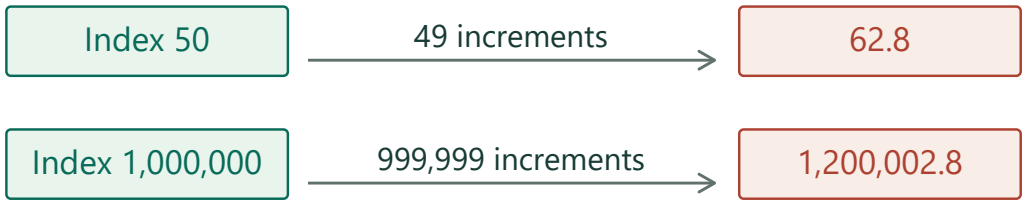
From index 1 to index n there are n−1 jumps. Thus a_n=4+1.2(n−1). Check the indexing on term 2: one jump produces 4+1.2=5.2. This prevents adding one extra increment when n is large.



Count gaps from the first term, not the term labels.

3. Evaluate the two indices

$a_{50}=4+1.2\cdot 49=62.8$. $a_{1000000}=4+1.2\cdot 999999=1{,}200{,}002.8$. For the millionth term, one million increments would add 1,200,000, but only 999,999 are needed. Subtract 1.2 from that increase and add the initial 4.



Apply the same gap count even at a very large index.

Check. The formula gives 4 at n=1 and its consecutive difference is always 1.2.

Exercise 724 · Recover the first term

Printed page 81 · Arithmetic sequences and series / Sequences and recursion

Suppose that $a_1, a_2, a_3, \ldots$ is an arithmetic sequence, in which $a_3 = 19$ and $a_{14} = 96$. Find a_1.

1. Count the intervals between known terms

Term 14 lies $14-3=11$ increments beyond term 3. Their values differ by $96-19=77$.



Eleven increments connect the two given terms.

2. Determine one increment

All increments are equal in an arithmetic sequence, so the common difference is 77/11=7.



Equal increments split the gain of 77 evenly.

3. Work back to term one

Term 3 has already gained two increments. Subtract twice: $a_1 = 19 - 2\cdot 7 = 5$. Adding 13 increments to this returns $a_{14} = 96$.



Go back two steps to recover term 1.

Check. Adding thirteen increments to 5 gives 96.

Exercise 725 · A six-digit repeating block

Printed page 81 · Geometric sequences and series / Limits and convergence

Express $0.138918\overline{138918}$ as a geometric series. Find the *rational* sum of the series.

1. Read one complete repeating block

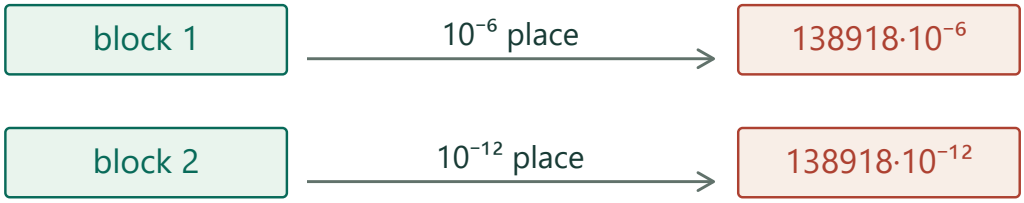
The block 138918 uses six decimal places. Its first appearance contributes $138918 \cdot 10^{-6}$. Each following appearance shifts six more places to the right.



Each slot is a complete six-digit decimal block.

2. Form the geometric series

The decimal is $138918 \cdot 10^{-6} + 138918 \cdot 10^{-12} + 138918 \cdot 10^{-18} + \cdots$, with ratio 10^{-6}.



Shifting a block six places multiplies it by 10^{-6}.

3. Sum and simplify

Divide the first term by $1 - 10^{-6}$, obtaining 138918/999999. Cancel the common factor 39 to get 3562/25641.



Summing all place-shifted copies gives an exact rational value.

1. $S = (138918 \cdot 10^{-6})/(1 - 10^{-6})$
Use the convergent geometric series; the six-place block has ratio 10^{-6}.

2. $S = 138918/(10^6 - 1) = 138918/999999$
Multiply numerator and denominator by one million.

3. $S = (138918/39)/(999999/39) = 3562/25641$
Cancel the same nonzero integer from both.

Check. If x is the decimal, $10^6 x - x$ is exactly 138918.

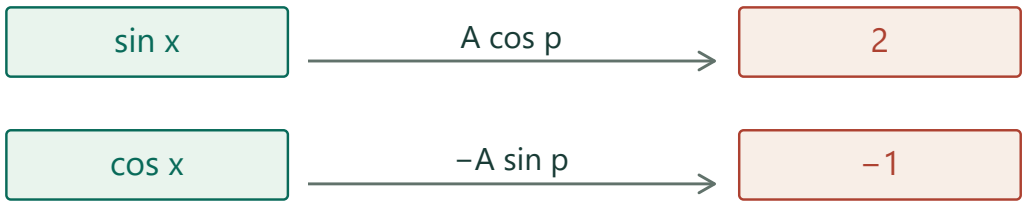
Exercise 726 · Combining sine and cosine

Printed page 81 · Sinusoidal functions / Angle addition formulas / Phase shifts

Show that the curve $y = 2\sin x - \cos x$ is sinusoidal, by writing it in an equivalent form $y = a\sin(x - p)$.

1. Expand a shifted sine

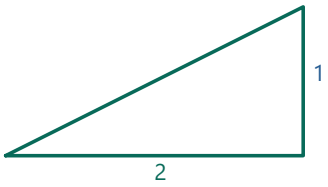
A sin(x−p)=A cos p·sin x−A sin p·cos x. Matching coefficients with 2sin x−cos x demands A cos p=2, A sin p=1.



The two coefficients must match for every x.

2. Find amplitude and phase

Squaring and adding gives $A^2=2^2+1^2=5$. With A positive and both coefficients positive, choose p in quadrant I with tan p=1/2.



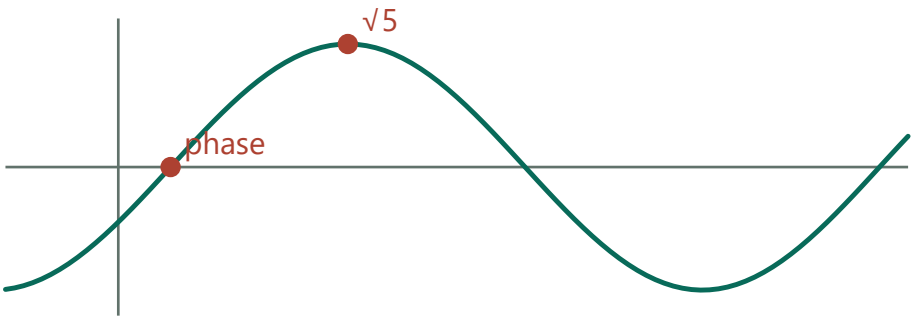
The coefficient triangle has hypotenuse √5 and tan p = 1/2.

1. $A^2(\cos^2 p+\sin^2 p)=4+1 \Rightarrow A=\sqrt{5}$
Square the matched coefficients, add, and select positive amplitude.

2. $\tan p=(A \sin p)/(A \cos p)=1/2$
The sine and cosine are both positive, so select the acute phase.

3. Write the equivalent formula

Using A=√5 and p=arctan(1/2) gives y=√5 sin(x−arctan(1/2)). The coefficient match proves it is sinusoidal at every x.



The shifted sine has the measured zero and peak.

Check. At x=0 and π/2 the expressions both return −1 and 2, identifying the two coefficients.

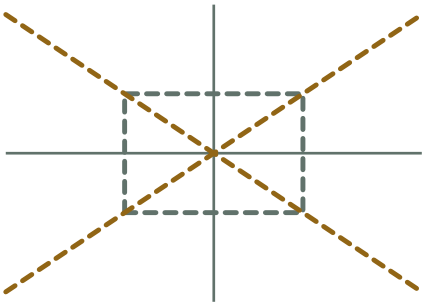
Exercise 727 · Two orientations of a hyperbola

Graph each hyperbola. Plot the focal points and write equations for the asymptotes.

(a) $9y^2 - 4x^2 = 36$ **(b)** $4x^2 - 9y^2 = 36$

1. Read the opening direction

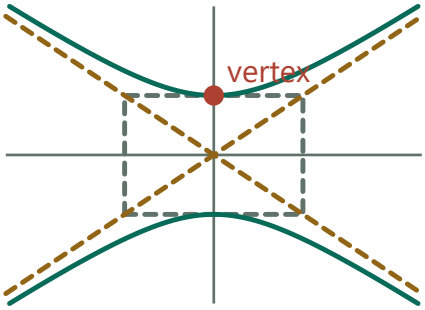
Divide both equations by 36. The variable with positive squared term identifies the opening axis. The squared semiaxes add to c^2 for the focal distance.



The same guide rectangle serves both orientations.

2. Construct the vertical branches (a)

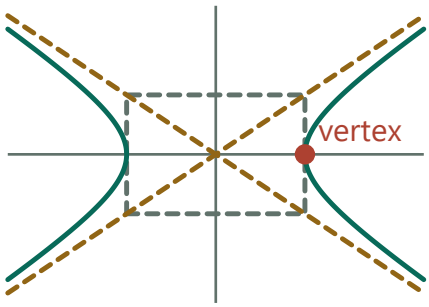
$y^2/4 - x^2/9 = 1$ opens up and down from $(0, \pm 2)$. With $c^2 = 4 + 9 = 13$ its foci are $(0, \pm\sqrt{13})$. The rectangle with corners $(\pm 3, \pm 2)$ has diagonal lines $y = \pm 2x/3$, the asymptotes.



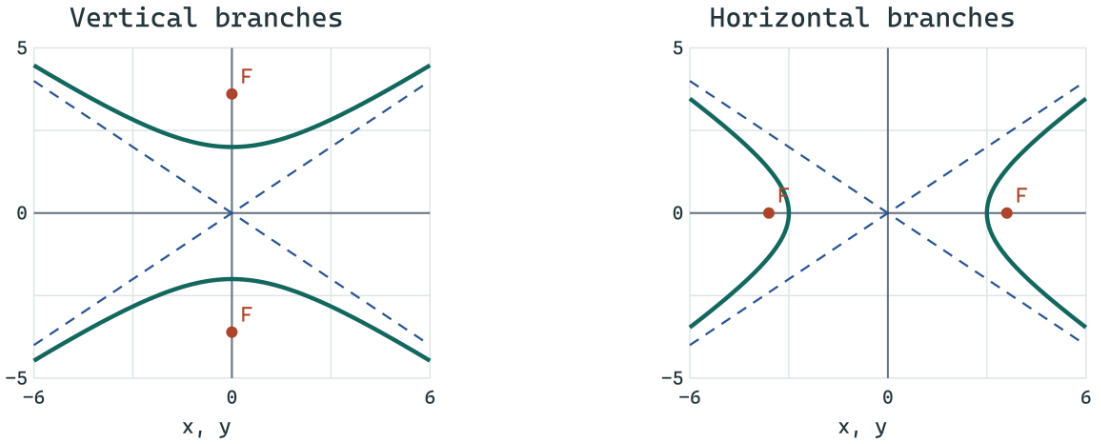
A positive y^2 term opens the upper and lower branches.

3. Construct the horizontal branches (b)

$x^2/9 - y^2/4 = 1$ opens right and left from $(\pm 3, 0)$, with foci $(\pm\sqrt{13}, 0)$. The rectangle has the same dimensions as before, so its diagonal asymptotes are again $y = \pm 2x/3$.



A positive x^2 term opens the left and right branches.



Check. For either curve $c^2 = 4 + 9 = 13$. Replacing the right side of either original equation by zero gives the same two lines.

Exercise 728 · An invariant vector sum

Printed page 81 · Transition matrices / Invariant quantities / Sequences and recursion

Let $\mathbf{M} = \begin{bmatrix} 0.8 & 0.6 \\ 0.2 & 0.4 \end{bmatrix}$ and $\mathbf{v} = \begin{bmatrix} 5 \\ 9 \end{bmatrix}$. Draw the vectors $\mathbf{v}, \mathbf{M}\mathbf{v}, \mathbf{M}^2\mathbf{v}, \mathbf{M}^3\mathbf{v}, \ldots$ with their tails at the origin. Explain why the heads define a sequence of collinear points. Find the property of matrix $\mathbf{M}$ that is responsible for the collinearity.

1. Trace each matrix step

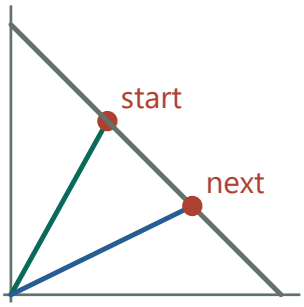
From a column (x,y), M gives x′=.8x+.6y and y′=.2x+.4y. Add these equations: x′+y′=(.8+.2)x+(.6+.4)y=x+y.



The first vector has coordinate sum 14.

2. Locate all arrowheads

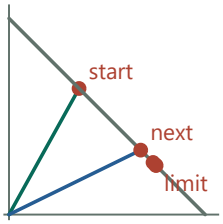
The first sum is 5+9=14; column sums of one preserve it forever. Draw the arrows from the origin to points on the line x+y=14.



Column sums of one keep every head on x + y = 14.

3. Compute successive coordinates

With y=14−x, x′=.2x+8.4. Starting at 5 gives x=9.4,10.28,10.456. Their partners 14−x are 4.6,3.72,3.544. The limiting head is the fixed point (10.5,3.5).

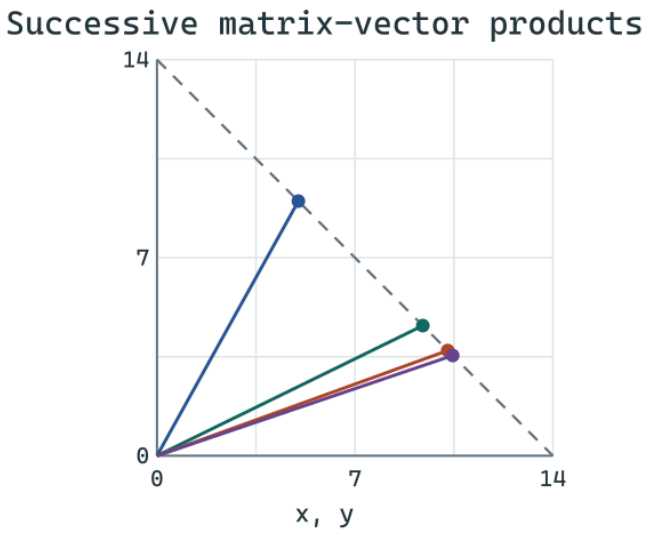


The heads approach the fixed vector (10.5, 3.5).

1. x′=.8x+.6(14−x)=.2x+8.4
Use the invariant coordinate sum to eliminate y.

2. x*=.2x*+8.4 ⇒ .8x*=8.4 ⇒ x*=10.5
At the fixed point the horizontal coordinate is unchanged.

3. x′−10.5=.2(x−10.5), y*=14−10.5=3.5
The deviation shrinks at each iteration, proving the claimed limit.



Heads in order: blue, green, orange, purple. The dashed line is x + y = 14.

Check. Multiplying (10.5,3.5) by the matrix leaves it fixed. Each x-distance from the limit shrinks by a factor .2.

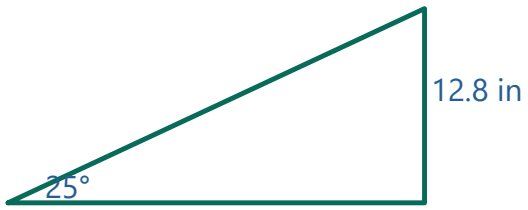
Exercise 729 · Hypotenuse from cosecant

Printed page 81 · Right-triangle trigonometry / Trigonometric ratios

The reciprocal of the sine is *cosecant* (abbreviated csc), useful for expressing answers to trigonometry problems without using a division sign. Use this function to express the hypotenuse of a right triangle that has a 12.8-inch side opposite a 25-degree angle.

1. Label the right triangle

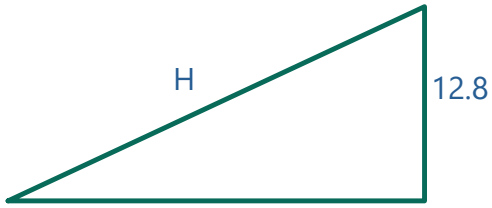
Let H be the hypotenuse opposite its right angle. The given 12.8-in side lies opposite the 25° angle.



Opposite means across the triangle from the marked angle.

2. Use the opposite-to-hypotenuse ratio

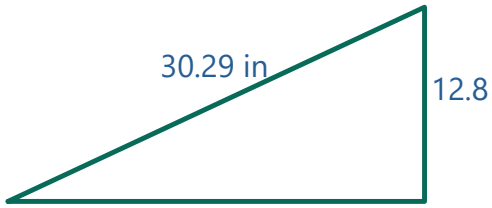
sin25°=12.8/H. Multiplying by H and dividing by sin25° gives H=12.8/sin25°.



Sine compares the opposite side with the longest side.

3. Write the requested reciprocal function

Since csc25°=1/sin25°, the length is 12.8 csc25°≈30.29 inches, longer than the known leg.



The hypotenuse is 12.8 csc 25°.

Check. The answer exceeds the opposite side and 30.2874 sin 25°≈12.8.

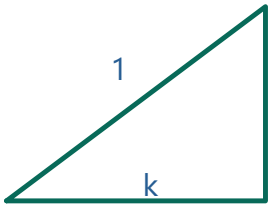
Exercise 730 · Three ratios from one cosine

Printed page 81 · Trigonometric ratios / Trigonometric identities

If the cosine of an acute angle is some number k, then what is the *sine* of the same angle? In terms of k, what is the cosecant of this angle? Simplify the expression $\tan(\cos^{-1} k)$.

1. Build an acute right triangle

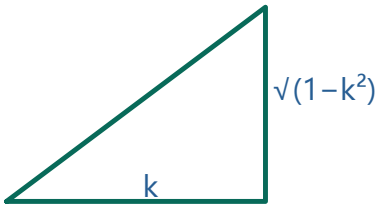
Let $\theta=\arccos k$. Give the triangle hypotenuse 1 and adjacent side k, so $\cos\theta=k$ with $0<k<1$.



Choose a unit hypotenuse and an acute angle with adjacent side k.

2. Recover the missing side

Pythagoras gives opposite length $\sqrt{(1-k^2)}$. The positive root is required because the angle is acute.



Pythagoras determines the positive opposite leg.

3. Form the three ratios

$\sin\theta=\sqrt{(1-k^2)}$, $\csc\theta=1/\sqrt{(1-k^2)}$, and $\tan(\arccos k)=\text{opposite/adjacent}=\sqrt{(1-k^2)}/k$.

opposite / 1	sine →	$\sqrt{(1-k^2)}$
1 / opposite	cosecant →	$1/\sqrt{(1-k^2)}$
opposite / k	tangent →	$\sqrt{(1-k^2)}/k$

The three ratios use the same labeled triangle.

Check. For $k=3/5$ these give 4/5, 5/4 and 4/3, the ratios of a 3–4–5 triangle.

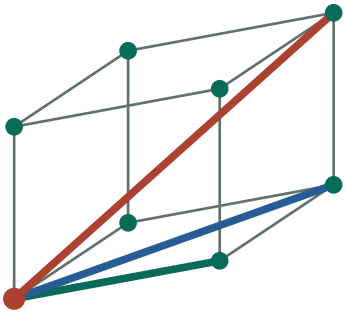
Exercise 731 · Equilateral triangles in a cube

Printed page 81 · Combinations / Probability / Cube symmetry

Three distinct vertices of a cube are to be randomly chosen. What is the probability that they will be the vertices of an equilateral triangle?

1. List the three vertex distances

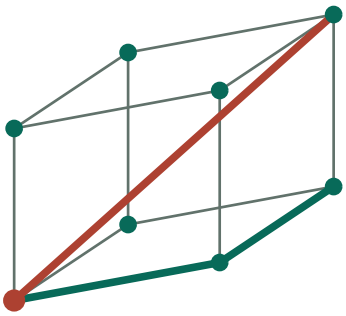
Using unit cube coordinates 0 and 1, two vertices differing in one, two or three coordinates are separated by 1,√2 or √3. All three sides of an equilateral triangle must use the same type.



Edges, face diagonals and body diagonals have three different lengths.

2. Eliminate edges and space diagonals

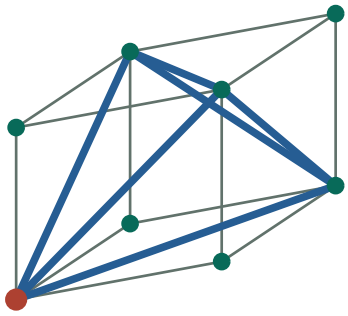
Each edge flips the parity of the coordinate sum, so an odd closed walk of three edges is impossible. Each vertex has only one opposite space-diagonal partner, ruling out three mutually opposite vertices.



Three edges cannot close. Each vertex has one opposite vertex.

3. Count all face-diagonal triples

Split the vertices by even or odd coordinate sum. Each class has four vertices. Distinct vertices in one class differ in exactly two coordinates; every triple in it is equilateral. Conversely all face diagonals join points of the same class.



Four vertices of equal parity form a regular tetrahedron inside the cube.

4. Divide by all unordered triples

There are 2·C(4,3)=8 favorable triples and C(8,3)=56 total. The probability is 8/56=1/7.



1. C(8,3)=(8·7·6)/(3·2·1)=56
Divide the ordered vertex choices by the six orders of each triple.

2. 2·C(4,3)=2·4=8 ⇒ P=8/56=1/7
Each parity class supplies four possible triples.

Choose three cube vertices uniformly among all 56 unordered triples.

Check. There are three such triangles through each vertex. Counting vertex incidences gives 8·3/3=8.

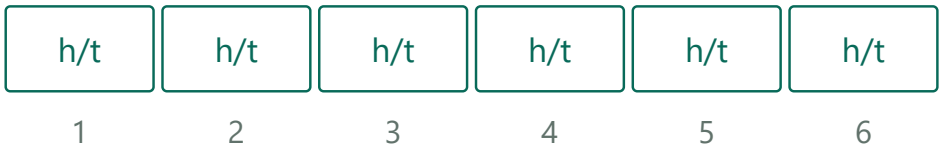
Exercise 732 · Where two heads occur

Printed page 81 · Binomial probability / Counting and permutations

When the *binomial* power $(h+t)^6$ is expanded, the product h^2t^4 appears several times. After like terms are collected, what is the numerical coefficient of h^2t^4? If six pennies are tossed onto the table, what is the probability of seeing two heads and four tails?

1. Interpret the six factors

In $(h+t)^6$ a term is made by picking one symbol from each of six factors. To get h^2t^4 choose exactly two factors for h; the rest must supply t.



Each factor represents one independent choice of head or tail.

2. Find the coefficient

Two positions can be chosen in $6\cdot5/2=15$ ways. Those 15 products combine into coefficient 15 on h^2t^4.



One possible placement of the two h factors among six.

3. Translate to coin tosses

Six fair independent pennies give $2^6=64$ equally likely ordered head/tail outcomes. The same 15 position choices give exactly two heads, so the probability is 15/64.

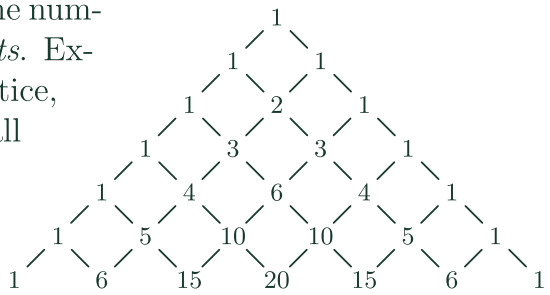


Every resulting ordered toss string has probability 1/64.

Check. Choosing the four tails instead gives $C(6,4)=15$.

Exercise 733 · Reading the binomial triangle

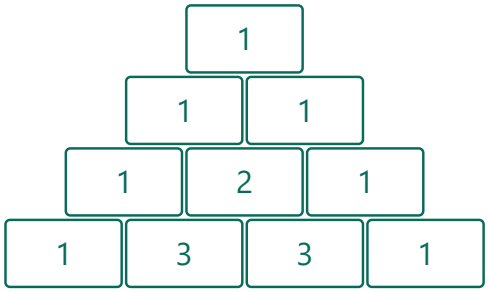
Printed page 81 · Binomial coefficients / Combinations



A special triangle is partially shown at right. The numbers that appear in it are called *binomial coefficients*. Explain this terminology. What patterns do you notice, and can you account for them? It is customary to call the top row of this array the 0th *row*. According to this convention, how many entries appear in the nth row, and what is the sum of all the entries in the nth row? This triangle is often referred to as *Pascal's Triangle*; however, centuries before Pascal's time it was known in China and India. It is known by different names around the world, such as *Yang Hui's Triangle* in China, *Meru-Prastāra* in India, *Khayyam's Triangle* in Iran, and *Tartaglia's Triangle* in Italy. Do you know of any other names for this special triangle?

1. Read a binomial expansion by choices

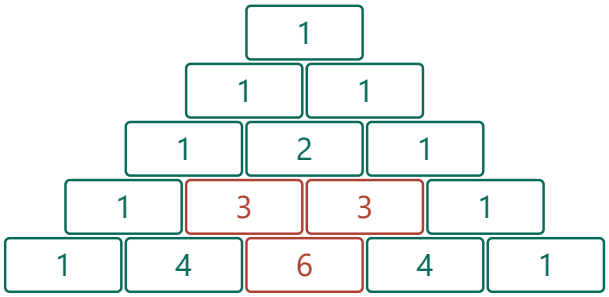
Choosing x from k of n factors in $(x+y)^n$ produces $x^k y^{(n-k)}$. Its coefficient C(n,k) occupies position k of row n. The indices 0 through n give n+1 entries and the two edge entries equal 1.



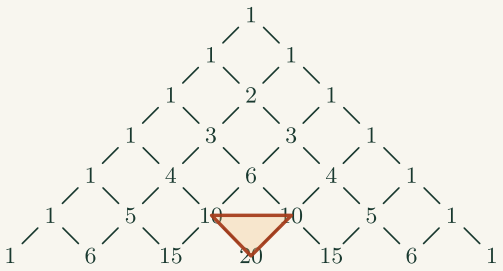
Each row is indexed by the number of binomial factors.

2. Explain the triangular addition

When choosing k items, distinguish whether a particular item is included. Including it leaves a choice of k−1 among n−1, excluding it a choice of k among n−1. Their sum is C(n,k), explaining addition of two neighboring entries above.



Two neighboring entries in the previous row add to the child.



Two parents of a coefficient

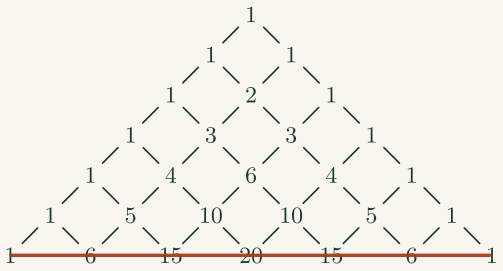
The two tens above the central entry provide two disjoint contributions. Follow the connectors from their places to their common child.

3. Explain symmetry and the row sum

Choosing k items or naming their n−k complements gives the same number, so rows are symmetric. Put x=y=1: the sum of row n is $(1+1)^n=2^n$. For row 6 this gives 64.



The symmetric sixth-row entries total 64.

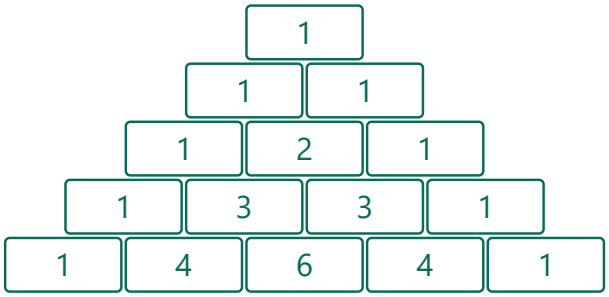


One full row

Read along the displayed bottom row from one edge entry to the other. Symmetric positions correspond to complementary selections.

4. Connect the name to the construction

These are binomial coefficients because they multiply terms in powers of a binomial. Another historical name for the displayed array is the arithmetic triangle.



These integers are the coefficients of successive binomial powers.

Check. The sixth row 1,6,15,20,15,6,1 sums to $64=2^6$.

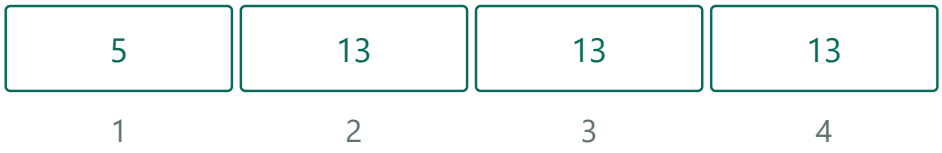
Exercise 734 · A repeating tail after one digit

Printed page 81 · Geometric sequences and series / Limits and convergence

Express the repeating decimal $0.513\overline{13}$ as a fraction in lowest terms.

1. Separate the nonrepeating digit

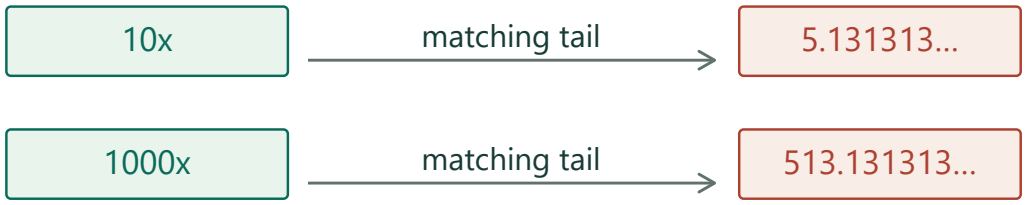
Let x=.5131313… . Multiply by 10 so the decimal following the integer is a pure repeating block: 10x=5.131313… .



The first digit after the decimal does not belong to the repeating block.

2. Line up two identical tails

The block has length two, so multiply that equation by 100 to get 1000x=513.131313… . Now its fractional tail matches the first exactly.



Shift by one two-place block after isolating the nonrepeating digit.

3. Cancel the repeat

Subtracting gives 990x=508, hence x=508/990=254/495. The numerator and denominator now have no common factor.



1. 1000x−10x=513.1313…−5.1313… ⇒ 990x=508
The shifted tails agree digit for digit, so their difference is zero.

2. x=508/990=(508/2)/(990/2)=254/495
Divide by 990 and remove the common factor 2.

Subtract first, then reduce the fraction.

Check. Long division starts .5 and repeatedly produces the pair 13.

Exercise 735 · Three graphs and their asymptotes

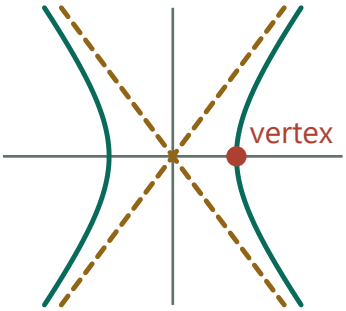
Printed page 82 · Hyperbolas / Asymptotes / Inverse functions

Without a graphing tool, graph the following curves, and find equations for their linear asymptotes:

(a) $16x^2 - 9y^2 = 144$ **(b)** $y = 2^x - 3$ **(c)** $x = 2^y - 3$

1. Sketch the hyperbola (a)

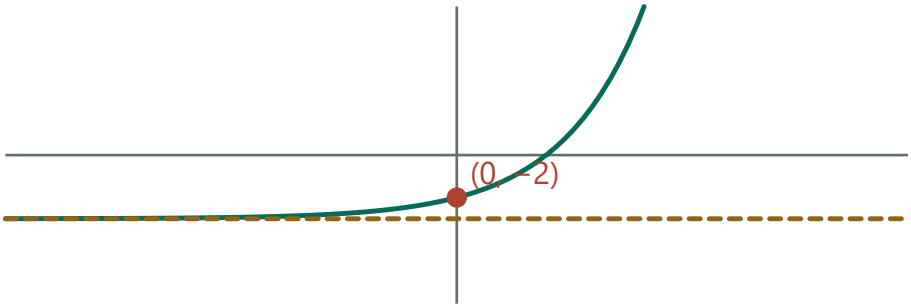
Divide by 144: $x^2/9 - y^2/16 = 1$. The branches open left and right from $(\pm 3, 0)$. Draw the rectangle with corners $(\pm 3, \pm 4)$; its extended diagonal lines $y = \pm 4x/3$ are the asymptotes.



The hyperbola opens horizontally toward diagonal asymptotes.

2. Sketch the translated exponential (b)

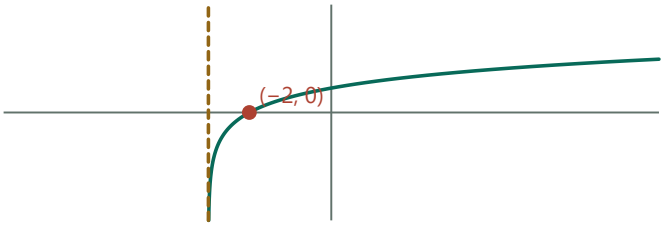
Subtracting 3 from 2^x lowers its graph, taking $(0, 1)$ to $(0, -2)$ and its horizontal asymptote to $y = -3$. It increases from just above that line toward the right.



The exponential has been moved down 3.

3. Sketch the reflected relation (c)

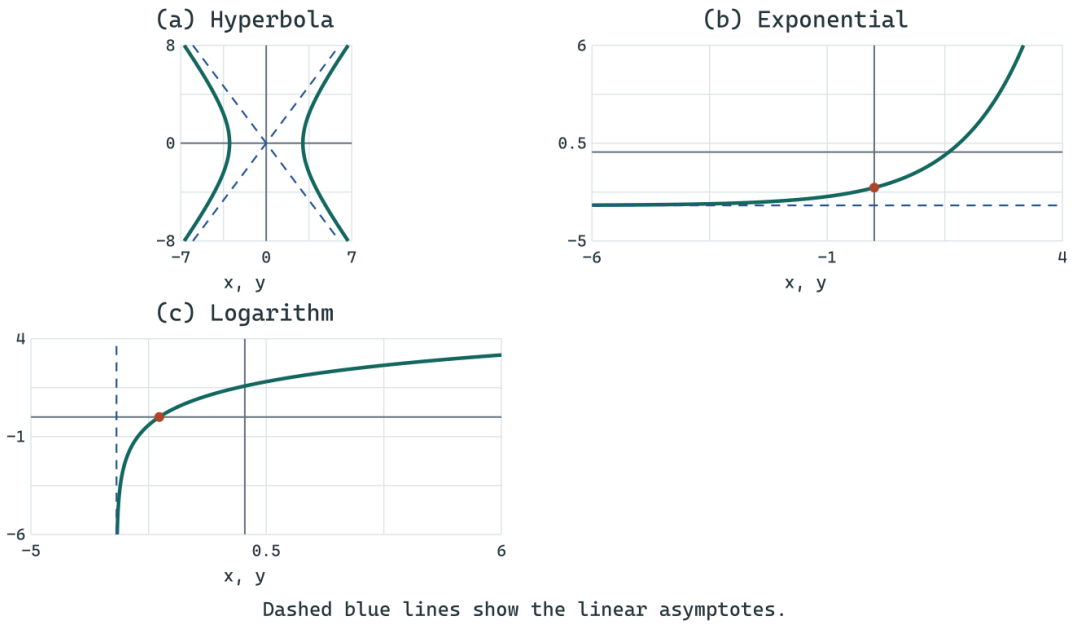
Exchanging x and y in (b) gives $x = 2^y - 3$ or $y = \log_2(x + 3)$. Reflect each point and the asymptote across $y = x$: the curve passes $(-2, 0)$, increases on $x > -3$, and approaches $x = -3$ vertically.



Inverse reflection turns the old horizontal boundary vertical.

1. $x = 2^y - 3 \Rightarrow x + 3 = 2^y$
After the coordinate swap add 3 to isolate the positive power.

2. $y = \log_2(x + 3)$, $x + 3 > 0 \Rightarrow x > -3$
Undo the power with base-two log and retain its input condition.

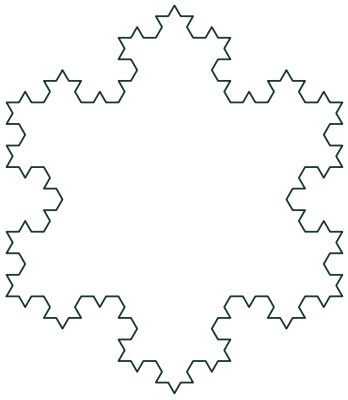


Check. The exponential and log points swap coordinates, as do their asymptotes.

Exercise 736 · Area inside a Koch snowflake

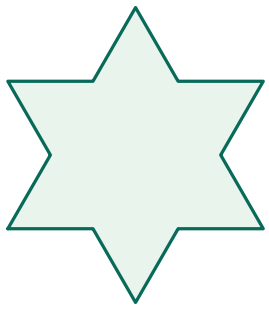
Printed page 82 · Fractals and iteration / Geometric sequences and series / Limits and convergence

The Koch snowflake is an example of a fractal curve of infinite length. As suggested by the figure, however, the area enclosed by this curve is finite. Suppose that the area enclosed by stage 0 (the initial equilateral triangle) is 1. What is the area enclosed by stage 1? by stage 2? by stage n? Show that the area enclosed by the completed snowflake can be obtained with the help of a geometric series.

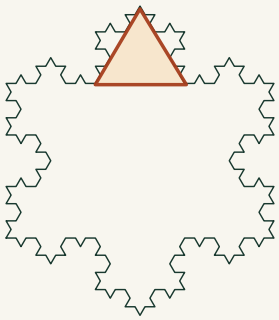


1. Calculate the first stage

Each of the three initial sides contributes one new equilateral triangle with side one third as large and area 1/9. Total area becomes 1+3/9=4/3.



Three new triangles each add 1/9 of the original area.

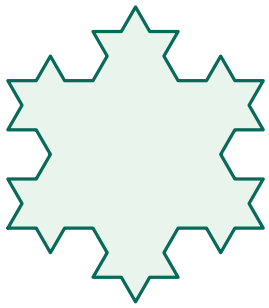


Large outward triangle

This construction traces the broad outline of one protrusion before the smaller refinements along its edges. Compare its side length with the side it replaces.

2. Calculate the second stage

Replacing each segment by four gives 12 segments. Their outward bumps have side 1/9 of the original and area 1/81 each. New total is 4/3+12/81=40/27.

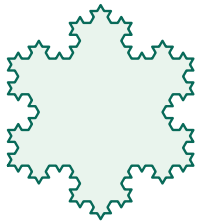


Each of twelve edges now contributes a triangle of area 1/81.

3. Track each later increment

From one stage to the next the number of new bumps multiplies by 4 and their individual areas divide by 9. Therefore the total new area multiplies by 4/9. Starting with increment 1/3, the first n increments sum to $(1/3)[1-(4/9)^n]/(1-4/9)$.

Snowflake stage · Stage 3

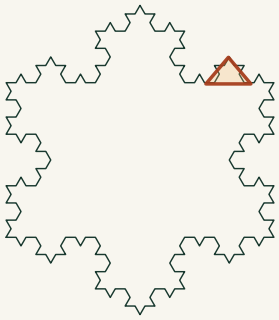


The number of bumps grows by 4 while each area scales by 1/9.

1. $\Delta A_{n+1}/\Delta A_n = 4\cdot(1/3)^2 = 4/9$
Four times as many triangles each have one ninth the preceding area.

2. $\Delta A_n=(1/3)(4/9)\text{^}(n-1)$
Index the first increase as n=1.

3. $\Sigma\Delta A_j=(1/3)[1-(4/9)^n]/(1-4/9)$
Sum exactly n increases using the finite geometric sum.

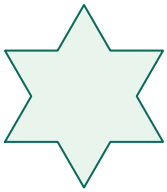


A finer outward bump

Farther along the upper-right boundary another outward triangle appears at a smaller scale. Each repetition changes both the number of bumps and their areas.

Compare settings · Snowflake stage

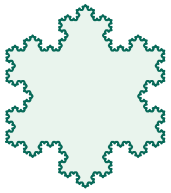
Stage 1



This stage adds 1/3 of the original triangle's area.

This stage adds 1/3 of the original triangle's area.

Stage 4

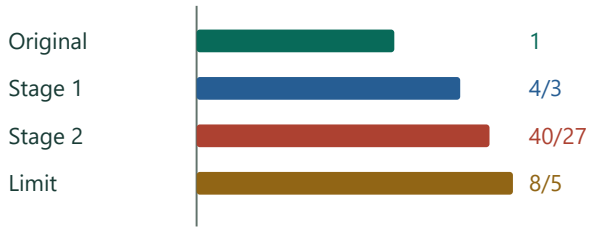


This stage adds 64/2187 of the original triangle's area.

This stage adds 64/2187 of the original triangle's area.

4. Find the finite area limit

Add the original unit area to get $A_n=1+(3/5)[1-(4/9)^n]$. Because $4/9<1$, the limit is $1+3/5=8/5$ square units.



The successive area increments shrink geometrically.

1. $(1/3)/(1-4/9)=(1/3)/(5/9)=3/5$
Multiply by the reciprocal of the retained denominator.

2. $A_n=1+(3/5)[1-(4/9)^n] \to 1+3/5=8/5$
Add the original triangle and take the limit as the power vanishes.

Check. Set n=0 and n=1 to recover 1 and 4/3. The positively increasing areas stay below 8/5.

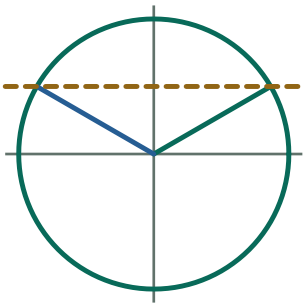
Exercise 737 · A family of parallel lines

Printed page 82 · Trigonometric equations / Line equations

Graph $\sin(x + y) = \frac{1}{2}$ by hand. Check your results with a graphing tool.

1. Find all angles with the required sine

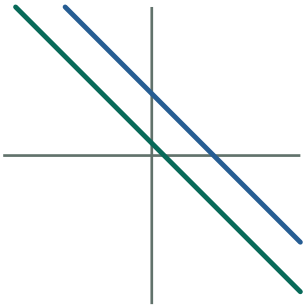
On the unit circle sin u=½ at u=π/6 and 5π/6 in one revolution. Add 2πk for every integer k to get all possible angles.



Two angles in each revolution give sine 1/2.

2. Allow x and y to share that angle

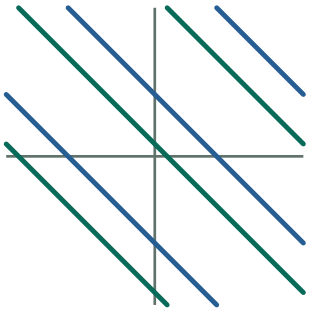
Here the angle is their sum x+y. Setting that sum to any allowed constant gives a straight line with slope −1.



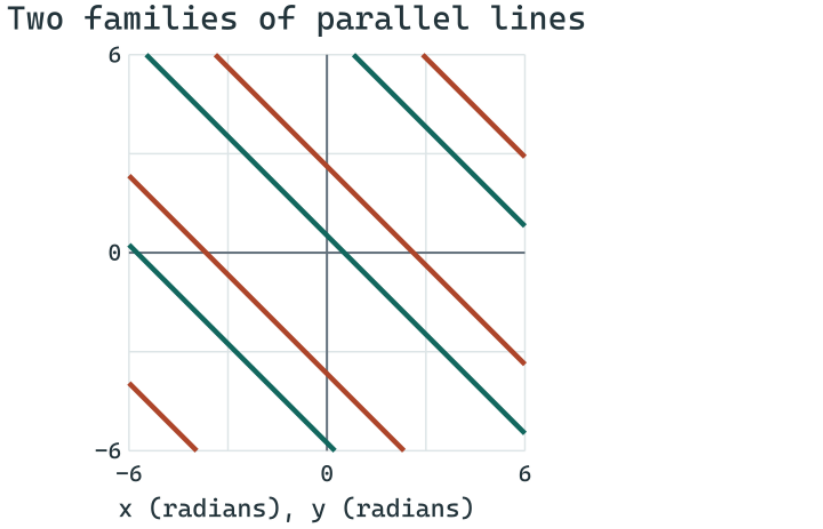
Each allowed angle is a fixed value of x + y.

3. Draw the two families

Plot $y=-x+\pi/6+2\pi k$ and $y=-x+5\pi/6+2\pi k$. The degree-mode intercept lists are $30°+360°k$ and $150°+360°k$. Every point on one line has the same required sine.



Adding full turns generates two families of parallel lines.



Check. Each entire line has constant $x+y$ and therefore gives the required sine.

Exercise 738 · Arrange five repeated digits

Printed page 82 · Counting and permutations

Using the five digits 7, 7, 8, 8, and 8 once each, how many positive five-digit integers can be formed?

1. Choose a feature that uniquely describes an integer

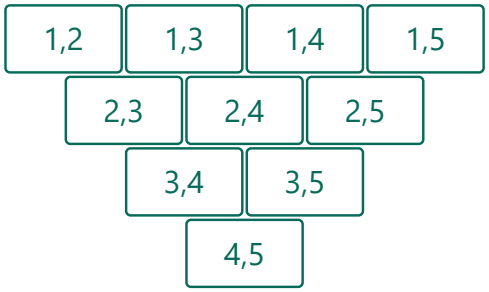
All five positions must contain two sevens and three eights. Once the positions of the sevens are specified, the eights fill every remaining position.



The locations of the two sevens determine the integer.

2. Count unordered positions

Choose any two of the five places. Ordered selections would count each pair twice, so the count is $5\cdot4/2=10$.



List each pair of positions exactly once.

3. Check distinctness

Changing the pair of seven-positions changes the integer, and rearranging identical copies in the same positions does not. All ten begin with a nonzero digit.



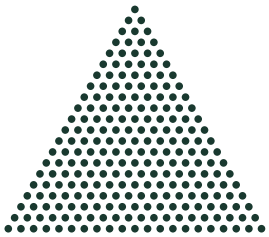
Choose two places for sevens. Their internal order has no effect.

Check. Distinguishing the copies gives $5!$ orders, then dividing by $2!3!$ also yields 10.

Exercise 739 · Pair the integers to 100

Printed page 82 · Arithmetic sequences and series / Pairing arguments

The Gauss problem. Find the sum $1 + 2 + 3 + \cdots + 100$.

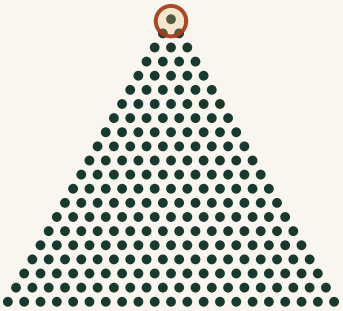


1. Pair the two ends

Add the smallest remaining number to the largest: 1+100, then 2+99, then 3+98. Each pair sums to 101.

1	2	3	…
100	99	98	…

Opposite ends give the same total in each displayed column.



Short row at the apex

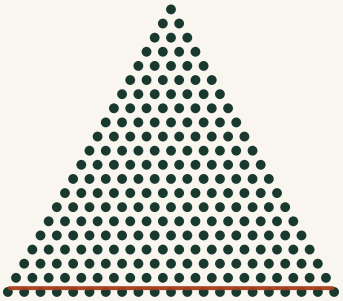
The top row represents one. Imagine a second copy of the triangle turned around so short rows can be paired with long rows.

2. Count how many pairs are used

The 100 integers can be grouped into 50 disjoint pairs. The final pair is 50+51, so every term has been counted once.

1	2	…	50
100	99	…	51

The fifty columns use all one hundred numbers exactly once.



Long row at the base

Count a whole base row as one group. A second, reversed triangle pairs this long group with a short one just as the arithmetic pairing does.

3. Calculate the sum

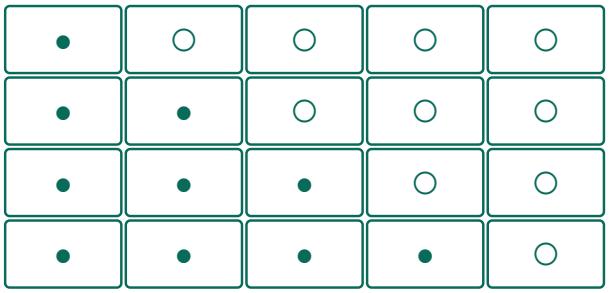
Multiplying gives 50·101=5050. Equivalently the mean of the numbers is (1+100)/2=50.5, times 100 terms.



Multiply the count of pairs by their common total.

4. Build a rectangle from two dot triangles

The drawing shows the pattern on a smaller scale. Imagine extending it to 100 rows of 1,2,...,100 dots. In a second copy reverse the row order. Put corresponding rows side by side so the first row holds 1+100 dots, the second 2+99, and so on. The combined rectangle holds 100·101 dots. One copy has half that many, confirming 5050.



A four-row example: opposite triangular copies fill a rectangle.

Check. There are 100 terms with average (1+100)/2=50.5.

Exercise 740 · Sum any arithmetic progression

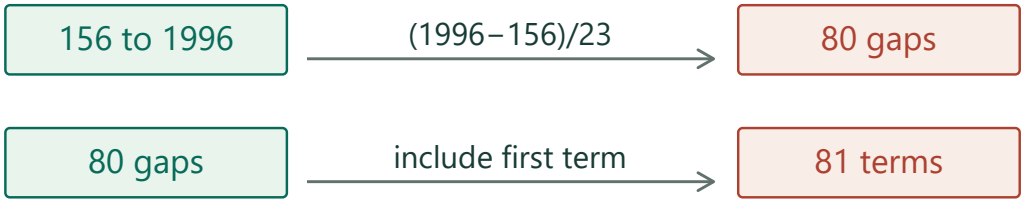
Printed page 82 · Arithmetic sequences and series / Pairing arguments

(Continuation) Find sums for the *arithmetic series*:

(a) $156 + 179 + 202 + \cdots + 1996$ **(b)** $1 + 2 + 3 + \cdots + n$

1. Count increments in (a)

The difference is 179−156=23. From 156 to 1996 there are (1996−156)/23=80 increments, or 81 terms including the starting one.



Count the gaps between the two printed endpoints.

2. Use the endpoint average

In an arithmetic sequence each pair of opposite terms has the same sum as the endpoints. Its mean is therefore (156+1996)/2=1076, also the middle term here. The sum is 81·1076=87156.

156	179	…
1996	1973	…

Each pair sums to 2,152, giving a common mean of 1,076.

1. $a_k=156+23(k-1)$, $a_{82-k}=156+23(81-k)$
Opposite positions in the list of 81 terms have indices adding to 82.

2. $a_k+a_{82-k}=312+23\cdot80=2152$
The k terms cancel, so every forward/reverse column has the same sum.

3. $2S=81\cdot2152 \Rightarrow S=81\cdot1076$
The forward and reverse lists each contain the full sum once.

3. Derive the general result in (b)

Write 1,…,n twice, with the second list reversed. Every one of the n columns adds to n+1. This counts the desired sum twice, so 1+2+⋯+n=n(n+1)/2.

1	2	…	n
n	n−1	…	1

Two copies have n columns each summing to n + 1.

Check. Eighty increments of 23 bridge the endpoints. The unpaired middle term also equals 1076.

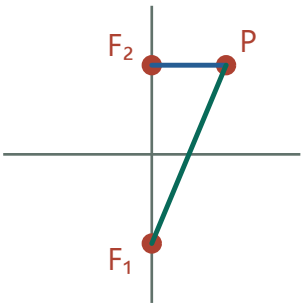
Exercise 741 · Hyperbola from two foci and a point

Printed page 82 · Hyperbolas / Foci and directrices

The focal points of a hyperbola are $(0, 6)$ and $(0, -6)$, and the point $(5, 6)$ is on one of its *branches*. Find coordinates for the points where the hyperbola intersects its major axis. Also find equations for the asymptotes, and use them to help you draw the curve.

1. Use the distance difference

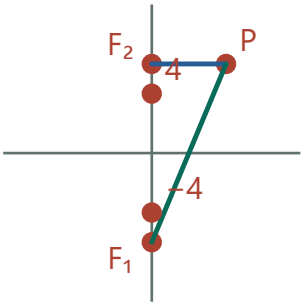
From (5,6) the upper focus (0,6) is 5 units away. The lower focus is $\sqrt{(5^2+12^2)}=13$ away. Their difference is $8=2a$, so $a=4$.



The distance difference is $13-5$.

2. Locate the branches

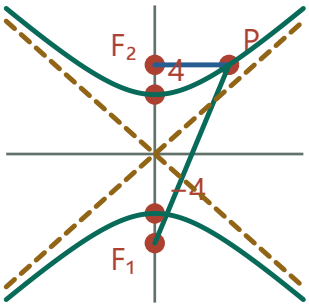
The focal axis is vertical and the center is (0,0), with $c=6$. The vertices are $(0,\pm4)$. Use $b^2=c^2-a^2=20$ to write $y^2/16-x^2/20=1$.



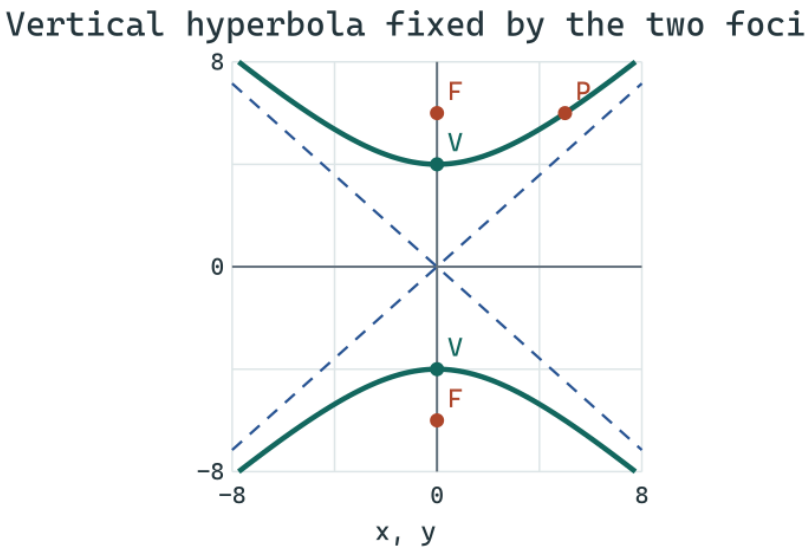
Vertical separation of the two vertices is the focal difference.

3. Draw the asymptotes and curve

A rectangle of horizontal half-width $\sqrt{20}$ and vertical half-height 4 has diagonal slopes $\pm4/\sqrt{20}$. Draw those lines through the origin, then draw the two branches away from the vertices toward the asymptotes, passing through (5,6) on the upper side.



The upper branch passes through P and approaches the guides.



Check. At P, 36/16−25/20=1. The two branches open vertically along the focal axis.

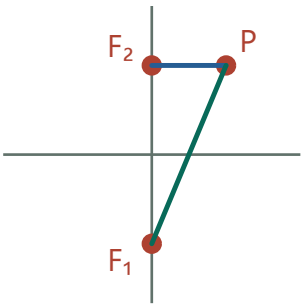
Exercise 742 · Ellipse from the same foci

Printed page 82 · Ellipses / Foci and directrices

(Continuation) The focal points of an ellipse are $(0, 6)$ and $(0, -6)$, and the point $(5, 6)$ is on the ellipse. Find coordinates for the points where the ellipse intersects its major axis.

1. Calculate the two distances to P

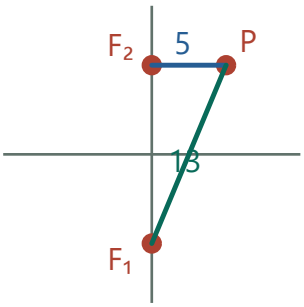
For P=(5,6), the upper focus is 5 units away horizontally. The lower focus is 5 horizontal and 12 vertical units away, a distance of 13.



Measure the same two focal radii as for the hyperbola.

2. Use the ellipse's defining addition

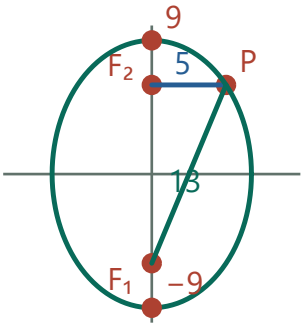
The sum of the focal distances is 5+13=18 at every point of this ellipse. At a major vertex that sum equals 2a, so a=9.



Add them to obtain the ellipse’s constant total of 18.

3. Place the major vertices

The midpoint of the foci is (0,0) and their line is the y-axis. Move a=9 in both vertical directions to reach (0,9) and (0,−9).



The vertical semimajor axis is half that total.

Check. From (0,9) the distances are 3 and 15, again totaling 18.

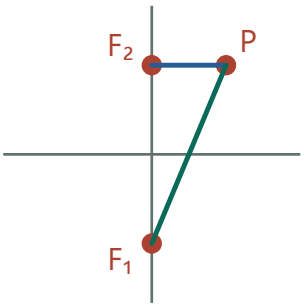
Exercise 743 · An ellipse tangent bisects angles

Printed page 82 · Ellipses / Conic reflection properties / Tangent lines

(Continuation) Recall the *reflection property of the ellipse* in order to find the slope of the line λ that is tangent to the ellipse at $(5, 6)$.

1. Use equal-length direction vectors

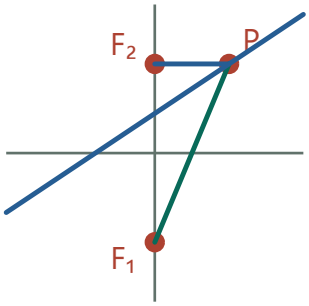
At P=(5,6), the directions from the foci toward P are (5,0) and (5,12). Divide by lengths 5 and 13 to obtain u=(1,0) and v=(5/13,12/13).



Normalize the two focal directions before comparing them.

2. Find the normal from the angle bisector

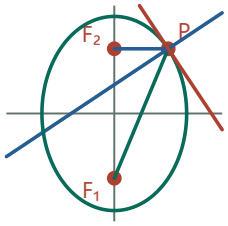
Two unit vectors make a rhombus. Its diagonal u+v bisects their angle and gives the ellipse normal by the reflection property. Here it is (18/13,12/13), of slope 12/18=2/3.



Adding the unit directions gives the outward normal.

3. Turn the normal through a right angle

The tangent is perpendicular to the normal and has slope $-3/2$. Independently the ellipse $x^2/45+y^2/81=1$ gives $y'=-81x/(45y)$, also $-3/2$ at P.



Through P draw the perpendicular tangent of slope $-3/2$.

1. $b^2=9^2-6^2=45 \Rightarrow x^2/45+y^2/81=1$
The focal axis is vertical, with a=9 and c=6.

2. $2x/45+(2y/81)y'=0 \Rightarrow y'=-81x/(45y)$
Use the chain rule and divide by the nonzero vertical coordinate at P.

3. $y'=-(81\cdot5)/(45\cdot6)=-3/2$
Evaluate at the point to confirm the normal-based construction.

Check. The ellipse has equation $x^2/45+y^2/81=1$. Implicit differentiation gives slope $-81x/(45y)=-3/2$ at P.

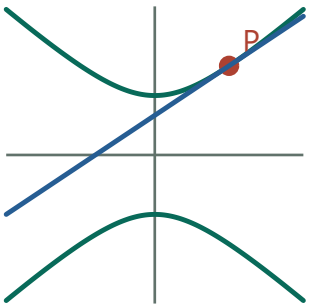
Exercise 744 · A perpendicular hyperbola tangent

Printed page 82 · Hyperbolas / Tangent lines / Orthogonality

(Continuation) *Reflection property of the hyperbola.* The line perpendicular to λ at $(5, 6)$ is in fact tangent to the hyperbola. Verify that this is true. Show also that the two tangent lines can be viewed as angle bisectors.

1. Compute the hyperbola tangent

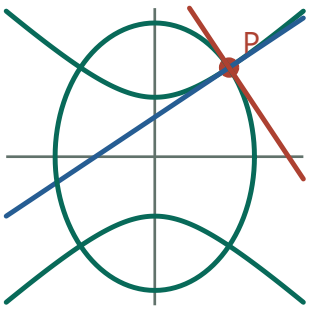
The hyperbola is $y^2/16 - x^2/20 = 1$. Differentiating gives $(2y/16)y' = 2x/20$, so at $P=(5,6)$ the slope is $16\cdot 5/(20\cdot 6) = 2/3$.



The hyperbola tangent through P has slope 2/3.

2. Compare the two tangents

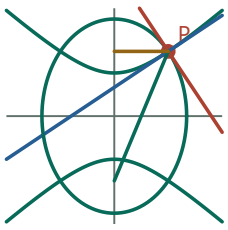
The ellipse tangent λ has slope $-3/2$. Their slope product is -1, hence the perpendicular through P is indeed the hyperbola tangent.



The ellipse tangent is perpendicular at their shared point.

3. Identify both angle bisectors

With the previous unit focal directions $u=(1,0)$, $v=(5/13,12/13)$, the rhombus diagonals $u+v$ and $u-v$ point along the two bisectors. Their slopes are $2/3$ and $-3/2$. Moreover $(u+v)\cdot(u-v) = \|u\|^2 - \|v\|^2 = 0$, explaining their right angle.



The two tangent directions bisect the focal angles.

1. $u - v = (1 - 5/13, -12/13) = (8/13, -12/13)$
Subtract unit vectors component by component.

2. slope$(u-v) = (-12/13)/(8/13) = -3/2$
Their common denominator cancels.

3. $(u+v)\cdot(u-v) = u\cdot u - u\cdot v + v\cdot u - v\cdot v = 1 - 1 = 0$
The mixed products cancel, and both vectors have unit length.

Check. The vector difference (8/13,−12/13) has slope −3/2 and its perpendicular has slope 2/3.

Exercise 745 · A bounded deer population

Printed page 83 · Sequences and recursion / Logistic growth

The logistic equation. Because resources are limited, populations cannot grow forever in an exponential fashion. For example, the wildlife refuge Deer Island can sustain a deer population of only 2000. A quadratic model for the growth of such a population is obtained by assuming that the yearly change in the population is proportional to the size of the population, and also proportional to the difference between the size of the population and its maximum sustainable size. Suppose that the population would grow at an annual rate of 28% if it were not for resource limitations, and that the 240 deer currently on the island represent 12 percent of capacity. The size of the deer population can be predicted recursively by $P_n = P_{n-1} + 0.28P_{n-1}\cdot(1 - P_{n-1})$, where P_n is the size in n years, expressed as a fractional part of capacity. The initial condition is $P_0 = 0.12$.

(a) What will the size of the population be after 1 year? after 10 years? after 20 years?

(b) What if the initial condition were $P_0 = 1.12$? What does this mean?

1. Distinguish population fraction from deer count

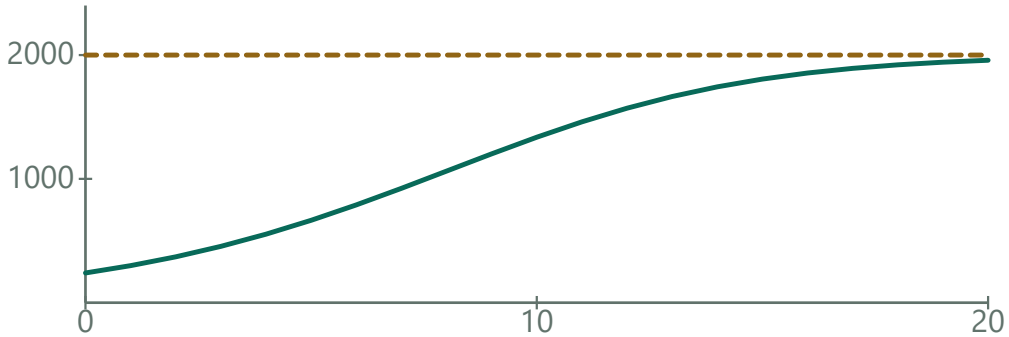
A fraction P=1 denotes the capacity of 2000 deer. With P=.12 there are 240. The extra growth at each step is .28P(1−P), smaller near capacity.



Convert an initial share of .12 to a count out of 2,000.

2. Update from below

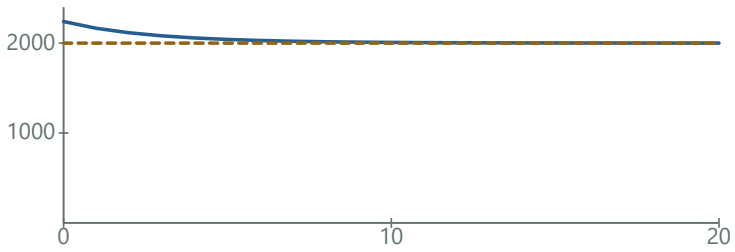
The first update gives P_1=.12+.28(.12)(.88)=.149568, or 299.136 expected deer. Iterate the same rule without rounding the intermediate shares. Rounding the final counts gives about 299,1336,1958 deer at years 1,10,20.



From below capacity the population grows toward 2,000.

3. Update from above

$P_0=1.12$ means 2240 deer, 12% over capacity. Since $1-P$ is then negative, the population falls. The same iteration gives about 2165,2008,2000 at years 1,10,20, approaching capacity from above.



Above capacity the increment is negative and the count falls.

1. $P_1=1.12+.28(1.12)(-.12)=1.082368$
Above capacity the growth correction subtracts from the share.

2. $2000P_1=2164.736\approx2165$
Multiply by capacity before rounding to an animal count.

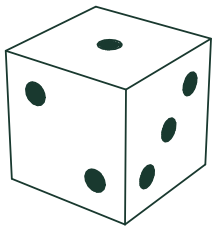
Check. The first update from .12 is .149568, giving 299.136 deer. The factor $1-P$ makes changes negative above capacity and positive below.

Exercise 746 · Aces on ten dice

Printed page 83 · Binomial probability / Probability

Una is going to roll ten standard (six-sided) dice, one after another. What is the probability that

(a) *none* of the dice land showing an ace (a single spot) on top?
(b) *some* (at least one) of the dice land showing an ace on top?
(c) the first die shows an ace on top, but none of the others do?
(d) the last die shows an ace on top, but none of the others do?
(e) *exactly one* of the ten dice shows an ace on top?



1. Set up fair independent rolls

Let $p=1/6$ for an ace and $q=5/6$ for any other face. For a specified pattern multiply the independent one-die probabilities.



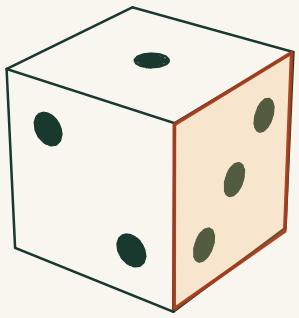
Each fair die has one ace face.

2. Use zero and its complement (a,b)

No ace means ten non-aces, probability q^{10}. At least one ace has the complementary probability $1-q^{10}$.



The two cases exhaust the outcomes of ten independent rolls.

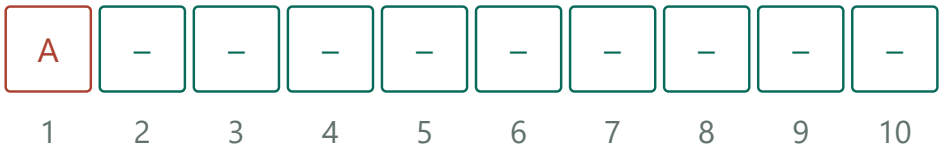


A face counted as a non-ace

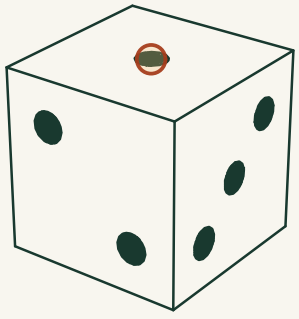
The three-pip face is one of the alternatives to an ace. Group all five such faces when a specified roll must avoid an ace.

3. Fix the sole ace's position (c,d)

For an ace first and nine non-aces, the probability is pq^9. For nine non-aces and an ace last the factors change order but give the same pq^9.



Fixing the position of the sole ace specifies one event.



One pip on the top face

This single-pip face is an ace. Each fair roll has one such face among six possible upper faces.

4. Allow any position (e)

There are ten disjoint ways to choose where the sole ace appears. Summing ten copies gives $10pq^9$.



Each of ten positions contributes the same disjoint probability.

Check. The ten choices for the sole ace are disjoint. These are the k=0 and k=1 terms of the binomial distribution.

Exercise 747 · Secant and tangent identity

Printed page 83 · Trigonometric identities / The tangent function

The *secant* is the reciprocal of the cosine: $\sec t = \dfrac{1}{\cos t}$. Show how to convert the Pythagorean identity $\cos^2 t + \sin^2 t = 1$ into the form $\sec^2 t - \tan^2 t = 1$.

1. Find the common domain first

The original Pythagorean identity holds for every real t. Introducing either tangent or secant requires dividing by cos t, so the transformed identity has a smaller domain.

1 **tan t = sin t / cos t, sec t = 1 / cos t**
Both definitions put cos t in a denominator.

2 **cos t ≠ 0 ⇔ t ≠ π/2 + kπ, k an integer**
Exclude the zeros of cosine in radian measure.

3 **cos²t > 0**
On the permitted domain the divisor is nonzero.

2. Divide each term by $\cos^2 t$

Write the division explicitly on both sides. Dividing each addend by the same nonzero number preserves the equality.

1 **$\cos^2 t + \sin^2 t = 1$**
Begin with the Pythagorean identity.

2 **$\cos^2 t/\cos^2 t + \sin^2 t/\cos^2 t = 1/\cos^2 t$**
Divide both sides by $\cos^2 t$.

3 **$1 + (\sin t / \cos t)^2 = (1 / \cos t)^2$**
Cancel the first quotient and regroup the squares.

3. Rewrite with secant and tangent

The ratios are exactly the two functions defined above. Subtracting $\tan^2 t$ from both sides isolates the stated difference.

1 **$1 + \tan^2 t = \sec^2 t$**
Substitute tangent and secant for the two ratios.

2 **$1 = \sec^2 t - \tan^2 t$**
Subtract $\tan^2 t$ from both sides.

3 **$\sec^2 t - \tan^2 t = 1$**
Reverse the equality, retaining the condition $\cos t \neq 0$.

Check. At 45° the left side is $2-1=1$. The excluded angles are precisely where both ratios are undefined.

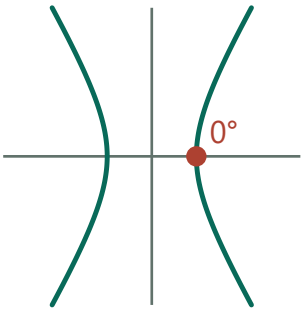
Exercise 748 · Travel on two unbounded branches

Printed page 83 · Hyperbolas / Parametric equations / Asymptotes

Describe the motion of a particle that is moving along the path described by the equation $(x, y) = (3\sec t, 5\tan t)$, for $0 \leq t \leq 360$.

1. Identify the track and breaks

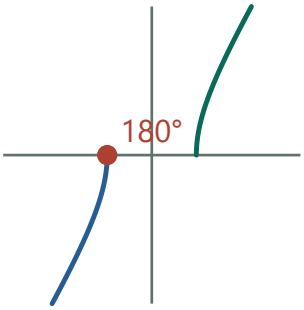
Using $\sec^2 t-\tan^2 t=1$ in $x=3\sec t, y=5\tan t$ gives $x^2/9-y^2/25=1$. Angles 90° and 270° have zero cosine, so no finite particle location exists there.



The positive or negative cosine determines the branch.

2. Follow the first half-turn

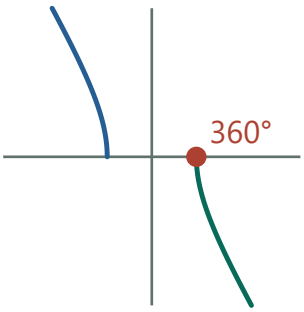
At 0° the location is (3,0). Up to 90° it moves up and right without bound. Just past 90° it is far down and left and then approaches (−3,0) at 180°.



The path breaks at 90° and approaches the left vertex.

3. Follow the second half-turn

From 180° to 270° it moves up and left without bound. Just after 270° it lies far down and right, and by 360° returns to (3,0). Each continuous interval stays on the branch determined by the sign of sec t.



After a second break at 270° the path returns to the right vertex.

Check. Eliminating t gives $x^2/9-y^2/25=1$. Signs of secant and tangent give the four directed pieces.

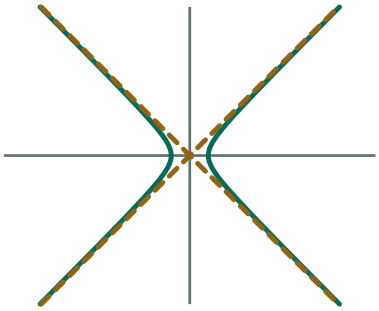
Exercise 749 · Conics seen from far away

Printed page 83 · Conic sections / Asymptotes

From a great distance, a hyperbola looks like an "X". Explain. What does a parabola look like from a great distance? Does a parabola have linear asymptotes?

1. Compare a hyperbola to its straight guides

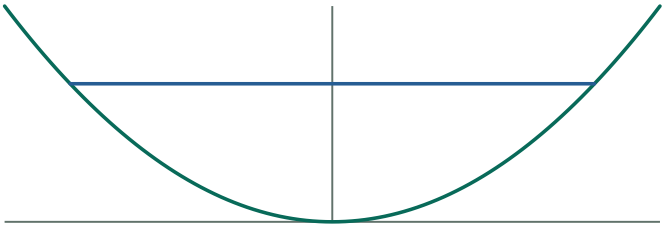
For a centered hyperbola its far branches approach the two intersecting diagonal asymptotes with vanishing gap. At a wide scale these four ends look like the arms of an X.



Hyperbola branches approach two straight asymptotes.

2. Inspect a representative parabola

For $y=x^2$, at large height H the two arms occur at $x=\pm\sqrt{H}$. They are separated by $2\sqrt{H}$, small relative to H, so the distant arms look increasingly aligned with the axis.



At height H, the parabola's horizontal span is $2\sqrt{H}$.

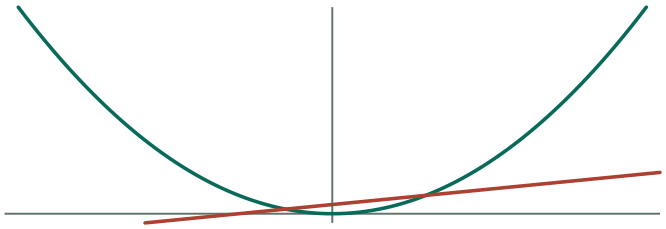
1. $H=x^2 \Rightarrow x=\pm\sqrt{H}$ for $H>0$
Slice the parabola horizontally at the positive height H.

2. width/height$=(\sqrt{H}-(-\sqrt{H}))/H=2/\sqrt{H}$
Subtract the x-coordinates and divide by the height.

3. $2/\sqrt{H}\to 0$ while $2\sqrt{H}\to\infty$
The picture narrows relative to its height even as its absolute width grows.

3. Test distance to any line

The gap from either arm to any fixed vertical line grows with √H. For a nonvertical line y=mx+b, the perpendicular distance is $|x^2-mx-b|/\sqrt{(m^2+1)}$, also growing without bound. Thus a parabola has no linear asymptote.



Against any fixed nonvertical line the quadratic term dominates.

1. distance=abs(x^2-mx-b)/√($1+m^2$)
Use the perpendicular distance from (x,x^2) to the line $mx-y+b=0$.

2. abs(x^2-mx-b)=x^2·abs($1-m/x-b/x^2$)
Factor out x^2 for nonzero x to compare growth rates.

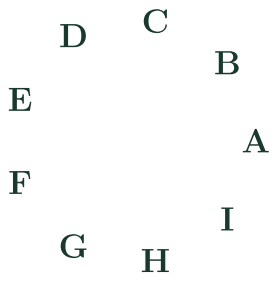
3. abs($1-m/x-b/x^2$)→1 as abs(x)→∞
The distance grows like x^2 divided by a fixed positive number.

Check. On $y=x^2$ the horizontal spread at height H is 2√H. Relative to H this goes to zero, although the spread itself keeps growing.

Exercise 750 · Ordered triples of letters

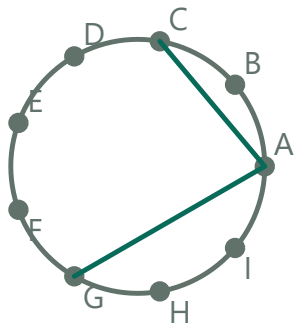
Printed page 83 · Counting and permutations

How many three-letter words can be formed from the letters A, B, C, D, E, F, G, H, and I, using each letter at most once per word? The words need not *mean* anything, so include GAC and CGA along with BIG and BAD in your count.

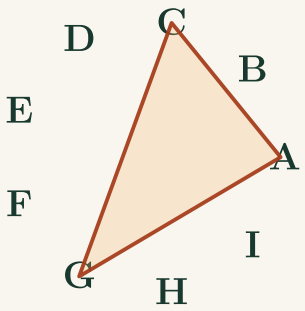


1. Decide whether order matters

GAC and CGA count separately, so view an entry as three ordered letter positions. A letter used in one position cannot be used again.



Read G, A, C in order; C, G, A is a different word.



Order G then A then C

Follow the selected letters in the named sequence. Read the starting letter first and the intermediate letter second.

2. Fill one position at a time

All nine letters are available for the first position, eight remain for the second, and seven remain for the third.



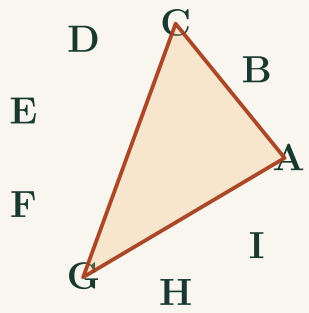
Each filled position removes one letter from later choices.

3. Multiply the numbers of choices

The count is 9·8·7=504. Equivalently choose three letters and then order them in 3! ways.



Multiply the choices for consecutive positions.



Order C then G then A

The same three letters appear in a different order. The endpoints of this two-leg route tell you which selections occurred first and last.

Check. C(9,3)·3!=84·6=504.

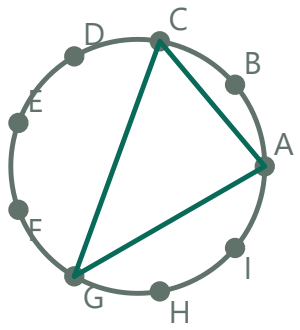
Exercise 751 · Triangles in a nonagon

Printed page 83 · Combinations / Regular polygons

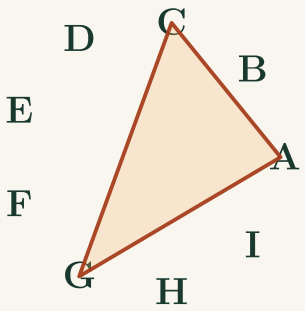
(Continuation) The points A, B, C, D, E, F, G, H, and I shown at right are vertices of a regular polygon. If three of the points are chosen, a triangle is determined. How many triangles can be formed in this way?

1. Identify which objects matter

Any three distinct vertices of the regular nine-sided polygon form one triangle. Starting at a different triangle vertex or reversing their listing leaves the same triangle.



Join three distinct vertices and close the triangle.

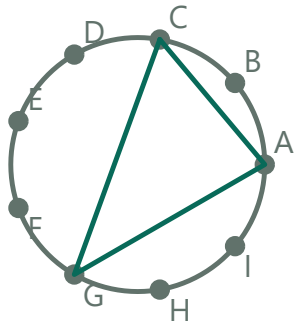


One set of three vertices · Figure from exercise 750

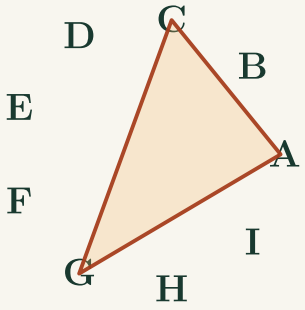
Join these three labeled positions and close the triangle. Fixing the three corners determines all three sides.

2. Use the ordered triples just counted

There are $9\cdot8\cdot7=504$ ordered triples. The same group of three vertices occurs in all $3!=6$ orders.



A reversed traversal draws the same three edges.

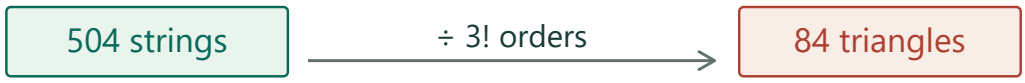


Another traversal of those vertices · Figure from exercise 750

Start at another corner and walk around in the other direction. The selected vertices and the three boundary segments are the same.

3. Divide out repeated descriptions

The count of triangles is C(9,3)=504/6=84. No three distinct vertices on this circle are collinear, so every selection works.



Every unordered triple has six ordered descriptions.

Check. Each triangle accounts for exactly six of the 504 ordered triples.

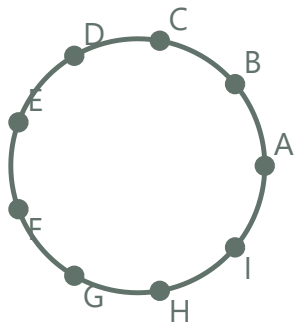
Exercise 752 · Special triangles in a nonagon

Printed page 83 · Combinations / Regular polygons / Triangle geometry

(Continuation) What is the probability that a randomly chosen triangle will be equilateral? What is the probability that a randomly chosen triangle will be a right triangle?

1. Use unordered triangles as outcomes

From the nine vertices there are C(9,3)=84 equally likely triples. Each step between adjacent vertices around the center is 360°/9=40°.

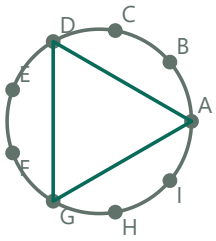


Adjacent vertices subtend 40° at the center.

2. Space three vertices evenly

An inscribed equilateral triangle needs three arcs of 120°, so take every third vertex. Nine possible starts describe each triangle three times, giving 3 distinct triples and probability 3/84=1/28.

Equilateral triangle · ADG

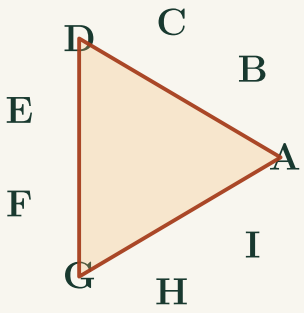


Skip two vertices between each selected pair for equal arcs.

1. 120°/40°=3 vertex steps
Equal sides have equal subtended arcs, each a third of the circle.

2. 9/3=3 distinct equilateral triples
A starting vertex and steps of three determine a triple. Each triple is named once from each of its three vertices.

3. 3/C(9,3)=3/84=1/28
Divide successful triples by all equally likely triples.

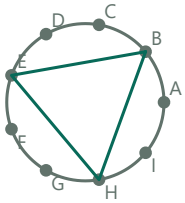


Three evenly spaced vertices · Figure from exercise 750

Trace A to D to G and back to A. Follow the cyclic labels to compare the three gaps.

Compare settings · Equilateral triangle

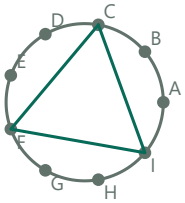
BEH



Two unused vertices lie between adjacent selected vertices.

Rotating the triangle by one vertex gives the next choice. BEH has three arcs of 120° each.

CFI

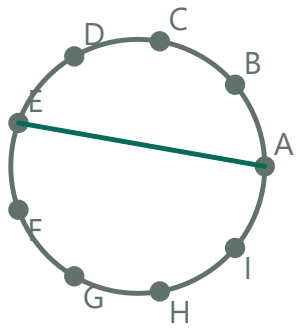


Two unused vertices lie between adjacent selected vertices.

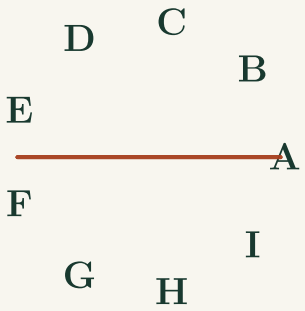
Rotating the triangle by one vertex gives the next choice. CFI has three arcs of 120° each.

3. Look for a diameter

An inscribed right triangle requires the side opposite its right angle to be a diameter. Opposite vertices would be 180°/40°=4.5 steps apart, impossible on the nine-gon. The right-triangle probability is zero.



Four steps reach 160°. The diameter would need 4½ steps.



Direction across the center · Figure from exercise 750

A line through A and the common center heads toward the gap between E and F. Check whether a labeled vertex lies at both ends of a diameter.

Check. A nine-gon step subtends 40°. Three steps give 120°, whereas 180° is not a whole number of steps.

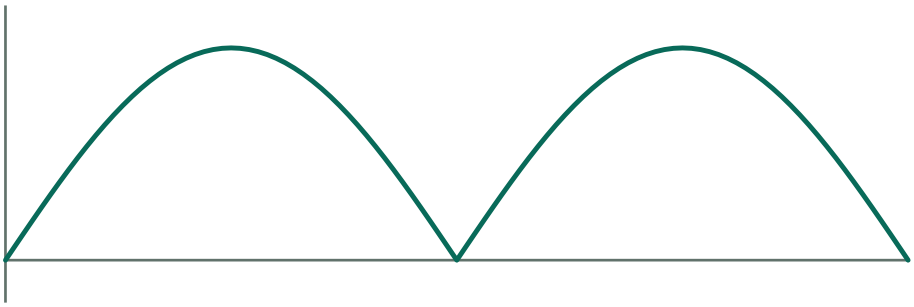
Exercise 753 · A periodic curve with sharp turns

Printed page 83 · Periodic functions / Absolute value

Invent an example of a periodic but *non-sinusoidal* function f, whose period is 50, and whose values $f(x)$ oscillate between the extremes -3 and 7. Draw a graph and write its equation.

1. Design a shape with a visible corner

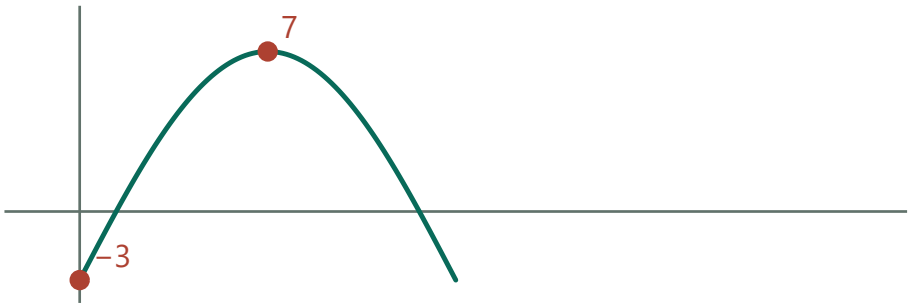
A folded sine, |sin u|, ranges from 0 to 1 and repeats after π radians. Its pointed minima keep it from being an ordinary sinusoid.



Folding sine gives sharp junctions at successive zeros.

2. Set the horizontal and vertical scales

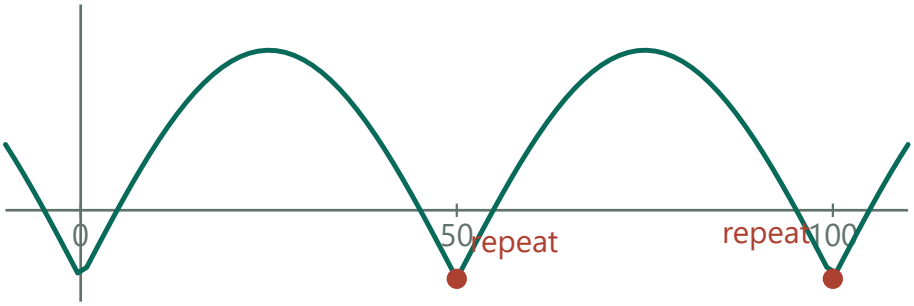
Use $u=\pi x/50$ for a 50-unit period. Multiply its height by 10 and subtract 3 to move the range to $[-3,7]$. Thus $f(x)=-3+10|\sin(\pi x/50)|$.



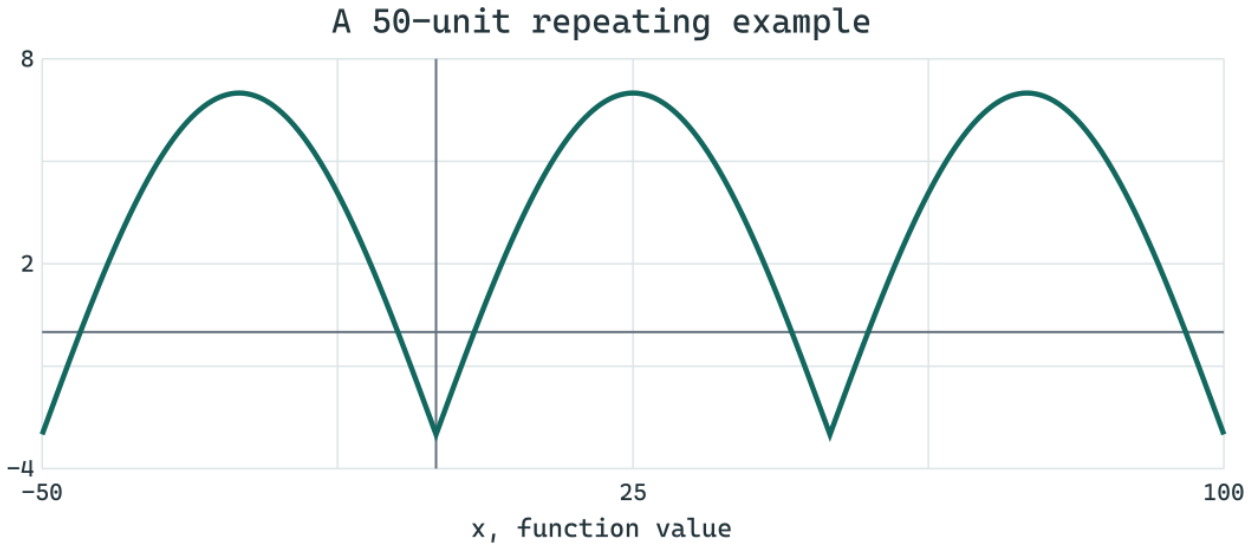
One scaled arch spans 50 units and rises from −3 to 7.

3. Draw one cycle and repeat

Sketch an arch from $(0,-3)$ through $(25,7)$ back to $(50,-3)$, then copy it left and right. Minima occur exactly 50 units apart, establishing the least period.



Copy the entire arch at each 50-unit displacement.



Check. The minima occur exactly 50 units apart. The range follows from $0 \le \text{abs}(\sin u) \le 1$.

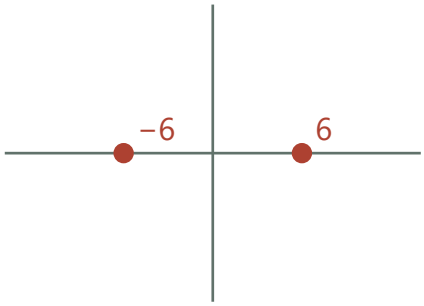
Exercise 754 · A hyperbola in parameters

Printed page 84 · Hyperbolas / Parametric equations

Find parametric equations for the graph of $4x^2 - 9y^2 = 144$.

1. Normalize the equation

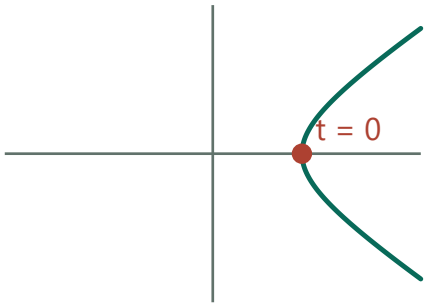
Divide by 144 to get $x^2/36 - y^2/16 = 1$. This has the pattern $(x/6)^2 - (y/4)^2 = 1$.



The normalized equation fixes the horizontal vertex distances.

2. Match a trigonometric identity

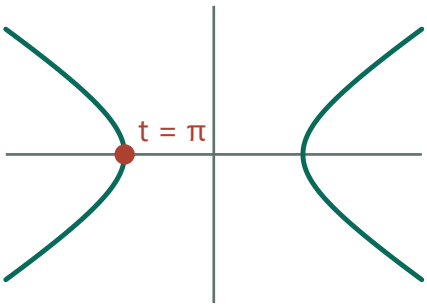
$\sec^2 t - \tan^2 t = 1$ for $\cos t \neq 0$. Assign $x/6 = \sec t$ and $y/4 = \tan t$, giving $x = 6\sec t$ and $y = 4\tan t$.



Choosing sec and tan gives a continuous sweep on one branch.

3. Cover both branches

As t moves between adjacent excluded angles, tangent takes every real value. Intervals where cosine has opposite signs supply both signs of x, hence both hyperbola branches.



Allowing both signs of cosine traces the two branches.

Check. Substituting leaves $\sec^2 t - \tan^2 t = 1$.

Exercise 755 · Prescribed vertices and asymptotes

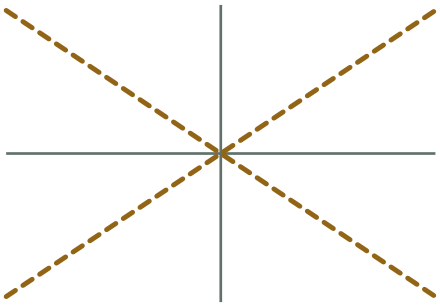
Printed page 84 · Hyperbolas / Asymptotes

There are many hyperbolas whose asymptotes are $y = \pm\frac{2}{3}x$. Sketch and write an equation for such a hyperbola, given that its vertices are

(a) $(-6, 0)$ and $(6, 0)$; **(b)** $(0, -5)$ and $(0, 5)$.

1. Use the rectangle diagonal slopes

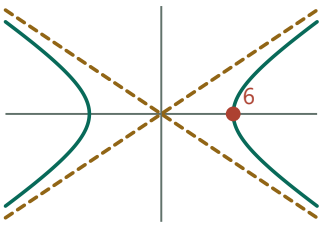
For horizontal semiaxis length H and vertical half-height V, the centered guide rectangle has slopes ±V/H. In both problems require V/H=2/3; vertices tell which half-length is known.



Both constructions must approach lines of slopes ±2/3.

2. Construct (a) with horizontal vertices

H=6, so V=4. Draw branches opening to the left and right through (±6,0) along the given guides. Their equation is $x^2/36 - y^2/16 = 1$.



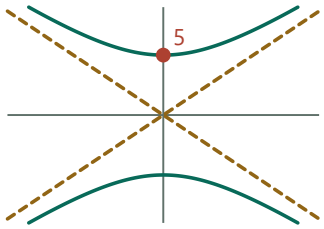
Horizontal vertices fix the half-width at 6 and half-height at 4.

1. V/6=2/3 ⇒ 3V=12 ⇒ V=4
The horizontal vertex gives H=6. Multiply the slope equation by 18, then divide by 3.

2. $H^2=36$; $V^2=16$
Use the squared horizontal and vertical semiaxes as the standard-form denominators.

3. Construct (b) with vertical vertices

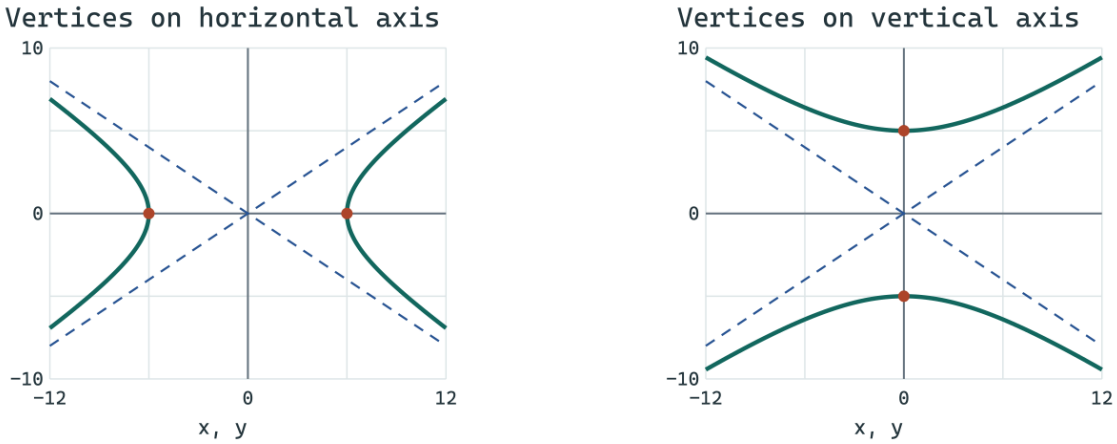
V=5, so H=15/2. Draw upward and downward branches through (0,±5). Their equation is $y^2/25-x^2/(225/4)=1$. Both constructions approach y=±2x/3.



Vertical vertices instead fix the half-height at 5.

1. $5/H=2/3 \Rightarrow 15=2H \Rightarrow H=15/2$
The vertical half-height is given and horizontal half-width is positive.

2. $H^2=225/4 \Rightarrow y^2/25-x^2/(225/4)=1$
The positive y term selects the vertically opening hyperbola.



Both pairs of dashed asymptotes have slopes ±2/3.

Check. Replacing 1 by 0 in either normalized equation gives exactly the required lines.

Exercise 756 · Transform a semicircle

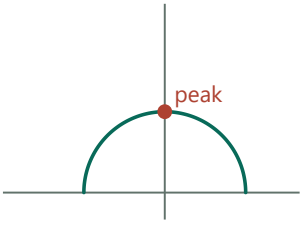
Printed page 84 · Graph transformations / Domain and range

Let $f(x) = \sqrt{9 - x^2}$ for $-3 \le x \le 3$. Sketch the graph $y = f(x)$, then compare that graph with the graphs of the following related functions. Be prepared to discuss the role of the *parameter* 2 in each; in particular, how does it affect the domain and the range?

(a) $y = 2f(x)$ **(b)** $y = f(2x)$ **(c)** $y = f(x) + 2$ **(d)** $y = f(x + 2)$

1. Sketch the original semicircle

Squaring $y=\sqrt{(9-x^2)}$ gives $x^2+y^2=9$ with $y\ge0$. Mark endpoints (−3,0),(3,0) and peak (0,3). These determine its domain [−3,3] and range [0,3].



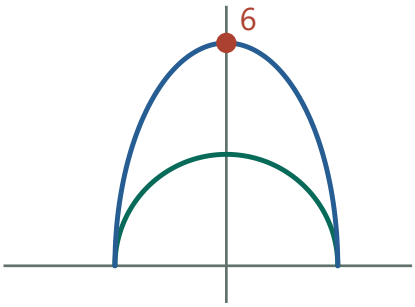
Start with the upper half of a radius-3 circle.

1. $9-x^2\ge0 \Leftrightarrow x^2\le9 \Leftrightarrow -3\le x\le3$
A real square root requires a nonnegative radicand.

2. $0\le9-x^2\le9 \Rightarrow 0\le\sqrt{(9-x^2)}\le3$
The square root increases on its nonnegative domain. The peak and endpoints attain the extremes.

2. Scale the output (a)

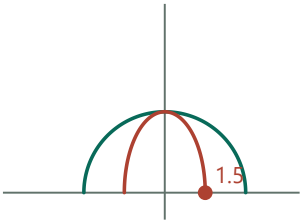
In 2f(x) every height doubles. Endpoints keep their positions and the peak reaches (0,6). Domain [−3,3], range [0,6].



Double every height while preserving its x-coordinate.

3. Scale the input (b)

To reproduce an old input u in f(2x), solve 2x=u. Horizontal coordinates halve while heights stay fixed, giving domain [−1.5,1.5] and range [0,3].



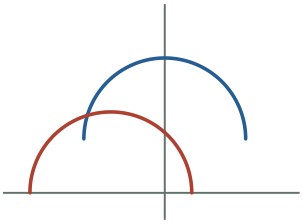
Halve each horizontal coordinate to keep the old input.

1. $9-(2x)^2 \geq 0 \Rightarrow 4x^2 \leq 9$
A real square root requires a nonnegative radicand.

2. $-3 \leq 2x \leq 3 \Rightarrow -3/2 \leq x \leq 3/2$
Both signs of the inner input are allowed; divide by positive 2.

4. Translate outside or inside (c,d)

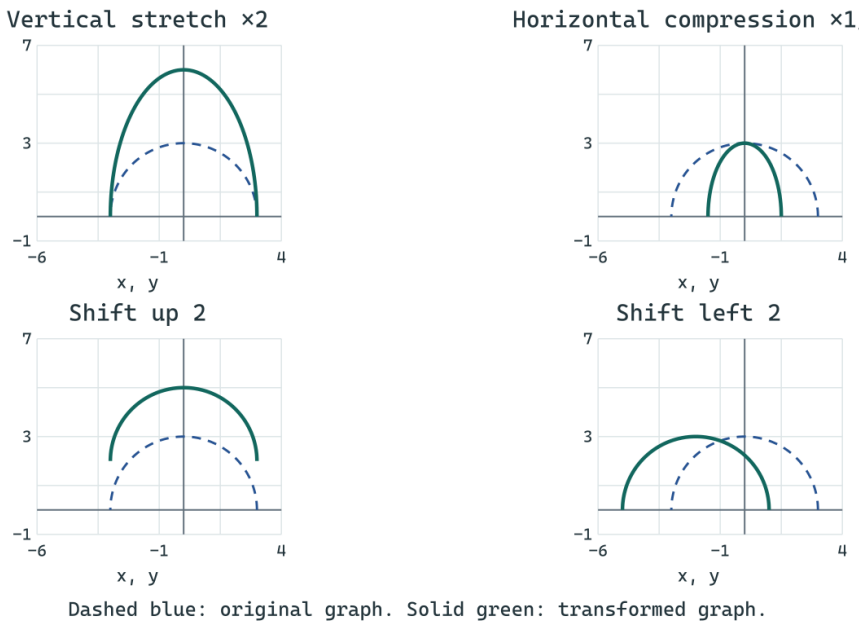
f(x)+2 raises every height by 2, giving domain [−3,3], range [2,5]. In f(x+2) the old input u occurs at x=u−2, shifting the semicircle 2 left: domain [−5,1], range [0,3].



Blue translates up 2; red translates left 2.

1. $-3 \leq x+2 \leq 3 \Rightarrow -5 \leq x \leq 1$
The shifted input must lie inside the old semicircle's domain.

2. $0 \leq f(x) \leq 3 \Rightarrow 2 \leq f(x)+2 \leq 5$
An outside addition shifts both output bounds instead.



Check. Track the original peak (0,3) to (0,6), (0,3), (0,5), (−2,3), respectively.

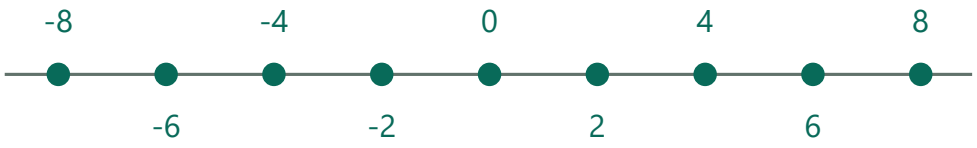
Exercise 757 · Eight jumps on a number line

Printed page 84 · Random walks / Binomial probability

Starting at the origin, a bug jumps randomly along a number line. Each second, it jumps one unit to the right or one unit to the left, either move being equally likely. Describe all the places that the bug could be after eight seconds, and tell how likely each of them is.

1. Count right jumps

Of eight jumps suppose k go right and 8−k left. Net position is k−(8−k)=2k−8. As k goes from 0 to 8 the positions are −8,−6,−4,−2,0,2,4,6,8.



The possible net positions have the same parity as 8.

2. Count the paths to one endpoint

Choose the k positions of the right jumps in C(8,k) ways. Each ordered path is equally likely with probability $1/2^8 = 1/256$.



Choose which ordered steps go right; the others go left.

3. List the probabilities

For those endpoints in increasing order, the numerators over 256 are 1,8,28,56,70,56,28,8,1. Reflecting every jump interchanges the probabilities at x and −x, explaining the symmetry.

-8	-6	-4	-2	0	2	4	6	8
1	8	28	56	70	56	28	8	1

Top: endpoint. Bottom: favorable paths out of 256.

1. $x = 2k - 8 \Rightarrow k = (x+8)/2$
A reachable point determines an integer number of right jumps from 0 to 8.

2. $x = 0 \Rightarrow k = 4 \Rightarrow C(8,4) = 8\cdot7\cdot6\cdot5/(4\cdot3\cdot2\cdot1) = 70$
For the central endpoint specifically choose four of eight jump slots.

3. $P(x=0) = 70/256 = 35/128$
Divide its count by the common number of ordered paths.

Check. The numerators sum to 256; reflecting every path explains the symmetry.

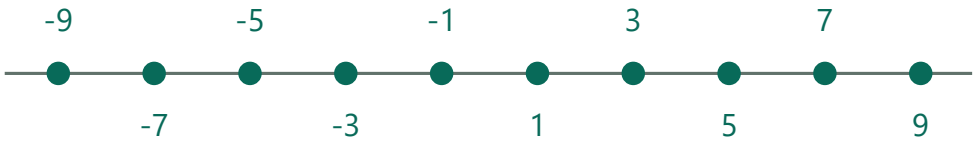
Exercise 758 · Nine jumps on a number line

Printed page 84 · Random walks / Binomial probability

(Continuation) Describe all the places that the bug could be after nine seconds, and tell how likely each of them is.

1. Track the displacement after nine jumps

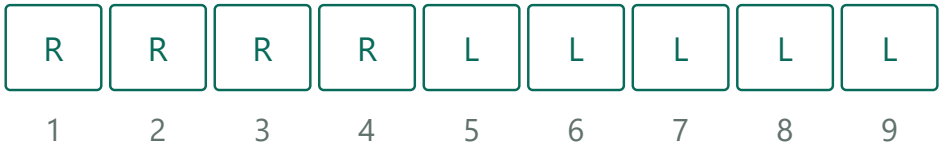
With k right jumps and 9−k left jumps, x=2k−9. The possible positions are the odd numbers −9,−7,−5,−3,−1,1,3,5,7,9.



The possible net positions have the same parity as 9.

2. Count ordered choices

The right-jump times may occupy C(9,k) possible sets of positions. Each sequence of nine fair jumps has probability 1/512.



Choose which ordered steps go right; the others go left.

3. Read the probability row

In increasing endpoint order, the probabilities have numerators 1,9,36,84,126,126,84,36,9,1 over 512. They can also be found by sending half of each eight-step probability to each adjacent point.

-9	-7	-5	-3	-1	1	3	5	7	9
1	9	36	84	126	126	84	36	9	1

Top: endpoint. Bottom: favorable paths out of 512.

Check. Taking each eight-step probability and distributing half to each neighbor reproduces the nine-step list.

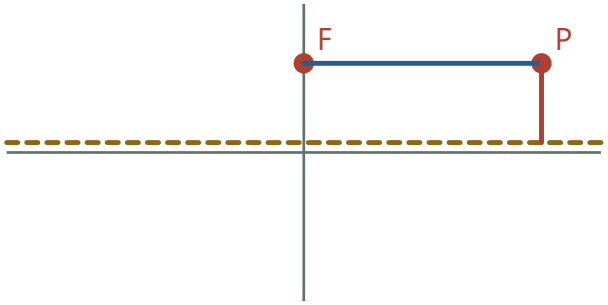
Exercise 759 · A focus and a directrix

Printed page 84 · Conic sections / Foci and directrices / Eccentricity

Let $F = (0, 9)$ be the focus and the line $y = 1$ be the directrix. Plot several points P that are three times as far from the focus as they are from the directrix, including the vertices on the y-axis. The configuration of all such P is a hyperbola of eccentricity 3. Write an equation for the hyperbola. Find the values of a, b, and c for this curve, then calculate the ratio c/a. Is the result what you expected?

1. Translate both distances

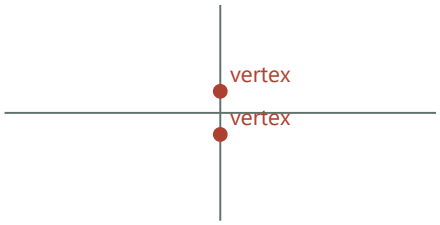
From P=(x,y) to F=(0,9) the distance is $\sqrt{(x^2+(y-9)^2)}$. Distance to the horizontal directrix is $|y-1|$. Equate the first to three times the second.



The directrix distance is vertical and the focal distance is Euclidean.

2. Simplify the locus

Both distances are nonnegative, so square: $x^2+(y-9)^2=9(y-1)^2$. Expansion cancels the $-18y$ terms and gives $8y^2-x^2=72$, or $y^2/9-x^2/72=1$.



Canceling the matched linear terms centers the conic at the origin.

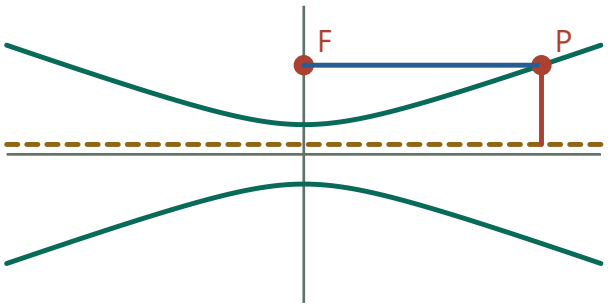
1. $x^2+y^2-18y+81=9y^2-18y+9$
Expand both squared binomials and distribute the factor 9.

2. $x^2+72=8y^2$
Add 18y, subtract y^2 and subtract 9 on both sides.

3. $y^2/9-x^2/72=1$
Subtract x^2 from both sides, then divide by 72.

3. Construct the graph

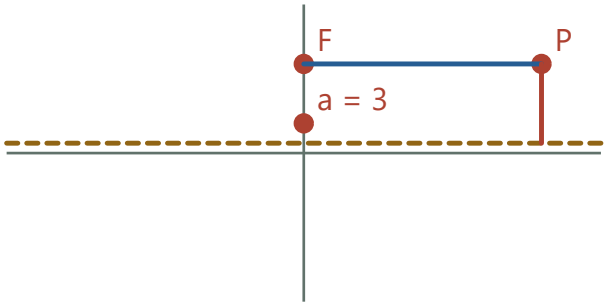
The vertical vertices are (0,±3). At y=9 the equation gives $x^2=576$, so mark (±24,9), then reflect to the lower branch. At (24,9) the focal distance is 24, three times the line distance 8.



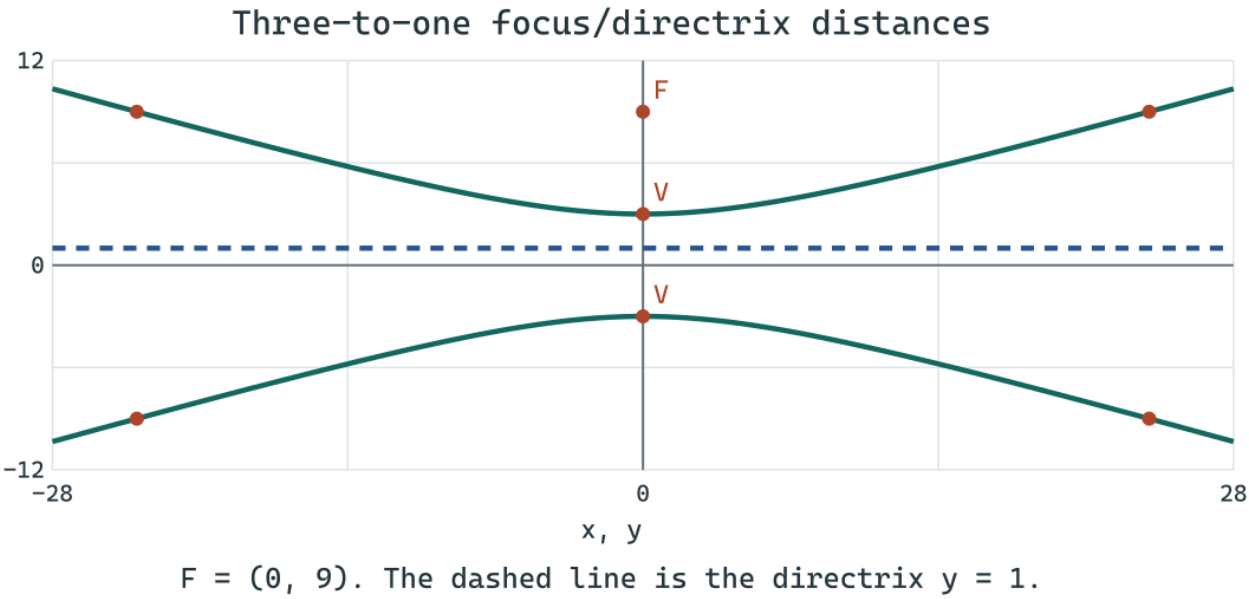
At (24, 9) the displayed focal and line distances are 24 and 8.

4. Recover the eccentricity

The parameters are a=3, b=√72 and $c=\sqrt{(a^2+b^2)}=9$. Thus c/a=3 agrees with the imposed focus-to-line distance ratio.



The 3-to-1 ratio agrees with the normalized semiaxes.



Check. At (24,9) the focus distance is 24 and the line distance is 8. At the lower vertex those distances are 12 and 4.

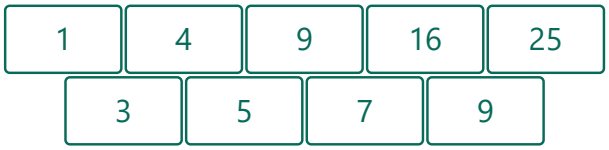
Exercise 760 · Partial sums of squares

Printed page 84 · Sums of squares / Sigma notation

Your experiments with sugar-cube pyramids led to series of consecutive squares, which look like $1 + 4 + 9 + \cdots + n^2$. This series is neither arithmetic nor geometric, thus a concise formula for its sum $S(n)$ is not readily available. After you explain why the series is neither arithmetic nor geometric, calculate the five specific values $S(1)$, $S(2)$, $S(3)$, $S(4)$, and $S(5)$.

1. Test an arithmetic pattern

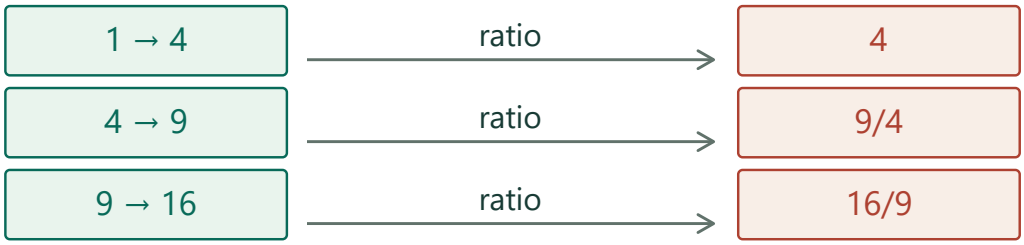
The square terms begin 1,4,9,16,25. Their first differences are 3,5,7,9, which vary; consecutive squares are not arithmetic.



Consecutive square differences grow through odd numbers.

2. Test a geometric pattern

The ratios 4/1,9/4,16/9 differ, so no fixed multiplier takes each term into the next.



Unequal ratios rule out a geometric sequence.

3. Accumulate the layer sizes

Start at 1 and add 4, then 9, then 16, then 25. The successive sums S(1) through S(5) are 1,5,14,30,55. Each increment is the number of cubes in one more square layer.



These totals add the successive square layers.

1. $S_1=1$; $S_2=1+2^2=5$
Start with the first layer and add the second layer.

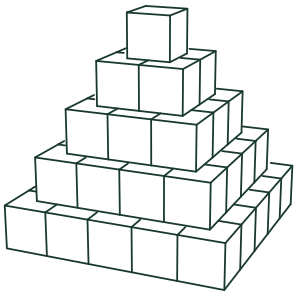
2. $S_3=5+3^2=14$; $S_4=14+4^2=30$
Each running sum adds the next square to the previous total.

3. $S_5=30+5^2=55$
The fifth layer adds 25 new units.

Check. The successive changes 5−1,14−5,30−14,55−30 reproduce 4,9,16,25.

Exercise 761 · A square pyramid of cubes

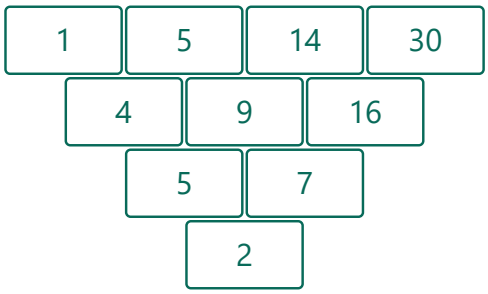
Printed page 84 · Sums of squares / Area and volume scaling / Polynomial interpolation



(Continuation) Sasha builds a sugar-cube pyramid by stacking centered square layers. The dimension of each layer is one less than the dimension of the layer immediately below it. The bottom layer is n-by-n. Sasha would like a formula for the total number of sugar cubes in such a pyramid. Sasha knows the formula $n(n+1)/2$ for the sum of consecutive integers $1+2+3+\cdots+n$. Because $n(n+1)/2$ is a quadratic function of n (it can be written in the form an^2+bn+c), Sasha guesses that the formula for $1+4+9+\cdots+n^2$ is a cubic function of n. In other words, $S(n)$ can be written an^3+bn^2+cn+d. Use the data $S(1)=1$, $S(2)=5$, $S(3)=14$, and $S(4)=30$ to determine values for a, b, c, and d. Test your formula on $S(5)$. Use the formula to express the volume of the sugar-cube pyramid as a fractional part of the volume of an n-by-n-by-n cube. What is this ratio when n is a very large number?

1. Use differences to identify the cubic coefficient

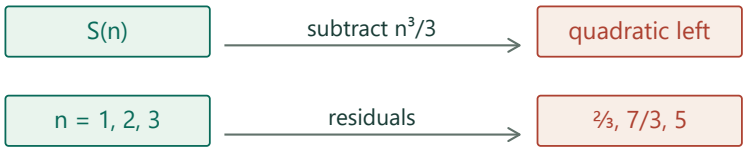
The values 1,5,14,30 have differences 4,9,16, then 5,7, then 2. For a cubic an^3+bn^2+cn+d its third forward difference is 6a, giving a=1/3.



The constant third difference determines the cubic coefficient.

2. Recover the remaining coefficients

Subtract $n^3/3$ from S(n) at n=1,2,3. The residuals are 2/3,7/3,5 from bn^2+cn+d. Consecutive differences give 3b+c=5/3 and 5b+c=8/3. Hence b=1/2, c=1/6 and d=0.



Fit the remaining coefficients to the residuals.

1. $(5b+c)-(3b+c)=8/3-5/3 \Rightarrow 2b=1$
Subtract the two differences to eliminate c.

2. b=1/2; c=5/3−3/2=1/6
Divide by two, then substitute into the first difference.

3. d=2/3−1/2−1/6=0
At n=1 the residual equals b+c+d.

3. Check and justify the sum

Factor $S(n)=n(n+1)(2n+1)/6$. At n=5 it gives 5·6·11/6=55. For all n the formula starts at S(0)=0 and $S(n)-S(n-1)=n^2$, precisely the addition of the next layer.

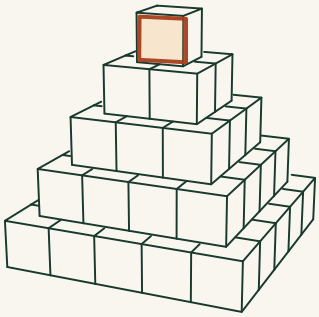


At n = 5 the square layers sum to 55.

1. $6S(n)=2n^3+3n^2+n=n(2n^2+3n+1)$
Multiply the cubic by six, then factor out its common n.

2. $6[S(n)-S(n-1)]=n[(n+1)(2n+1)-(n-1)(2n-1)]$
Subtract the factored expressions and take out n.

3. $=n[(2n^2+3n+1)-(2n^2-3n+1)]=6n^2$
Expand and cancel. Dividing by six gives the square layer exactly.

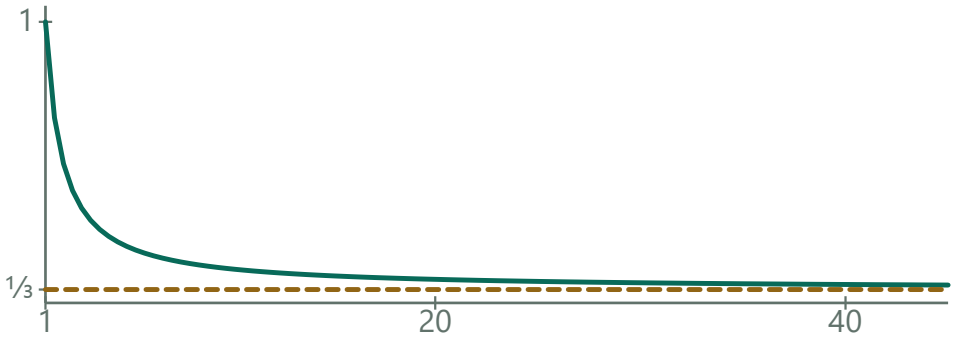


The first cube

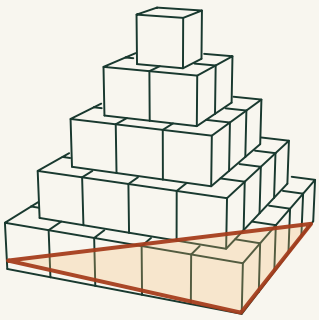
The top layer consists of a single unit cube. Start the running count here, then inspect successively wider square layers below it.

4. Compare with the full cube

Divide by n^3 to obtain $(n+1)(2n+1)/(6n^2)$. Dividing numerator and denominator by n^2 shows the volume ratio tends to $2/6=1/3$.



The ratio of stack volume to n^3 tends to one third.



Wide base layer

Follow the front edge of the broadest layer. A layer with n cubes across and n cubes deep contributes a square number.

Check. The candidate has $S(0)=0$ and $S(n)-S(n-1)=n^2$, proving it for all nonnegative integers.

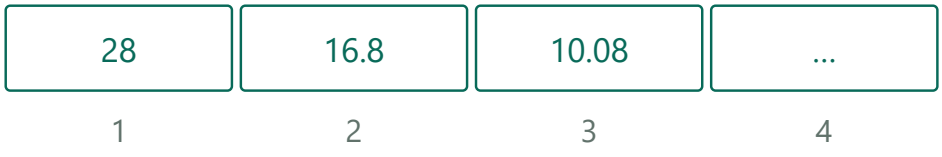
Exercise 762 · A sixteen-term geometric sum

Printed page 84 · Geometric sequences and series / Sigma notation

Express the geometric series $28 + 16.8 + \cdots + 28(0.6)^{15}$ in sigma notation. Then find the sum of these sixteen terms.

1. Find the summand

Since $16.8/28=.6$, term k with zero-based indexing is $28(.6)^k$. The last term uses $k=15$.



Each term is multiplied by 0.6.

2. Count and encode

The inclusive integers 0 through 15 number sixteen. Thus the sigma expression is $\Sigma\ 28(.6)^k$, k=0,…,15.



Inclusive endpoints give sixteen powers.

3. Sum by cancellation

If the sum is S, .6S shifts every exponent one higher. Subtraction leaves $.4S=28[1-(.6)^{16}]$, hence $S=70[1-(.6)^{16}]$.

28	28r	…	$28r^{15}$	
	28r	…	$28r^{15}$	$28r^{16}$

With r = 0.6 only the outer powers survive subtraction.

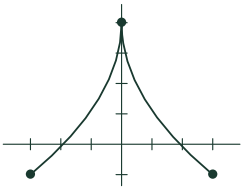
Check. The infinite sum is 70 and its omitted tail is exactly $70(.6)^{16}$.

Exercise 763 · Transform a curve with a pointed crest

Printed page 85 · Graph transformations / Domain and range

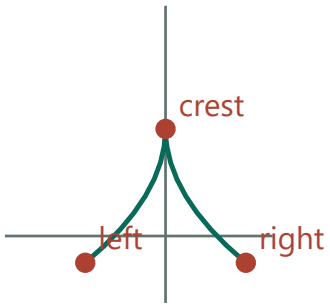
The graph of $y = g(x)$ for $-3 \le x \le 3$ is shown at right. Notice that $(0, 4)$, $(3, -1)$, and $(-3, -1)$ are points on the graph. Sketch the graph of each of the following related functions. Be prepared to discuss the role of the parameter 2 for each function. Write the domain and range of the functions.

(a) $y = 2g(x)$ **(b)** $y = g(2x)$ **(c)** $y = g(x) + 2$ **(d)** $y = g(x + 2)$

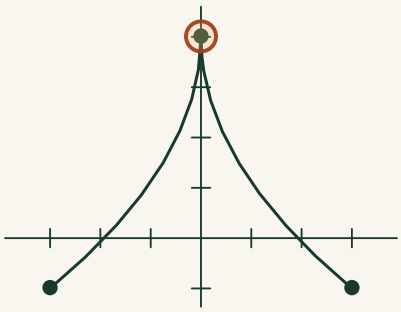


1. Read the original outline

The curve starts at (−3,−1), crests at (0,4), and ends at (3,−1). Apply each coordinate change to the entire drawn shape, keeping the intervening portions as well as these landmarks.



Track the two endpoints and the pointed crest of the printed outline.

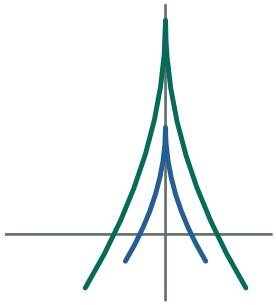


Pointed crest

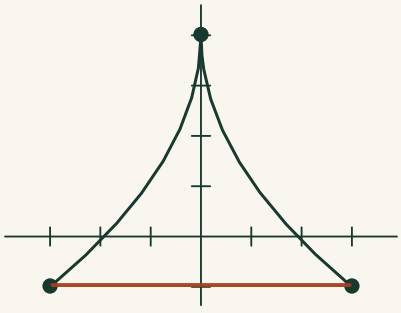
Track the marked peak independently of the endpoints. A vertical transformation changes its height, and an input transformation changes its horizontal coordinate.

2. Compare multiplication outside and inside (a,b)

For 2g(x) double all heights: endpoints (±3,−2), crest (0,8), domain [−3,3], range [−2,8]. For g(2x), recover old input u at x=u/2. Endpoints move to (±1.5,−1), crest stays (0,4), domain [−1.5,1.5], range [−1,4].



Green doubles heights; blue halves horizontal coordinates.

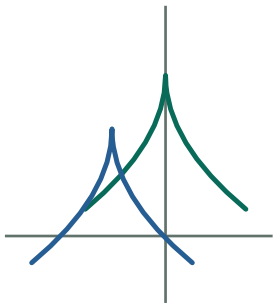


Ends of the domain

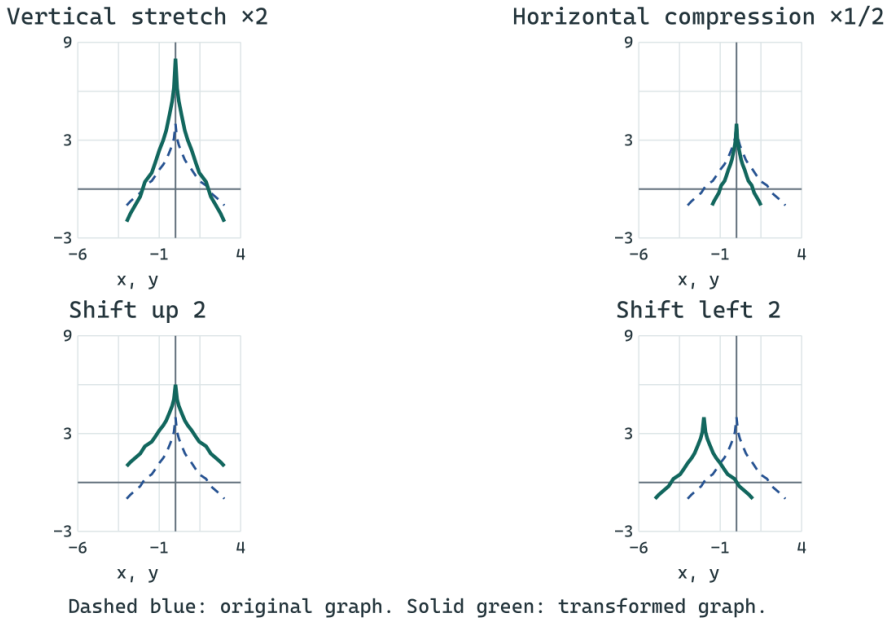
The left and right marked points give the limits of the input interval. Carry both along when graphing a transformation.

3. Compare addition outside and inside (c,d)

g(x)+2 raises endpoints to (±3,1) and crest to (0,6): domain [−3,3], range [1,6]. For g(x+2), solve x=u−2; the curve moves left 2. Endpoints have x=−5,1 and the crest is (−2,4): domain [−5,1], range [−1,4].



Green moves up 2; blue moves left 2.



Check. Follow the crest through (0,8), (0,4), (0,6), (−2,4). Each transformed endpoint follows the same coordinate rule.

Exercise 764 · A second trigonometric hyperbola

Printed page 85 · Hyperbolas / Parametric equations / Trigonometric identities

Show that csc and cot (the reciprocals of sin and tan) can be used to parametrize a hyperbola, just as sec and tan can.

1. Create a new identity

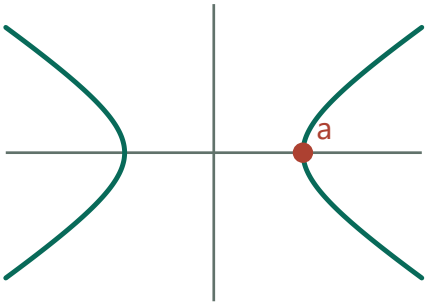
Divide $\sin^2 t+\cos^2 t=1$ by $\sin^2 t$ where $\sin t\neq 0$. This gives $1+\cot^2 t=\csc^2 t$, or $\csc^2 t-\cot^2 t=1$.



Only angles with nonzero sine can be used.

2. Introduce the scale factors

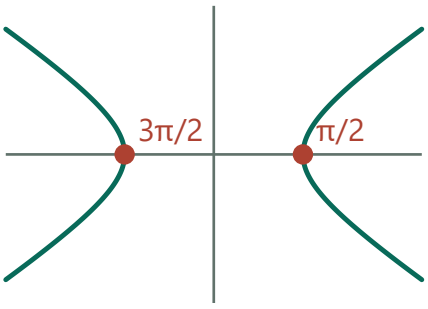
Set $x=a\csc t$, $y=b\cot t$ with $a,b>0$. Then $(x/a)^2-(y/b)^2=1$, the equation of a horizontal hyperbola.



Example $a = 3$, $b = 2$. The squared coordinates have difference 1.

3. Check the coverage

At $t=\pi/2$ and $3\pi/2$ the vertices are $(a,0)$ and $(-a,0)$. Between consecutive multiples of π, cotangent takes every real value, and the sign of sine selects which branch is traced.



The sine sign selects one of the two branches.

Check. At $t=\pi/2$ and $3\pi/2$ the vertices are $(a,0)$ and $(-a,0)$. On each interval between excluded angles cotangent attains all real values.

Exercise 765 · A standing wave on a string

Printed page 85 · Sinusoidal functions / Amplitude and period

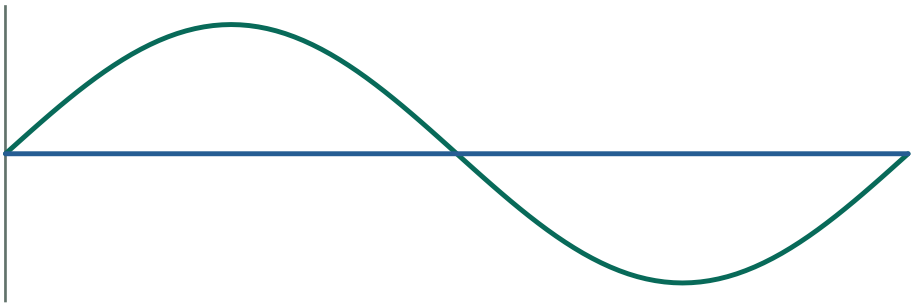
Interpreted in degree mode, the equation $y = \cos(72000t)\sin(2x)$ models the motion of a stretched string that is 180 centimeters long ($x = 0$ is one end of the string and $x = 180$ is the other), and that is vibrating 200 times per second.

(a) There are times when the model predicts that the string is straight. Find two consecutive such times.

(b) The center of the string never moves. Find a point on the string that is moving the fastest at an instant when the string appears straight. How fast is this point moving?

1. Recognize when the entire string is straight

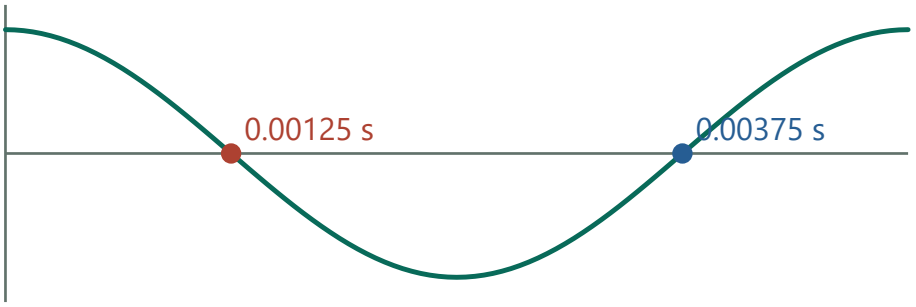
The fixed spatial factor sin(2x°) varies with x. For all heights to vanish together, the time multiplier cos(72000t°) must be zero.



All points are simultaneously flat when the time factor vanishes.

2. Find consecutive crossings

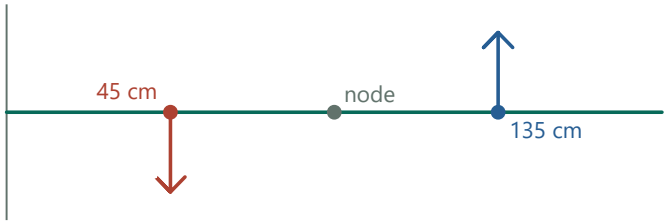
Cosine vanishes at 90°+180°k. Dividing by 72000 gives consecutive times .00125 s and .00375 s, separated by half of the .005-s vibration period.



Two consecutive zeros of the time factor are half a cycle apart.

3. Locate the fastest points

Their spatial multiplier must have magnitude 1. On $0 \le x \le 180$, these are $x=45$ and $x=135$ cm; the center $x=90$ is a node. Write the time factor as $\cos(400\pi t)$. Its rate has magnitude 400π at zero crossings, giving speed $400\pi \approx 1256.6$ cm/s if y is measured in cm.



At 0.00125 s the string is flat and its two antinodes move fastest.

1. $72000° \cdot \pi/180° = 400\pi$
Convert to radians before differentiating with respect to seconds.

2. $\partial y/\partial t = -400\pi \sin(400\pi t)\sin(2x°)$
The position x is fixed as the string moves. The chain rule supplies 400π.

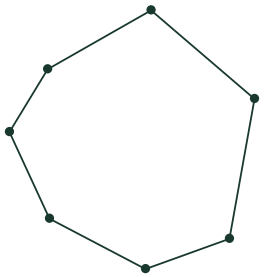
3. $2x = 90 + 180j \Rightarrow x = 45 + 90j$
The spatial sine has maximum magnitude there. Only $j = 0, 1$ places x on the string.

Check. The full period is .005 s, matching 200 Hz. The straight configurations are half a period apart and the nodes $x = 0, 90, 180$ stay fixed.

Exercise 766 · Label a seven-sided figure

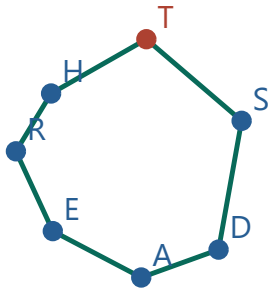
Printed page 85 · Counting and permutations / Symmetry

How many different labeled versions of the heptagon shown at right could you create by applying the seven letters T, H, R, E, A, D, and S (in any order) to the seven vertices? Would the answer be the same with a regular heptagon?

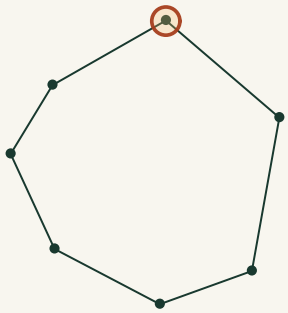


1. Assign distinct corners on the pictured heptagon

For the irregular shape, recognizable individual corners distinguish positions. Seven choices for the first letter, then six, down to one give 7!=5040 assignments.



On the irregular outline individual corner positions can be distinguished.

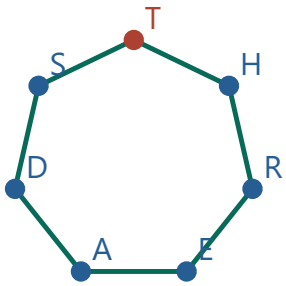


One recognizable corner

The upper corner can serve as a reference for a particular letter. Its location in the irregular outline can be distinguished from the other corners.

2. Consider rotations of a regular heptagon

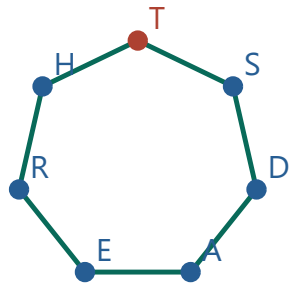
Its cyclic ordering can be viewed starting from any vertex. If rotated copies count as the same, fix T at a reference corner, then order the other letters: 6!=720.



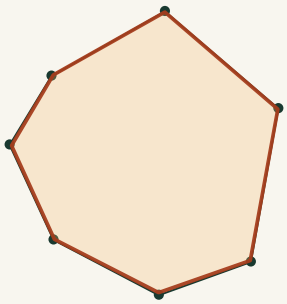
On a regular heptagon fix T as a reference when rotations count alike.

3. Specify whether flips also identify drawings

If reversal is allowed, each distinct-letter cyclic order pairs with its reverse and the count becomes 360. If instead positions stay fixed on the page, both shapes permit 5040 assignments.



Reflection reverses the cyclic reading while leaving T fixed.



A cyclic order

Follow neighboring vertices around the edge in succession. A different starting place or a reversal changes the description depending on the equivalence rule used.

Check. Fix the position of T to remove rotational choice. Arrange the remaining six letters in 6!=720 cyclic orders, paired by reversal if reflections count.

Exercise 767 · Fit waves to a crest and a trough

Printed page 85 · Sinusoidal functions / Phase shifts / Amplitude and period

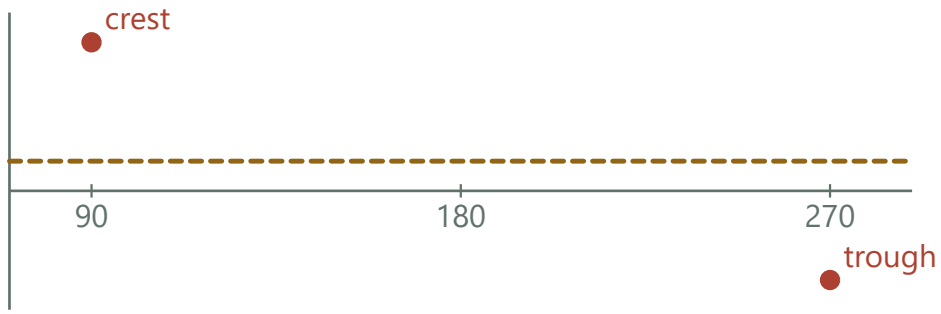
Answer the following without using a graphing device.

(a) Write an equation for a sinusoidal curve that has a crest at the point $(90, 10)$ and an adjacent valley at $(270, -6)$.

(b) If we remove the word adjacent from part (a) there are many different sinusoidal curves that have a crest at point $(90, 10)$ and valley at point $(270, -6)$. Write equations for at least two such curves.

1. Set the vertical parameters

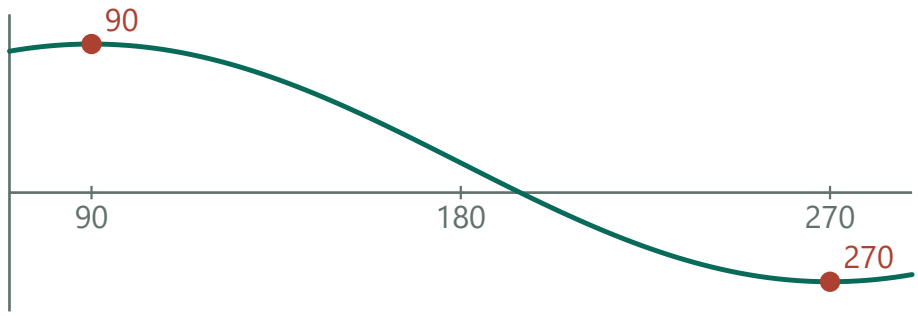
The midpoint height of crest 10 and valley −6 is (10−6)/2=2; amplitude is (10−(−6))/2=8.



The midline lies halfway between −6 and 10.

2. Use the word “adjacent”

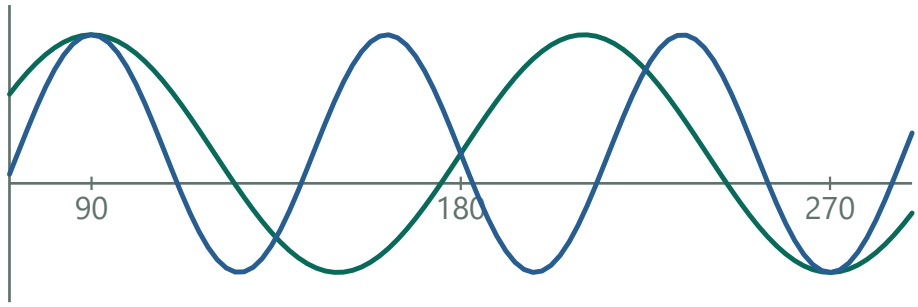
The 180 horizontal units from x=90 to 270 then span half a period. The period must be 360. A matching curve is y=2+8sin(πx/180), which has its crest at 90 and next trough at 270.



Adjacent opposite extrema are separated by half a period.

3. Insert more half-cycles

To keep a crest at 90 use y=2+8cos(kπ(x−90)/180). At 270 the cosine is cos(kπ), equal to −1 for every odd k. Taking k=3 and k=5 supplies two distinct cases with intervening extrema.



Allowing extra odd half-cycles gives different frequencies.

Check. At x=90 the cosine is 1. At x=270 it is cos(kπ)=−1 for odd k, placing both required endpoints.

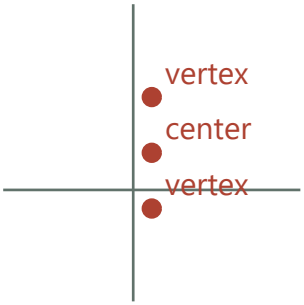
Exercise 768 · A translated vertical hyperbola

Printed page 85 · Hyperbolas / Translations

Write an equation for the hyperbola whose vertices are $(1, 5)$ and $(1, -1)$, and whose focal points are $(1, 7)$ and $(1, -3)$. Draw a graph and find equations for the asymptotes.

1. Find the common midpoint

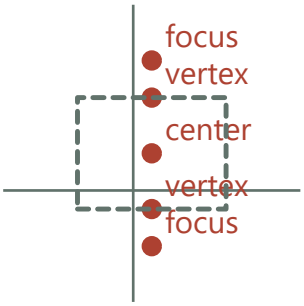
The two vertices average to (1,2), also the focal midpoint. Their shared x-coordinate means a vertical opening axis.



The two vertices have a common x-coordinate and midpoint (1, 2).

2. Measure a, c and b

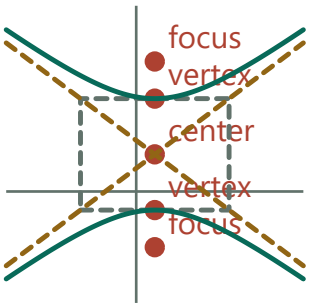
Each vertex is 3 units from the center, so a=3. Each focus is 5 away, giving c=5, hence $b^2=c^2-a^2=16$.



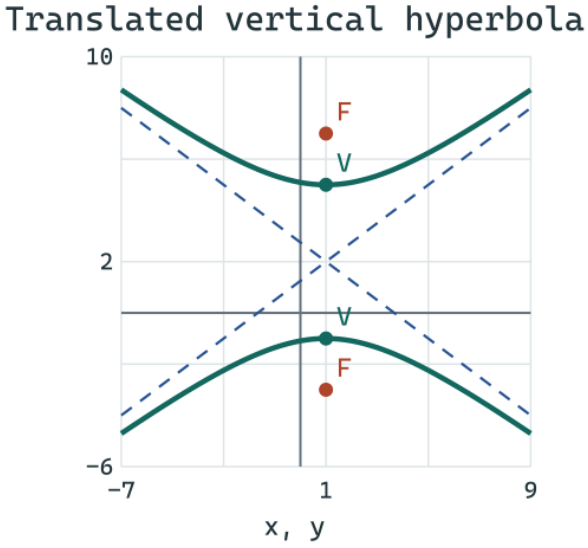
The vertical half-height is 3; the focal displacement is 5.

3. Draw with shifted coordinates

The equation is $(y-2)^2/9-(x-1)^2/16=1$. Draw the centered guide rectangle with half-width 4 and half-height 3. Its diagonal lines are $y-2=\pm 3(x-1)/4$. The branches open above and below the marked vertices.



Shift both branches and both diagonal guides to the same center.



Check. The foci lie five units above and below the center and the vertices three units above and below.

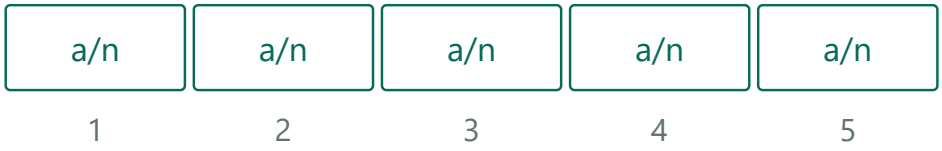
Exercise 769 · Choose two factors out of five

Printed page 85 · Combinations / Binomial coefficients

Without using a calculator to expand $(a+n)^5$, find the coefficient of its a^2n^3 term.

1. View the power as five choices

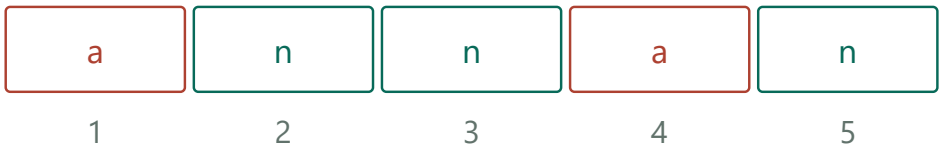
Expanding $(a+n)^5$ picks a or n from each of five factors. The monomial a^2n^3 uses exactly two a choices.



Choose one term independently from each factor.

2. Choose their places

There are 5·4 ordered ways to point to two different positions. Choosing the same two positions in reverse order describes the same product.



This choice contributes one product with exactly two a factors.

3. Read the coefficient

Divide by 2 to obtain $C(5,2)=10$. Each of these 10 selections contributes one a^2n^3 term.



Each pair of positions contributes the same a^2n^3 term.

Check. Choosing positions for n instead gives $C(5,3)=10$.

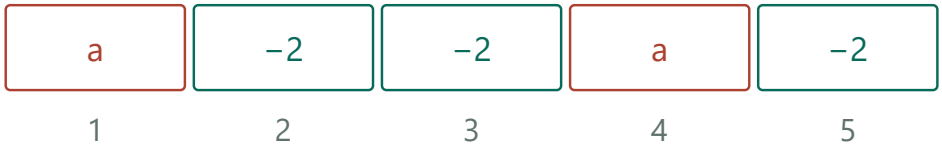
Exercise 770 · A negative binomial coefficient contribution

Printed page 85 · Binomial coefficients / Algebraic manipulation

(Continuation) What is the coefficient of the a^2 term in the expansion of $(a-2)^5$?

1. Locate terms that have power a²

From five factors (a−2) select a exactly twice. There are C(5,2)=10 ways to choose those two factors.



Choose exactly two a positions in the five factors.

2. Multiply the remaining constants

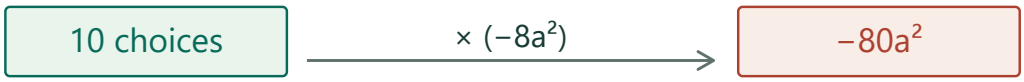
The other three factors supply −2 each. Their product is $(-2)^3=-8$, so every selected product contributes $-8a^2$.



The three unchosen factors multiply to −8.

3. Collect the like terms

Ten copies of $-8a^2$ give $-80a^2$. The numerical coefficient is −80.



Collecting like products adds their negative coefficients.

Check. Substituting n=−2 in the term $10a^2n^3$ from the previous expansion returns $-80a^2$.

Exercise 771 · Two aces in five rolls

Printed page 85 · Binomial probability / Combinations

Five standard (six-sided) dice are rolled, one at a time. What is the probability that
(a) the *first two* dice show aces, and the next three do not?
(b) two of the five dice show aces, and the other three do not?

1. Use the one-roll probabilities

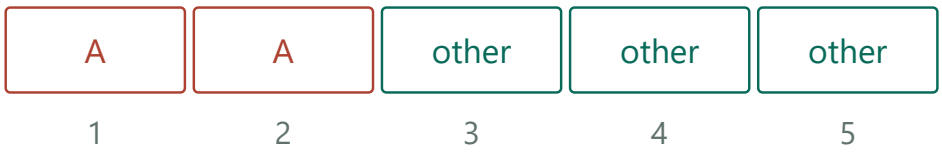
An ace is face 1, probability 1/6; a non-ace has probability 5/6. Rolls are independent.



An ace is one specified face of the die.

2. Require a specified order (a)

For two aces first and three non-aces afterward, multiply $(1/6)^2(5/6)^3=125/7776$. No position choices remain.



One exact order contributes $(1/6)^2(5/6)^3$.

3. Allow any two ace locations (b)

C(5,2)=10 pairs of locations are possible. Each has the same probability and distinct pairs are disjoint. Thus the probability is 10·125/7776=625/3888.



Sum the probabilities for these ten possible pairs of ace positions.

Check. The ten choices of ace positions are equally probable and disjoint.

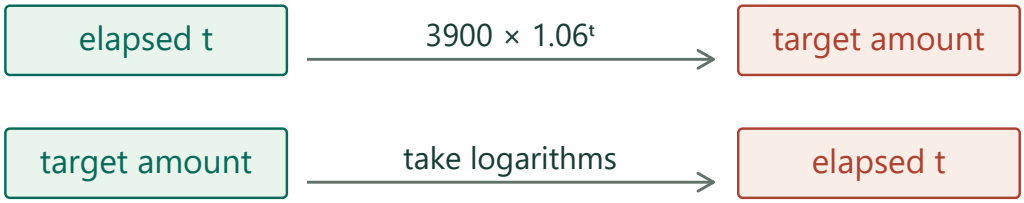
Exercise 772 · Invert exponential growth

Printed page 85 · Inverse functions / Logarithms / Exponential growth and decay

Given that $P(x) = 3900(1.06)^x$, find a formula for the *inverse function* P^{-1}. In particular, calculate $P^{-1}(5200)$, and invent a context for this question. Graph $y = P^{-1}(x)$.

1. Reverse the input-output question

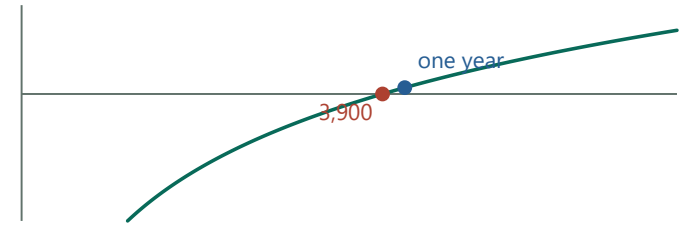
Let x denote a target amount and t the input required to reach it. From x=3900(1.06)ᵗ divide by 3900, then take logarithms.



Reverse which of the two quantities is supplied.

2. Write the inverse

$t=P^{-1}(x)=\ln(x/3900)/\ln(1.06)$, for $x>0$. For its mathematical graph mark (3900,0),(4134,1). It is increasing and approaches the vertical asymptote $x=0$.



On the inverse plot the amount is horizontal and time is vertical.

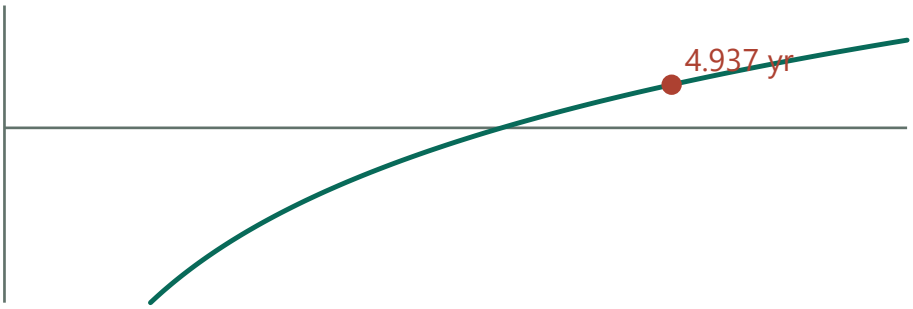
1. $x/3900=(1.06)^t>0$
The growth factor and starting amount are positive, so x must be positive to invert the formula.

2. $\ln(x/3900)=t\ln(1.06)$
Take the natural log and use its power rule.

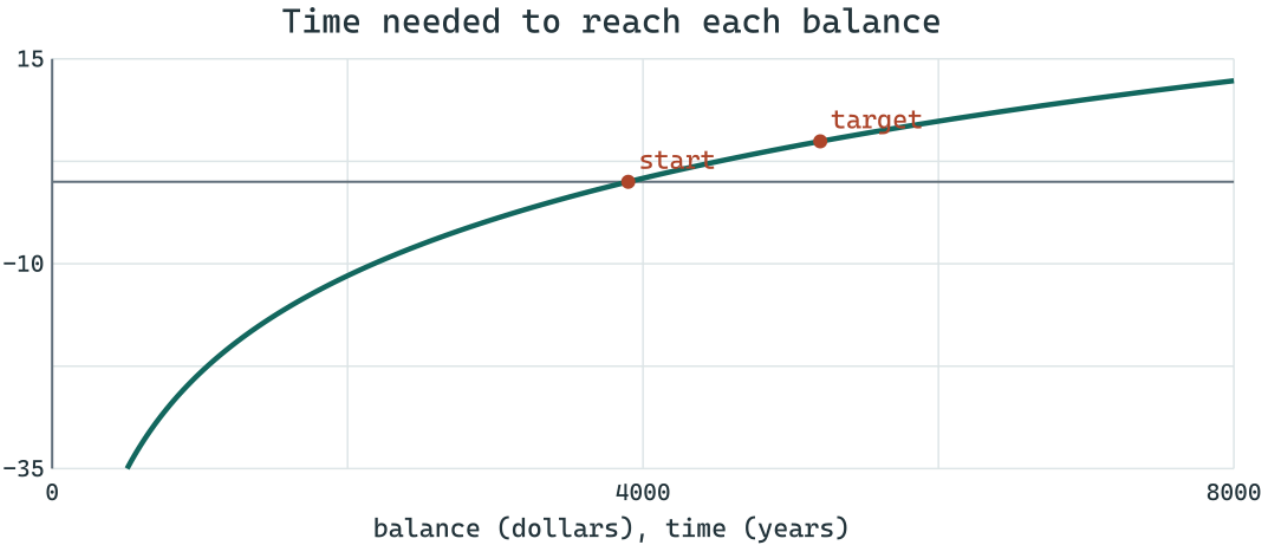
3. $t=\ln(x/3900)/\ln(1.06)$
The base exceeds one, so its logarithm is positive and division is valid. A time model restricted to $t\geq 0$ has $x\geq 3900$.

3. Interpret the target value

For a \$3900 investment growing by a factor 1.06 per year, $P^{-1}(5200)\approx 4.937$ years under the smooth interpolation. If balances are credited only on annual dates, year 5 is the first posted balance above \$5200.



A smooth model reaches \$5,200 shortly before the fifth anniversary.



Check. 3900·1.06^(4.9371)≈5200. If interest is only credited annually, the first posted balance exceeding \$5200 is in year 5.

Exercise 773 · Flight time across two impacts

A fact from physics: The time required to fall from a height of h feet (or to rise to that height after a bounce) is $\sqrt{h}/4$ seconds. Suppose that a ball, whose rebound ratio is 64 percent, is dropped from a height of 25 feet.

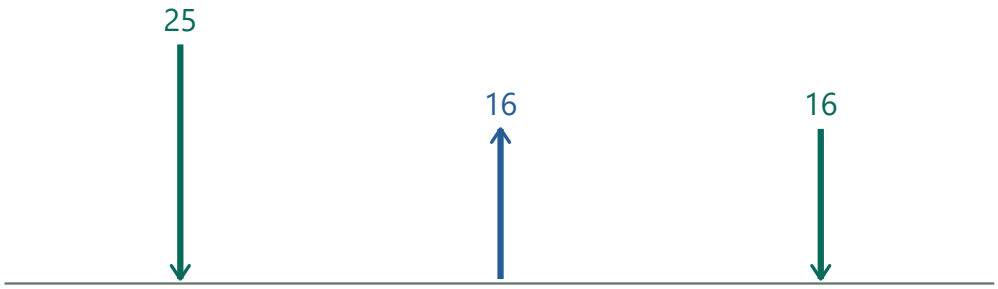
(a) When the ball strikes the ground for the second time, it will have traveled 57 feet in total. Confirm that this is a true statement.

(b) How much time passes between the initial drop and the second impact?

(c) Avery responds to the preceding question, "That's easy, you just divide $\sqrt{57}$ by 4 and get 1.89 seconds." Brooks responds, "I think you meant to divide $\sqrt{25+16}$ by 4 and get 1.60 seconds." What do you think of these remarks, and why?

1. Divide the journey at the reversal

The original drop is 25 ft. The first rebound reaches .64·25=16 ft before the ball falls those same 16 ft. Thus at impact 2 its total path is 25+16+16=57 ft.



The first two impacts contain one 25-ft drop and a 16-ft round trip.

2. Time each leg independently

Using the supplied law, the initial drop needs √25/4=1.25 s. The rise and fall each need √16/4=1 s. The complete time is 3.25 s.



Take the square root of each separate height.

1. $t_1=\sqrt{25}/4=5/4$ s
The 25-foot first descent is a single leg.

2. $t_2=t_3=\sqrt{16}/4=4/4=1$ s
The rebound includes an upward leg and a downward leg of equal length.

3. $t=5/4+1+1=13/4=3.25$ s
Add the three elapsed intervals.

3. Diagnose the two suggested answers

Applying one square root to 57 treats a broken journey with reversals as one uninterrupted fall. Applying it to 25+16 also omits one 16-ft leg. Add the individual leg times rather than combining heights first.



The ball reverses direction. Add the time for each separate leg.

Check. After 1.25 s the ball first hits the ground. It spends another 2 s rising to and falling from 16 ft.

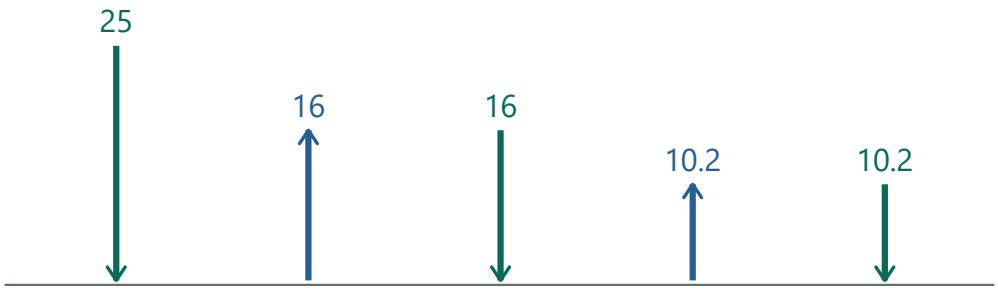
Exercise 774 · A hundred bounces and the limit

Printed page 86 · Geometric sequences and series / Limits and convergence

(Continuation) How much time passes between the initial drop and the hundredth impact? How far has the ball traveled by then? How far does the ball travel if it is left to bounce "forever"? How much time does all this bouncing actually take?

1. Count round trips after the first drop

Impact 1 follows one 25-ft descent. Impacts 2 through 100 add 99 rises and matching falls, with successive rebound heights $25(.64)^j$, $j=1,\ldots,99$.



Start the 100-impact count with one drop, then add round trips.

2. Sum lengths

The full path at impact 100 is $25+50(.64)[1-(.64)^{99}]/.36$ ft. In the infinite limit the distance becomes $25+32/.36=1025/9\approx113.889$ ft. For example, impact 2 adds a rise of $25\cdot.64=16$ ft and a fall of 16 ft, hence the factor 50 times the first rebound ratio. The powers .64 through $.64^{99}$ count 99 full up-and-down trips.



After the initial drop each pair of legs contributes twice a rebound height.

1. $G=.64+.64^2+\cdots+.64^{99}$
Collect one rebound-height ratio for each round trip.

2. $G-.64G=.64-.64^{100}$
Subtract the shifted sum. Every shared power cancels.

3. $G=.64(1-.64^{99})/.36$
Factor the right side and divide by $1-.64$. Multiply G by 50 ft for both legs and add the initial 25 ft.

3. Sum times on their own scale

Time varies with square root of height, so the factor per rebound is $\sqrt{.64}=.8$. One initial leg takes 1.25 s. The 99 round trips add $2.5(.8)[1-(.8)^{99}]/.2$. The time is $1.25+10[1-(.8)^{99}]$ s, tending to 11.25 s. The initial free fall satisfies $25=16t^2$ in feet and seconds, giving $t=1.25$. Each later rise and fall at a given rebound height take equal time; this accounts for the factor 2 on the round trips.



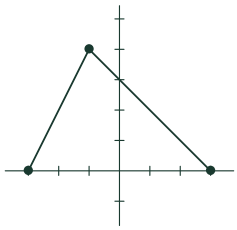
Flight-time ratios are square roots of the height ratios.

Check. Using just the first rebound in these geometric sums gives 57 ft and 3.25 s, agreeing with impact two.

Exercise 775 · Move a triangular graph

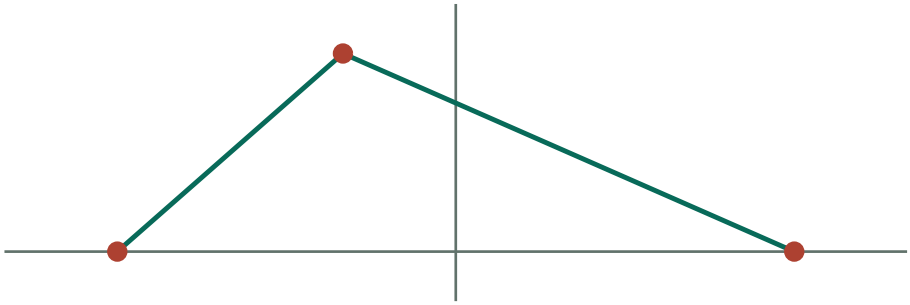
Printed page 86 · Graph transformations / Domain and range

A graph of $y = h(x)$ for $-3 \le x \le 3$ is shown. For each of the following related functions, sketch a graph, explain the role of the numerical parameter, and write the domain and range.
(a) $y = -4h(x)$ **(b)** $y = h(x/2)$ **(c)** $y = h(x) - 1$ **(d)** $y = h(x - 3)$

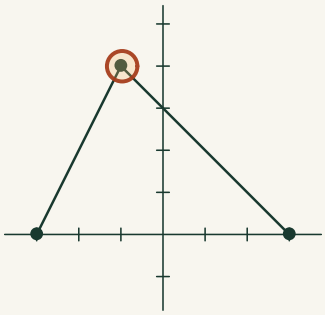


1. Read the three corners

The printed graph is two straight segments joining $(-3,0)$, $(-1,4)$ and $(3,0)$. Mark those three points. Each transformation maps straight segments to straight segments, so moving the corners and joining them in the same order gives the complete new graph.



The corners determine both straight segments.

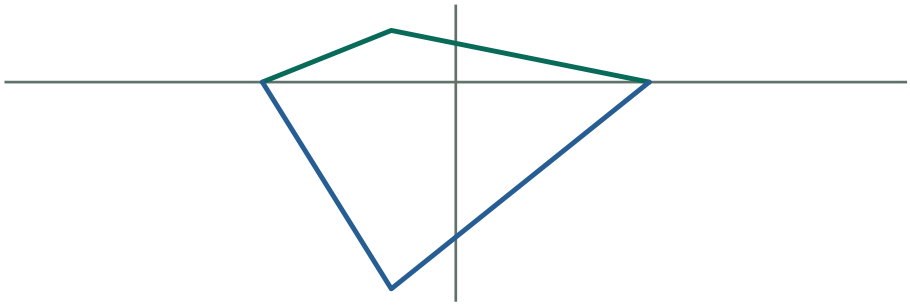


Corner where the slopes change

Both segments meet at this corner. Locate its transformed position before drawing either new segment.

2. Act on the heights (a)

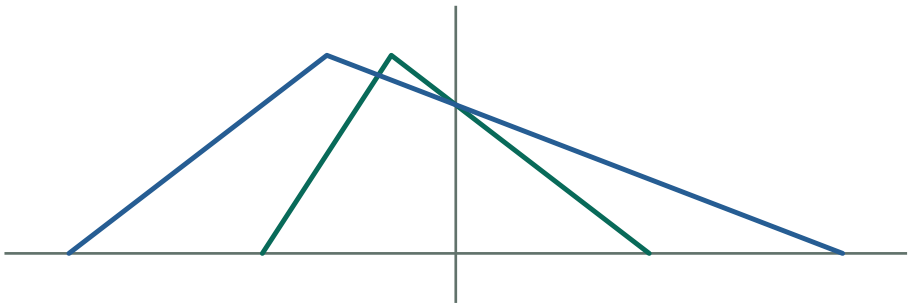
For $y=-4h(x)$, the input stays in place. Multiply each old height by -4: the corner heights $0,4,0$ become $0,-16,0$. Plot $(-3,0),(-1,-16),(3,0)$. The negative sign reflects in the x-axis and 4 stretches vertically. Domain $[-3,3]$; range $[-16,0]$.



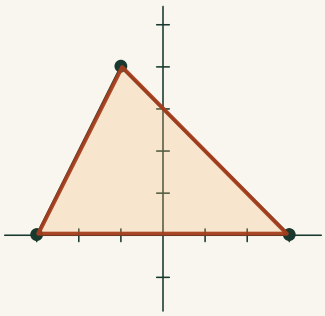
Multiply every height by -4. The peak becomes a trough.

3. Stretch the horizontal coordinates (b)

Let u be an old input. To produce h(u) from h(x/2), set x/2=u, so x=2u. Double only the horizontal coordinates. The corners become (−6,0),(−2,4),(6,0); domain [−6,6], range [0,4].



For h(x/2), double each horizontal coordinate.

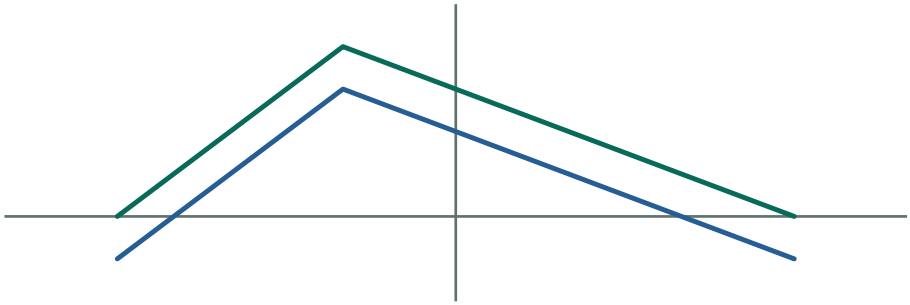


Two linear pieces

Trace a straight segment from the first endpoint to the peak, then another to the last endpoint. Their order helps you reconstruct each transformed graph.

4. Lower the whole graph (c)

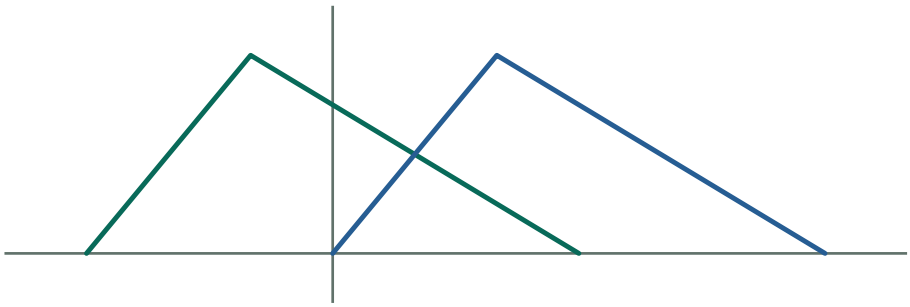
For h(x)−1 subtract 1 from every old height, giving (−3,−1),(−1,3),(3,−1). The graph moves down by one; domain [−3,3], range [−1,3]. The peak and both endpoints move by the same vertical amount.



For h(x) − 1, lower all three corners equally.

5. Move the graph to the right (d)

For h(x−3) to equal h(u), choose x=u+3. Each point moves three units right while its height stays fixed. Join (0,0),(2,4),(6,0); the domain moves to [0,6] and the range stays [0,4].



For h(x − 3), slide the graph right three units.



Check. Read the smallest and largest x and y among each set of corners. Straight connecting segments add no new extremes.

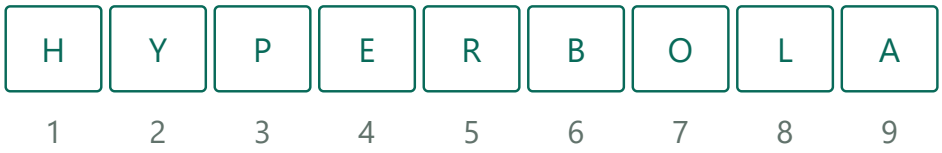
Exercise 776 · Arrange the letters of HYPERBOLA

Printed page 86 · Counting and permutations

How many nine-letter words can be formed from the letters of *hyperbola*, using each letter once per word? The words do not have to actually spell anything, of course.

1. Inspect the letter inventory

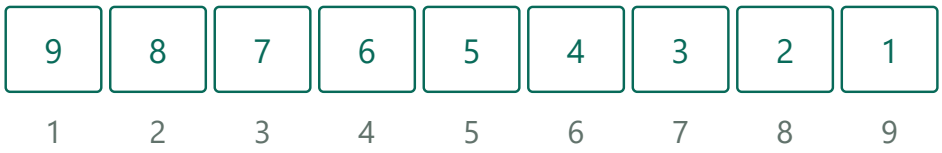
H,Y,P,E,R,B,O,L,A are nine different letters. Every nine-letter word must use each one once, and changing order changes the word.



All nine letters in HYPERBOLA are distinct.

2. Choose successive positions

The first position permits 9 choices, the next 8, and so on down to one choice for the last position.



Each place uses a letter unavailable to later places.

3. Evaluate the factorial

The number of arrangements is $9! = 9\cdot8\cdot7\cdot6\cdot5\cdot4\cdot3\cdot2\cdot1 = 362880$. Equivalently each of nine possible first letters has $8!$ completions.



The product of all nine choice counts is $9!$.

Check. Fixing the first letter gives nine disjoint groups, each containing $8! = 40320$ arrangements.

Exercise 777 · Size of a large factorial

Printed page 86 · Counting and permutations / Logarithms / Digits and place value

The product $9 \cdot 8 \cdot 7 \cdot 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1$ is usually abbreviated by 9!, and read *9 factorial*. Because factorials are often large, they are a challenge to compute and display. Consider 1000!, for example. What does the information $\log(1000!) = 2567.604644$ tell us about this gigantic number? Invent a question that has 1000! as its answer.

1. Separate the two parts of the common logarithm

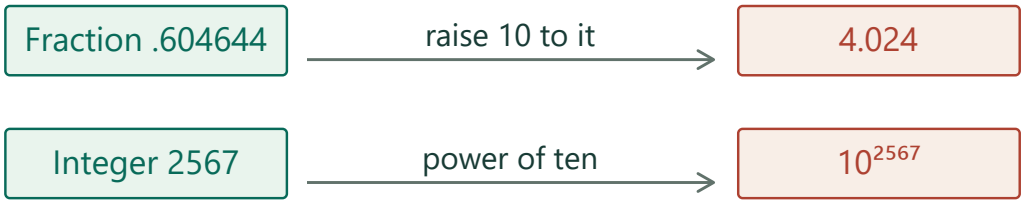
$\log_{10}(1000!) \approx 2567.604644$ lies between 2567 and 2568. Thus $10^{2567} \le 1000! < 10^{2568}$ and the integer has 2568 decimal digits.



The common logarithm lies between consecutive whole exponents.

2. Recover the leading scale

Exponentiating the fractional part gives $10^{.604644} \approx 4.024$. Hence the number is approximately 4.024×10^{2567}.



The fractional part sets the leading digits. The integer sets their place.

3. Give a counting interpretation

For 1000 distinct books placed left-to-right on one shelf, the position choices are 1000,999,...,1. Their product, the number of arrangements, is 1000!.



Arranging distinct books multiplies these successive choice counts.

Check. Summing $\log_{10} k$ from k=1 to 1000 computes the same logarithm without constructing the full integer.

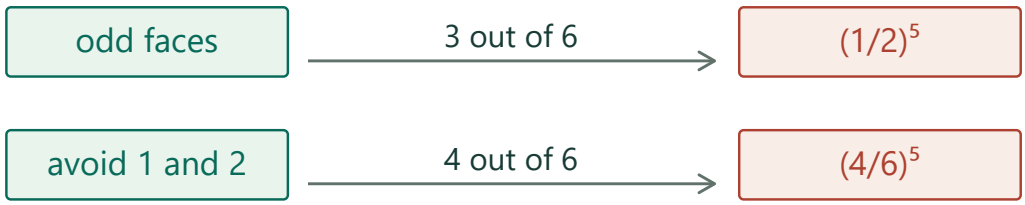
Exercise 778 · Three five-die events

Printed page 86 · Binomial probability / Probability

If five standard (six-sided) dice are tossed onto the table, what is the probability that
(a) all of them will show an odd number on top?
(b) no aces or deuces (that means ones or twos) will show on top?
(c) the five dice will show five different values on top?

1. Count favorable faces (a,b)

Three of six faces are odd, so all five fair dice odd has probability $(1/2)^5=1/32$. Four faces avoid ace and deuce, giving probability $(4/6)^5=32/243$ that all avoid them.



Raise each one-die probability to a fifth power.

2. Require pairwise distinct values (c)

Any face is allowed on die 1. After one face is used, die 2 has 5 choices, then the remaining dice have 4,3,2 choices. There are 6·5·4·3·2 favorable ordered outcomes.



Avoid every face already used when all five results must differ.

3. Divide by all rolls

All 6^5 ordered rolls are equally likely. Thus P(five distinct)=6·5·4·3·2/6^5=5/54.



Divide 720 favorable strings by all 6^5 equally likely strings.

Check. Alternatively for (c) choose the omitted face in six ways and permute the five others in 5! ways.

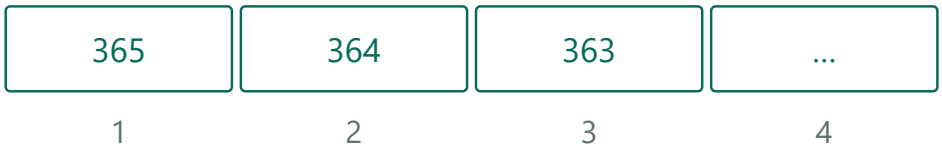
Exercise 779 · Birthdays among thirty-two residents

Printed page 86 · Probability / Counting and permutations

What is the probability that the thirty-two residents of Lamont have thirty-two different birthdays? What is the probability that there is at least one birthday coincidence in Lamont?

1. Count birthdays one resident at a time

Assume independent, uniformly distributed birthdays among 365 days. For all 32 to differ, the first is unrestricted. The second must avoid one day, the third must avoid two, and so on.



Count unused dates after each distinct birthday.

2. Multiply the conditional chances

For the 32nd person, 31 days are already used and 334 remain. The probability of all distinct is $(365/365)(364/365)\cdots(334/365)\approx.2466525$.



The 32nd person must avoid the 31 previously used dates.

3. Complement to include every kind of match

At least one shared birthday means the all-distinct event failed. Its probability is $1-.2466525\approx.7533475$, including assignments with multiple pairs.

Group size · 32 residents



The complement includes any number of matching birthdays.

Compare settings · Group size

22 people



22 people: 47.57% chance of at least one shared date.

P(all distinct) = 0.524305. Subtracting from 1 gives P(at least one match) = 0.475695.

33 people



33 people: 77.50% chance of at least one shared date.

P(all distinct) = 0.225028. Subtracting from 1 gives P(at least one match) = 0.774972.

Check. The same numerator counts ordered selections without replacement, 365!/(333!), out of 365^{32} equally likely assignments.

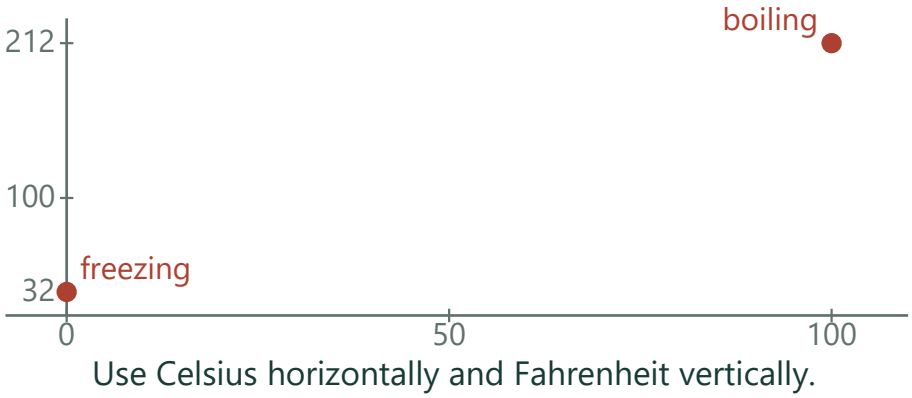
Exercise 780 · Invert a temperature conversion

Printed page 86 · Inverse functions / Unit conversion

To convert Celsius temperatures to Fahrenheit temperatures, the function F defined by $F(x) = 32 + 1.8x$ is useful. Confirm this, then find a formula for the function C (which could also be called F^{-1}) that converts Fahrenheit temperatures to Celsius temperatures.

1. Check the conversion against two reference points

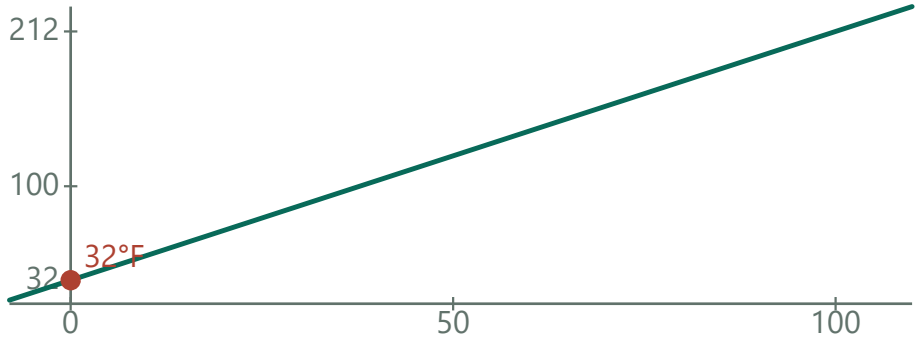
At standard pressure, freezing corresponds to 0°C and 32°F, and boiling to 100°C and 212°F. The Fahrenheit rise per Celsius degree is (212−32)/(100−0)=1.8.



Use Celsius horizontally and Fahrenheit vertically.

2. Use slope and intercept

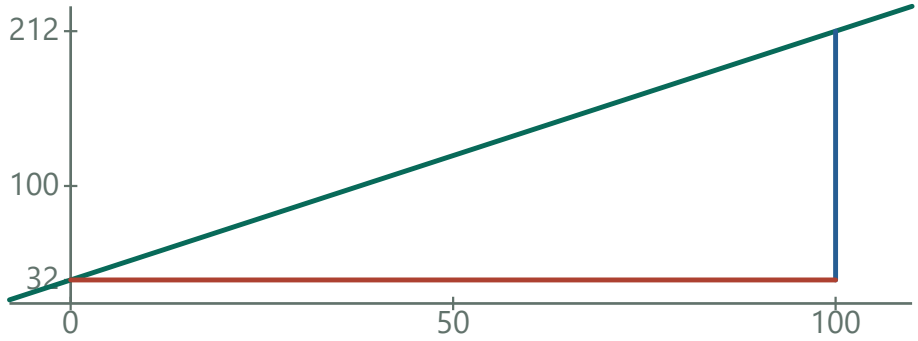
Starting from 32°F at 0°C yields F(x)=32+1.8x. Insert x=100 to recover 212°F.



The line rises 180°F in 100°C and starts at 32°F.

3. Undo the two operations

For a Fahrenheit input f, solve f=32+1.8x by subtracting 32 and dividing by 1.8. Thus C(f)=5(f−32)/9. Applying it to 32 and 212 gives 0 and 100 respectively.



Reading backward subtracts 32 before dividing by 1.8.

Check. Substitution into the original map gives 32+1.8·5(F−32)/9=F.

Exercise 781 · Solve an exponential for its exponent

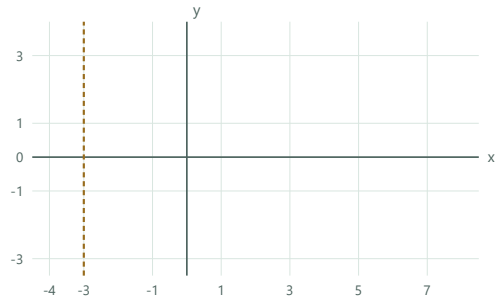
Printed page 86 · Exponential functions / Logarithms / Change of base

To graph the equation $x = 2^y - 3$, one approach is to rewrite the equation so that y is expressed as a function of x. Do so.

1. Isolate the exponential

Add 3 to both sides of $x=2^y-3$ to get $x+3=2^y$. The right side is positive for every real y, so $x>-3$.

Solve for the height on a shifted base-two logarithm

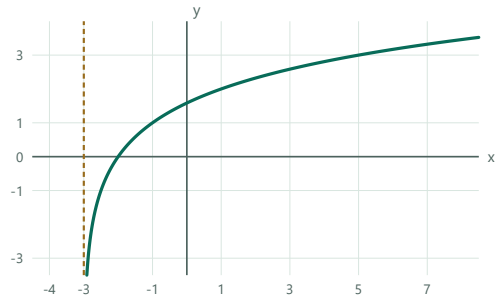


The boundary $x = -3$
Rearrange to $2^y = x + 3$.
Every exponential output is positive, so only $x > -3$ is possible.

2. Undo the base-two power

By definition, the exponent needed to produce $x+3$ is $\log_2(x+3)$. Hence $y=\log_2(x+3)$.

Solve for the height on a shifted base-two logarithm



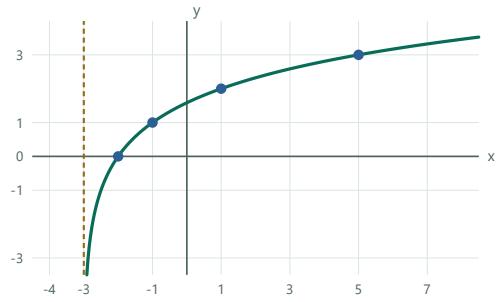
The boundary $x = -3$
Rearrange to $2^y = x + 3$.
Every exponential output is positive, so only $x > -3$ is possible.

$y = \log_2(x + 3)$

3. Locate points for the sketch

At $x=-2$ the argument is 1 and $y=0$. At $x=-1$ the argument is 2 and $y=1$. The logarithm increases with a vertical asymptote at $x=-3$.

Solve for the height on a shifted base-two logarithm



The boundary $x = -3$
Rearrange to $2^y = x + 3$.
Every exponential output is positive, so only $x > -3$ is possible.

$y = \log_2(x + 3)$

Generate points by choosing y

y	0	1	2	3
x	−2	−1	1	5

Going down one unit halves $x + 3$, bringing the curve closer to its vertical asymptote.

Check. Exponentiating the proposed expression returns $x+3$.

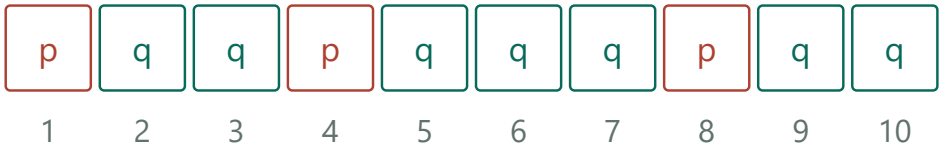
Exercise 782 · Thread three p’s into seven q’s

Printed page 86 · Counting and permutations / Combinations

Using the ten letters p, p, p, q, q, q, q, q, q, q only, how many ten-letter words can be formed? What is the coefficient of p^3q^7 when the binomial $(p+q)^{10}$ is expanded?

1. Choose three locations for p

In every word all seven remaining positions must be q. Thus only the set of p positions matters; choosing them in a different order does not create another word.



One example fixes the three p locations before filling the q locations.

2. Evaluate the combination

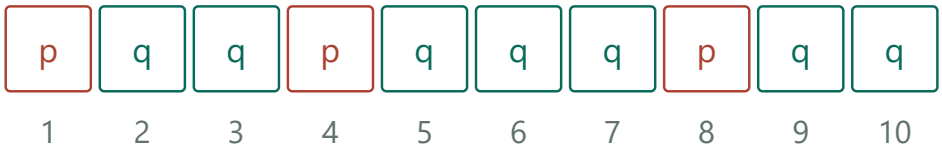
C(10,3)=10·9·8/(3·2·1)=120. Exactly 120 ten-letter words use the specified multiplicities.



Reordering the three selected places leaves the word unchanged.

3. Read the same count in the expansion

Each of ten factors (p+q) contributes one symbol. Selecting the same three p positions yields the same product p^3q^7, so 120 is its coefficient.



Read the same selections as factors of p^3q^7 in the expansion.

Check. The multiset count 10!/(3!7!) produces the same answer.

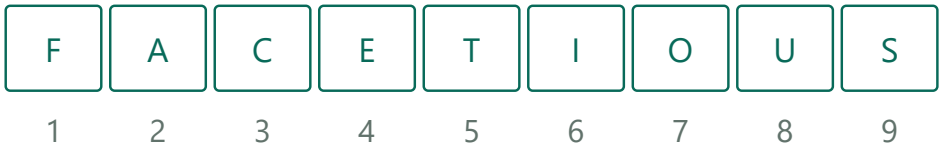
Exercise 783 · Four letters from FACETIOUS

Printed page 87 · Counting and permutations / Combinations

How many four-letter words can be formed by using the letters in *facetious*? There are nine letters available, and each one can be used at most once per word.

1. Check the inventory

The nine letters of "facetious" are distinct. Four-letter words are ordered and using a letter removes it from further consideration.



FACETIOUS supplies nine distinct letters.

2. Multiply the four choices

The first position has 9 options, the second 8, the third 7, and the fourth 6. Therefore 9·8·7·6=3024 words.



A four-letter ordered result uses four successive unrepeated choices.

3. Check by choosing a set first

There are C(9,4)=126 different sets of four letters, each with 4!=24 orderings, again giving 3024.



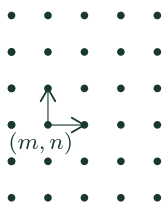
Select four letters first, then order them; the count agrees.

Check. Choose four letters without order and arrange them: C(9,4)·4!=126·24=3024.

Exercise 784 · Two biased lattice jumps

Printed page 87 · Lattice paths / Probability

A bug jumps from lattice point to lattice point on a piece of graph paper, one jump per second, as follows: From (m, n), there is a 60 percent chance that the bug jumps to $(m+1, n)$ and a 40 percent chance that it jumps to $(m, n+1)$. Find all the places the bug could be, two seconds after it leaves the origin $(0, 0)$. Are they equally likely?

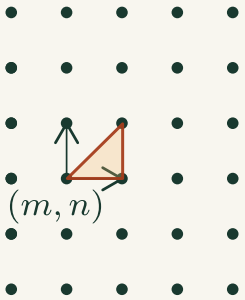


1. Assign probabilities to individual moves

Write R for right with probability .6 and U for up with probability .4. A two-step ordered path receives the product of its two move probabilities.



Weight the two possible moves at every jump.



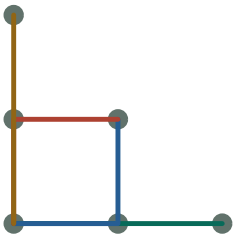
Route with a right move first

Follow the right arrow and then move one row up at its endpoint. Multiply the probabilities for these particular successive directions.

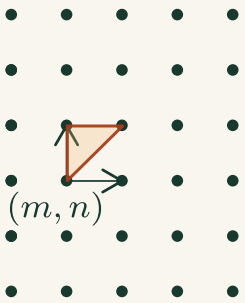
2. Follow each path from the origin

RR ends at (2,0) with $.6^2=.36$. UU ends at (0,2) with $.4^2=.16$. RU and UR both end at (1,1), each with $.6\cdot.4=.24$.

Two-jump route · All four routes



RR and UU each have one route; the mixed endpoint has two.

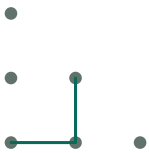


Route with an up move first

Following the up arrow first and then moving right leads to the same mixed endpoint. The two ordered routes contribute separately.

Compare settings · Two-jump route

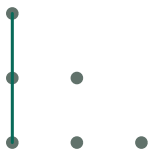
RU



RU ends at (1,1) with probability 0.24.

RU ends at (1,1) with probability 0.24.

UU

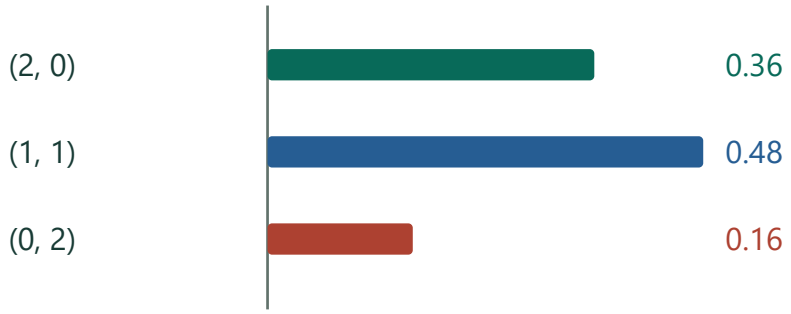


UU ends at (0,2) with probability 0.16.

UU ends at (0,2) with probability 0.16.

3. Collect paths sharing a destination

The mixed endpoint has probability .48. Thus probabilities at (2,0),(1,1),(0,2) are .36,.48,.16. They sum to 1 but differ because both directions and route counts matter.



Add the two mixed route probabilities at their common destination.

Check. The three probabilities sum to 1 and their weighted mean x-coordinate is 1.2=2(.6).

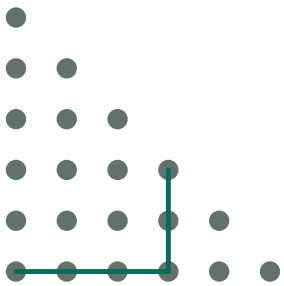
Exercise 785 · A biased path to (3,2)

Printed page 87 · Lattice paths / Binomial probability

(Continuation) It would take the bug five seconds to reach $(3, 2)$ from the origin. How likely is it that this will actually happen?

1. Read the desired endpoint

At (3,2) the bug must have made three right moves and two up moves. No other mix of the five moves gives this endpoint.



The target fixes the counts at three right and two up.

2. Calculate one route's probability

For instance RRRUU has probability $(.6)^3(.4)^2$. Any permutation of the same moves has the same product.



One ordered route contributes $(0.6)^3(0.4)^2$.

3. Count and add the routes

Select the two up-move positions in C(5,2)=10 ways. These alternatives are disjoint, giving $10(.6)^3(.4)^2=.3456$.



Ten route orders contribute the same product to the target.

Check. The coefficient of r^3u^2 in $(.6r+.4u)^5$ is .3456.

Exercise 786 · Read asymptotes from factors

Printed page 87 · Hyperbolas / Asymptotes / Algebraic manipulation

A hyperbola equation can be written in *factored form*, as in $(4x + 3y)(4x - 3y) = 72$. This enables the *asymptotes* to be written down easily: $4x + 3y = 0$ and $4x - 3y = 0$. What is the reasoning behind this statement? Apply this reasoning to sketch the following graphs (all actually hyperbolas). Draw the asymptotes first, then plot one convenient point (an axis intercept, for example), then use symmetry to freehand the rest of a rough sketch.

(a) $(x + 2y)(x - 2y) = 36$

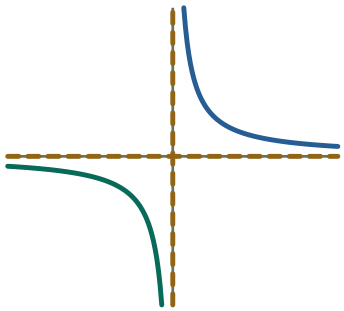
(b) $(x + 2y)(x - 2y) = -36$

(c) $(x + 3y)(x - y) = 9$

(d) $xy = 18$

1. Read an asymptote from a product

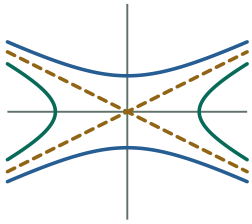
If U·V=c≠0 and |U| grows without bound, then V=c/U tends to zero. For a nonconstant linear function V, |V| divided by its normal's length is distance to the line V=0. Interchange U and V to obtain both asymptotes.



In factor coordinates V = c/U tends to zero as |U| grows.

2. Sketch the opposite signs (a,b)

In both cases the factors vanish on y=±x/2. With product 36 put y=0 to mark (±6,0), then draw the branches opening left and right. With product −36 put x=0 to mark (0,±3), then open upward and downward.



Changing the product sign switches which sectors hold the branches.

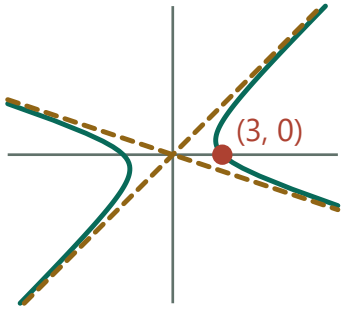
1. $(x+2y)(x-2y)=x^2-4y^2$
Multiply the conjugate factors. The mixed terms cancel.

2. $y=0 \Rightarrow x^2=36 \Rightarrow x=\pm 6$
The positive constant gives two crossings of the horizontal axis.

3. $x=0 \Rightarrow -4y^2=-36 \Rightarrow y^2=9 \Rightarrow y=\pm 3$
The negative constant gives two crossings of the vertical axis.

3. Sketch the oblique pair (c)

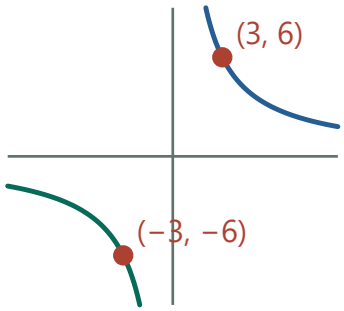
Zero factors give y=−x/3 and y=x. Use (3,0) and (−3,0) since each gives product 9. Draw through them in the two sectors where the factors have the same sign, approaching the bounding lines.

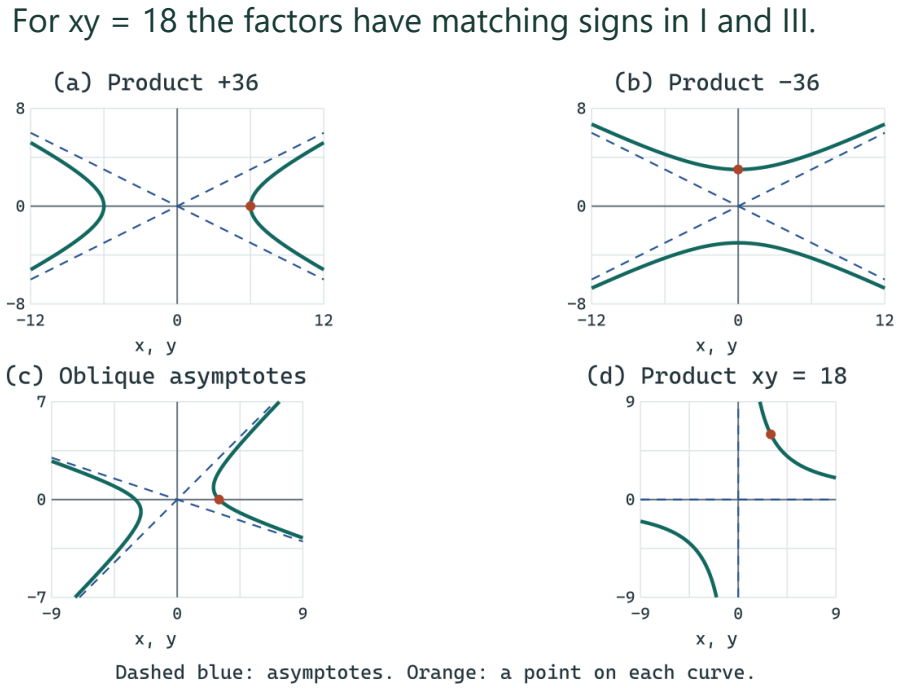


For positive product 9, the two oblique factors have the same sign.

4. Sketch the axis pair (d)

For xy=18 the zero-factor lines are the axes. Mark (3,6) and (−3,−6); the branches occupy quadrants I and III, where x and y have the same sign.





Check. All four curves have origin symmetry. The sign of the two factors selects which two regions between the asymptotes contain the branches.

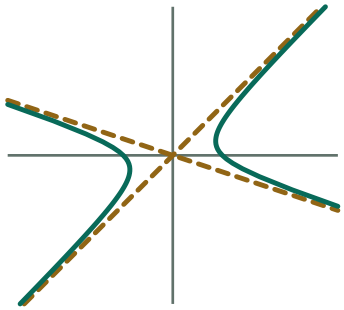
Exercise 787 · Vertices of rotated hyperbolas

Printed page 87 · Hyperbolas / Coordinate transformations / Symmetry

(Continuation) To find coordinates for the vertices of examples (c) and (d), a special approach is needed. One familiar method uses angle bisectors. After you find their coordinates, add the vertices to your sketches.

1. Locate the symmetry axis of the oblique branches

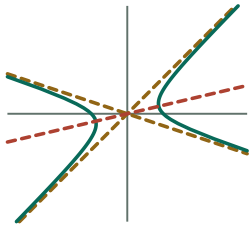
Vertices occur on the bisector running through the branches. Points on either bisector have equal perpendicular distance to $x+3y=0$ and $x-y=0$. Divide each expression by its normal length: $(x+3y)/\sqrt{10}=\pm(x-y)/\sqrt{2}$.



Find the bisectors of the actual oblique asymptotes.

2. Use the bisector with positive product

For (c) the factors must have the same sign, selecting the plus relation. Solving gives $y=kx$, $k=\sqrt{5}-2$. Substitute into $(x+3y)(x-y)=9$ to get $x^2(1+3k)(1-k)=9$.



Only the bisector through the occupied sectors meets the curve.

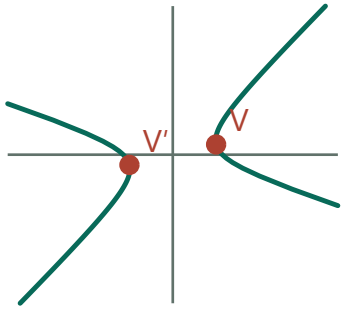
1. $x+3y=\sqrt{5}(x-y)$
Multiply the positive-sign bisector equation by $\sqrt{10}$.

2. $(3+\sqrt{5})y=(\sqrt{5}-1)x$
Collect the y terms on the left and the x terms on the right.

3. $y=[(\sqrt{5}-1)/(3+\sqrt{5})]x=(\sqrt{5}-2)x$
Divide by the positive coefficient. Multiplying $\sqrt{5}-2$ by $3+\sqrt{5}$ gives $\sqrt{5}-1$.

3. Mark both vertices of (c)

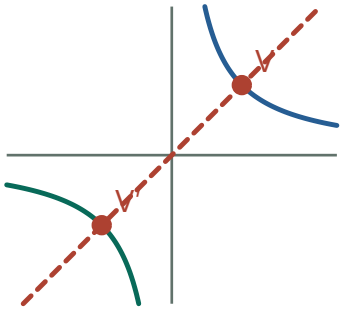
Write $v=3/\sqrt{[(1+3k)(1-k)]}$. The two opposite vertices are (v,kv) and $(-v,-kv)$. The other bisector places the two factors at opposite signs and misses these branches.



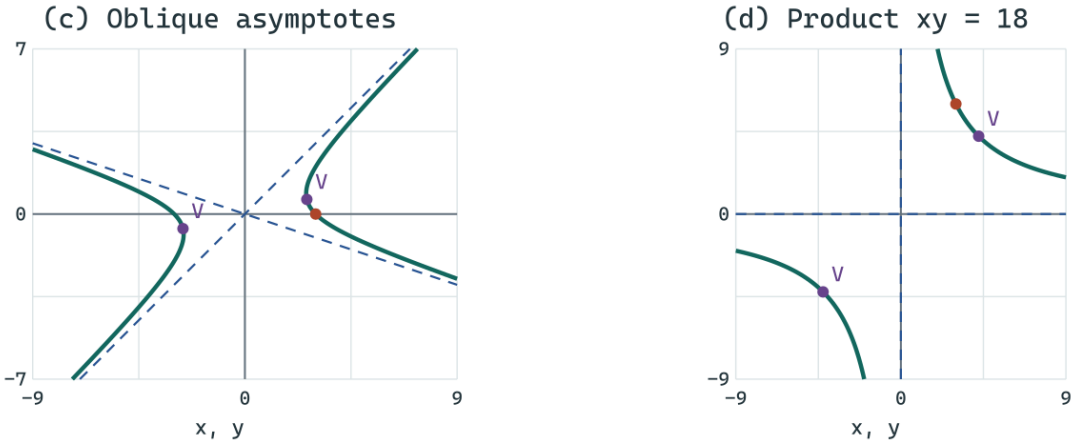
Its two intersections with the curve are the symmetric vertices.

4. Repeat with coordinate axes (d)

Their angle bisectors are y=x and y=−x. Only y=x reaches xy=18. Thus $x^2=18$, giving $(3\sqrt{2},3\sqrt{2})$ and $(-3\sqrt{2},-3\sqrt{2})$.



For coordinate-axis asymptotes the relevant bisector is y = x.



Check. In (c) the other bisector gives a negative product and cannot contain a point of the hyperbola. In (d) its rival y=−x likewise gives a negative product.

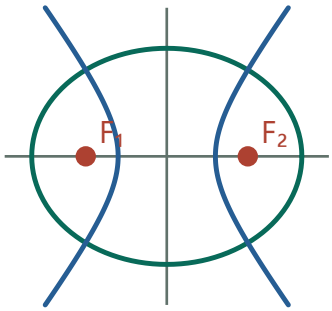
Exercise 788 · Confocal conics meet at right angles

Printed page 87 · Ellipses / Hyperbolas / Orthogonality

Show that the ellipse $16x^2 + 25y^2 = 400$ and the hyperbola $400x^2 - 225y^2 = 1296$ are *confocal*. Find coordinates for the four points where the graphs intersect each other. Then find the size of the *angle of intersection* at one of these points, to the nearest 0.1°.

1. Compare the focal parameters

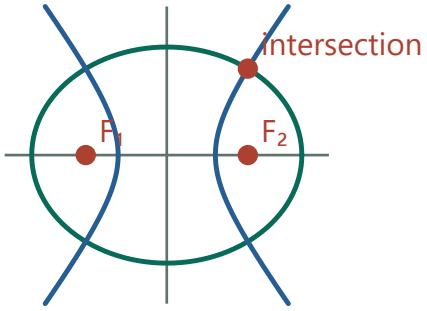
The ellipse is $x^2/25+y^2/16=1$, so $c^2=25-16=9$. The hyperbola is $x^2/(81/25)-y^2/(144/25)=1$, so $c^2=(81+144)/25=9$. Both open or extend along the x-axis and share $(\pm3,0)$.



The ellipse and hyperbola share foci at $x = \pm3$.

2. Solve for the squared coordinates

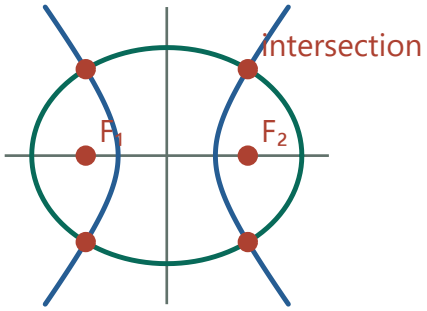
Let $X=x^2, Y=y^2$. The original equations give $16X+25Y=400$ and $400X-225Y=1296$. Multiply the first by 9 and add the second to get $544X=4896$, so $X=9$ and $Y=256/25$.



In quadrant I the common coordinates satisfy $x^2 = 9$ and $y^2 = 256/25$.

3. Recover all four signs

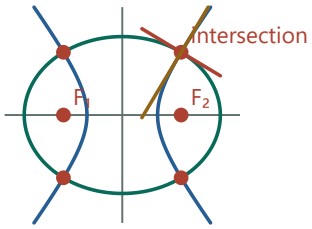
$x=\pm3$ and $y=\pm16/5$, independently. Each pair lies on both curves because the equations use only squares.



Both equations depend only on squares, so all four signs occur.

4. Compare their tangent slopes

Implicit differentiation gives ellipse slope −16x/(25y) and hyperbola slope 16x/(9y). At the intersections their product is −256·9/[225·(256/25)]=−1. The curves meet at 90.0°.



The two tangent slopes multiply to −1.

1. 32x+50y y′=0 ⇒ y′=−32x/(50y)
Differentiate the ellipse term by term. Each intersection has y≠0, so divide by 50y.

2. 800x−450y y′=0 ⇒ y′=800x/(450y)
Apply the same chain rule to the hyperbola, then divide by 450y.

Check. At (3,16/5), the two original left sides are 144+256=400 and 3600−2304=1296.

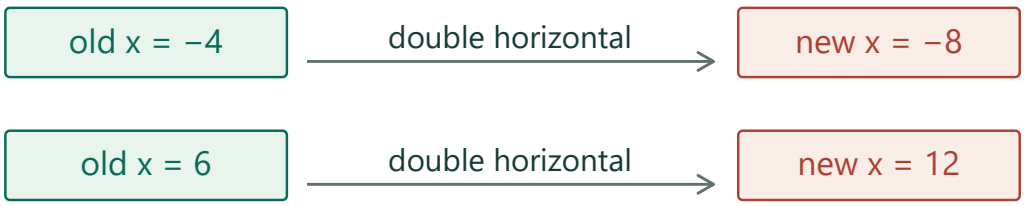
Exercise 789 · Domain and range under scaling

Printed page 87 · Domain and range / Graph transformations

The domain of function f is all numbers between -4 and 6, inclusive, and the range of f is all numbers between -3 and 5, inclusive. Find the domain and range of $g(x) = 3f(x/2)$.

1. Trace an admissible input

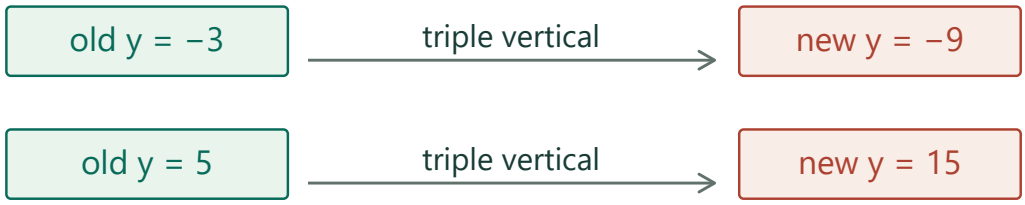
In f(x/2), the argument x/2 must belong to the original domain [−4,6]. Multiplying the inequality by 2 gives x∈[−8,12].



A stretched graph needs twice as much input to reach the same old x.

2. Trace the possible outputs

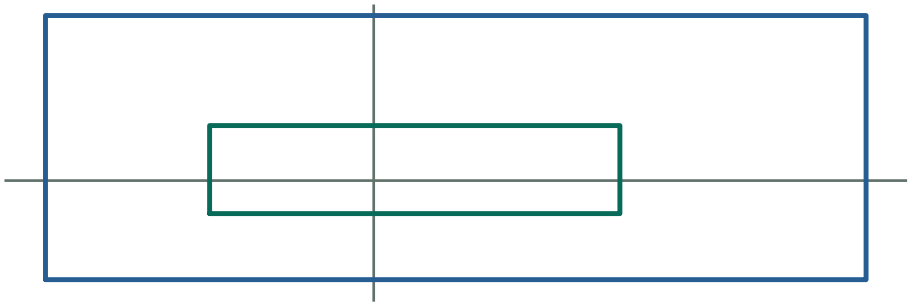
On that enlarged domain every allowable f value occurs. Multiplication by 3 maps the full interval [−3,5] to [−9,15].



The output interval stretches independently of the input interval.

3. Read the geometric transformation

An original point (u,v) is sent to (2u,3v): the horizontal stretch accounts for the domain change, while the vertical stretch accounts for the range change.



These boxes show the allowable coordinate extents, not the graph itself.

Check. The point map (u,v)↦(2u,3v) sends the endpoints of both coordinate ranges to the same bounds.

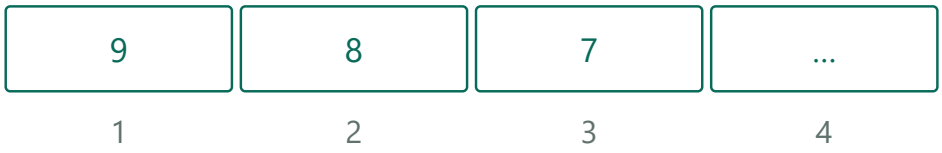
Exercise 790 · Factorial entries of the ninth row

Printed page 87 · Binomial coefficients / Counting and permutations

Pascal's Triangle is associated with binomial situations (right versus left, heads versus tails, etc). Its ninth row is $1 \;\; 9 \;\; 36 \;\; 84 \;\; 126 \;\; 126 \;\; 84 \;\; 36 \;\; 9 \;\; 1$. The zeroth and ninth entries in this row are both 1. Notice that the third entry in this row can be calculated in the form $\frac{9 \cdot 8 \cdot 7}{3 \cdot 2 \cdot 1}$. Find analogous presentations for the other entries in this row.

1. Associate entry k with choosing positions

Pick k positions from nine without regard to the order of selection. The ordered choices number 9·8·…·(9−k+1); divide by k! to remove the order.



Choosing k distinct positions gives k declining factors.

2. Use the same rule for every entry

At k=0 an empty choice gives 1. At k=1,2,3,4 the products give 9, 9·8/2, 9·8·7/6, 9·8·7·6/24. For k=5 through 9 use the same numerator pattern divided by 5!,6!,7!,8!,9!, respectively.

k	0	1	2	3	4
C(9,k)	1	9	36	84	126

Divide each ordered selection count by k! to obtain the first half.

3. Read the completed row

Equivalently each entry equals 9!/[k!(9−k)!], so k and 9−k agree. The row is 1,9,36,84,126,126,84,36,9,1.

0	1	2	3	4	5	6	7	8	9
1	9	36	84	126	126	84	36	9	1

The same row reads identically from both ends.

Check. Changing k to 9−k interchanges the two denominator factorials and leaves the entry unchanged.

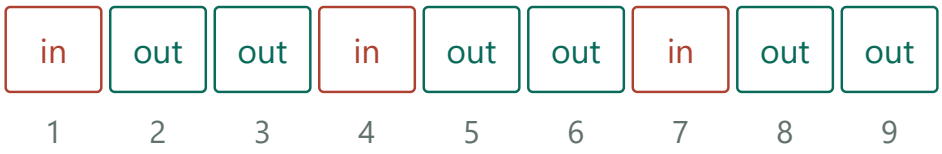
Exercise 791 · Triangle selections are binomial

Printed page 87 · Binomial coefficients / Combinations

(Continuation) Recall #751, about counting the number of triangles that can be formed by choosing three dots from nine available dots. *This is a binomial question.* Explain why. This justifies the appearance of the answer in Pascal's Triangle. The third entry in the ninth row is usually denoted by ${}_9C_3$ or by $\binom{9}{3}$, and read "nine choose three."

1. Make one binary decision per dot

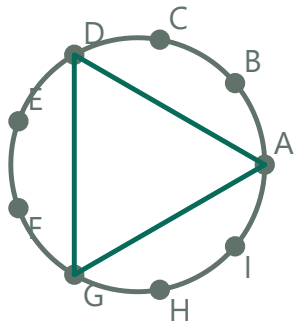
A chosen triangle corresponds to three vertices included and six left out. A factor (1+t) records the options for one dot: 1 means exclude, t means include.



Each of nine points makes one include-or-exclude selection.

2. Read the exponent

Expanding $(1+t)^9$, a term t^3 requires exactly three included dots. Its coefficient counts each possible included set once, independently of ordering.



Exactly three included points determine one triangle.

3. Identify the Pascal entry

The count is C(9,3)=9·8·7/(3·2·1)=84, the entry at index 3 in row 9. This is why the triangle-counting problem is binomial.

0	1	2	3	4	5	6	7	8	9
1	9	36	84	126	126	84	36	9	1

Select entry k = 3 in the ninth binomial row.

Check. The ordered count 9·8·7 counts each triangle 3! times, also giving 84.

Exercise 792 · A loan balance as a straight line

Printed page 88 · Loan amortization / Sequences and recursion

Repaying loans, part I. The bank has just granted Jordan a \$10 000 loan, which will be paid back in 48 equal monthly installments, each of which includes a 1 percent charge on the unpaid balance. The problem is to calculate the correct monthly payment.

(a) Suppose that the monthly payments are all \$150. The first payment must include \$100 just for interest on the \$10 000 owed. The other \$50 reduces the debt. That leaves a debt of \$9950 after the first payment. Let A_n be the amount owed after n payments (so that $A_0 = 10\,000$). Justify the recursion $A_n = 1.01A_{n-1} - 150$, then apply it repeatedly to show that Jordan still owes \$6938.87 after 48 payments of \$150 have been made.

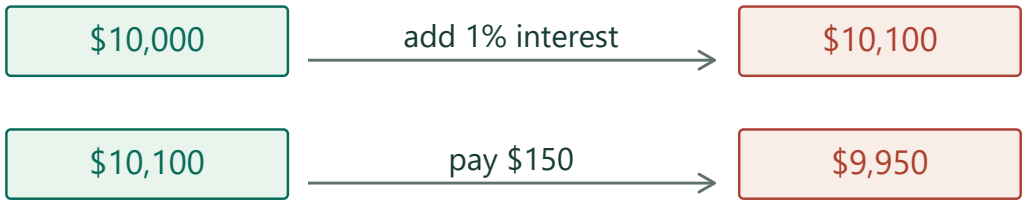
(b) Because $P = 150$ is not large enough to pay off the loan in 48 installments, try the following set of values of P: 200, 250, 300, 350. For each value of P, apply the recursion $A_n = 1.01A_{n-1} - P$ with $A_0 = 10\,000$ until you have A_{48}.

(c) Plot the five points (P, A_{48}) on a graph, one of which is $(150, 6938.87)$. You should notice that these five points are collinear.

(d) Find an equation for the line containing the points from (c). Use this line to find the payment P that makes $A_{48} = 0$. This is the payment that pays off the loan.

1. Update a balance in the right order

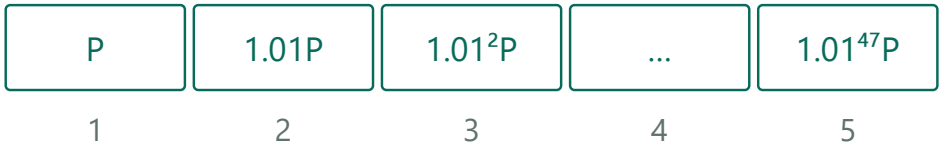
Let A_n denote the amount after payment n. Interest first multiplies A_{n-1} by 1.01, then payment P is subtracted: $A_n = 1.01A_{n-1} - P$ with $A_0 = 10000$. At $P = 150$ the first balance is $10100 - 150 = 9950$.



Within each month, interest comes before the fixed payment.

2. Unroll the monthly rule

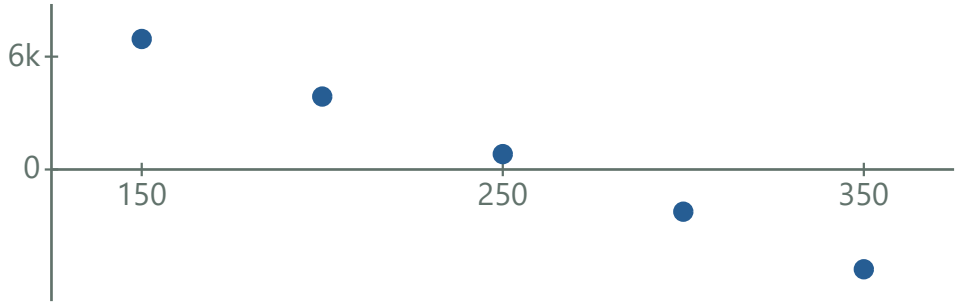
Two months give $A_2 = 10000(1.01)^2 - P(1.01 + 1)$. After 48, $A_{48} = 10000(1.01)^{48} - P[1 + 1.01 + \cdots + 1.01^{47}]$. Each early payment avoids more months of interest.



Earlier payments avoid more later interest.

3. Calculate the five plotted points

For $P=150,200,250,300,350$, respectively, the 48th balances are $\$6938.87, \$3877.74, \$816.61, -\$2244.52, -\$5305.65$. Plot each balance against its payment. Negative amounts mean the fixed-payment model has pushed beyond zero.



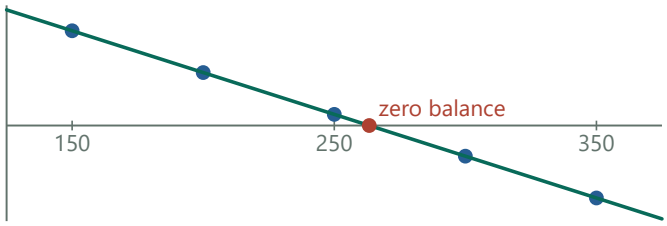
Monthly payment is horizontal; the 48th balance is vertical.

Model balances after 48 end-of-month payments

Monthly payment ($)	Balance after payment 48 ($)
150.00	6,938.87
200.00	3,877.74
250.00	816.61
300.00	−2,244.52
350.00	−5,305.65

4. Derive the line and its intercept

The geometric coefficient of P is $[(1.01)^{48}-1]/.01 \approx 61.22261$. Thus $A_{48} \approx 16122.26078 - 61.22261P$, explaining collinearity. Setting $A_{48}=0$ gives $P \approx \$263.33835$, about $\$263.34$ monthly before the final cent adjustment.

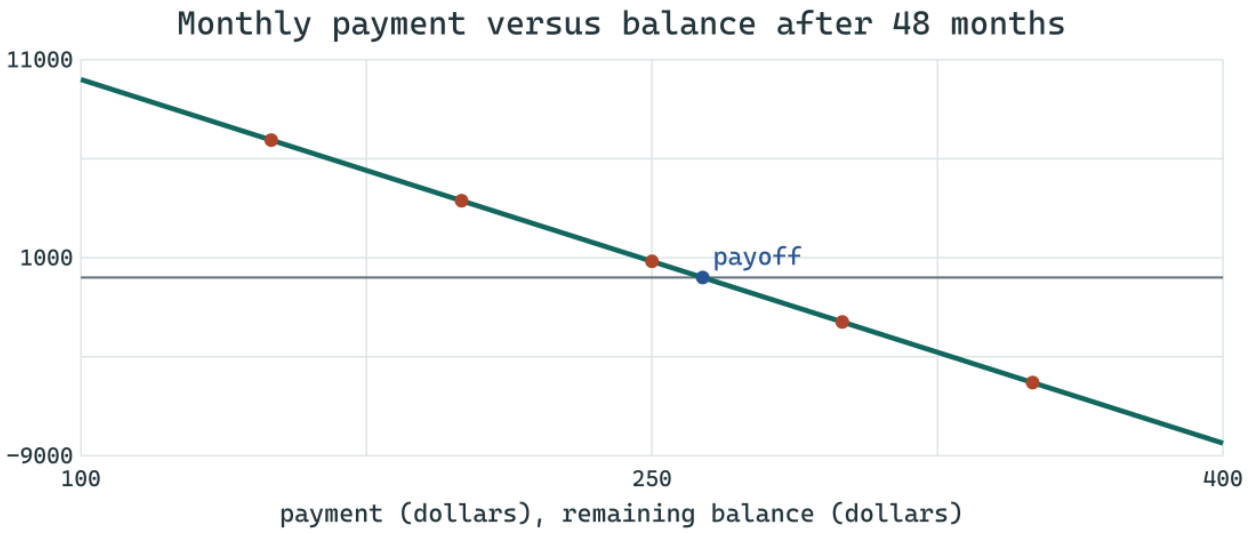


The straight line's horizontal intercept is the amortizing payment.

1. $G = 1 + 1.01 + \cdots + 1.01^{47} \Rightarrow 1.01G - G = 1.01^{48} - 1$
Subtract the original geometric sum from its shifted version.

2. $G = (1.01^{48} - 1)/.01$
The difference on the left is $.01G$.

3. $P = 10000(1.01)^{48}/G$
A zero balance requires PG to equal the accrued initial principal. Every term of G is positive.



Orange: five trial payments. Blue: payment that makes the last balance zero.

Check. Every additional $50 of monthly payment reduces the final balance by about $3061.13. Discounting the 48 equal payments at 1% gives the initial $10000.

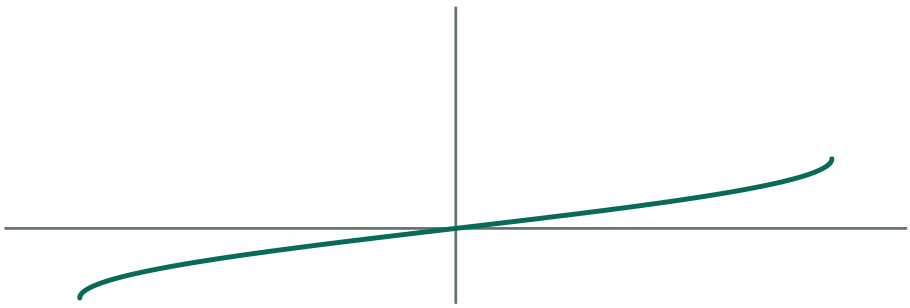
Exercise 793 · Two branches with equal sine

Printed page 88 · Inverse trigonometric functions / Trigonometric equations

In radian mode, graph $y = \sin^{-1} x$ and $y = \pi - \sin^{-1} x$ on the same coordinate-axis system. What do you notice?

1. Start with the principal sine inverse

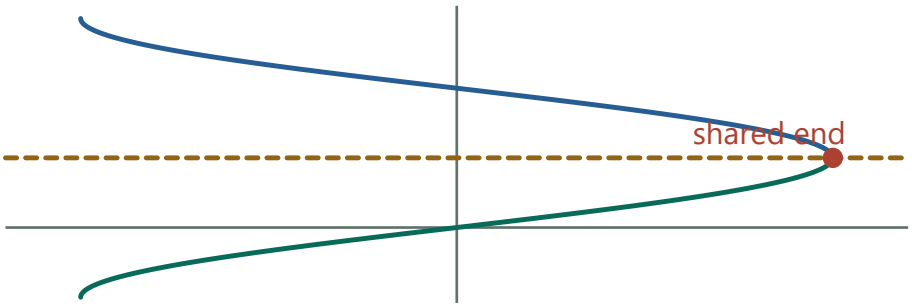
arcsin x is defined on [−1,1] and takes heights from −π/2 to π/2, increasing through (0,0).



Begin with the restricted arcsine graph.

2. Reflect each height

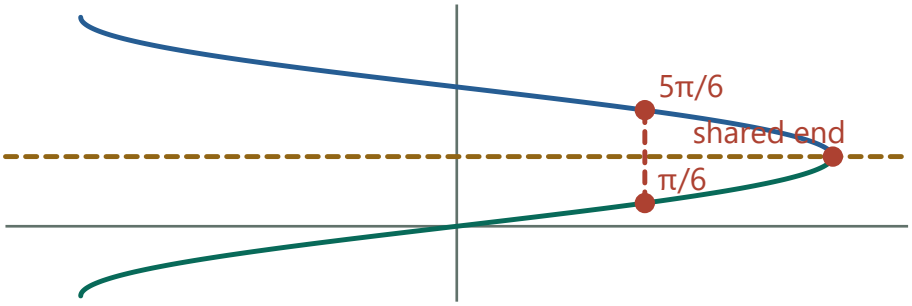
The second graph replaces a height y by π−y. Their average is π/2, so the two are reflections across y=π/2. The second falls from 3π/2 to π/2 and the curves touch at (1,π/2).



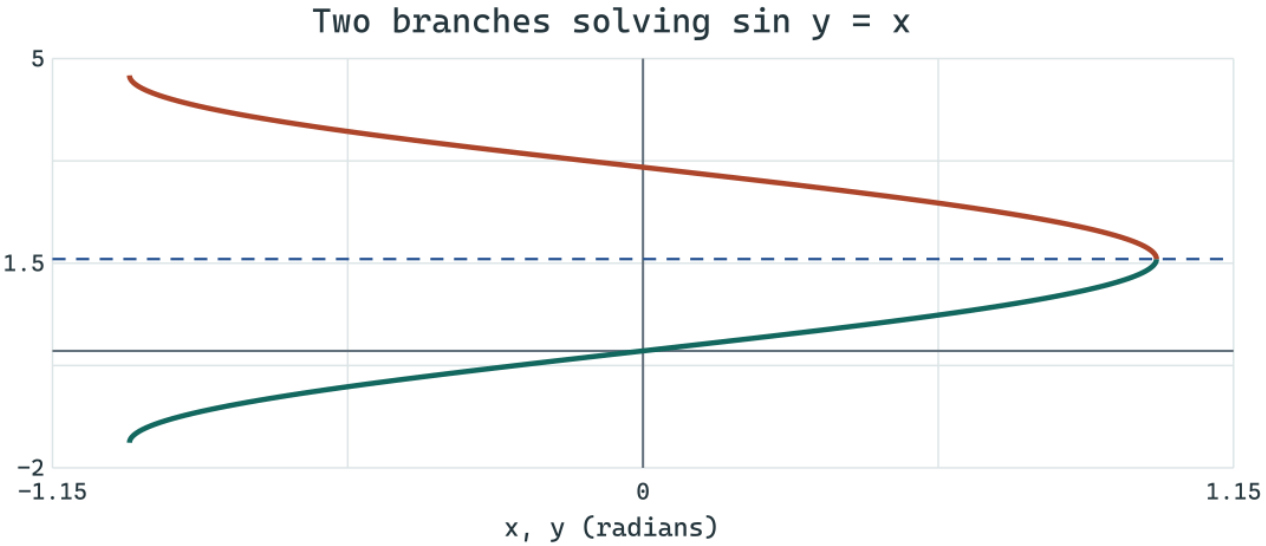
Replace each height y with π − y across y = π/2.

3. Interpret the two choices of angle

sin(π−y)=sin y. Thus both curves give angles with sine x, together covering all such y in [−π/2,3π/2]. Shifting both vertically by multiples of 2π gives all remaining angles.



A single sine input can correspond to two angles in this window.



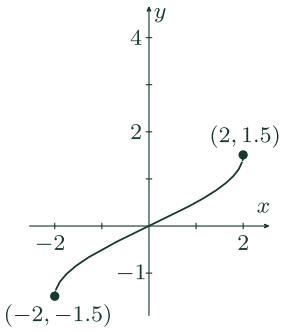
Green: arcsin x. Orange: π − arcsin x. The dashed line is y = π/2.

Check. At x=0 the two outputs are 0 and π. At any x their sum is π, and sin(π−y)=sin y.

Exercise 794 · Reflect a curve around a horizontal line

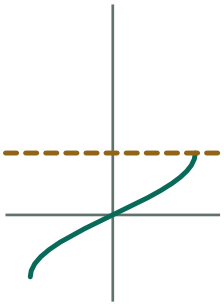
Printed page 88 · Graph transformations / Reflections

The diagram shows a graph $y = f(x)$. Add the graph of $y = 3 - f(x)$ to the diagram.

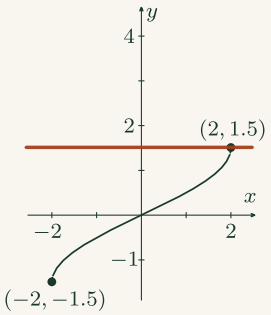


1. Find a fixed reference height

The old height v changes to 3−v at the same x. A height is unchanged when v=3−v, hence the horizontal line y=1.5 is the mirror.



The unchanged height solves y = 3 − y.

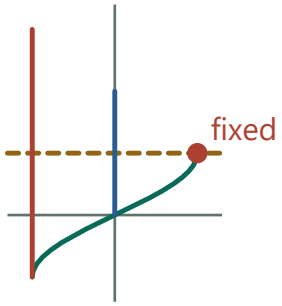


Mirror height

Construct this horizontal line through the right endpoint. The height rule leaves all points on that line fixed.

2. Transform marked points

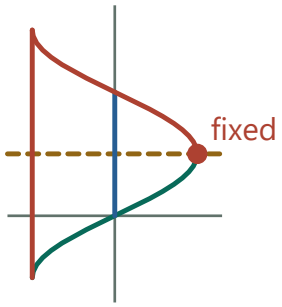
The left endpoint (−2,−1.5) goes to (−2,4.5); (0,0) goes to (0,3); the right endpoint (2,1.5) stays on the mirror.



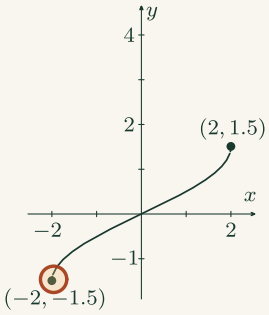
Reflect selected points vertically, keeping x fixed.

3. Trace the transformed curve

Place each original arc or point the same vertical distance on the opposite side of y=1.5. Reconnect in horizontal order to obtain the decreasing reflected curve. Applying the rule again restores f because 3−(3−v)=v.

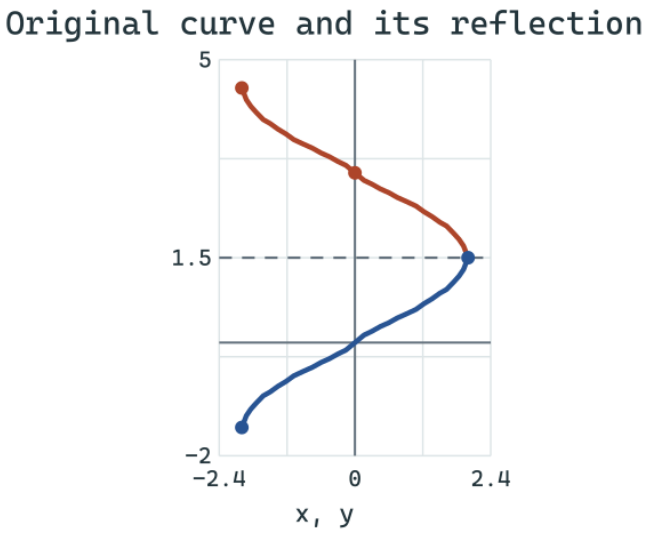


Apply the same height rule along the entire printed outline.



Low starting point

Inspect the marked left endpoint. Its distance below the mirror determines an equal distance upward for its new position.



Blue: original. Orange: reflected. The common endpoint lies on the mirror y = 1.5.

Check. A second reflection sends 3−(3−y) back to y. The endpoint on the mirror stays fixed.

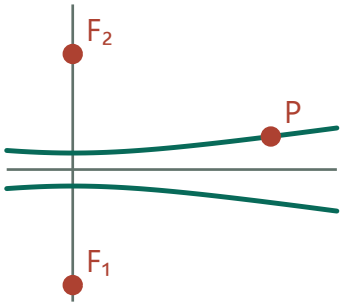
Exercise 795 · Focal reflection at a hyperbola tangent

Printed page 88 · Hyperbolas / Conic reflection properties / Tangent lines

Verify that the line $8y - x = 4$ is tangent to the hyperbola $48y^2 - x^2 = 48$ at $P = (12, 2)$ and that the focal points for this hyperbola are $F_1 = (0, -7)$ and $F_2 = (0, 7)$. Calculate the size of the two acute angles formed by the tangent line and the focal radii.

1. Verify the point and foci

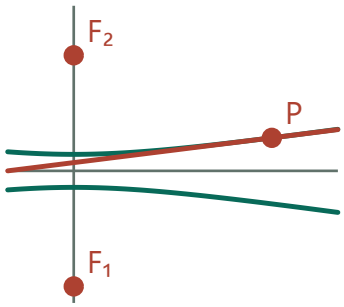
At P=(12,2), $48\cdot2^2-12^2=48$ and $8\cdot2-12=4$. Normalize the hyperbola as $y^2-x^2/48=1$; $c^2=1+48=49$ gives its vertical foci (0,±7).



The hyperbola has a vertical focal axis and contains (12, 2).

2. Verify the tangent

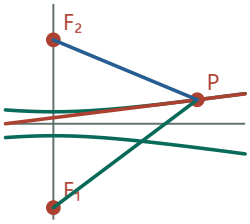
Differentiate $48y^2-x^2=48$: $96y\,y'=2x$, so $y'=x/(48y)=1/8$ at P. The given line $y=(x+4)/8$ passes through P and has that slope.



The tangent through P rises one for eight units of run.

3. Measure the two acute angles

The slopes of F_1P,F_2P are $(2+7)/12=3/4$ and $(2-7)/12=-5/12$. For tangent slope $1/8$ use $|(m-1/8)/(1+m/8)|$. The first ratio is $(5/8)/(35/32)=4/7$; the second is $(13/24)/(91/96)=4/7$. Both angles are $\arctan(4/7)\approx29.745°$.



Compare the two focal slopes against that same tangent.

1. $(3/4-1/8)/(1+3/32)=(5/8)/(35/32)=20/35$
Use the positive focal slope, put each sum over one denominator, then multiply by the reciprocal.

2. $(-5/12-1/8)/(1-5/96)=(-13/24)/(91/96)=-52/91$
Repeat with the negative focal slope. An acute angle uses the absolute value.

3. $20/35=52/91=4/7$
Reduce the fractions. Equality of these positive tangents gives equal acute angles.

Check. Putting $x=8y-4$ into the conic gives $(y-2)^2=0$ after simplification, independently verifying tangency.

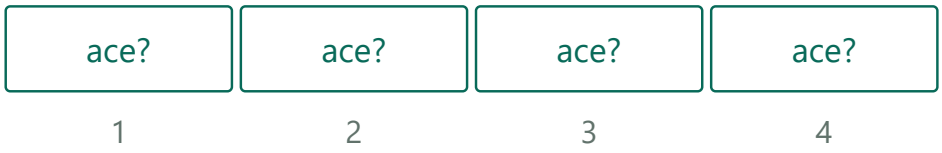
Exercise 796 · Four dice and five binomial terms

Printed page 88 · Binomial probability / Binomial coefficients

When the binomial power $\left(\frac{5}{6}+\frac{1}{6}\right)^4$ is expanded, five terms should appear. What are they, and what does each one mean?

1. Attach an outcome to each summand

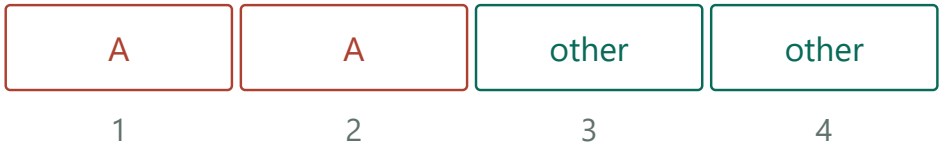
On one fair die call a 1 an ace (probability 1/6) and any other face a non-ace (5/6). Four factors correspond to four independent dice.



Each of four dice contributes either one ace or one non-ace.

2. Group by number of aces

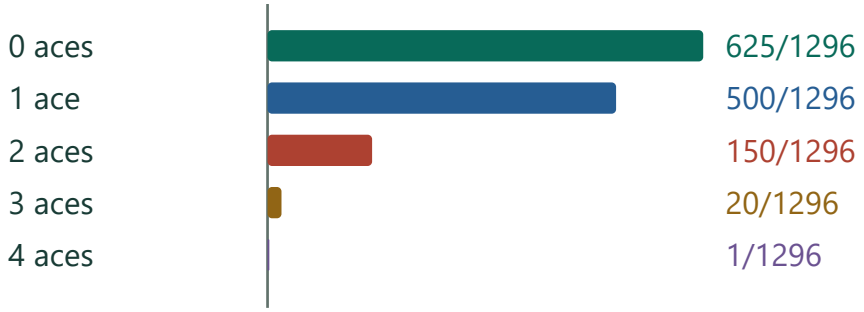
For exactly k aces choose their k positions in C(4,k) ways. Multiply by $(1/6)^k(5/6)^{(4-k)}$, the probability for any one such pattern.



One placement for exactly two aces contributes p^2q^2.

3. Read the five terms

For k=0,1,2,3,4 the binomial coefficients are 1,4,6,4,1. The respective probabilities are 625/1296,500/1296,150/1296,20/1296,1/1296. Their sum is 1, covering all outcomes.



Grouping all ordered rolls by ace count produces five probabilities.

Check. Their numerators sum to $1296=6^4$, exhausting all outcomes.

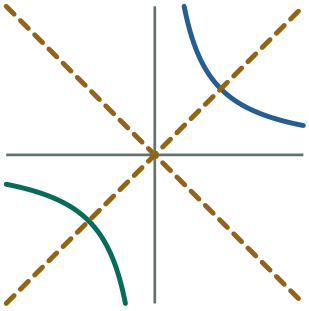
Exercise 797 · Rotate the rectangular hyperbola

Printed page 88 · Hyperbolas / Rotations / Coordinate transformations

Find the asymptotes, vertices, focal points, and eccentricity for the hyperbola $xy = 16$.

1. Choose axes along the bisectors

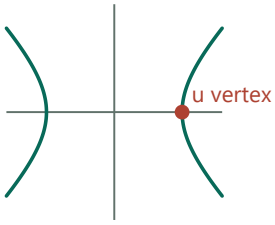
xy=16 has asymptotes x=0 and y=0, and positive and negative diagonal branches along y=x. Use length-preserving coordinates u=(x+y)/√2, v=(x−y)/√2.



Original x,y axes: the branches occupy the positive-product quadrants.

2. Find the rotated equation

Then $u^2-v^2=2xy=32$. The form $u^2/32-v^2/32=1$ gives a=b=4√2, c=√(32+32)=8 and eccentricity c/a=√2.



Rotated u,v axes: the transverse axis is horizontal.

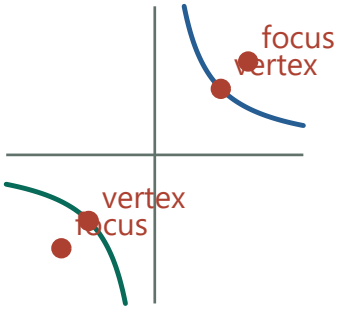
1. $u^2-v^2=[(x+y)^2-(x-y)^2]/2$
Square the two coordinate definitions and subtract.

2. $=[(x^2+2xy+y^2)-(x^2-2xy+y^2)]/2=2xy$
The pure squares cancel and the two mixed terms add.

3. $u^2-v^2=32$
Insert xy=16. Dividing by 32 gives the standard hyperbola.

3. Convert vertices and foci back

They lie on v=0, where x=y=u/√2. The values u=±4√2 give vertices (4,4),(−4,−4); u=±8 gives foci (4√2,4√2),(−4√2,−4√2).



Return the vertices and foci to the original diagonal.

Check. Both vertices satisfy xy=16. The transformed asymptotes u=±v are precisely the two original coordinate axes.

Exercise 798 · Permutations with repeated letters

Printed page 89 · Counting and permutations

Counting. As you have seen, counting rearrangements of items requires special consideration when the items are not all different.

(a) Explain why the seven letters A, B, C, D, E, F, G can be rearranged in 7! ways, while A, A, A, D, E, F, G can be arranged in $\frac{7!}{3!}$.

(b) How many ways can A, A, A, D, D, G, G be rearranged? What about A, A, D, D, E, E, F ? Hint: Writing out all the possibilities for similar but smaller examples like A, A, A, B and A, A, B, B might prove helpful.

(c) Rearrangements of r, r, r, r, u, u, u represent the number of ways our jumping bug of problem #719 can reach $(4, 3)$ seven seconds after leaving the origin. Write this number using factorial notation and compute its value. Where have you seen this value before?

(d) Use factorial notation to write all eight numbers in the seventh row of Pascal's triangle.

(e) If our bug jumped for twenty seconds, how many different paths are there for it to reach the point $(8, 12)$? What about the point $(15, 5)$?

1. Count visible arrangements (a)

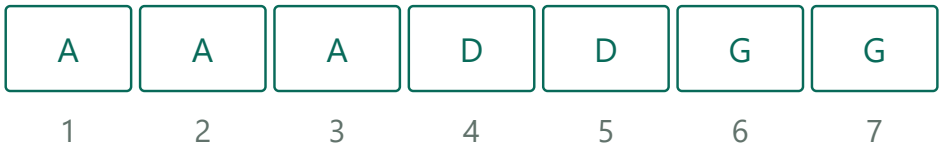
Seven distinct letters have 7!=5040 orders. Tag three identical A's temporarily. Each visible word now corresponds to 3! tag orders, so removing the tags gives 7!/3!=840.



Labeling identical copies temporarily overcounts each word 3! times.

2. Repeat for several duplicate groups (b)

Independent permutations of equal copies must all be divided out. Three A, two D, two G give 7!/(3!2!2!)=210. Two A, two D, two E and one F give 7!/(2!2!2!)=630.



Divide again for each additional repeated-letter group.

3. Connect paths to coefficients (c,d)

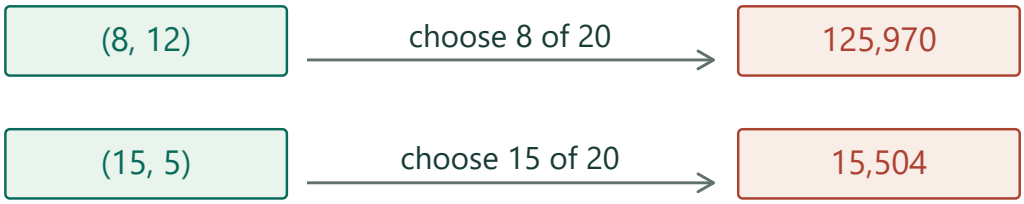
Four right and three up moves give 7!/(4!3!)=35 orders. This appears in Pascal's seventh row. Its entries for k=0,…,7 are 7!/[k!(7−k)!], giving 1,7,21,35,35,21,7,1.



The 35 at index 4 counts choices for four right moves.

4. Apply the rule after twenty jumps (e)

For (8,12) select eight right positions among 20: 20!/(8!12!)=125970. For (15,5) select fifteen right positions: 20!/(15!5!)=15504.



In each twenty-jump route, the right positions determine the rest.

Check. Choosing positions for each repeated type, one type at a time, gives the same counts. The seventh row sums to $128=2^7$.

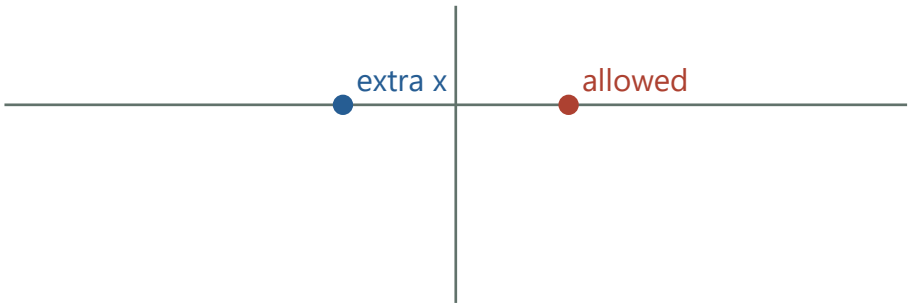
Exercise 799 · When a log identity changes domain

Printed page 89 · Logarithm rules / Domain and range

Compare the domains and ranges of the functions $f(x) = 2\log x$ and $g(x) = \log(x^2)$.

1. Test each log argument

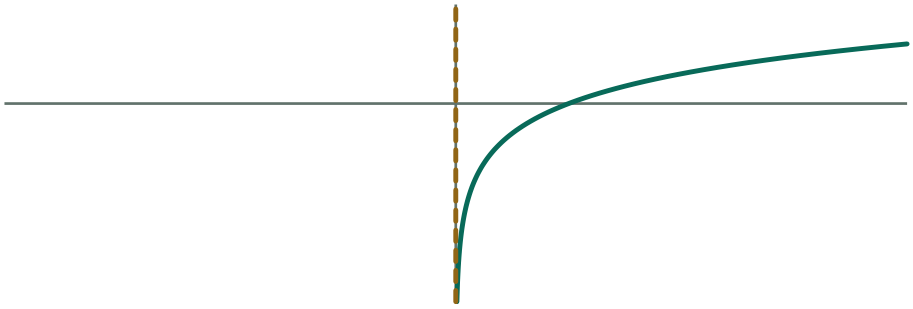
For $f(x)=2\log x$, require $x>0$. For $g(x)=\log(x^2)$, require $x^2>0$, equivalent to $x\neq 0$. Thus g has an extra negative-input branch.



Both expressions accept positive x; $\log(x^2)$ also accepts negative x.

2. Describe the common positive branch

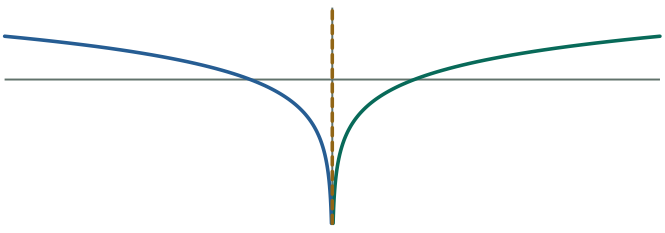
For $x>0$ the power rule gives $\log(x^2)=2\log x$. The logarithm takes every real value, and multiplying by 2 still covers $\mathbb{R}$. Both functions consequently have range $\mathbb{R}$.



The two formulas coincide throughout the positive branch.

3. Explain the additional branch

For $x<0$ use $x^2=|x|^2$ and obtain $g(x)=2\log|x|$. The two input signs give equal heights on g, reflecting its positive branch across the vertical axis.



Opposite inputs have the same square and the same logarithm.

1. $x<0 \Rightarrow -x>0$ and $x^2=(-x)^2$
The positive number $-x$ is an admissible input for a logarithm.

2. $\log(x^2)=\log((-x)^2)=2\log(-x)$
Apply the power rule to the positive base $-x$.

3. $-x=\text{abs}(x)$ for $x<0$
This gives the same compact form $2\log(\text{abs}(x))$ on both nonzero branches.

Check. At $x=-1$ the first is undefined and the second equals $\log 1=0$.

Exercise 800 · Condense two sums

Printed page 89 · Sigma notation / Logarithm rules / Geometric sequences and series

Find a simple expression for each of the sums: **(a)** $\sum_{n=1}^{96} \log n$ **(b)** $\sum_{n=0}^{\infty} \sqrt{0.9^n}$

1. (a) Combine the finite sum of logs

Use the same valid logarithm base throughout. Every integer from 1 to 96 is positive, including the first input 1, whose log is zero.

1 **log 1 + log 2 = log(1 · 2)**
Use the product law on the first two terms.

2 **log(1 · 2) + log 3 = log(1 · 2 · 3)**
Apply the same law to append the third factor.

3 **$\Sigma_{n=1}^{96}$ log n = log(1 · 2 · ⋯ · 96) = log(96!)**
Continue through n=96 and recognize the factorial.

2. (b) Identify the geometric series

The square root covers the whole power 0.9^n. Its base is positive, so square-rooting each power is equivalent to raising the positive square root to n.

1 **$\sqrt{(0.9^n)} = (\sqrt{0.9})^n$**
The two expressions both equal 0.9^(n/2).

2 **r = √0.9, 0 < r < 1**
Successive terms are multiplied by this fixed ratio.

3 **$r^0 = 1$**
The printed lower index is n=0, so the first term is 1.

3. Pass to the limit and rationalize

The finite geometric formula applies because r≠1. Since 0<r<1, the remaining power tends to zero and the infinite sum exists.

1 **S_n = (1−r^(n+1))/(1−r)**
Add terms from exponent 0 through n.

2 **S = 1/(1−√0.9)**
Take n to infinity. The numerator tends to 1.

3 **S = (1+√0.9)/(1−0.9)**
Multiply top and bottom by the positive conjugate.

4 **S = 10 + 3√10**
Divide by 0.1 and use √0.9 = 3/√10.

Check. Multiplying the second answer by 1−√.9 returns 1. Logarithms of products give the first answer term by term.

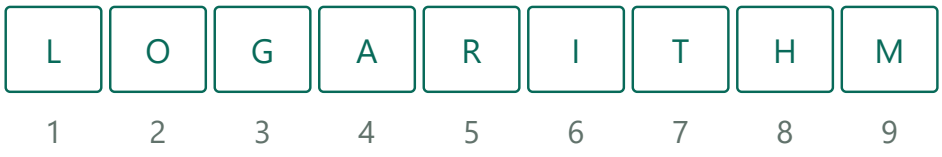
Exercise 801 · Four places for nine letters

Printed page 89 · Counting and permutations

Explain why there are $9 \cdot 8 \cdot 7 \cdot 6$ different four-letter words that can be formed by using only the letters of *logarithm*. This product is often denoted ${}_9P_4$, and read "nine permute four." A *permutation* is an arrangement of things (letters, for example).

1. Identify the items being permuted

The word "logarithm" has nine distinct letters. Each four-letter result is an ordered list drawn without reusing a letter.



LOGARITHM supplies nine distinct letters.

2. Fill the positions successively

There are 9 choices for the first letter, 8 after that, then 7 and 6. The multiplication rule gives 9·8·7·6=3024.



A four-letter ordered result uses four successive unrepeated choices.

3. Interpret the notation

This count is ${}_9P_4$, or nine permute four. Selecting an unordered set first in C(9,4) ways and arranging it in 4! ways yields the same count.



Select four letters first, then order them; the count agrees.

Check. Alternative count: select a set of four in C(9,4)=126 ways, then order it in 24 ways.

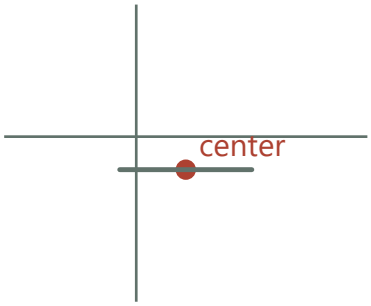
Exercise 802 · Shift a parametric hyperbola

Printed page 89 · Hyperbolas / Parametric equations / Translations

Find a parametric description for the hyperbola $\dfrac{(x-3)^2}{16} - \dfrac{(y+2)^2}{25} = 1$.

1. Identify the normalized coordinates

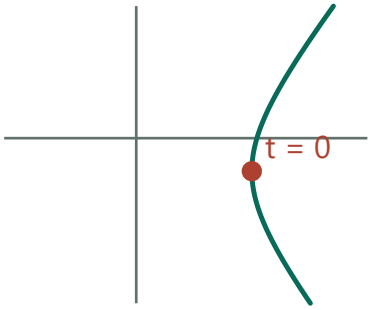
The relation is $[(x-3)/4]^2-[(y+2)/5]^2=1$. Its center is $(3,-2)$ with a horizontal transverse axis.



Shift the origin to the stated center before reading the horizontal reach.

2. Use the hyperbola identity

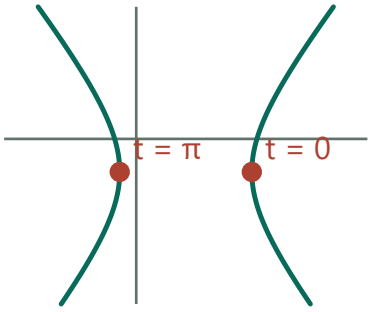
Since $\sec^2 t-\tan^2 t=1$, set $(x-3)/4=\sec t$ and $(y+2)/5=\tan t$. This gives $x=3+4\sec t$ and $y=-2+5\tan t$, with $\cos t\neq 0$.



For $x = 3 + 4 \sec t$, positive cosine selects the right branch.

3. Check both sides of the locus

Positive $\sec t$ reaches the right branch and negative $\sec t$ the left. At $t=0$ and π one obtains the vertices $(7,-2)$ and $(-1,-2)$, and substitution recovers the given equation.



The other sign of cosine gives the missing branch and vertex.

Check. At $t=0$ and π the points are $(7,-2)$ and $(-1,-2)$, the two vertices.

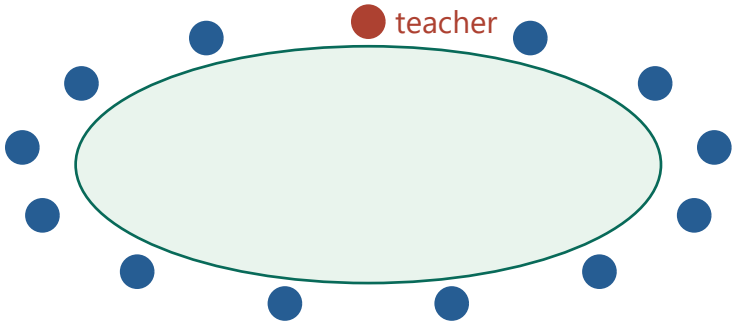
Exercise 803 · Seats around a circular table

Printed page 89 · Counting and permutations / Rotations

In how many ways can a mathematics class of twelve students and one teacher seat itself at a circular Harkness table?

1. Decide what identifies a circular order

In the usual circular seating count, rotating everyone by the same number of places is the same seating. Fix the teacher as a reference point; clockwise and counterclockwise orders remain distinguishable.



Anchor the teacher, then read clockwise around the table.

2. Arrange the remaining people

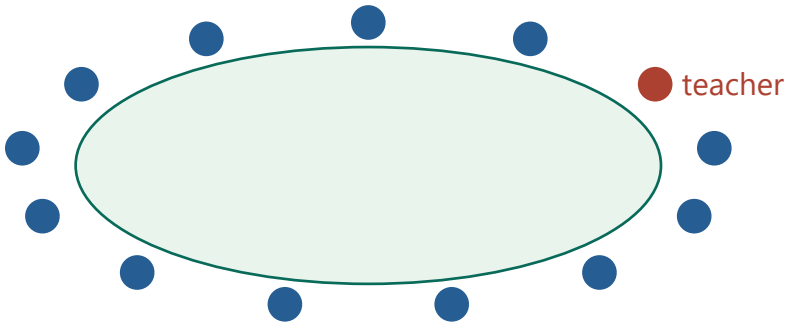
Place the twelve distinct students clockwise after the teacher. The number of orders is 12·11·…·1=12!=479001600.



Twelve remaining people fill successive positions.

3. Check the convention

If the individual chairs had fixed names, there would instead be 13! assignments. For unlabeled chairs each circular arrangement appears in 13 rotated versions of that count; dividing by 13 recovers 12!.



Rotate the entire table: no one changes neighbors.

Check. Each cyclic seating is represented exactly 13 times among the 13! linear lists, giving 13!/13=12!.

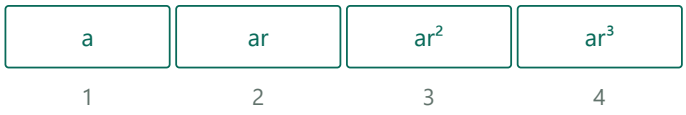
Exercise 804 · Gaps in a geometric sequence

Printed page 89 · Geometric sequences and series / Finite differences

Given a geometric sequence ar^0, ar^1, ar^2, …, form the sequence of gaps y_1, y_2, y_3, …, where $y_n = ar^n - ar^{n-1}$. Show that the y-sequence is also geometric. What is its ratio?

1. Write a gap as one factored quantity

Gap n is $y_n=ar^n-ar^{(n-1)}$, n≥1. Factoring out the smaller power gives $y_n=a(r-1)r^{(n-1)}$.



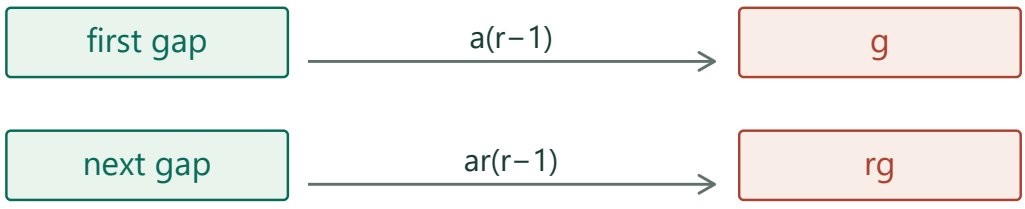
Each gap subtracts one term from the next.

1. $ar^n-ar^{n-1}=ar^{n-1}\cdot r-ar^{n-1}\cdot 1$
Express both adjacent terms with the common factor ar^{n-1}.

2. $=ar^{n-1}(r-1)$
The distributive law in reverse gives the gap without division.

2. Advance the index

$y_{n+1}=a(r-1)r^n=r\,y_n$. The same multiplier that advances the original sequence therefore advances its gaps.



Advancing one gap multiplies it by the original ratio.

3. Check zero cases

If every gap is zero, this recurrence still works with the stated r, although a ratio formed by dividing consecutive zero gaps is undefined. For a=2,r=3 the gaps 4,12,36 demonstrate the nonzero case.



With a = 2 and r = 3 these are gaps between 2, 6, 18 and 54.

Check. For a=2 and r=3 the sequence 2,6,18,54 gives gaps 4,12,36, which also triple.

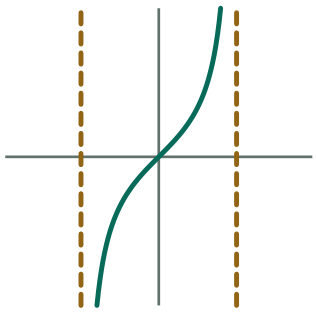
Exercise 805 · Principal inverse tangent

Printed page 89 · Inverse trigonometric functions / The tangent function

What are the domain and range of the function $\tan^{-1}$?

1. Choose the branch being inverted

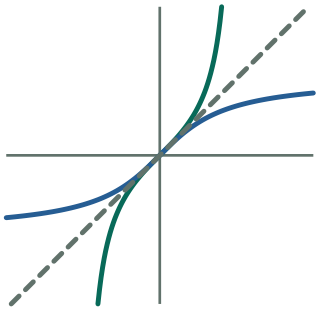
Ordinary tangent on (−π/2,π/2) is continuous and strictly increasing. Near the left endpoint its values are arbitrarily negative; near the right they are arbitrarily positive.



Only the central tangent branch is inverted.

2. Exchange inputs and outputs

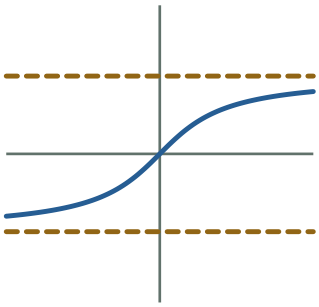
Thus arctan accepts every real tangent value and returns the unique angle from this interval. Its domain is $\mathbb{R}$ and its range is $(-\pi/2, \pi/2)$.



Swap coordinates using equal scales on both axes.

3. Check the endpoints

Tangent is undefined at $\pm\pi/2$, so arctan never outputs either endpoint. In degrees its range is $(-90°, 90°)$.



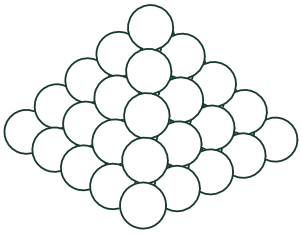
The horizontal output boundaries are approached, never attained.

Check. The endpoint angles are excluded because tangent is undefined there. In degrees the range is $(-90°, 90°)$.

Exercise 806 · A pyramid of grapefruit

Printed page 89 · Packing / Sums of squares / The Pythagorean theorem

A grocer has 1015 spherical grapefruit, which are to be stacked in a square pyramid — one in the top layer, four in the next layer, etc. How many layers will the completed pyramid have? The diameter of each grapefruit is 6 inches. Find the height of the completed pyramid.



1. Count square layers

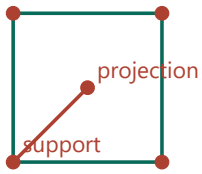
If the stack has n layers, the number of fruit is $1^2+2^2+\cdots+n^2=n(n+1)(2n+1)/6$. For n=14 this gives $14\cdot15\cdot29/6=1015$. The counts strictly increase with n, fixing 14 layers.



Stack successive square layers until the total reaches 1,015.

2. Find the horizontal offset of centers

Below an upper grapefruit four supporting centers form a square of side 6 in. The upper center projects to the square center, 3 in in each horizontal direction from a supporting center. Its horizontal offset from that center is $3\sqrt{2}$.

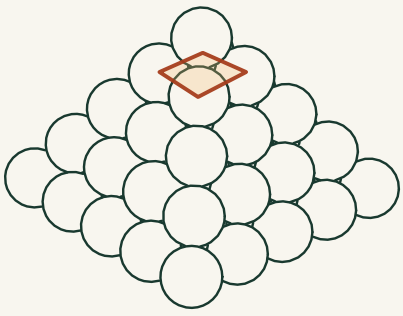


Top view: the horizontal offset from a support is $3\sqrt{2}$ inches.

1. $s^2=3^2+3^2=18 \Rightarrow s=\sqrt{18}=3\sqrt{2}$
The two horizontal offsets are perpendicular, so their squared lengths add.

2. $h^2+s^2=6^2 \Rightarrow h^2=36-18=18$
The horizontal displacement and vertical gap form a second right triangle with the center-to-center segment.

3. $h=\sqrt{18}=3\sqrt{2}$ inches
The upper layer lies above the lower one, selecting the positive root.

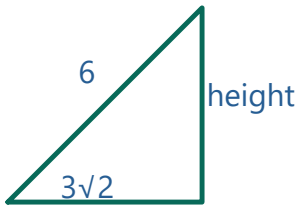


Four fruits supporting the top

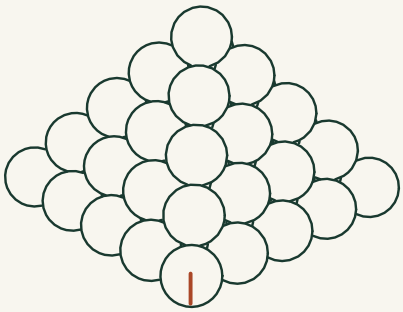
Use the centers in the layer just below the top. Some centers overlap in projection; in a horizontal plan those four form a square.

3. Use tangency to get vertical spacing

Touching centers are 6 in apart. Pythagoras gives vertical spacing $\sqrt{(6^2-(3\sqrt{2})^2)}=3\sqrt{2}$ in. This spacing is less than a diameter because the fruit nest between layers.



Use the 6-inch center separation as a hypotenuse.

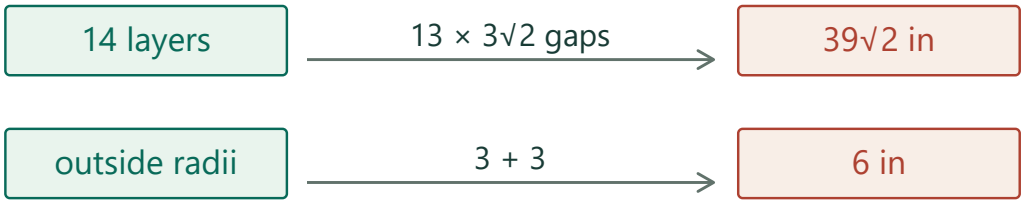


Lowest visible fruit

In the base layer locate a center and then its bottom. The drop to the bottom contributes an additional radius after the center-to-center gaps are counted.

4. Include both outside radii

Fourteen center levels give thirteen gaps. Add the 3-in radius below the lowest centers and another above the top center: total $6+39\sqrt{2}\approx 61.15$ in.



Center-to-center gaps and the two end radii give the full height.

Check. The 13-layer stack uses 819 spheres, and the fourteenth layer adds 196, reaching 1015.

Exercise 807 · Undo a base-four log inequality

Printed page 89 · Logarithms / Inequalities

Given the information $-1 < \log_4 p < 1.5$, what can you say about the possible values of p?

1. Interpret the bounds as exponents

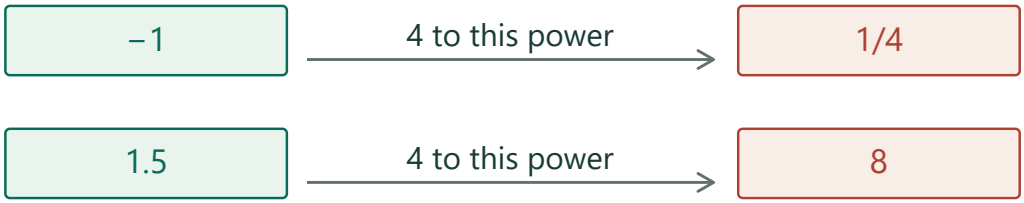
The inequality is $-1<\log_4 p<1.5$. Because base 4 exceeds one, the exponential 4^x increases throughout its domain.



The logarithm must be strictly between the two exponents.

2. Raise 4 to each part

This preserves the order: $4^{-1}<p<4^{1.5}$.



An increasing exponential keeps the inequality direction.

3. Evaluate the endpoints

$4^{-1}=1/4$ and $4^{(3/2)}=(\sqrt{4})^3=8$. Therefore $1/4<p<8$. Both endpoints are excluded by the strict original inequalities.



The original strict inequalities exclude both endpoints.

Check. Taking $\log_4$ of the endpoints gives exactly -1 and $3/2$. Strict inequalities exclude both.

Exercise 808 · From a log–log line to a power law

Printed page 89 · Data linearization / Power functions / Logarithm rules

Rewrite the equation $\log y = b + m\cdot\log x$ in a form that makes no reference to logarithms.

1. Invert the logarithm

Here log denotes base 10. The original log x and log y restrict x and y to positive reals. The slope m and intercept b may be any real numbers.

1 **$\log y = b + m \log x$**
Start with the given line in logarithmic coordinates.

2 **$10^{(\log y)} = 10^{(b + m \log x)}$**
Raise the same positive base 10 to each side.

3 **$y = 10^{(b + m \log x)}$**
Inverting the log is valid for the original $y>0$.

2. Separate the intercept from the power of x

The first factor will be a positive constant. The second factor simplifies to x raised to m. Keeping $x>0$ permits every real exponent m.

1 **$y = 10^{b} \cdot 10^{(m \log x)}$**
Split the sum in the exponent into two factors.

2 **$y = 10^{b} \cdot (10^{(\log x)})^{m}$**
Regroup the product in the exponent.

3 **$y = 10^{b} x^{m}, \quad x > 0$**
Since $x>0$, exponentiating log x returns x.

3. Check that the rewritten equation is equivalent

The right side of the power law is positive, so $y>0$ follows automatically. On this domain the logarithm recovers the original equation.

1 **$\log(10^{b} x^{m}) = \log(10^{b}) + \log(x^{m})$**
Take the log of the positive product.

2 **$\log(10^{b}) + \log(x^{m}) = b + m \log x$**
Apply the inverse and power rules to the two terms.

Check. Taking common logarithms recovers $b+m \log x$. The log-domain restrictions are preserved.

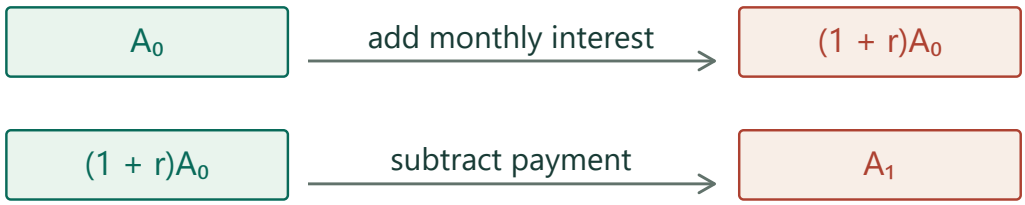
Exercise 809 · Derive the payment on a loan

Printed page 90 · Loan amortization / Geometric sequences and series

Repaying loans, part II. The bank has just granted Jordan a \$10 000 loan, which will be paid back in 48 equal monthly installments, each of which includes a 1 percent interest charge on the unpaid balance. Previously, Jordan has solved the loan payment problems through a guess-and-check procedure involving a linear equation. Now Jordan, who knows about geometric series, would like to develop a general formula for the correct loan payment.
(a) Let A_n be the amount owed after n payments (so that $A_0 = 10\,000$), let $r = 0.01$ be the monthly interest rate, and let P be the monthly payment. Explain why $A_1 = A_0 - (P - rA_0) = (1 + r)A_0 - P$, and $A_2 = (1 + r)A_1 - P$, then write a recursive equation that expresses A_n in terms of A_{n-1}.
(b) Apply the recursive equation to express A_n in terms of A_0. For example, you can write $A_2 = (1 + r)A_1 - P = (1 + r)[(1 + r)A_0 - P] - P$, which can be reorganized as $A_2 = (1 + r)^2 A_0 - (1 + r)P - P$. You should see a pattern developing! It involved the finite geometric series $P + (1 + r)P + (1 + r)^2 P + \ldots + (1 + r)^{n-1} P$.
(c) Explain why $A_{48} = 0$ for Jordan's loan. Then set $A_{48} = 0$ in your answer to (b), and solve for P. This expresses the monthly payment in terms of A_0 (which is 10 000), r (which is 0.01), and n (which is 48).

1. Separate interest from principal (a)

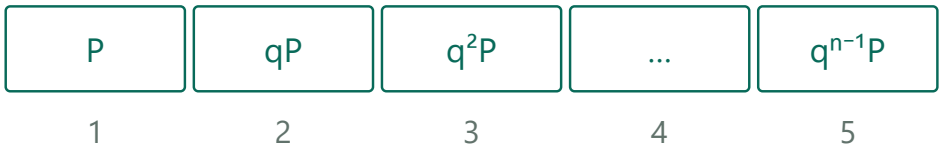
In the first month interest rA_0 is due. The principal reduction is only $P - rA_0$, hence $A_1 = A_0 - (P - rA_0) = (1+r)A_0 - P$. The same reasoning applies to $A_2 = (1+r)A_1 - P$ and to $A_n = (1+r)A_{n-1} - P$.



Post interest to the opening balance before subtracting the payment.

2. Follow one payment forward (b)

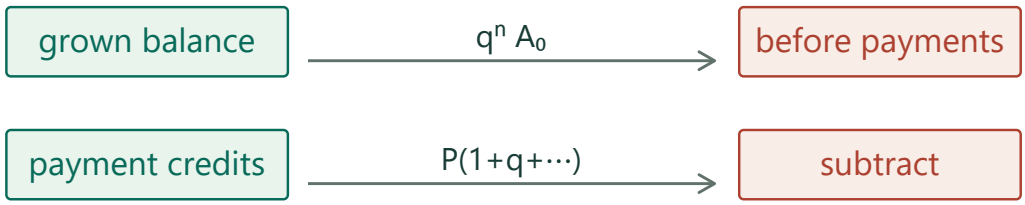
Write $q=1+r$. Expanding gives $A_2=q^2A_0-qP-P$. After n payments the initial amount has been multiplied by q^n. The subtractions are P,qP,…,q^(n−1)P because each earlier payment avoids more interest.



Earlier payments prevent more future interest.

3. Use the geometric sum

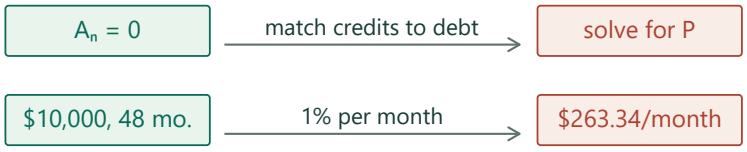
Therefore $A_n=q^nA_0-P(1+q+\cdots+q^{(n-1)})=q^nA_0-P(q^n-1)/r$ for $r\neq0$.



Apply a geometric sum to the accumulated payment credits.

4. Impose full repayment (c)

The balance just after the last installment is zero. Set $A_n=0$ and divide through by q^n: $P=A_0r/[1-(1+r)^{(-n)}]$. For $A_0=\$10{,}000$, r=.01, n=48 this is \$263.33835…, about \$263.34 monthly with a possible final cents adjustment.



Full repayment sets the ending balance to zero.

1. $P(q^n-1)=rq^nA_0$
Set the balance to zero and multiply by r, assuming $r\neq0$.

2. $P=A_0r/(1-q^{-n})$
Divide by q^n-1, then divide numerator and denominator by q^n. For $n\geq1$ and $q=1+r>0$, q^n-1 is nonzero whenever $r\neq0$.

3. If r=0: $A_n=A_0-nP \Rightarrow P=A_0/n$
Zero interest leaves n equal subtractions, so this case needs no geometric-sum division.

Check. The present value $P/q+\cdots+P/q^n$ equals the initial principal. Actual last payments can adjust for cents.

Exercise 810 · Committees with and without a teacher

Printed page 90 · Combinations / Counting and permutations

The twenty-person Mathematics Department is forming a four-person committee to draft a technology proposal. How many possible committees are there? How many include your mathematics teacher? How many do *not* include your mathematics teacher? Can you come up with more than one way to solve this final question?

1. Count committees without order

Four teachers chosen from twenty form one committee regardless of their order of selection. Thus C(20,4)=20·19·18·17/(4·3·2·1)=4845. The numerator counts ordered selections of four people. Dividing by 4! removes the 24 ways of listing the same four names.



A committee has four members without an internal order.

2. Require your teacher

One slot is filled; choose three of the other nineteen. This gives C(19,3)=969 committees.



Fixing the teacher leaves three choices among nineteen.

3. Exclude your teacher in two ways

Either choose all four from the nineteen others, C(19,4)=3876, or subtract the 969 including your teacher from all 4845. These cases partition the entire collection. In symbols this split is C(20,4)=C(19,3)+C(19,4): every committee falls into exactly one of the two groups.



Every committee belongs to one of the two disjoint groups.

Check. The partition gives 969+3876=4845, an instance of Pascal's identity.

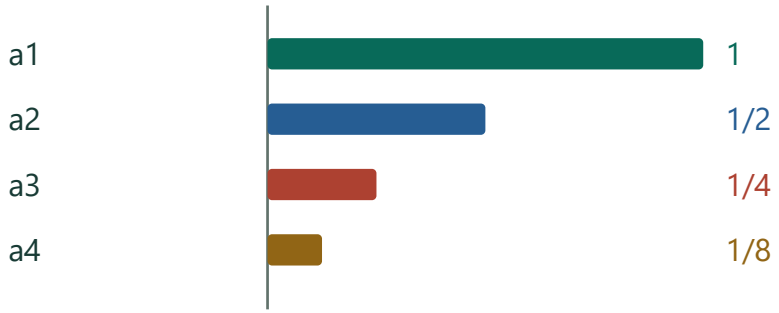
Exercise 811 · Sequence versus series

Printed page 90 · Sequences and recursion / Sigma notation / Geometric sequences and series

What is the difference between the mathematical uses of the words *series* and *sequence*?

1. List terms in order

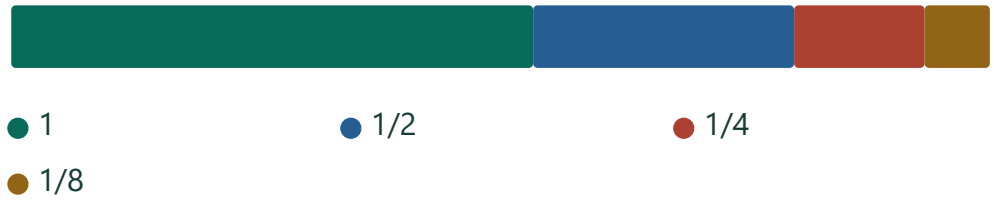
A sequence assigns a term to each allowed index. For example $a_1=1, a_2=1/2, a_3=1/4, \ldots$ specifies a sequence tending to zero.



Each index selects an individual term.

2. Add the same terms

A series is the sum formed from a sequence, here 1+1/2+1/4+⋯. For finite lists one simply adds the listed terms.



Lay the same four lengths end to end for a partial sum.

3. Define an infinite total

Form partial sums by stopping after 1 term, 2 terms, and so on. Their limit, when it exists, is the series sum. In this example the partial sums approach 2 even though the individual terms approach 0.



The partial sums leave gaps of 1, 1/2, 1/4, 1/8, … from 2.

1. $S_n = 1 + 1/2 + \cdots + (1/2)^{n-1}$
Write the first n terms of this particular series.

2. $S_n - S_n/2 = 1 - (1/2)^n$
The shifted sum cancels all terms except the first and the new last term.

3. $S_n = 2[1 - (1/2)^n] \to 2$
Double both sides. Successive halvings make the remaining power approach zero.

Check. The example sequence tends to zero, while its series sums to 2. These are different operations on the same terms.

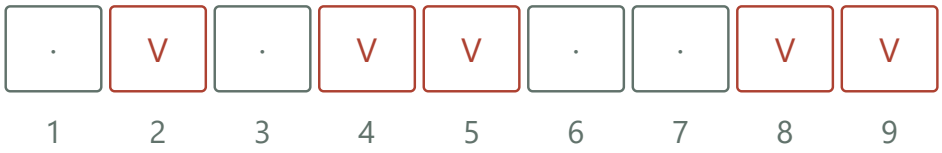
Exercise 812 · Alphabetical order among vowels

Printed page 90 · Counting and permutations / Probability

In how many of the 9! permutations of *facetious* do the vowels occur in alphabetic order? If you get stuck on this one, you can test your approach by trying it on an easier version of the question — the 4! permutations of *face*.

1. Choose positions for the vowels

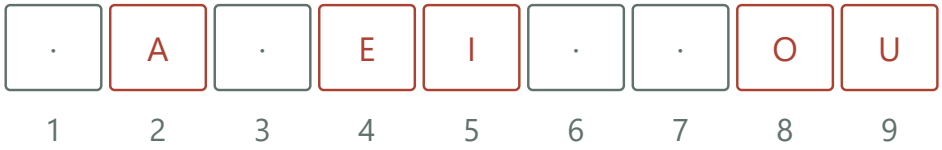
The distinct vowels A,E,I,O,U occupy five of the nine positions in “facetious.” Choose those positions in C(9,5)=126 ways.



Choose the five vowel slots. This pattern is one selection.

2. Impose the alphabetical order

Once their positions are fixed there is exactly one permitted order for the vowels. Arrange the four distinct consonants in the four remaining places in 4!=24 ways.



Fill the selected slots from left to right: A, E, I, O, U.

3. Combine the independent choices

The count is 126·24=3024. Equivalently, among all 9! permutations exactly one of every 5! vowel orders is alphabetical, so 9!/5!=3024.



The four consonant positions can be filled in 24 orders.

Check. For FACE the same reasoning gives 4!/2!=12, exactly half of its permutations.

Exercise 813 · A general combination formula

Printed page 90 · Combinations / Counting and permutations

We have seen that ${}_9C_3$ can be calculated as $\frac{9 \cdot 8 \cdot 7}{3 \cdot 2 \cdot 1}$. Show that this can be expressed more compactly as $\frac{9!}{3!\,6!}$. Write a general formula for ${}_nC_r$ in terms of factorials.

1. Convert a descending product to factorials

Multiplying 9·8·7 by 6! yields 9!, so 9·8·7=9!/6!. This counts ordered selections of three items from nine.



Removing the shared tail 6! leaves three descending factors.

2. Remove ordering

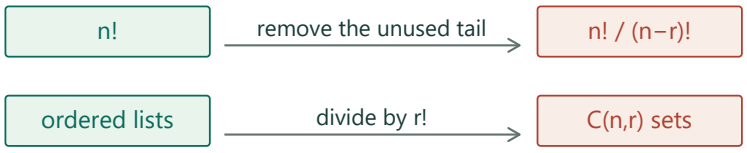
Every three-item set occurs in 3! orders, hence C(9,3)=9!/(3!6!)=84.



The six orders of a triple describe a single choice.

3. Generalize to r selections

The ordered count from n distinct items is n!/(n−r)!. Dividing by r! gives C(n,r)=n!/[r!(n−r)!] for integers 0≤r≤n, with 0!=1.



Removing the unused tail and reordering are different operations.

1. n(n−1)⋯(n−r+1)·(n−r)!=n!
The appended factorial supplies all missing factors down to 1.

2. ordered count=n!/(n−r)!
Divide away the appended factors, using 0!=1 if r=n.

3. unordered count=n!/[r!(n−r)!]
Divide by the r! orders of each subset. For r=0 the unique empty selection also has count one.

Check. Exchanging r and n−r leaves the expression fixed, agreeing with taking complements.

Exercise 814 · Four algebraic cancellations

Printed page 90 · Counting and permutations / Logarithm rules / Geometric sequences and series

Simplify by hand: (a) $\frac{20!}{18!}$ (b) $\frac{\log 32}{\log 8}$ (c) $\frac{\log\left(m^5\right)}{\log\left(m^3\right)}$ (d) $\frac{1+r+r^2+\cdots+r^{39}}{1+r+r^2+\cdots+r^{19}}$

1. (a) Cancel the shared factorial

Stop expanding at the factorial in the denominator. It is a product of positive integers, so cancellation is valid.

1 **20! = 20 · 19 · 18!**
Peel off the two factors larger than 18.

2 **20!/18! = (20 · 19 · 18!)/18!**
Replace the numerator by that equal product.

3 **20!/18! = 20 · 19 = 380**
Cancel the nonzero 18! and multiply.

2. (b) Rewrite both inputs as powers of 2

Both log arguments are positive. Since $2 \neq 1$, its logarithm in any valid base is nonzero and can be cancelled.

1 $32 = 2^5, \quad 8 = 2^3$

Use the same base for the two inputs.

2 $\frac{\log 32}{\log 8} = \frac{5 \log 2}{3 \log 2}$

Bring down the exponents using the power rule.

3 $\frac{\log 32}{\log 8} = \frac{5}{3}$

Cancel the common nonzero factor log 2.

3. (c) Derive the domain before cancelling log m

Both real logarithms require positive arguments. The odd powers m^5 and m^3 have the sign of m. Even with positive inputs, the original denominator must stay nonzero.

1 $m^5 > 0 \text{ and } m^3 > 0 \iff m > 0$

These are the two logarithm input tests.

2 $\log(m^3) \neq 0 \iff m^3 \neq 1 \iff m \neq 1$

The denominator vanishes precisely when its argument equals 1.

3 $\frac{\log(m^5)}{\log(m^3)} = \frac{5 \log m}{3 \log m}$

Use the power rule on positive m.

4 $\frac{\log(m^5)}{\log(m^3)} = \frac{5}{3}, \quad m > 0, m \neq 1$

Those restrictions make the cancelled factor nonzero.

4. (d) Factor out the entire denominator

Call the twenty-term denominator D. Split the forty numerator terms into two blocks of twenty. Each term in the second block is r^{20} times the matching term in the first.

1 $\mathbf{D = 1 + r + \cdots + r^{19}}$
Define the common block of terms.

2 $\mathbf{1 + r + \cdots + r^{39} = D + r^{20}D}$
The second block runs from r^{20} through r^{39}.

3 $\mathbf{(D + r^{20}D)/D = D(1 + r^{20})/D}$
Factor the entire common block.

4 $\mathbf{D(1 + r^{20})/D = 1 + r^{20} \quad \text{if } D \neq 0}$
The original denominator must be checked before cancellation.

5. (d) Determine exactly when D is zero

Pair successive terms, starting with 1+r. The remaining factor is a sum of even powers and includes 1, so it is strictly positive for every real r.

1 $\mathbf{D = (1+r) + r^2(1+r) + \cdots + r^{18}(1+r)}$
Group the original twenty terms in consecutive pairs.

2 $\mathbf{D = (1+r)(1+r^2+\cdots+r^{18})}$
Factor the shared 1+r from the ten pairs.

3 $\mathbf{1+r^2+\cdots+r^{18} \geq 1 \;\Rightarrow\; D = 0 \text{ only at } r = -1}$
Every even power is nonnegative. At r=−1 the first factor is zero.

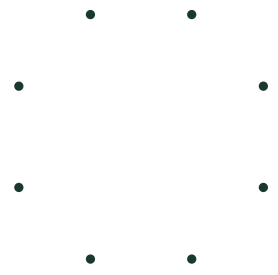
4 $\mathbf{\text{quotient} = 1+r^{20}, \quad r \neq -1}$
Keep the original restriction. At r=1 the ratio is 40/20=2.

Check. At r=1 the original ratio is 40/20=2, as $1+1^{20}$ predicts. At m=2, part (c) reproduces part (b).

Exercise 815 · Polygons chosen from eight points

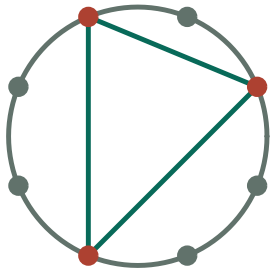
Printed page 90 · Combinations / Regular polygons

Choose three of the eight dots shown in the figure at right; they determine a triangle. There are 56 such triangles possible. How many quadrilaterals can be obtained by choosing four dots? How many pentagons are possible? There are several questions of this sort that can be asked about these eight dots. Where are the answers to all these questions found?

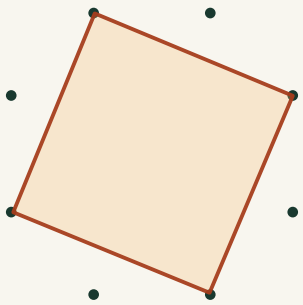


1. Translate a polygon into its vertex set

The eight marked points are in convex position. Pick k of them and connect them in their order around the circle to form one convex k-gon.



Three selected vertices determine one triangle.



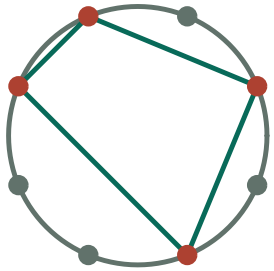
One selection of four vertices

Connect the selected points in cyclic order and return to the start. Each set of selected points gives a boundary this way.

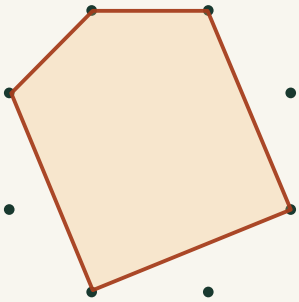
2. Count quadrilaterals and pentagons

Four selected vertices give C(8,4)=8·7·6·5/(4·3·2)=70. Five selected vertices give C(8,5)=56, also C(8,3) since three points are left unused.

Polygon size · 4 vertices



A quadrilateral uses four of eight points. One choice is outlined.

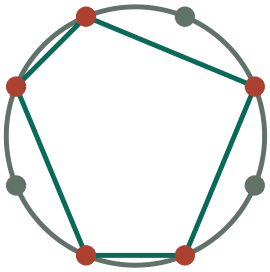


One selection of five vertices

Follow these five selected points around the same circle. Notice the three points that the boundary bypasses.

Compare settings · Polygon size

5 vertices



Five selected vertices form a pentagon.

Choose five of the eight points in any order, then join them cyclically.

3. Locate the whole family of counts

Row 8 of Pascal's triangle is 1,8,28,56,70,56,28,8,1. The entry at index k counts the ways to select k of the eight points, including the requested polygon counts for k≥3.

1	8	28	56	70	56	28	8	1

Row 8 counts every possible size of vertex selection.

Check. Complementing the five selected points gives a three-point set, so the pentagon and triangle counts agree.

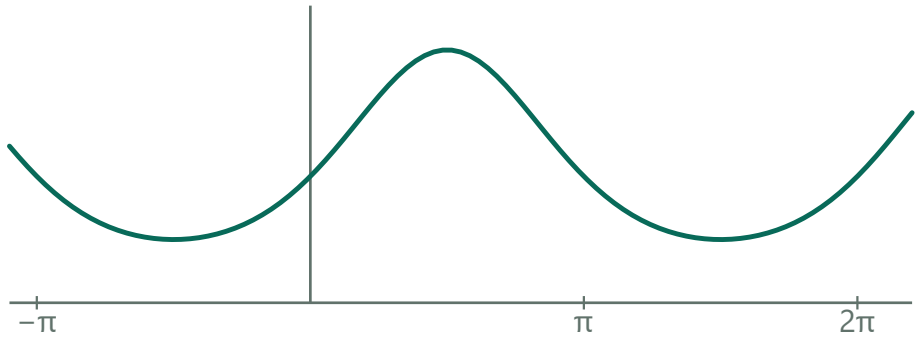
Exercise 816 · A periodic exponential of sine

Printed page 90 · Periodic functions / Exponential functions

The function E defined by $E(x) = 2^{\sin x}$ is periodic, but not sinusoidal. Explain.

1. Use the exponent's repetition

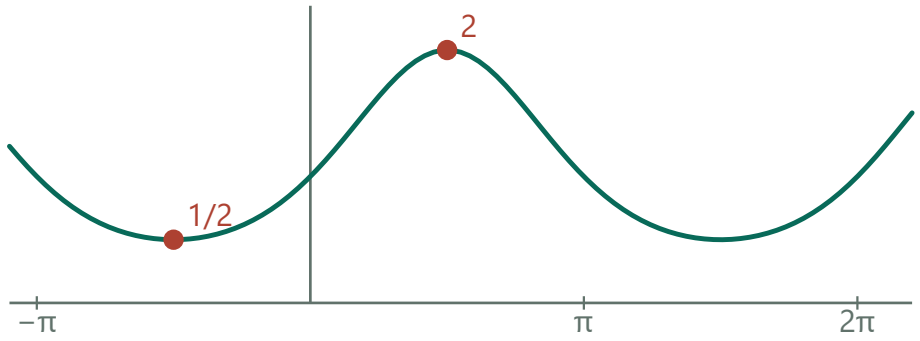
E(x)=2^(sin x) repeats after 2π because sine repeats. Since the map u→2ᵘ is one-to-one, any repetition of the whole graph must also repeat sine; its least period is 2π.



A shift of 2π repeats the exponent and the height.

2. Find adjacent extremes

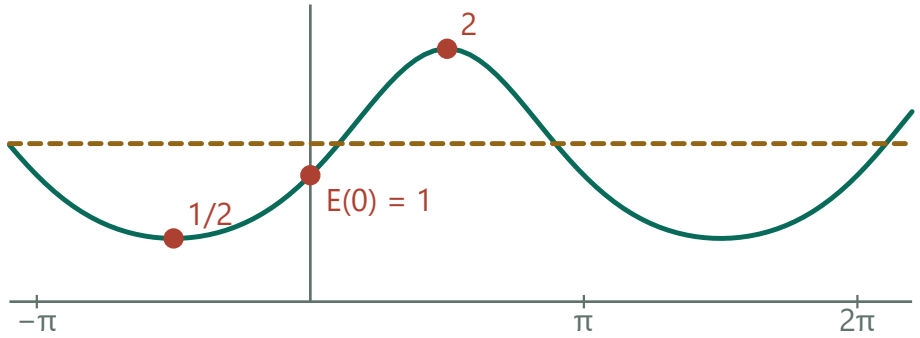
The exponential increases with its exponent. Thus the trough at $x=-\pi/2$ has height $2^{-1}=1/2$ and the adjacent crest at $x=\pi/2$ has height $2^{1}=2$.



The nearest trough and crest lie π apart.

3. Test the required midline crossing

A sinusoid must cross the average of its extreme heights halfway in x between adjacent trough and crest. Here that average is $(1/2+2)/2=5/4$, yet $E(0)=2^{0}=1$. This proves the wave cannot be sinusoidal.



At their halfway time, the curve lies below the midheight 5/4.

Check. Another test: $E(x)E(x+\pi)=1$, whereas a sinusoid with this period has a constant half-period sum. The actual sums at 0 and $\pi/2$ are 2 and 2.5.

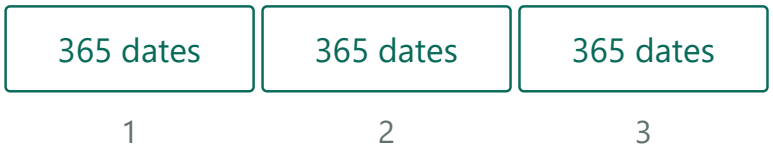
Exercise 817 · A birthday collision in a class

Printed page 91 · Probability / Complementary events

What is the probability that all the members of your mathematics class have different birthdays? What is the probability that there is a birthday coincidence in the group?

1. Specify the number and model

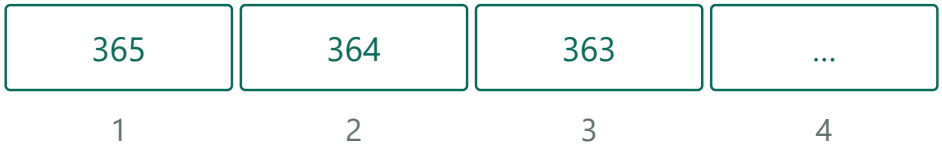
Let n denote the class size. Treat each birthdate as independently and uniformly selected from 365 non-leap dates.



Each independent student can land on any of the 365 dates.

2. Build the no-match probability

The first person can have any date. For the second to differ there are 364 acceptable dates, for the third 363 if the first two differed, and so on. For $n \le 365$ multiply $D_n = \prod(365-j)/365$ for $j=0,\ldots,n-1$.



To stay distinct, each new birthday has one fewer date available.

3. Obtain the probability of any coincidence

At least one duplicate is the complement, $1-D_n$. Insert the actual class size to evaluate. If $n>365$ then two people must share a date, so the probabilities are 0 and 1 respectively.

Group size · Example with 3 students



Example with three students. The two probabilities add to one.

Compare settings · Group size

10 people



10 people: 11.69% chance of at least one shared date.

P(all distinct) = 0.883052. Subtracting from 1 gives P(at least one match) = 0.116948.

30 people



30 people: 70.63% chance of at least one shared date.

P(all distinct) = 0.293684. Subtracting from 1 gives P(at least one match) = 0.706316.

Check. For two people the formulas give 364/365 and 1/365, respectively.

Exercise 818 · The birthday threshold

Printed page 91 · Probability / Complementary events / Inequalities

(Continuation) How large a group is needed, in order that there be at least a 50 percent chance of finding at least one birthday coincidence somewhere in the group?

1. Track the complementary event

Let D_n be the probability of n distinct birthdays in the independent uniform 365-day model. With one person $D_1 = 1$.

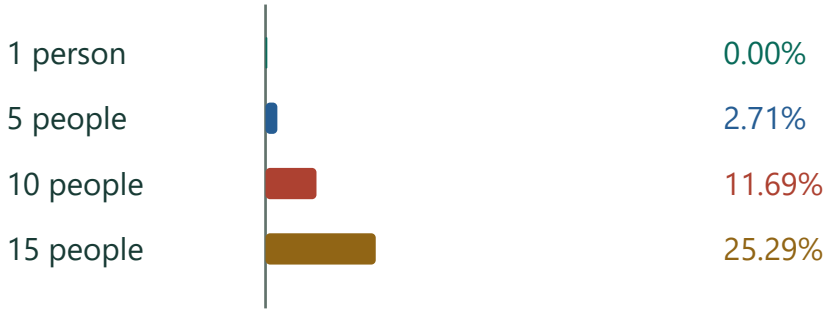


● all distinct

One student cannot make a pair.

2. Add one person at a time

If the n dates so far differ, the next person must avoid all n. Thus $D_{n+1}=D_n(365-n)/365$. Iterate until the complement $1-D_n$ reaches 0.5.

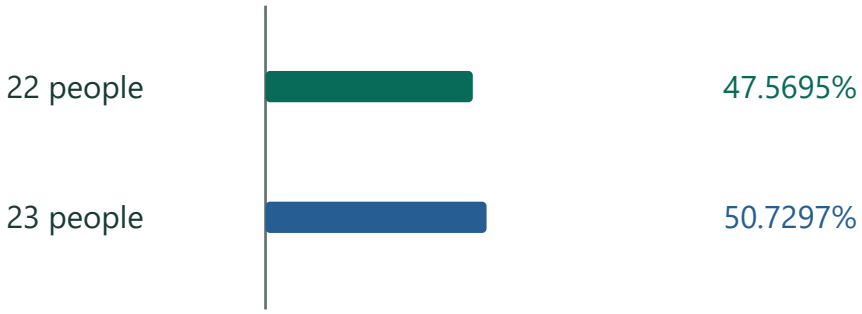


Multiply the distinct-date probability by the new unused-date fraction.

3. Check both sides of the threshold

At 22 people the coincidence probability is about .475695; at 23 it is .507297. Adding another person cannot undo any match, so 23 is the smallest qualifying group.

Group size · 22 and 23 together



At 22 the match chance is below one half. At 23 it exceeds one half.

Compare settings · Group size

22 people



22 people: 47.57% chance of at least one shared date.

P(all distinct) = 0.524305. Subtracting from 1 gives P(at least one match) = 0.475695.

25 people



25 people: 56.87% chance of at least one shared date.

P(all distinct) = 0.431300. Subtracting from 1 gives P(at least one match) = 0.568700.

Check. The two no-match values, about .524305 and .492703, lie on opposite sides of .5.

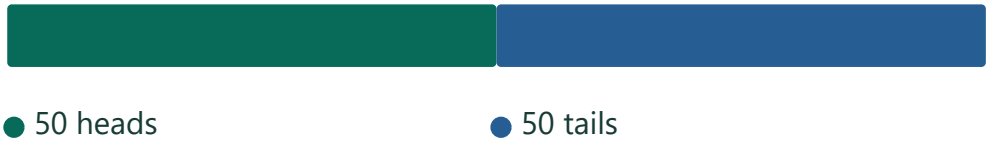
Exercise 819 · Exactly fifty heads

Printed page 91 · Binomial probability / Combinations

If a hundred pennies are tossed onto the floor, the *most likely* outcome is that fifty of them land heads and fifty land tails. What is the probability of this actually happening, however?

1. Count favorable toss sequences

With fair independent pennies the 100 outcomes each have two equally likely choices. There are 2^{100} equally probable ordered sequences. Choose which 50 positions show heads in C(100,50) ways.



A favorable length-100 sequence contains exactly fifty of each.

2. Calculate the chance

The desired probability is $C(100,50)/2^{100} \approx .07958924$, about 7.96%. A single count can be the most likely and still have a small probability.



The event occupies about 7.96% of the outcome space.

3. Verify the mode

The factor from C(100,k) to C(100,k+1) is (100−k)/(k+1). This is greater than 1 through k=49 and less than 1 from k=50 onward, placing the largest count at k=50.



The probability increases up to 50 and decreases after 50.

1. C(100,k+1)/C(100,k)=[100!/((k+1)!(99−k)!)]·[k!(100−k)!/100!]
Divide the two factorial formulas for integers 0≤k<100.

2. =(100−k)/(k+1)
Cancel 100! and expand each of the two larger factorials by one factor.

3. 100−k>k+1 ⇔ k<49.5
Subtract k and 1 and divide by 2. The last increasing step begins at k=49.

Check. The distribution is symmetric about k=50. Its central point is the most likely single result even though most outcomes are elsewhere.

Exercise 820 · Ratio of successive gaps

Printed page 91 · Geometric sequences and series / Finite differences

Simplify the expression $\dfrac{a \cdot r^{n+1} - a \cdot r^{n}}{a \cdot r^{n} - a \cdot r^{n-1}}$, then explain the significance of the result.

1. Recognize successive differences

The numerator subtracts two neighboring geometric terms: ar^(n+1)−ar^n=ar^n(r−1). Likewise the denominator is ar^(n−1)(r−1).



Example a = 2, r = 3. Consecutive gaps have lengths 4 and 12.

2. Cancel where possible

When the denominator is nonzero, division leaves r. Thus each consecutive gap is r times the previous gap.



Factoring out the matching lower power leaves the common ratio.

3. Handle zero gaps

Whenever the powers are defined, the undivided equality ar^(n+1)−ar^n=r[ar^n−ar^(n−1)] still holds even if the ratio itself cannot be formed.



Zero divided by zero is undefined. The undivided gap identity still holds.

Check. Even in zero-gap cases the undivided identity $ar^{n+1}-ar^n=r(ar^n-ar^{n-1})$ remains valid when the powers are defined.

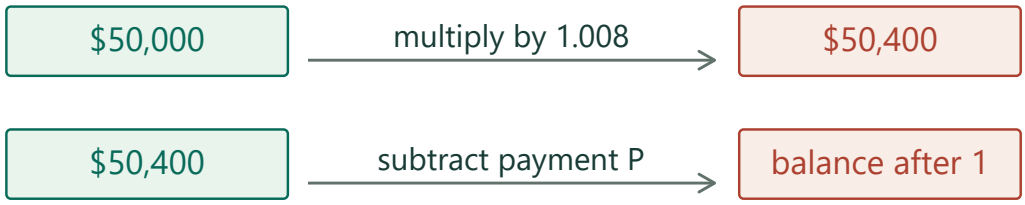
Exercise 821 · Ten years of monthly payments

Printed page 91 · Loan amortization / Compounding

What is the monthly payment needed to repay a $50000 loan in ten years, if the bank charges 0.8 percent per month?

1. Synchronize rate and payment frequency

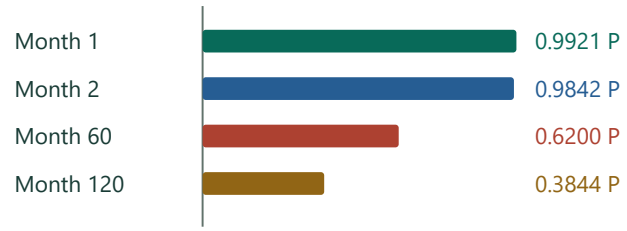
Ten years mean 120 monthly payments. A monthly rate of 0.8% is r=.008, so a balance multiplies by 1.008 before each end-of-month payment.



Interest is posted before the monthly payment.

2. Discount all installments to time zero

If each is P, their present values sum to $P(1-1.008^{-120})/.008$. This must equal the borrowed $50,000.



Each payment loses one discount factor for each month back to today.

1. $q=1.008$; $S=q^{-1}+q^{-2}+\cdots+q^{-120}$
Discount each future installment back to the loan date, counting one term per payment.

2. $qS-S=1-q^{-120}$
Multiplying by q shifts every exponent up by one, so the overlapping powers cancel.

3. $PS=P(1-q^{-120})/.008$
Divide by $q-1=.008$ and multiply by the payment amount.

3. Solve for the regular payment

$P=50000(.008)/(1-1.008^{-120})\approx\$649.72846$, or $649.73 per month. A final cents adjustment can account for rounding individual payments.



One regular payment times the 120-month factor repays the principal.

Check. The discounted value $P(1-1.008^{-120})/.008$ equals $50000.

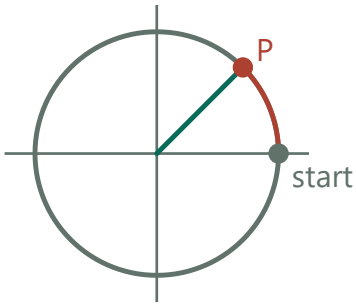
Exercise 822 · Why inverse sine is called arcsine

Printed page 91 · Inverse trigonometric functions / Radian measure / Arc length

Another common name for the inverse sine function is *arcsin*, which is an abbreviation of *find the arc whose sine is*. Explain this terminology.

1. Connect an angle to an arc

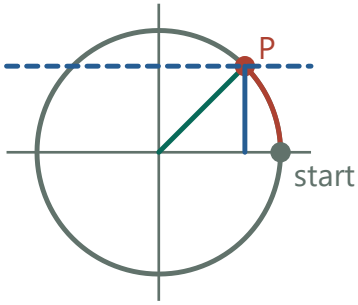
On the unit circle start at (1,0), move counterclockwise for positive angles or clockwise for negative angles, and measure the signed arc length. Because the radius is 1, the signed length equals the angle in radians.



Travel along the arc from (1,0). On a unit circle its length equals θ.

2. Read sine geometrically

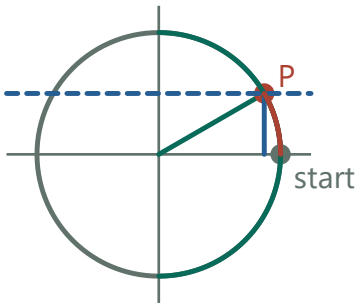
At the end of that arc the y-coordinate is sin θ. Giving a sine value x therefore fixes the endpoint height, but many arcs can reach that height.



Two circle points can share the same vertical height.

3. Select one arc

arcsin x returns the signed angle or arc length between $-\pi/2$ and $\pi/2$ whose endpoint has height x. For $x=1/2$ this chosen arc is $\pi/6$. The construction works for $-1 \le x \le 1$.



Use the right half-circle. Height 1/2 selects the arc of length $\pi/6$.

Check. The height 1/2 gives $\pi/6$, although many other arcs have the same endpoint height. The principal interval makes it unique.

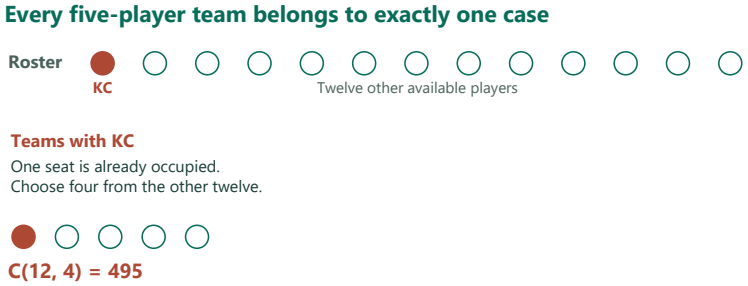
Exercise 823 · Teams that include a chosen player

Printed page 91 · Combinations / Counting and permutations

From the thirteen available Preps, the P.E. teacher is about to choose five to form a basketball team. There are $1287 = {}_{13}C_5$ possible teams. How many of these teams include KC, who is the tallest in the group? How many of the teams do not include KC? The answers to these questions illustrate a familiar property of Pascal's triangle — what is it?

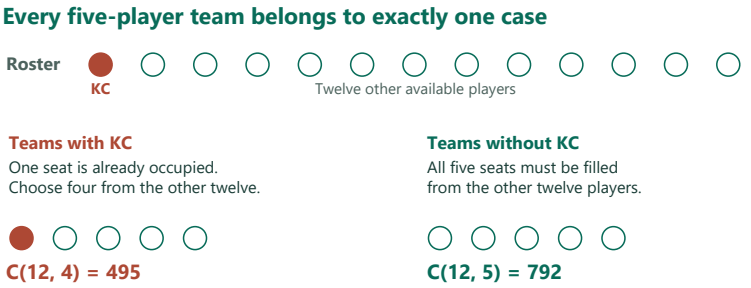
1. Include the designated player

With KC already selected, only four teammates remain to be picked from the other twelve. C(12,4)=495 teams include KC.



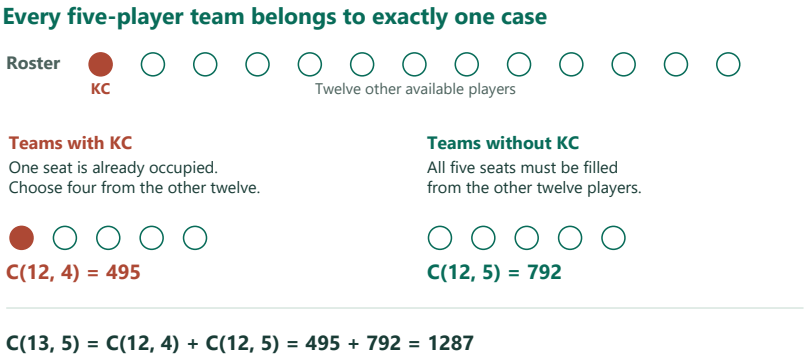
2. Exclude the designated player

All five must then come from the remaining twelve, giving C(12,5)=792 teams.



3. Recover the Pascal relation

A team either includes KC or does not, so the two disjoint counts add: 495+792=1287. This is C(13,5)=C(12,4)+C(12,5), the adjacent-entry addition rule.



Check. The two cases are disjoint and exhaust the five-player teams.

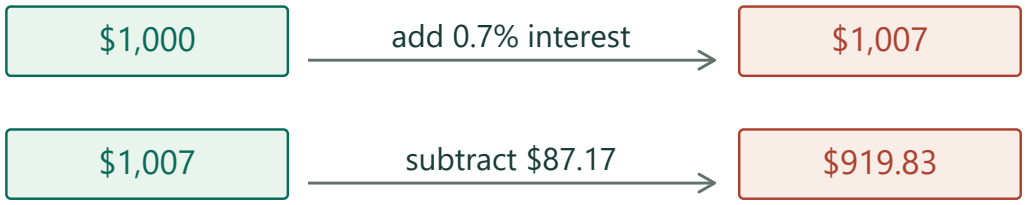
Exercise 824 · A small unpaid balance after rounding

Printed page 91 · Loan amortization / Rounding and precision

In December Corey took out a 1000-dollar loan with monthly interest rate 0.7 percent. In order to pay back the loan, Corey has been paying $87.17 a month since January. Explain why the sequence defined recursively by $x_0 = 1000$ and $x_n = 1.007x_{n-1} - 87.17$ for positive n describes the balance of Corey's debt throughout the repayment of the loan. How many payments does Corey need to pay everything back? How much does this loan cost? What would the monthly payment have been if Corey had been scheduled to pay back the loan in twenty-four months?

1. Model one monthly cycle

The balance starts at $x_0=1000$. Each month adds interest $.007x_{n-1}$ and then subtracts \$87.17, giving $x_n=1.007x_{n-1}-87.17$. Interest applies only to the amount still owed.



The next month starts from the reduced balance.

2. Locate the last payment

Iterate without rounding each intermediate balance. After eleven installments about \$86.61 remains. With the next month's interest, \$87.21346 is due immediately before payment 12. Paying about \$87.21 then settles it to the nearest cent.



The last payoff includes interest through the twelfth payment date.

1. $x_1=1000(1.007)-87.17=919.83$
Apply interest to the starting balance, then subtract the installment.

2. $x_{11}=1000(1.007)^{11}-87.17[(1.007^{11}-1)/.007]$
Expand the recurrence eleven times. The subtracted payments acquire powers from zero through ten.

3. amount due $=1.007x_{11}\approx 87.21346$
Apply month twelve's interest before its payment, using the unrounded balance.

3. State the rounding convention

An adjusted twelfth payment gives total approximately $11\cdot 87.17+87.21=\$1046.08$, or \$46.08 interest. If exactly \$87.17 is required for each of the first twelve, about \$.04346 remains, requiring a small thirteenth payment. Lender rounding can change the final cent.



A regular payment in month 12 leaves a few cents outstanding.

4. Compare the 24-month alternative

Set the present value of 24 equal monthly payments at rate .007 to \$1000. The regular amount is $1000(.007)/[1-1.007^{-24}] \approx \$45.41$.



For a two-year loan, discount 24 payments at the same monthly rate.

Check. After eleven payments the outstanding principal is about \$86.61. Multiplying it by 1.007 before the next payment independently reveals the cents discrepancy. Lender rounding conventions can affect the last cent.

Exercise 825 · Birthdays on graduation day

Printed page 91 · Probability / Binomial probability

What is the probability that this year's graduation will fall on the birthday of:
(a) exactly one of the 336 Seniors?
(b) exactly two of the 336 Seniors?
(c) more than two such Seniors?

1. Set up successes and misses

Model the 336 seniors' birthdays as independent and uniform over 365 dates. On the fixed graduation date each senior has match probability $p=1/365$ and miss probability $q=364/365$.



Each student gets one matching date and 364 nonmatching dates.

2. Count one or two matches

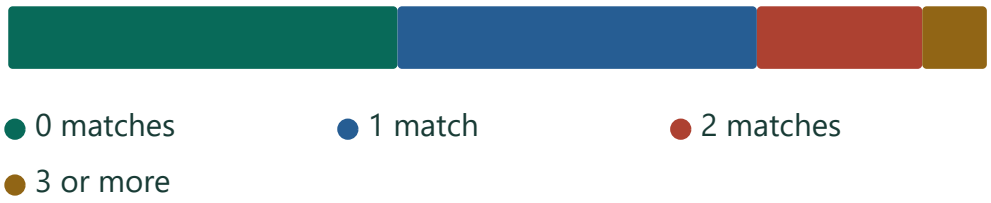
For exactly one senior, choose that person in 336 ways: $P_1 = 336pq^{335} \approx .367198$. For two, choose an unordered pair: $P_2 = C(336,2)p^2q^{334} \approx .168972$.



Count the eligible individuals or pairs, then weight their dates.

3. Complement the small counts

No matching senior has probability $q^{336} \approx .397798$. More than two means none of the cases 0,1,2 occurred, so the probability is $1 - q^{336} - P_1 - P_2 \approx .0660325$.



The remaining probability is the event “more than two.”

Check. $P_0 \approx .397798$. The four disjoint probabilities sum to one.

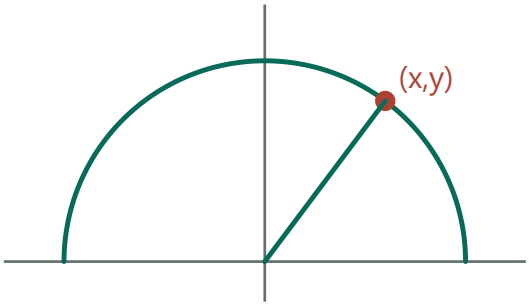
Exercise 826 · Sine of an arccosine

Printed page 91 · Inverse trigonometric functions / Trigonometric identities

What is the domain of the function f defined by $f(x) = \sin(\cos^{-1} x)$? What is its range of values? Find an equivalent way to describe the graph $y = f(x)$.

1. Choose the principal angle

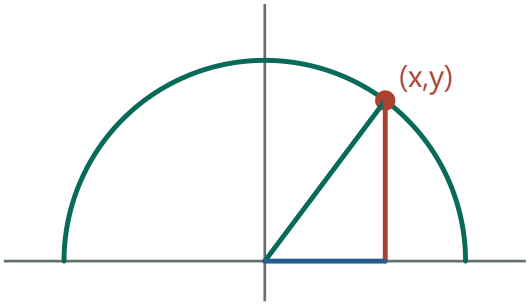
Put $\theta = \arccos x$. Such an angle exists for $-1 \le x \le 1$ and lies in $[0, \pi]$, the upper half of the unit circle.



Principal arccos selects the upper-half-circle point with coordinate x.

2. Use the sign of sine

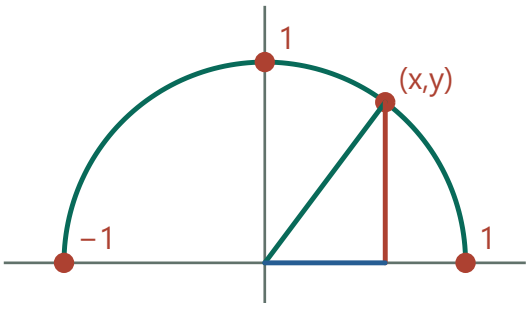
$\cos\theta = x$ and $\sin^2\theta = 1 - x^2$. On the upper half-circle sine is nonnegative, hence $f(x) = \sin\theta = \sqrt{1 - x^2}$.



The triangle gives $y^2 = 1 - x^2$. The upper point has $y \ge 0$.

3. Read the graph and its range

The equation $x^2 + y^2 = 1$ with $y \ge 0$ is the upper unit semicircle. It reaches 0 at $x = \pm 1$ and 1 at $x = 0$, and every height between, so its range is $[0, 1]$.



Moving along the arc produces every height from zero through one.

Check. At $x = -1, 0, 1$ the outputs are $0, 1, 0$, matching both descriptions.

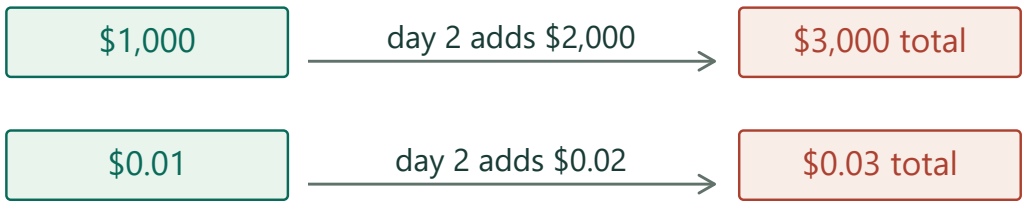
Exercise 827 · Compare two salary plans

Printed page 92 · Arithmetic sequences and series / Geometric sequences and series

For the first 31 days of your new job, your boss offers you two salary options. The first option pays you \$1000 on the first day, \$2000 on the second day, \$3000 on the third day, and so forth — \$1000n on the nth day, in other words. The second option pays you one penny on the first day, two pennies on the second day, four pennies on the third day — the amount doubling from one day to the next. Let A_n be the total number of dollars earned in n days under the first plan, and G_n be the total number of dollars earned in n days under the second plan. Write recursive and explicit descriptions of both A_n and G_n. Which plan is preferable? Explain.

1. Track cumulative earnings

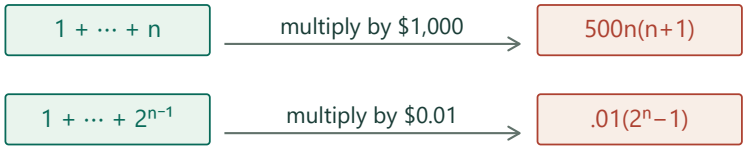
Set $A_0=G_0=0$ dollars. On day n the first plan adds \$1000n, so $A_n=A_{n-1}+1000n$. The second adds .01·2^(n−1) dollars, so $G_n=G_{n-1}+.01\cdot2^{(n-1)}$.



Balance each plan by adding the new day's pay to the previous total.

2. Sum each payment pattern

The arithmetic total is $1000(1+\cdots+n)=500n(n+1)$. The geometric total is $.01(1+2+\cdots+2^{(n-1)})=.01(2^n-1)$.



Summing daily amounts turns both recurrences into explicit balances.

1. $T=1+2+\cdots+n$; $2T=(1+n)+(2+n-1)+\cdots+(n+1)$
Add the forward and reversed lists term by term. There are n identical pair totals n+1.

2. $T=n(n+1)/2$
Divide the paired total by two.

3. $U=1+2+\cdots+2^{n-1}$; $2U-U=2^n-1$
Doubling the geometric sum shifts all powers by one. Only the outer powers survive subtraction.

3. Compare the full 31 days

At n=31, A_{31}=\$496,000 and G_{31}=\$21,474,836.47. The doubling plan pays more over the stated term despite its small first payment.



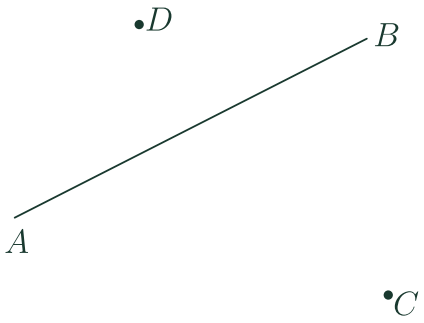
Both bars give cumulative dollars through day 31.

Check. Subtract each explicit formula at n−1 from its value at n to recover the daily amount, including the first day.

Exercise 828 · A reflected point maximizes a difference

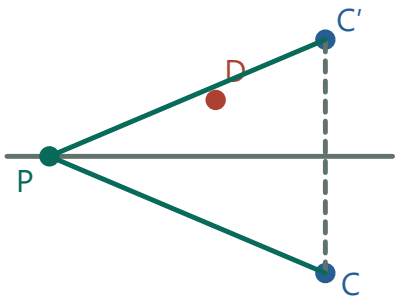
Printed page 92 · Reflections / Optimization / Distance and loci

Suppose that P is the point on line AB that makes the difference of distances $|CP-PD|$ as large as possible. Explain why the angles CPB and DPB must be the same size.

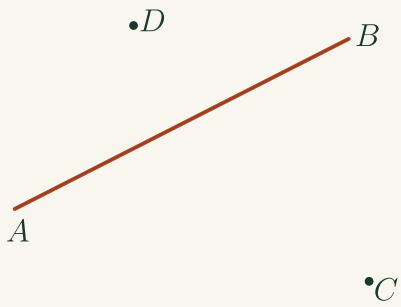


1. Reflect one endpoint

Reflect C in line AB to C′. Every P on line AB is equidistant from C and C′, so the target difference |PC−PD| equals |PC′−PD|.



Turning AB horizontal makes the two equal lengths from P easier to see.

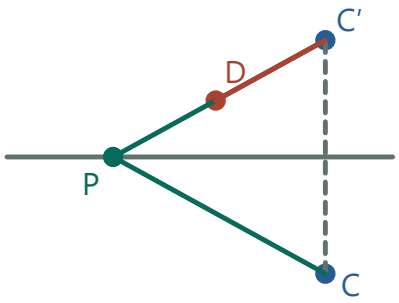


Reflecting line AB

The two endpoints label the mirror line. Perpendicular distances to this line are preserved by reflection.

2. Obtain a universal upper bound

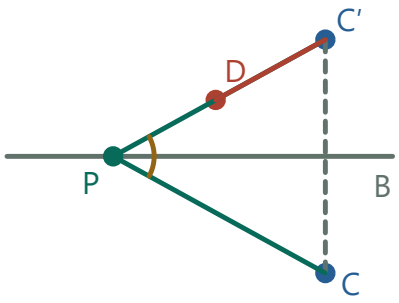
By the reverse triangle inequality, the absolute difference cannot exceed C′D. In the pictured configuration extend the line C′D to meet AB. From that P the points C′ and D lie on the same ray, so their distance equals the absolute difference and the bound is attained.



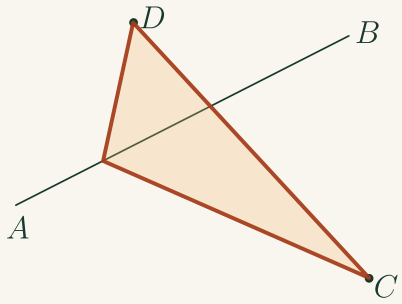
Slide P to the intersection of AB and the line through C′ and D.

3. Read the equal angles

Since PD shares the direction PC′, and PC′ is the mirror image of PC about AB, the two rays make the same angle with PB. Thus ∠CPB=∠DPB at the maximizing point.



Reflection sends one ray into the other, preserving its angle to PB.



Rays from the extremizing contact

For the pictured configuration, trace from C to the contact on the mirror and then to D. Compare the angles that the two rays make with the direction toward B.

Check. Moving P off the intersection makes the triangle PC′D nondegenerate and the reverse triangle inequality strict.

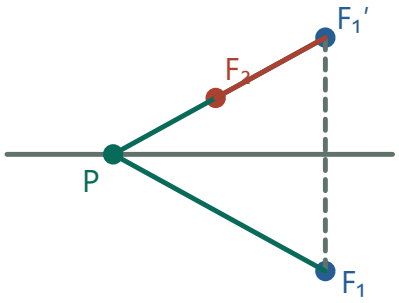
Exercise 829 · The hyperbola's reflecting tangent

Printed page 92 · Hyperbolas / Conic reflection properties / Tangent lines

Reflection property of the hyperbola. Suppose that P is a point on a hyperbola whose focal points are F_1 and F_2. Draw the intersecting lines PF_1 and PF_2, as well as the bisectors of the four angles they form. This problem is about the bisector that separates F_1 and F_2. Prove the following: Given any point Q *other than* P on this line, the difference $|QF_1 - QF_2|$ is smaller than the difference $|PF_1 - PF_2|$. Conclude that the line intersects the hyperbola only at P.

1. Reflect a focus in the separating bisector

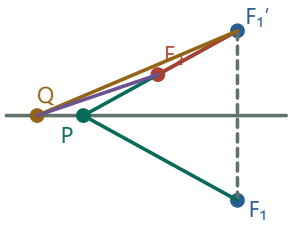
At P, reflect F_1 across the internal angle bisector to F_1' on ray PF_2. Reflection preserves PF_1. Because the two reflected segments point in the same direction, $F_1'F_2 = |PF_1 - PF_2|$.



Mirror F_1 across the bisector through P. Its image lies on ray PF_2.

2. Bound the focal difference elsewhere

For any Q on this bisector, $QF_1 = QF_1'$. The reverse triangle inequality yields $|QF_1 - QF_2| = |QF_1' - QF_2| \le F_1'F_2$.



From Q the two lengths form sides of a triangle with third side $F_1'F_2$.

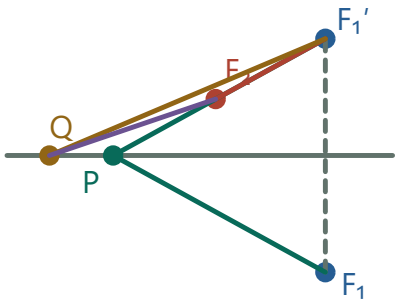
1. $QF_1' \le QF_2 + F_2F_1' \Rightarrow QF_1' - QF_2 \le F_1'F_2$
Use one triangle inequality and subtract the common length.

2. $QF_2 \le QF_1' + F_1'F_2 \Rightarrow QF_2 - QF_1' \le F_1'F_2$
Interchange the two distances. The same bound works when the difference has the opposite sign.

3. $\text{abs}(QF_1' - QF_2) \le F_1'F_2$
Together the two inequalities bound the absolute difference, and reflection gives $QF_1' = QF_1$.

3. Show why the bound is strict

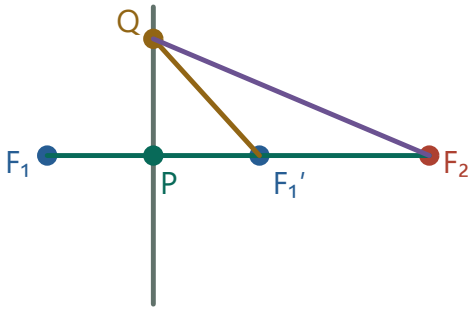
Equality requires Q, F_1', F_2 to be collinear. With distinct focal lines the line through F_1' and F_2 meets the bisector only at P. Every $Q \neq P$ therefore has a smaller difference and cannot lie on this hyperbola.



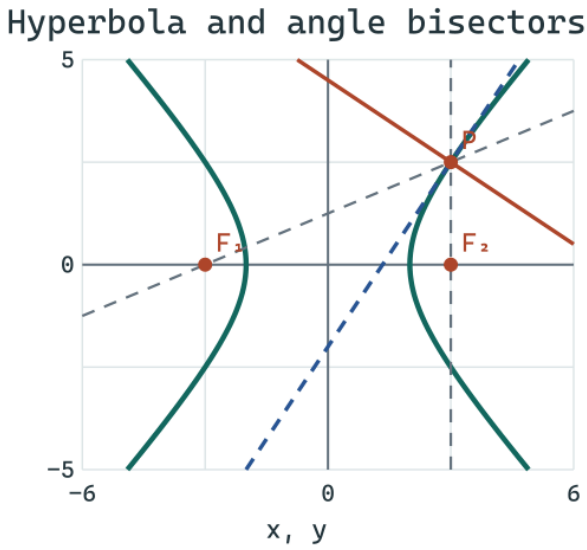
Moving Q off P makes the triangle nondegenerate, so the bound is strict.

4. Treat the vertex geometry

At an axial vertex use the perpendicular to the focal axis through P as the tangent. Reflecting one focus through it again puts both focal points on one axial ray from P. A point Q≠P on that perpendicular makes the triangle nondegenerate, giving strict inequality.



At an axial vertex, use the perpendicular mirror through P.



Check. At an axial vertex, the tangent is perpendicular to the focal axis. For another point on it, the reflected focal comparison is again a strict triangle inequality.

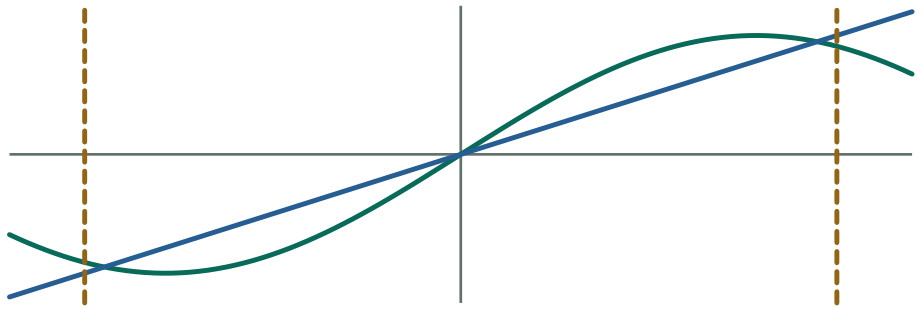
Exercise 830 · Three intersections with sine

Printed page 92 · Trigonometric equations / Intersections of graphs

Work in *radian* mode for this one. The line $y = 0.5x$ intersects the sine curve in three places. What are the coordinates of these intersection points? What would happen if the graphs were drawn in degree mode?

1. Bound the possible roots

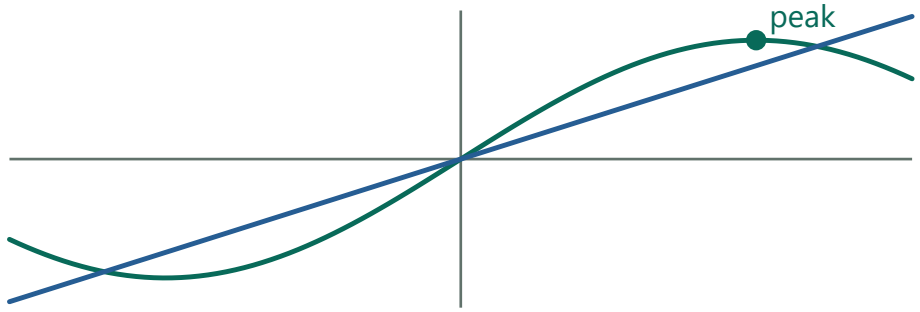
The equation is sin x=x/2. Since |sin x|≤1, any intersection satisfies |x|≤2. Both sides vanish at zero and are odd, so nonzero roots occur in opposite pairs.



Sine stays in [−1,1]. The line goes beyond that band when |x| > 2.

2. Count positive crossings in radians

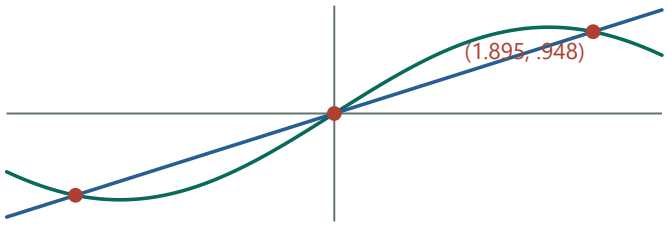
For g(x)=sin x−x/2, g′=cos x−1/2. It is positive from 0 to π/3 and negative from π/3 to 2. The curve starts at zero, rises above it, then falls below by 2, giving exactly one positive root.



On (0,2) the sine bends down while the line keeps constant slope.

3. Compute the coordinates

Bisection between 1.89549 and 1.89550 gives x≈1.895494267, y≈.947747134. The three intersections are this point, its negative, and (0,0).



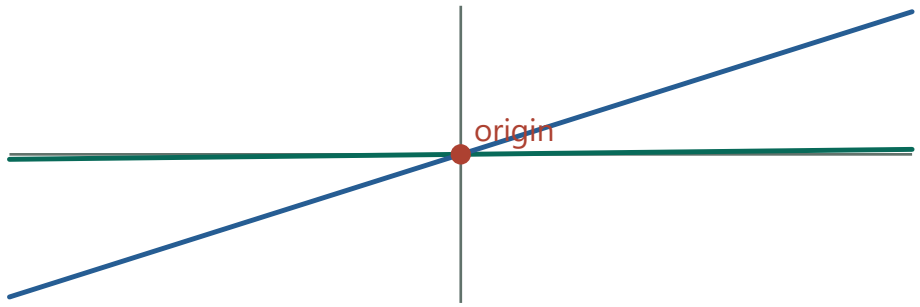
Odd symmetry supplies the negative partner of the positive crossing.

1. g(1.89549)≈0.000003495; g(1.89550)≈−0.000004695
Opposite signs bracket a root because g is continuous.

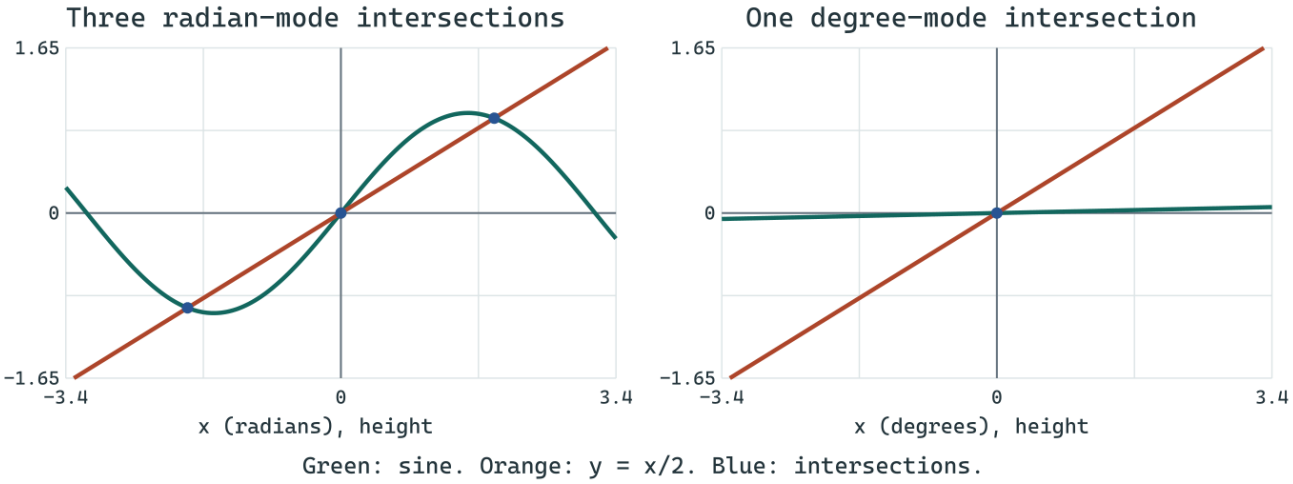
2. m=(1.89549+1.89550)/2=1.895495
Test the midpoint. Its negative sign retains the left half; continuing halves the uncertainty each time.

4. Change the angle convention

In degree mode the left side is sin(πx/180). For x≠0 its magnitude is at most π|x|/180, strictly smaller than |x|/2. Thus only the origin survives.



In degree mode sine is much flatter near zero.



Check. The positive root lies between 1.89549 and 1.89550, where sin x−x/2 changes sign.

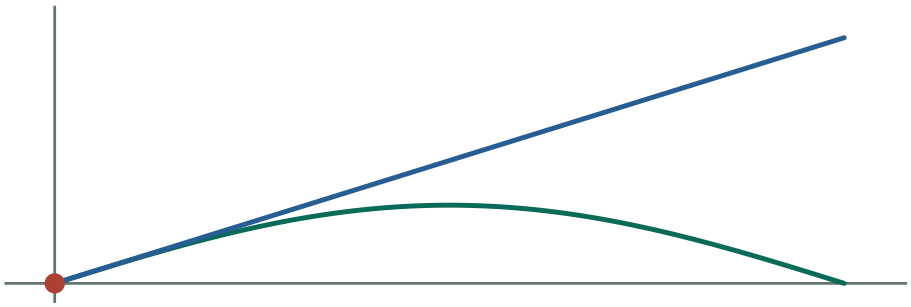
Exercise 831 · Random lines through the origin

Printed page 92 · Probability / Slopes and tangent lines / Sinusoidal functions

(Continuation) Again working in radian mode, draw the graph of $y = \sin x$. Then consider all straight lines of *positive* slope that can be drawn through the origin. If one of these lines is randomly chosen, what is the probability that it will intersect the sine graph more than once?

1. Characterize slopes too steep to cross again

Write the line as y=mx with m>0. In radians sin x<x for x>0, so if m≥1 the line lies above the positive sine graph. Odd symmetry then leaves only the origin.

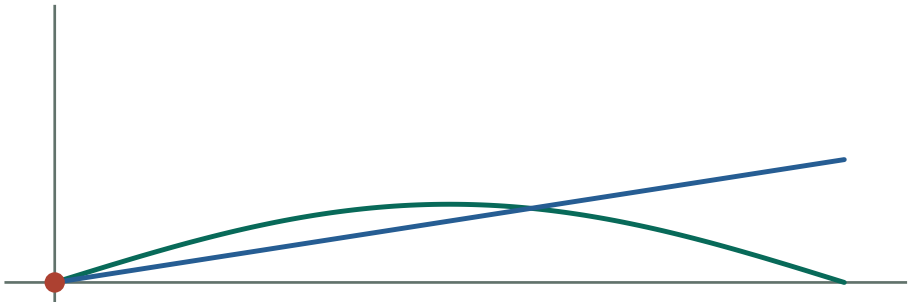


The slope-one line bounds sin x above for positive x.

2. Characterize the slopes that do cross

If 0<m<1, sine has larger initial slope and lies above mx just right of zero. At π its value 0 lies below mπ. Continuity forces another crossing and its reflected negative one.

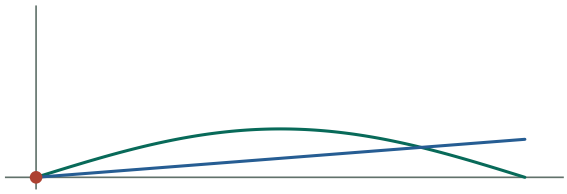
Positive slope below one · Slope 0.5



A slope below one begins below sine and ends above it by x = π.

Compare settings · Positive slope below one

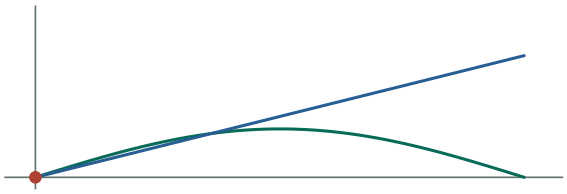
Slope 0.25



The blue line has slope 0.25.

The slope 0.25 is between zero and one. At π the line has height 0.785, above sine's zero.

Slope 0.8

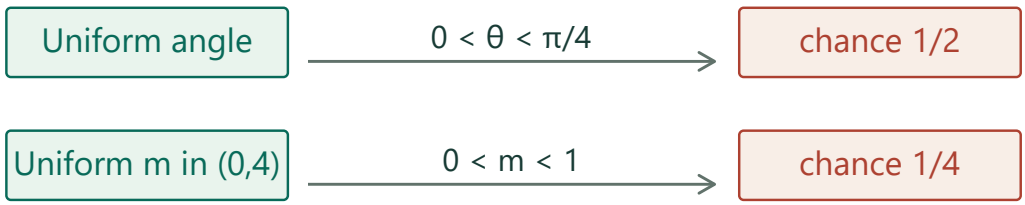


The blue line has slope 0.8.

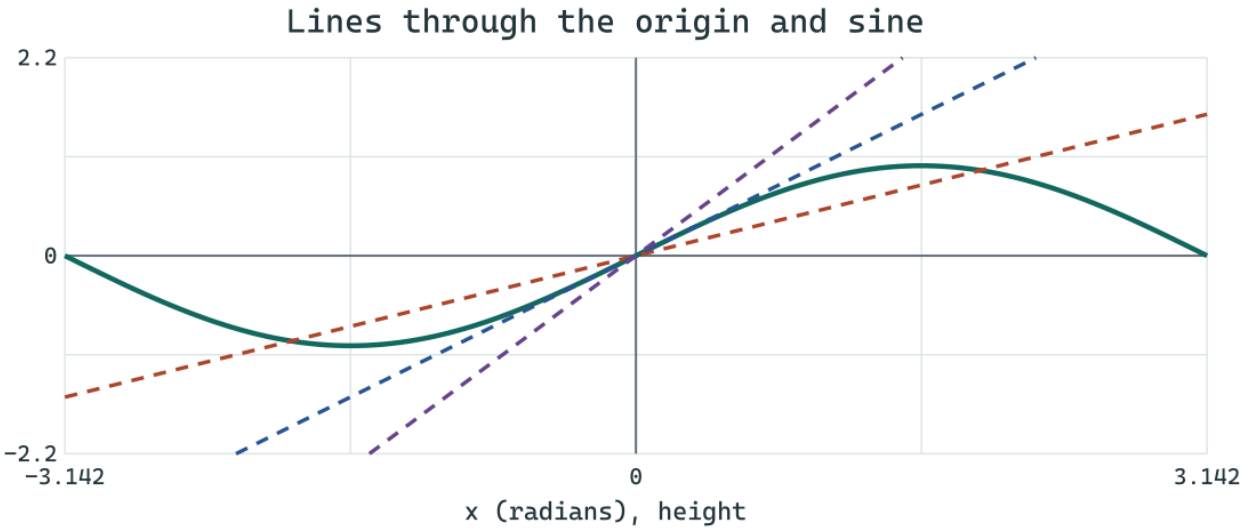
The slope 0.8 is between zero and one. At π the line has height 2.513, above sine's zero.

3. Specify how the line is chosen

A uniform random direction angle θ in (0,π/2) has m=tan θ<1 exactly when θ<π/4, an interval of half the total length. This model gives probability 1/2. A different rule changes the probability: for uniform slope on (0,4) the chance is 1/4.



Two different random-choice rules assign different probabilities.



Dashed slopes: orange 0.5, blue 1, purple 1.5. Only the orange line crosses again.

Check. Uniform m on (0,2) also gives 1/2, while uniform m on (0,4) gives 1/4. The event is fixed but its probability depends on the sampling model.

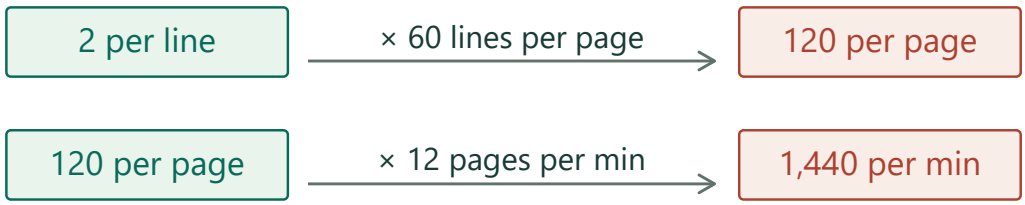
Exercise 832 · Time to print every alphabet arrangement

Printed page 92 · Counting and permutations / Unit conversion

There are 26! permutations of the word *abcdefghijklmnopqrstuvwxyz*. At two permutations per line and 60 lines per page and 12 pages per minute, how much time will be needed to print them all?

1. Compute the printer's throughput

Two permutations per line, 60 lines per page and 12 pages per minute give $2\cdot 60\cdot 12 = 1440$ permutations/minute.



Use units to build the printer's rate.

2. Divide total work by rate

All 26 letters are distinct, so there are 26! permutations. Time required is $26!/1440 \approx 2.8006 \times 10^{23}$ minutes.



Time equals the total number of permutations divided by the rate.

3. Convert the time scale

Using $60\cdot 24\cdot 365.25$ minutes per year, divide again to obtain about 5.325×10^{17} years.



1. 1 year$\approx 365.25\cdot 24\cdot 60 = 525960$ minutes
Multiply days per year by hours per day and minutes per hour.

2. years$= 26!/(1440\cdot 525960) \approx 5.325\times 10^{17}$
Divide the arrangement count by the number printed in one year; the time units cancel.

Convert from minutes to years using 365.25 days per year.

Check. Multiplying the unrounded time in years by the minutes per year and permutations per minute recovers 26!.

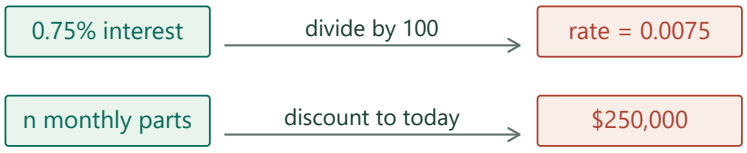
Exercise 833 · Two mortgage durations

Printed page 92 · Loan amortization / Compounding

Sasha takes out a \$250 000 mortgage loan. The bank charges 0.75 percent monthly interest on the unpaid balance. Calculate the monthly payments and the total amount that Sasha will pay the bank, assuming that the loan duration is **(a)** 30 years; **(b)** 15 years.

1. Use the monthly rate consistently

The problem gives .75% per month, so r=.0075. For n end-of-month payments, the sum of discounted payments is P[1−(1+r)^(−n)]/r; set it equal to $250,000.



All future installments are valued at the loan date.

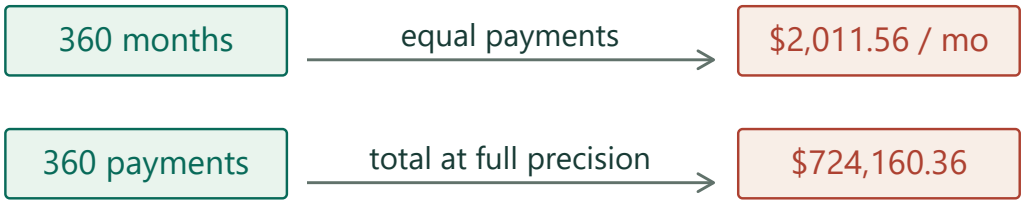
1. $q=1+r$; $G=q^{-1}+\cdots+q^{-n}$
Discount each month's payment to its value at the start of the loan.

2. $qG-G=1-q^{-n} \Rightarrow G=(1-q^{-n})/r$
Shifting the geometric sum cancels the overlapping terms, and the given r is positive.

3. $PG=250000 \Rightarrow P=250000r/(1-q^{-n})$
The payments fund the borrowed principal. The discount sum G is positive so division is valid.

2. Apply the 30-year term

Here $n=30\cdot12=360$. The payment is $P=250000(.0075)/(1-1.0075^{-360})\approx\$2{,}011.56$. Using the unrounded P, the 360 payments total $724,160.36.



The 30-year term has 360 monthly installments.

3. Apply the 15-year term

Now n=180, giving $P\approx\$2{,}535.67$ and total $456,419.96 computed before rounding installments. The larger monthly payment shortens the time interest accrues. A real schedule can adjust the last payment for cents.



The shorter term repays principal faster and accumulates less interest.

Check. Discounting either payment stream to time zero returns $250000. The shorter loan costs less in total despite its higher monthly payment.

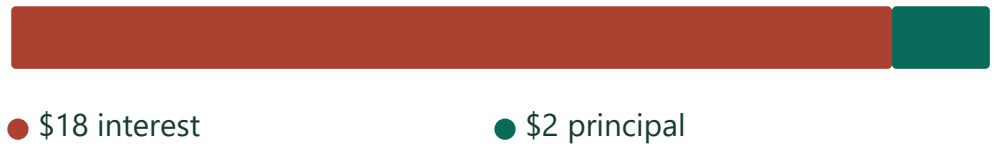
Exercise 834 · Minimum payments on a credit card

Printed page 92 · Loan amortization / Sequences and recursion

Tyler buys a new laptop for \$1200, and pays for it using a credit card. This particular card charges 1.5 percent monthly interest on the unpaid balance, and requires a minimum payment of \$20 each month. Suppose that Tyler pays only the minimum amount each month. How long will it take Tyler to pay off the debt in this way, and how much will the bank eventually receive for its \$1200 loan to Tyler?

1. Read the starting month's cash flow

The first month's interest is .015·1200=\$18. Of the first \$20 payment only \$2 reduces principal. Let A_n be the balance after n payments; then $A_n = 1.015A_{n-1} - 20$.



Split the first \$20 payment using the \$1,200 starting balance.

2. Solve for the payoff time

The geometric recurrence gives $A_n = 1200(1.015)^n - 20[(1.015)^n - 1]/.015$. Setting zero yields $(1.015)^n = 10$, hence $n = \ln 10/\ln 1.015 \approx 154.654$. The final payment falls in month 155.



The continuous root lies between payment dates 154 and 155.

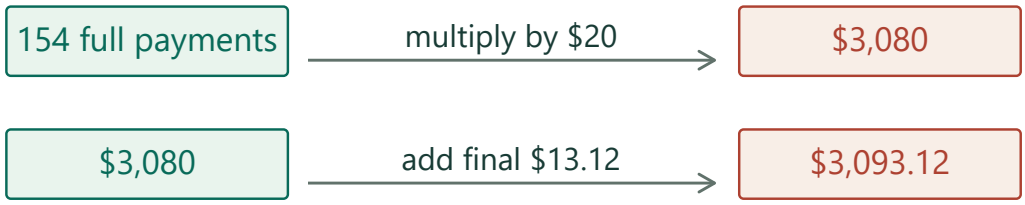
1. $u=(1.015)^n$; $0=18u-20(u-1)$
At payoff multiply the balance equation by .015, using .015·1200=18.

2. $0=20-2u \Rightarrow u=10$
Distribute −20 and combine the two terms in u.

3. $n\ln(1.015)=\ln(10)$
Take logs of the positive quantities. Dividing by the positive ln(1.015) gives the fractional crossing time.

3. Compute the final partial payment

After 154 full installments multiply the outstanding A_{154} by 1.015 once for final interest. The amount due is about \$13.12. The bank receives $154 \cdot \$20 + \$13.12 \approx \$3{,}093.12$ with full-precision accrual.



Only the net amount due is paid in the final month.

Check. The first \$20 payment covers \$18 interest and reduces principal by only \$2. An additional full \$20 payment in month 155 would exceed the debt.

Exercise 835 · An inverse cosine folded into one interval

Printed page 92 · Inverse trigonometric functions / Periodic functions

Let $f(x) = \cos^{-1}(\cos x)$, which is valid for all x. Sketch the graph of this periodic function f. Find its period and the range of its values $f(x)$.

1. Use the principal cosine angle

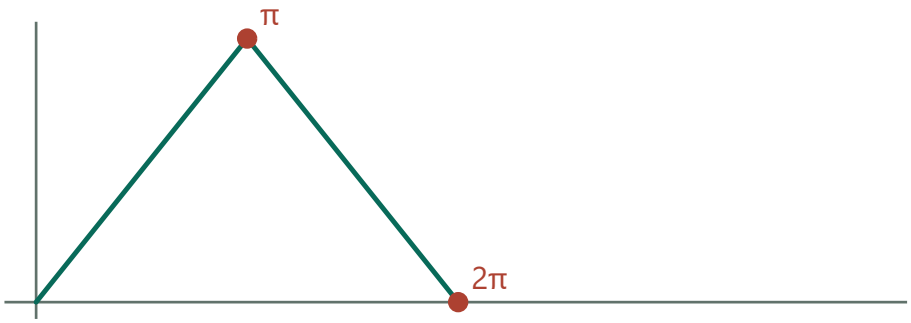
arccos always returns an angle in [0,π]. If x already lies between 0 and π, f(x)=arccos(cos x)=x.



Angles from 0 through π already lie in the principal output range.

2. Fold the second half-cycle

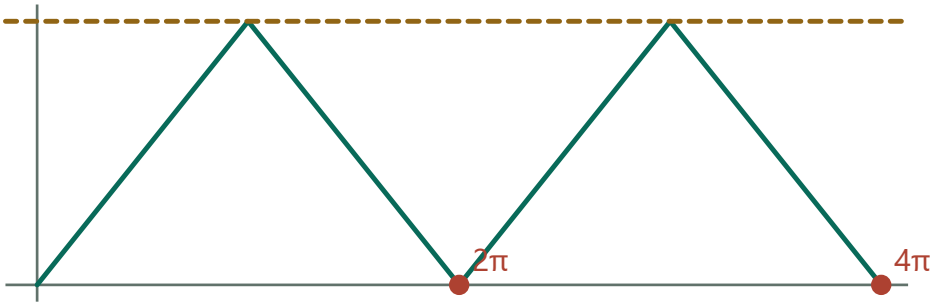
For $\pi \le x \le 2\pi$, the angle $2\pi - x$ lies in $[0,\pi]$ and has the same cosine as x. Thus $f(x) = 2\pi - x$ there. Draw a line rising from $(0,0)$ to (π,π), then falling to $(2\pi,0)$.



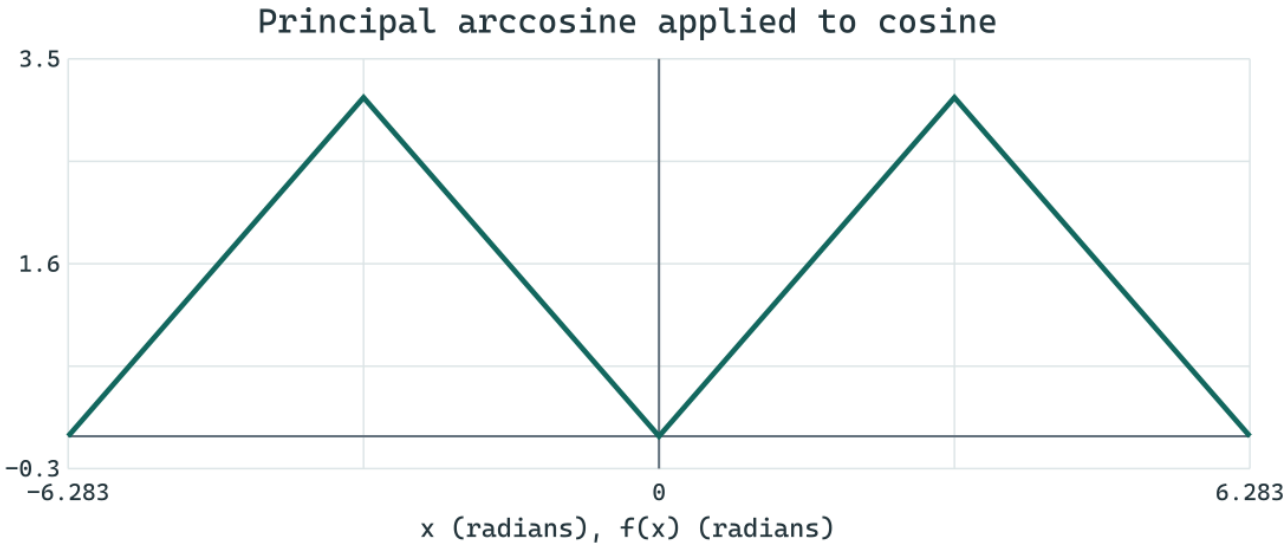
The second half-cycle folds back down.

3. Repeat and read the range

Cosine repeats every 2π; copy this triangle in both directions. The range is $[0,\pi]$. Consecutive minima are exactly 2π apart, so the least positive period is 2π.



The fold repeats every 2π and remains between 0 and π.



Check. Every output has the original cosine and lies in $[0,\pi]$. At $x = 3\pi/2$ it returns $\pi/2$.

Exercise 836 · A pair of reciprocal logarithms

Printed page 93 · Change of base / Logarithm rules

Simplify $(\log_a b)(\log_b a)$.

1. Determine the shared domain

The first logarithm uses a as base and b as argument. The second exchanges those roles. Each symbol must therefore pass both the base test and the input test.

1 **$\log_a b$ exists $\Leftrightarrow$ $a > 0, a \neq 1, b > 0$**
A logarithm needs a positive base other than 1 and a positive argument.

2 **log_b a exists $\Leftrightarrow$ $b > 0, b \neq 1, a > 0$**
Exchange base and argument for the second logarithm.

3 **$a > 0,\quad b > 0,\quad a \neq 1,\quad b \neq 1$**
Intersect the two sets of conditions.

2. Rewrite and cancel

Change both logs to natural logs. Their nonzero numerators and denominators appear in opposite positions, making the two factors reciprocals.

1 **$\log_a b = (\ln b)/(\ln a)$**
The change-of-base rule uses ln a in the denominator.

2 **log_b a = (ln a)/(ln b)**
Interchanging the base and input inverts the quotient.

3 **$(\log_a b)(\log_b a) = (\ln b \cdot \ln a)/(\ln a \cdot \ln b)$**
Multiply the two fractions.

4 **$(\log_a b)(\log_b a) = 1$**
Both logs being cancelled are nonzero on the shared domain.

Check. If $a^x=b$ then $b^{(1/x)}=a$, so the two exponents are reciprocals.

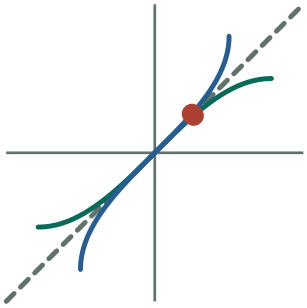
Exercise 837 · A window for sine and inverse sine

Printed page 93 · Inverse trigonometric functions / Inverse functions / Graph interpretation

Sasha tried to graph both $y = \sin x$ and $y = \sin^{-1} x$ on the same coordinate-axis system, using degree mode for the calculations. Sasha found this exercise to be difficult and confusing. Explain why it is clearer to use radian mode. How large a graphing window is needed to see all the necessary detail?

1. Track how coordinates exchange

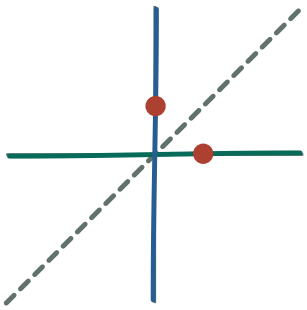
On the principal sine branch, pairs (angle,height) switch to (height,angle) for arcsin. Equal horizontal and vertical scales make that reflection across y=x accurate.



Swap the red point's coordinates to obtain the other red point.

2. Compare degree scales

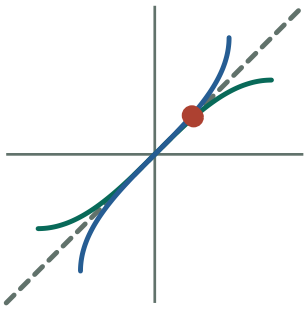
In degrees the extreme principal angle is ±90 while the sine heights lie only between −1 and 1. A square wide and tall enough for both curves stretches 180 units, compressing all heights or inputs of interest into a two-unit strip.



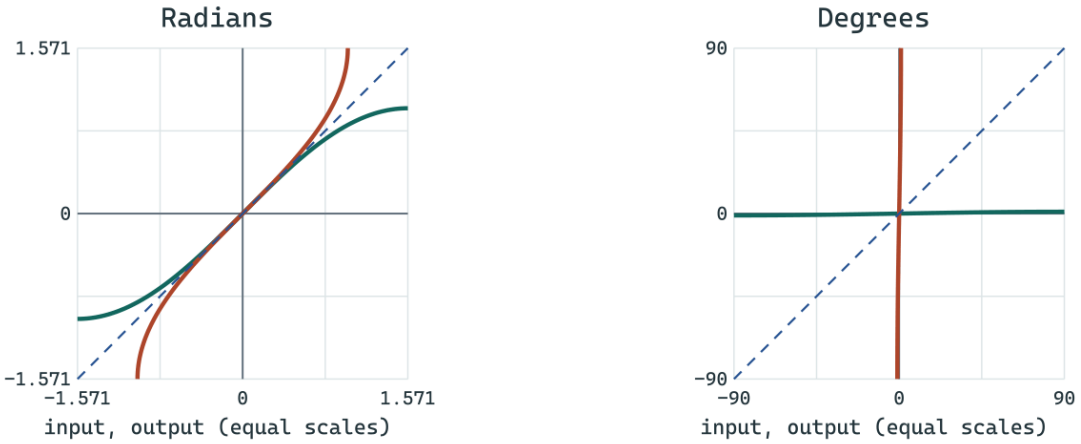
With axes in [−92,92], a height of one occupies a narrow strip.

3. Choose a radian window

In radians the principal angles span −π/2 to π/2. An equal-scale square $[-\pi/2,\pi/2]^2$ contains both full principal branches; $[-2,2]^2$ adds margin. Points such as (π/6,1/2) and (1/2,π/6) visibly reflect into each other.



A square [−2,2] window fits both principal branches in radians.



Green: principal sine branch. Orange: inverse sine. The degree window compresses detail.

Check. In the radian square, the points (π/6,1/2) and (1/2,π/6) visibly mirror each other across y=x.

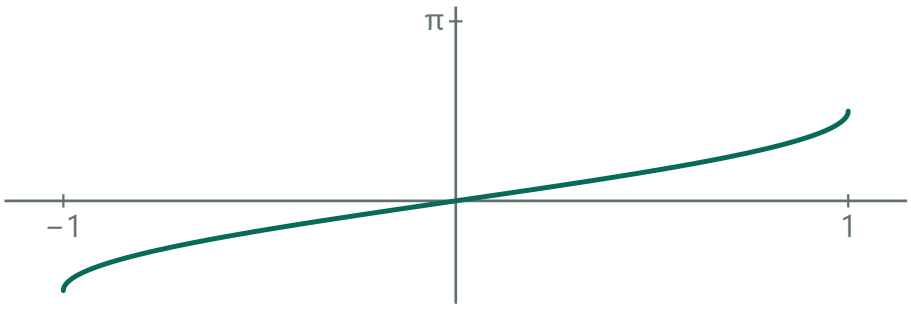
Exercise 838 · Complementary inverse angles

Printed page 93 · Inverse trigonometric functions / Trigonometric identities

Working in radian mode, graph both $y = \sin^{-1} x$ and $y = \frac{\pi}{2} - \sin^{-1} x$ on the same coordinate-axis system. What is the customary name for the second function?

1. Identify the complementary angle

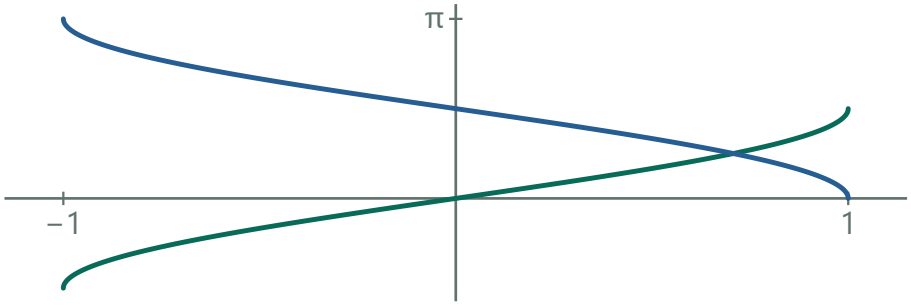
For $-1 \le x \le 1$ let $\theta = \arcsin x$, so $\sin\theta = x$ and $-\pi/2 \le \theta \le \pi/2$. The cofunction identity gives $\cos(\pi/2 - \theta) = x$.



Use the signed principal angle as the first height.

2. Verify that the principal branch matches

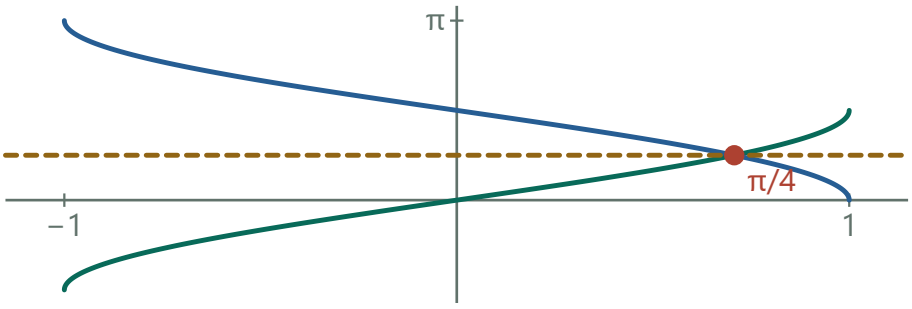
The angle $\pi/2 - \theta$ lies in $[0,\pi]$, precisely the range of arccos. Thus $\pi/2 - \arcsin x = \arccos x$ over the whole domain.



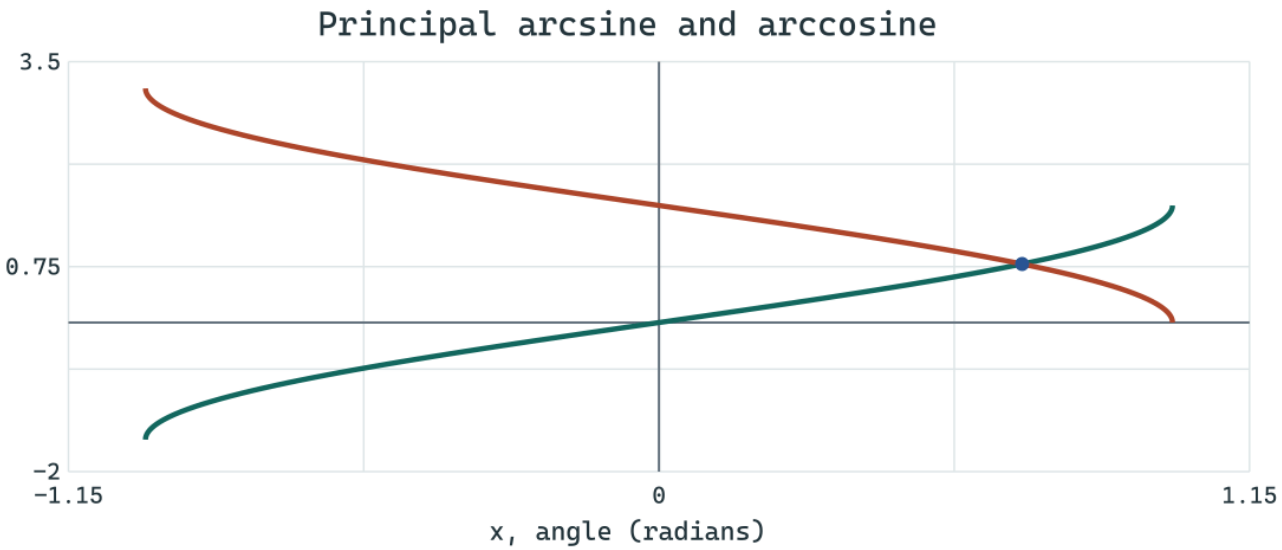
Subtract that height from $\pi/2$ to obtain the complementary angle.

3. Sketch and compare

arcsin increases from $-\pi/2$ to $\pi/2$ and arccos decreases from π to 0. Each pair of heights averages $\pi/4$, so the graphs reflect across $y = \pi/4$. They meet when $\theta = \pi/4$, at $(\sqrt{2}/2, \pi/4)$.



Paired heights average $\pi/4$. Equality occurs at $x = \sqrt{2}/2$.



Green: arcsin x. Orange: arccos x. They meet at (√2/2, π/4).

Check. At x=0 the values are 0 and π/2. At x=1 they are π/2 and 0.

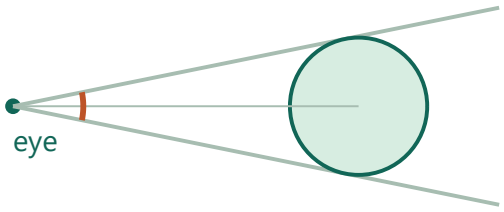
Exercise 11-1 · Two disks in the sky

Printed page 115 · Angular diameter / Similarity and scaling / Right-triangle trigonometry

Why might an Earthling believe that the sun and the moon are the same size?

1. What a distant object fills in your view

Imagine drawing two rays from your eye, one grazing each edge of the visible disk. The angle between the rays is the apparent width. Moving an object farther away narrows that angle even when its actual diameter stays fixed.



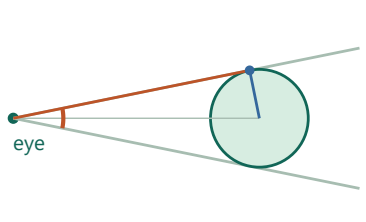
Sight lines

Edges define the angle.

Both rays start at the eye. Their opening measures apparent diameter.

2. Construct a right triangle

Join the eye to the center and to one point where a ray grazes the sphere. The radius to that point is perpendicular to the grazing ray. The side opposite half the viewing angle is the radius r. The hypotenuse is the center distance d, so sin(half-angle)=r/d.



Tangent ⊥ radius

sin(half angle) = r/d

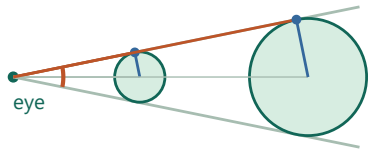
1. sin(θ/2) = r/d ⇒ θ/2 = arcsin(r/d)
Use the tangent right triangle, with the eye outside the sphere so 0 < r/d < 1.

2. θ = 2 arcsin(r/d)
Double the angle at the eye to span both sides of the visible disk.

The radius ends at the point of tangency, making a right triangle.

3. Compare the two ratios

The Sun is roughly 400 times as wide as the Moon and roughly 400 times as far away. Those two factors cancel in r/d. The two viewing angles are therefore very similar even though the physical sizes differ enormously.



Same opening

Equal r/d gives equal angle.

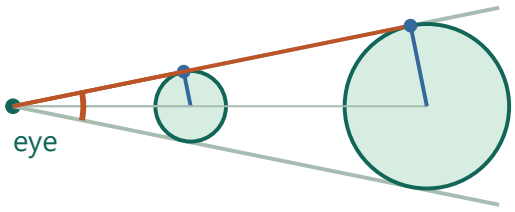
1. sin(θS/2) = (400r)/(400d)
Use the same approximate scale factor for solar radius and distance.

2. (400r)/(400d) = r/d = sin(θM/2)
Cancel 400. Both half-angles are acute, so their sines determine their sizes.

Increasing radius and center distance by the same factor preserves both bounding rays. The distances here are schematic.

4. Account for an eclipse

Both disks span about half a degree. Their angular sizes vary as orbital distances change.



Along the same sight line

The nearer disk may cover

the more distant disk.

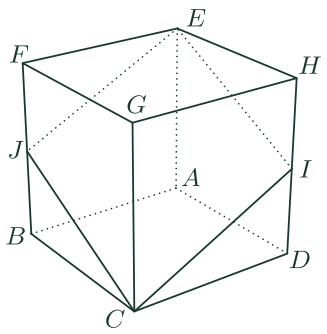
When the centers line up in the sky, the nearer disk can conceal the farther one; their angular sizes vary slightly with distance.

Check. A solar eclipse makes the comparison directly: the Moon can cover almost exactly the visible solar disk.

Exercise 27-1 · A tilted slice through a cube

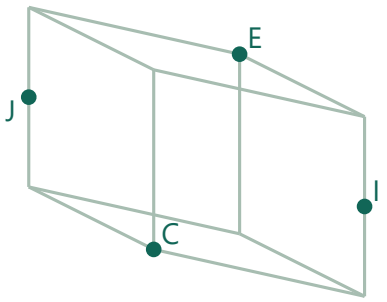
Printed page 115 · Cross sections / Triangle area / Three-dimensional geometry

The figure at right shows a $2 \times 2 \times 2$ cube $ABCDEFGH$, as well as midpoints I and J of its edges DH and BF. It so happens that C, I, E, and J all lie in a plane. Can you justify this statement? What kind of figure is quadrilateral $CIEJ$, and what is its *area*? Is it possible to obtain a polygon with a larger area by slicing the cube with a different plane? If so, show how to do it. If not, explain why it is not possible.



1. Pick coordinates along cube edges

Use C as the origin and three perpendicular edge directions as axes. Each edge has length 2. A midpoint lies halfway along its edge, so its changing coordinate differs by 1 from the corresponding endpoint. Choose axes so the four points are C=(0,0,0), I=(2,0,1), E=(2,2,2), J=(0,2,1).



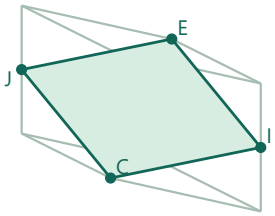
Edge length 2

I and J are midpoints.

The two midpoints sit one unit up their two-unit edges.

2. Identify the plane and its sides

Check z=(x+y)/2 for every one of the four points. This linear equation describes a single plane. Along CI and JE the displacement is (2,0,1). Along CJ and IE it is (0,2,1). These two side vectors have the same length √5, making the parallelogram a rhombus.



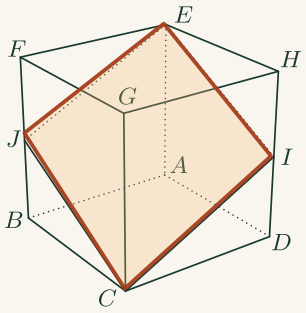
z = (x + y)/2

Opposite sides are parallel.

1. CI = (2,0,1), CJ = (0,2,1)
Subtract C from each neighboring vertex.

2. CI·CI = 4+0+1 = 5 = CJ·CJ
Both side lengths are √5. The opposite edges are parallel translations.

Joining C–I–E–J makes a plane section. The two different side directions both have length √5.

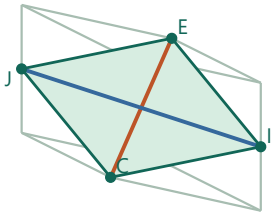


Section C–I–E–J

Trace the slice in boundary order. Two of its vertices are cube corners and the other two are halfway along vertical edges.

3. Measure its diagonals

CE has length $\sqrt{(2^2+2^2+2^2)}=2\sqrt{3}$. IJ has length $\sqrt{(2^2+2^2)}=2\sqrt{2}$. The diagonal vectors (2,2,2) and (−2,2,0) have dot product zero. Thus the four small right triangles formed by the diagonals have total area half their product, or 2√6.



CE = 2√3

IJ = 2√2

Area = ½ · CE · IJ

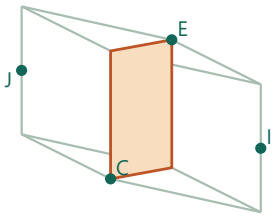
1. A = 4 × ½ × (CE/2) × (IJ/2)
The perpendicular diagonals bisect one another, making four right triangles.

2. A = (2√3)(2√2)/2 = 2√6
Cancel one factor of two and combine the square roots.

The diagonals meet at right angles; their four right triangles account for half the product of diagonal lengths.

4. Exhibit a larger slice

Slice vertically along a diagonal of the square floor. The section is a rectangle of height 2 and width $2\sqrt{2}$, so its area is $4\sqrt{2}$. Both areas are positive and $(4\sqrt{2})^2=32$ exceeds $(2\sqrt{6})^2=24$. One example settles the existence question.



Vertical diagonal slice

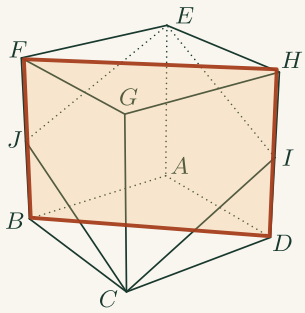
Width $2\sqrt{2}$, height 2

Area $4\sqrt{2} > 2\sqrt{6}$

1. $A_{vertical}/A_{tilted} = (4\sqrt{2})/(2\sqrt{6}) = 2/\sqrt{3}$
Divide the two positive areas and cancel $\sqrt{2}$.

2. $2/\sqrt{3} > 1$ since $4 > 3$
Squaring positive numbers preserves their order, so the vertical cut is larger.

Replace the rhombus with a vertical rectangle. The full base diagonal supplies its longer side.



Alternative vertical slice

Follow a diagonal across the floor and lift its two ends straight up to the top face. The resulting rectangle can be compared with the sloping section.

Check. The side vectors $(2,0,1),(0,2,1)$ have cross product $(-2,-2,4)$, of length $2\sqrt{6}$.

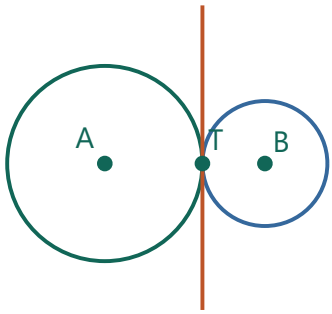
Exercise 31-1 · One tangent, two radii

Printed page 115 · Circle tangency / Orthogonality

Circles centered at A and B are tangent at T. Prove that A, T, and B are collinear.

1. Use the same tangent at T

Picture the straight line touching both circles at their common contact point T. A radius drawn to a point of tangency is perpendicular to the tangent. Draw AT and BT.



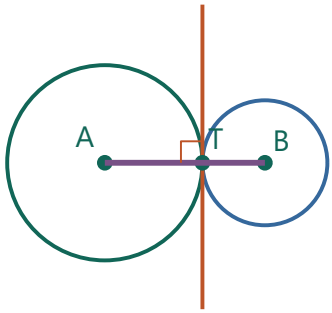
One point of contact

One common tangent.

A common tangent touches both circles at T.

2. Recover the line of centers

Through T there is just one line perpendicular to the tangent. Both AT and BT must lie on that line. Therefore A, T and B are collinear.



Radii to T

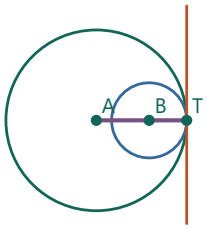
Both are normal to

the same straight line.

Draw both radii to the point of contact. The perpendicular through T is unique, so the centers lie on the same line.

3. Check both kinds of tangency

For externally tangent circles T lies between the centers and AB is the sum of the radii. For internally tangent circles the centers are on the same side of T and AB is the difference of the radii. Both arrangements give the same collinearity.



Internal contact

AB = larger r − smaller r

1. T = 0, A = −r, B = s ⇒ AB = s−(−r) = r+s
Use signed coordinates on the common perpendicular. The external centers are on opposite sides.

2. T = 0, A = r, B = s ⇒ AB = abs(s−r)
For internal tangency the coordinates have the same sign; take the nonnegative distance.

Moving the small circle inside reverses the center order along that line; the center separation changes from a sum to a difference.

Check. For external tangency the center separation equals the sum of the radii; equality in the triangle inequality again forces collinearity. Internal tangency gives their difference.

Exercise 122-1 · Two ways around the circle

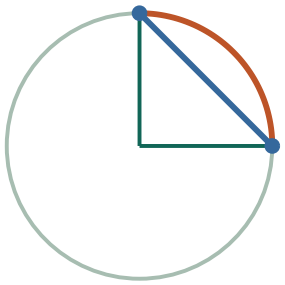
Printed page 115 · Chords and inscribed angles / Arc length / Graph interpretation

The table at right shows measurements made on a circle with a one-meter radius. Each entry in the s-column is an arc length, and the adjacent entry in the c-column is the corresponding chord length, both in meters. Explain why $c < s$, and determine the range of values for c and for s. With s on the horizontal axis and c on the vertical axis, sketch an approximate graph of c vs s.

s	c
0.256	0.255
0.618	0.608
1.234	1.157
1.571	1.414
1.896	1.625

1. Read one measurement

The s-column tracks distance traveled along the circle. The c-column measures the direct segment joining the same endpoints. For example 1.571 m around the rim corresponds to a shorter segment of about 1.414 m.



Radius 1 m

Orange: arc s

Blue: chord c

The rim and straight segment join the same pair of endpoints but measure different paths.

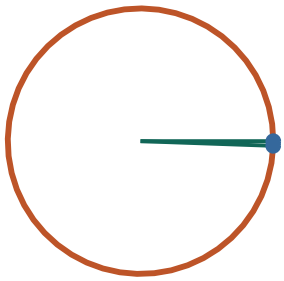
s	c
0.256	0.255
0.618	0.608
1.234	1.157
1.571	1.414
1.896	1.625

Arc length column

Read the left column as travel along the circumference. Its entry belongs on the horizontal axis of the requested graph.

2. Compare the limits of travel

Arc and chord both have length zero at s=0. Every nonzero arc is longer than its chord: for distinct endpoints the straight segment is the shortest path, and a full turn has positive length with chord zero. For one traversal $0 \le s \le 2\pi$ and $0 \le c \le 2$. For minor arcs alone $0 \le s \le \pi$.



Near one full turn

Arc approaches 2π m.

Chord shrinks toward zero.

Keep the first endpoint fixed and move the other almost all the way around. The orange length grows while the endpoints come together again.

s	c
0.256	0.255
0.618	0.608
1.234	1.157
1.571	1.414
1.896	1.625

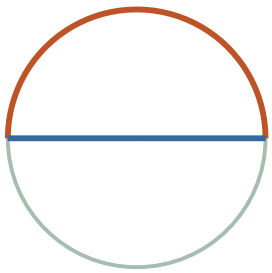
Chord length column

The right entry in each row measures the straight-line distance between the endpoints of the corresponding arc. Plot it vertically.

3. Locate the turning point

Start with both endpoints together. Moving one end to the far side increases the chord until it becomes the diameter, which is 2 m. Continuing around the other half shortens the chord to zero even as the arc length keeps increasing.

Central angle · 180°



Sweep 180°

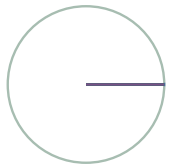
Arc s = 3.142 m

Chord c = 2.000 m

Put the endpoints opposite one another to attain the maximum possible chord.

Compare settings · Central angle

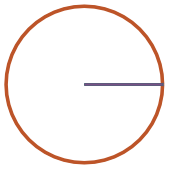
0°



Sweep 0°
Arc s = 0.000 m
Chord c = 0.000 m

The radius is one meter. At 0°, the arc is 0.000 m and its chord is 0.000 m.

360°

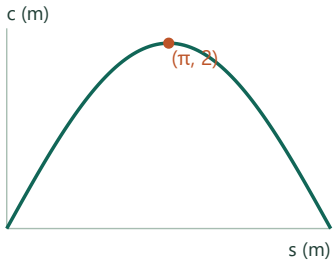


Sweep 360°
Arc s = 6.283 m
Chord c = 0.000 m

The radius is one meter. At 360°, the arc is 6.283 m and its chord is 0.000 m.

4. Place the measurements on the axes

Plot (0,0), (π/2,√2), (π,2), (3π/2,√2), (2π,0). Join with a smooth concave arch; the measurements sample its increasing portion.

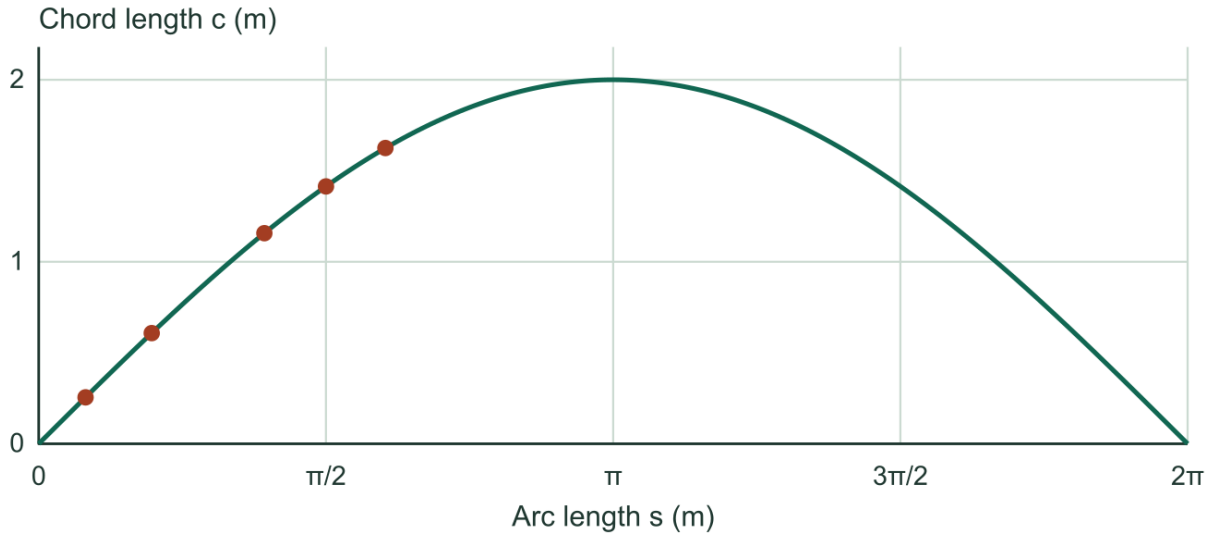


Start: c = 0
Half-turn: c = 2 m
Full turn: c = 0

1. $c^2 = 1^2 + 1^2 - 2\cos(s)$
For unit radii the central sweep is s radians; cosine gives the squared endpoint separation.

2. $c(\pi/2) = \sqrt{(2-2\cdot 0)} = \sqrt{2}$
Use the nonnegative root for length. The same cosine occurs at 3π/2.

Plot the two measured lengths on different axes. One sweep gives a rising and then falling graph.



Check. On a semicircle the arc is π and the chord is a diameter of length 2. Complementary major and minor arcs have the same chord.

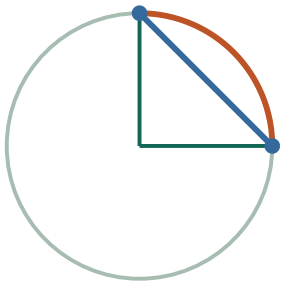
Exercise 122-2 · Half the central angle

Printed page 115 · Chords and inscribed angles / Arc length / Radian measure

(Continuation) Express s and c in terms of θ, the central angle that intercepts s and c. Combine these equations to express c as a function of s. Graph this relationship.

1. Turn distance into angle

Let θ be the angle swept by the radius in radians. The arc rule s=rθ gives s=θ on a circle of radius 1. Half a turn gives θ=π, a useful check on units.



Radius 1

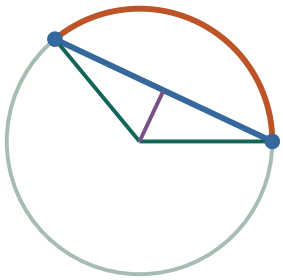
Arc length s = θ

θ is measured in radians.

A radius-one arc has the same numerical length as its central angle in radians.

2. Bisect the central triangle

For a minor arc, draw radii to its endpoints and drop a perpendicular to the chord midpoint. Each resulting right triangle has hypotenuse 1, opposite side c/2, and central angle θ/2. Hence c/2=sin(θ/2).



Two equal halves

Half-chord = sin(θ/2)

The perpendicular from the center bisects the chord, exposing the opposite side of a unit right triangle.

s	c
0.256	0.255
0.618	0.608
1.234	1.157
1.571	1.414
1.896	1.625

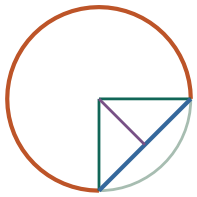
Chord length column · Figure from exercise 122-1

The right entry in each row measures the straight-line distance between the endpoints of the corresponding arc. Plot it vertically.

3. Extend the formula around the circle

For a major arc the smaller interior angle is $2\pi-\theta$. Its half-angle sine is $\sin(\pi-\theta/2)=\sin(\theta/2)$. Both routes between the same endpoints give $c=2\sin(s/2)$ on $0\le s\le 2\pi$.

Central angle · 270°



Sweep 270°

Arc s = 4.712 m

Chord c = 1.414 m

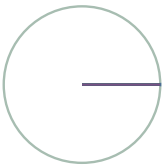
1. $c/2 = \sin((2\pi-\theta)/2) = \sin(\pi-\theta/2)$
Bisect the smaller angle when the sweep takes the long way around.

2. $c = 2\sin(\theta/2) = 2\sin(s/2)$
Supplementary angles have equal sines; multiply by two and use radius one.

The reflex angle and the minor angle give supplementary half-angles, hence the same half-chord.

Compare settings · Central angle

0°



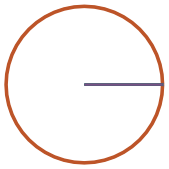
Sweep 0°

Arc s = 0.000 m

Chord c = 0.000 m

The radius is one meter. At 0°, the arc is 0.000 m and its chord is 0.000 m.

360°



Sweep 360°

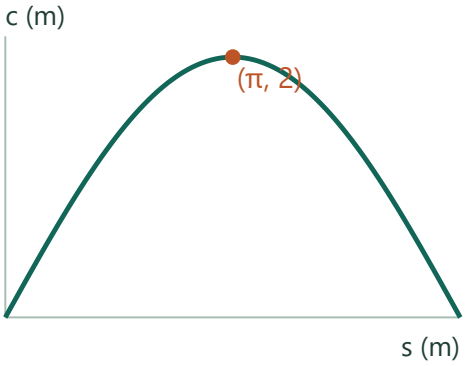
Arc s = 6.283 m

Chord c = 0.000 m

The radius is one meter. At 360°, the arc is 6.283 m and its chord is 0.000 m.

4. Read the graph

Its graph is a sine half-wave with maximum 2 at $s=\pi$.

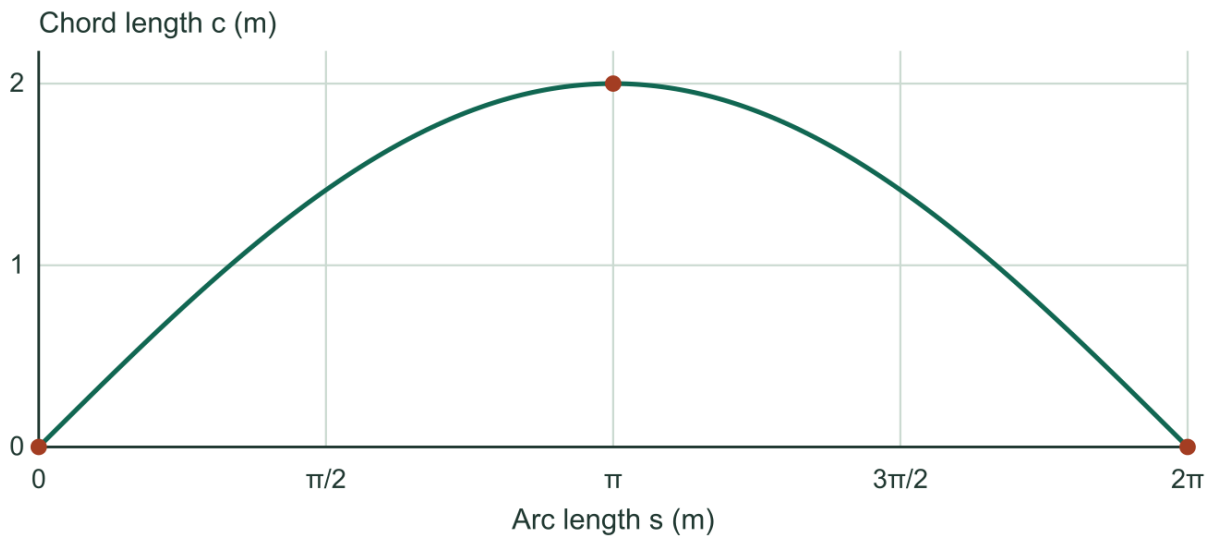


Start: c = 0

Half-turn: c = 2 m

Full turn: c = 0

Vary the swept angle continuously; doubling its half-angle sine yields the full chord.



Check. The cosine rule gives $c^2 = 2 - 2\cos\theta = 4\sin^2(\theta/2)$, which yields the same nonnegative length.

Exercise 168-1 · Tracking a tack

Printed page 115 · Circular motion / Radian measure / Unit conversion

The wheels on Devon's bike have r-inch radii. After the front wheel picks up a tack, Devon rolls another d feet and stops. How far above the ground is the tack?

1. Choose the tack's starting direction

At the instant the tire picks up the tack, the tack lies directly under the hub. The hub stays r inches above level ground. Relative to the hub the tack starts at vertical displacement −r.



At pickup

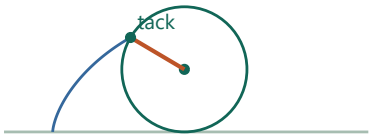
Hub height = r

Tack height = 0

The marked radius points down, placing the tack on the ground.

2. Measure the turn in radians

With rolling and no slipping, the distance traveled equals the arc length swept at the rim. Convert d feet to 12d inches. Since arc length is radius times angle, θ=12d/r.



No slipping

Distance = radius × angle

12d = rθ (inches)

1. rθ = 12d
Match the no-slip distance in inches to the rim arc in inches.

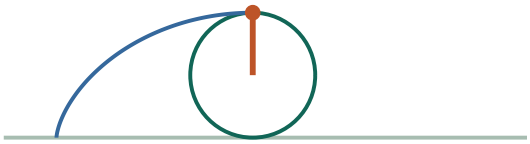
2. θ = 12d/r
Divide by the positive radius. The angle in the arc formula is in radians.

As the wheel rolls, the blue trace follows the tack while the hub travels rθ horizontally.

3. Project the rotating radius vertically

The starting radius points down, so after a turn θ its vertical component is −r cos θ. Add the hub height: h=r−r cos(12d/r) inches. At θ=0 it gives ground level. At θ=π it gives the top of the tire, 2r inches above the ground.

Wheel turn · 180°



Turn 180°

Height/r = 2.000

Travel/r = 3.142

Add the hub height to the vertical component of the rotating radius. At half a turn those two lengths add.

Compare settings · Wheel turn

0°



Turn 0°

Height/r = 0.000

Travel/r = 0.000

After 0°, travel equals 0.000 radii and the tack is 0.000 radii above the ground.

360°



Turn 360°

Height/r = 0.000

Travel/r = 6.283

After 360°, travel equals 6.283 radii and the tack is 0.000 radii above the ground.

Check. A half-turn gives 2r, and one full turn returns the tack to the road.

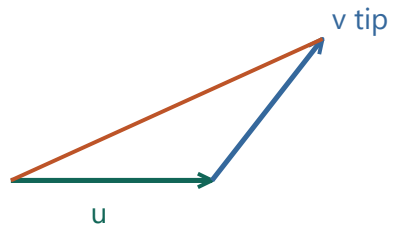
Exercise 191-1 · The triangle inequality

Printed page 115 · Vectors and components / Inequalities / The dot product

Use the vector form of the Law of Cosines to show that $|\mathbf{u}+\mathbf{v}|^2 \leq (|\mathbf{u}|+|\mathbf{v}|)^2$ holds for any vectors $\mathbf{u}$ and $\mathbf{v}$. What does this prove?

1. Compare squared lengths

Both lengths are nonnegative, so an inequality between their squares gives the same ordering. Replace the squared length of a vector by its dot product with itself.



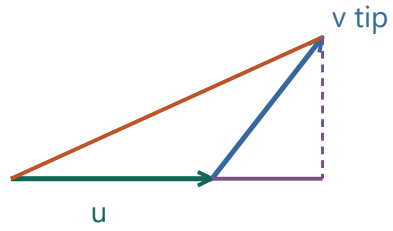
Head to tail

Orange = resultant

Placing v at the tip of u makes u+v the direct route from start to finish.

2. Expand the dot product

Distribute $(u+v)\cdot(u+v)$. The two mixed terms agree because $u\cdot v=v\cdot u$, so the result is $\|u\|^2+2u\cdot v+\|v\|^2$.



Squared length

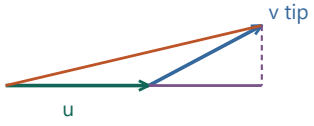
Two cross terms add

to 2u·v.

The squared length includes a mixed term determined by the angle between the two arrows.

3. Bound the only mixed term

For nonzero vectors the cosine rule gives $u\cdot v=\|u\|\|v\|\cos\theta$. Cosine is at most 1. Substituting the greatest possible mixed term gives exactly $(\|u\|+\|v\|)^2$.



Reduce the bend

A smaller angle increases

the resultant length.

1. $2u\cdot v \le 2\|u\|\|v\|$
The cosine factor is at most one and the two lengths are nonnegative.

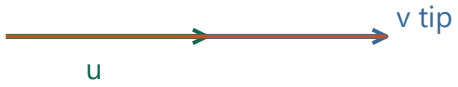
2. $\|u+v\|^2 \le \|u\|^2+2\|u\|\|v\|+\|v\|^2$
Replace only the mixed term by its upper bound.

3. $\|u+v\|^2 \le (\|u\|+\|v\|)^2$
Factor the perfect-square trinomial.

For fixed lengths, straighten the angle to increase the mixed term toward its largest possible value.

4. Interpret the bound

Taking square roots yields ‖u+v‖≤‖u‖+‖v‖, the triangle inequality. A zero vector gives equality immediately. Otherwise equality requires cos θ=1, so the two arrows point the same way.



Equality

Same direction gives

‖u + v‖ = ‖u‖ + ‖v‖.

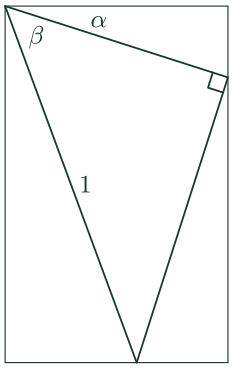
When the two arrows point the same way the direct route and the broken route have equal length.

Check. In a head-to-tail construction a direct segment cannot exceed the length of the two-leg path.

Exercise 230-1 · Reconstructing the angle-addition identities

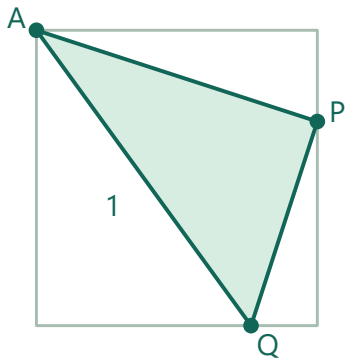
Printed page 115 · Angle addition formulas / Right-triangle trigonometry / Vectors and components

The rectangle shown has been formed by fitting together four right triangles. As marked, the sizes of two of the angles are α and β (*Greek* "alpha" and "beta"), and the length of one segment is 1. Find the two unmarked angles whose sizes are α and $\alpha+\beta$. By labeling all the segments of the diagram, discover formulas for $\sin(\alpha+\beta)$ and $\cos(\alpha+\beta)$, written in terms of $\sin\alpha$, $\cos\alpha$, $\sin\beta$, and $\cos\beta$.



1. Identify the unit segment

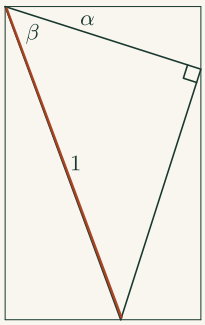
Call the rectangle corners A,B,C,D clockwise from the upper left, P the point on the right edge, and Q the point on the bottom edge. The diagonal segment AQ has length 1. The central triangle APQ has its right angle at P.



AQ = 1

Right angle at P.

Mark the unit hypotenuse and the right-angled broken route A–P–Q.

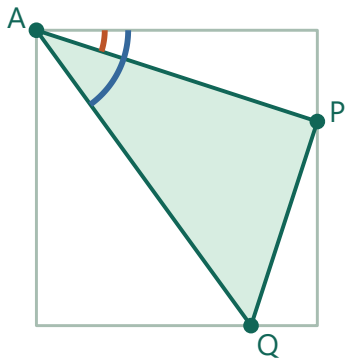


Unit hypotenuse AQ

The top-left point and the point on the bottom edge are joined by the segment labeled one. Its horizontal and vertical displacements are the two quantities sought.

2. Recover the two missing angles

At the top left the unit diagonal makes α+β with the top edge. The acute angle between the lower leg and the right side at their intersection is α, since both sides of that angle are perpendicular to the sides of the marked α.

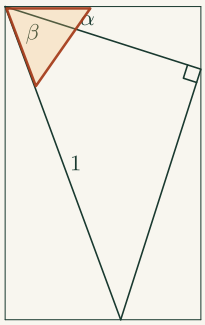


Outer angle at A

α from the top to AP;

α + β from top to AQ.

Compare each ray with the top horizontal edge. Adding the two adjacent angles gives the angle of AQ.

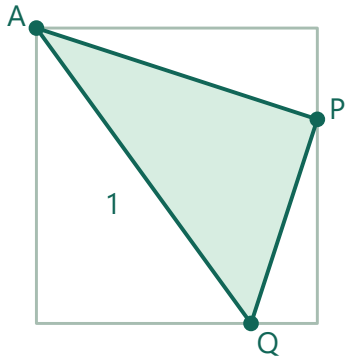


Combined angle at A

Compare the top of the rectangle with the unit diagonal. The shallower diagonal from A splits this full angle into the two angles printed in the drawing.

3. Read the two legs

At A the angle between AP and AQ is β. In triangle APQ the adjacent leg AP is cos β times the unit hypotenuse, while the opposite leg PQ is sin β times the unit hypotenuse.



Central triangle

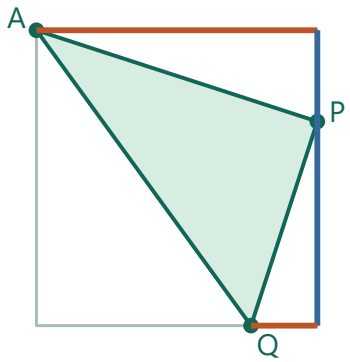
AP = cos β

PQ = sin β

With AQ fixed at one, the legs adjacent and opposite to β have lengths cos β and sin β.

4. Resolve AP and PQ

Use rightward and downward as positive directions. AP moves right by cos β cos α and down by cos β sin α. PQ points left and down. Its horizontal contribution is −sin β sin α, and its vertical contribution is sin β cos α.



Resolve both legs

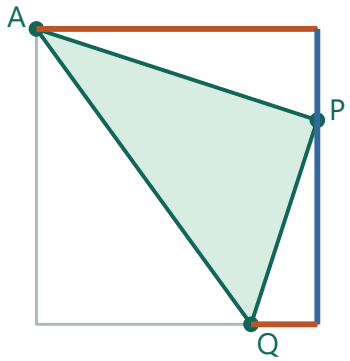
Orange: horizontal

Blue: vertical

Project AP and PQ onto perpendicular horizontal and vertical directions.

5. Label the remaining segments

For a full labeling, call the top-left, top-right, bottom-right and bottom-left corners A,B,C,D. Let P be the right-edge intersection and Q the bottom-edge intersection. Then AQ=1, AP=cos β, PQ=sin β, AB=cos β cos α, BP=cos β sin α, PC=sin β cos α, QC=sin β sin α, DQ=cos(α+β) and AD=sin(α+β).



PQ returns to the left

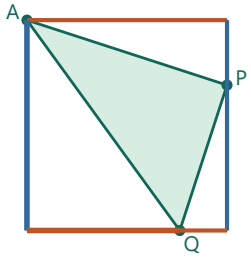
Its horizontal part

subtracts from AP's.

Horizontal motion reverses on the second leg; both downward components have the same sign.

6. Combine the two displacements

Add components: cos(α+β)=cos α cos β−sin α sin β; sin(α+β)=sin α cos β+cos α sin β.



Net displacement

Horizontal = cos(α + β)

Downward = sin(α + β)

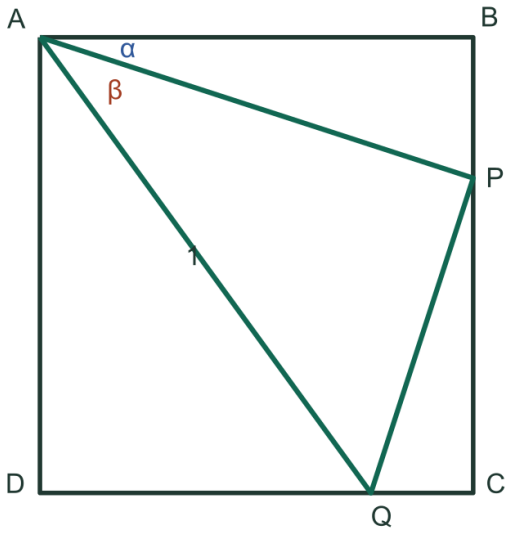
1. DQ = AB − QC
Subtract the leftward part of the second leg from the first leg's rightward travel.

2. cos(α+β) = cos β cos α − sin β sin α
The diagonal is one unit long, so DQ equals its horizontal component.

3. AD = BP + PC
Both contributions point downward in these diagram coordinates.

4. sin(α+β) = cos β sin α + sin β cos α
Substitute the two vertical projections.

Adding the two vectors gives the displacement along the unit segment AQ, whose components are the angle-addition formulas.



AQ = 1
AP = cos β
PQ = sin β
AB = cos β cos α
BP = cos β sin α
PC = sin β cos α
QC = sin β sin α
DQ = cos(α + β)
AD = sin(α + β)
∠BAQ = α + β
∠QPC = α

Check. Setting α=β=45° gives cos 90°=0 and sin 90°=1.

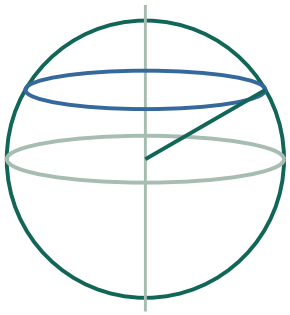
Exercise 288-1 · Speed along a parallel

Printed page 115 · Spherical coordinates / Angular and linear speed

What is the earth's rotational speed (in miles per hour) at a site whose latitude is θ degrees?

1. Follow a point for one rotation

A point on Earth's surface travels in a circle centered on the polar axis. Except at the equator this circle has a smaller radius than Earth itself.



One parallel

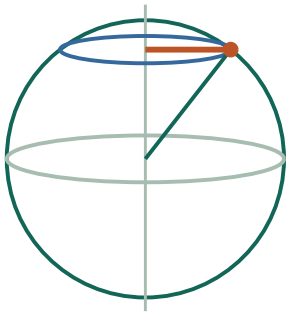
Rotation follows the

blue latitude circle.

A point turns around the polar axis, tracing its own latitude circle.

2. Find the horizontal projection

In a cross section through the poles, the radius to latitude θ makes angle θ with the equatorial plane. Its horizontal component, the distance to the rotation axis, is R cos θ.



Distance to the axis

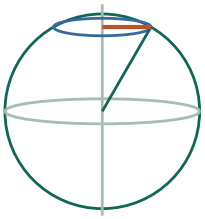
Parallel radius = R cos θ

The orange horizontal segment is the radius of that circle, the horizontal projection of Earth's radius.

3. Divide one circumference by one day

The daily path has radius R cos θ, so v=2πR cos θ/24 mph with R in miles and a 24-hour day. For R=3960 mi this is about 1037 cos θ mph.

Latitude · 60°



Latitude 60°

Parallel radius = 0.500 R

Speed scales with radius.

1. v = (2πρ)/(24 hours)
One revolution traverses the circumference at latitude θ.

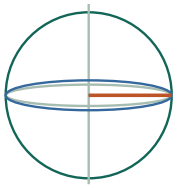
2. ρ = 3960 cos θ miles
Take the component perpendicular to the polar axis.

3. v = (2π·3960/24) cos θ ≈ 1037 cos θ mph
Divide the distance by its travel time.

Moving farther from the equator makes the circuit shorter but leaves the rotation time unchanged.

Compare settings · Latitude

0°



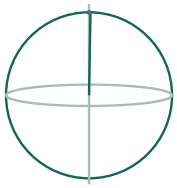
Latitude 0°

Parallel radius = 1.000 R

Speed scales with radius.

The point is 1.000 Earth radii from the axis and moves at 100.0% of equatorial rotational speed.

89°



Latitude 89°

Parallel radius = 0.017 R

Speed scales with radius.

The point is 0.017 Earth radii from the axis and moves at 1.7% of equatorial rotational speed.

Check. At the equator this gives the full equatorial speed; at either pole the radius and speed vanish.

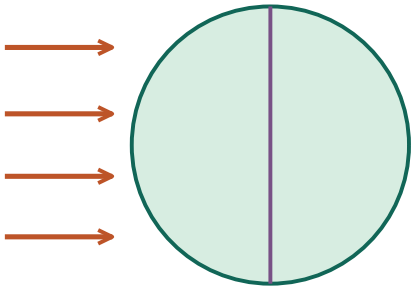
Exercise 310-1 · The circle of day and night

Printed page 115 · Sphere geometry / Planes in space / Orthogonality

At any given moment, the *terminator* is the circle on the earth's surface that separates night from day. Is the terminator a great circle? Explain.

1. Locate a point at sunrise

In the spherical model a ray from the Sun just grazes Earth at the edge of the lit region. The local surface normal at a point on a sphere is the radius from its center.



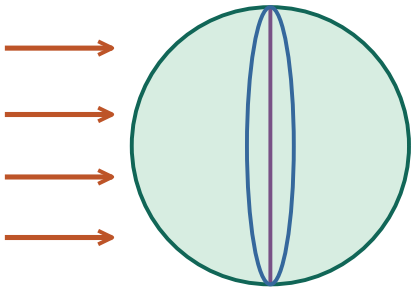
Grazing rays

Earth shown in section

Incoming sunlight is parallel in the distant-source model; a surface radius at sunrise is perpendicular to a grazing ray.

2. Describe the plane of the terminator

Model Earth as a sphere and sunlight as parallel rays. A boundary point has a surface normal perpendicular to the incoming direction. The radius is that normal, so all boundary points lie in the plane through the center perpendicular to the rays.



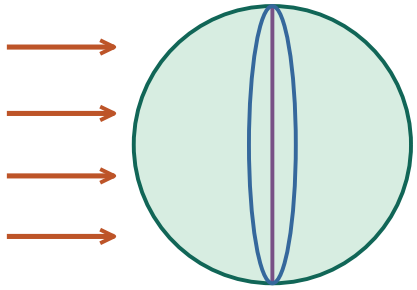
Radii at the boundary

are normal to the light.

The boundary radii lie in one plane perpendicular to the rays.

3. Intersect the plane with the sphere

The plane cuts out a great circle. Atmospheric twilight makes the physical boundary gradual.



Through Earth's center

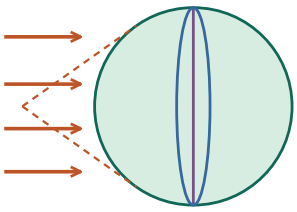
Plane section is

a great circle.

A plane through the sphere's center intersects its surface in a great circle, shown edge-on and obliquely here.

4. Test the parallel-ray assumption

For a point source at a finite position S measured from Earth's center, the ray to a horizon point P is tangent when $P\cdot(S-P)=0$. Thus $P\cdot S=R^2$, a plane offset from the center by $R^2/\|S\|$. Its intersection is a small circle. The great-circle answer becomes exact in the parallel-ray model.



Parallel-ray idealization

Dashed rays show how

a nearby source differs.

1. $0 = P\cdot(S-P) = P\cdot S - P\cdot P$
The tangent ray is perpendicular to the radius. Distribute the dot product.

2. $P\cdot P = R^2$, so $P\cdot S = R^2$
Every surface point P has length R.

3. $P\cdot(S/\|S\|) = R^2/\|S\|$
Dividing by the length of the Sun's position vector produces a unit normal; its projection is the plane offset.

If the light source were nearby, the incident directions would vary across Earth. The former great-circle argument would no longer apply.

Check. The ideal plane splits Earth into equal illuminated and dark hemispheres, as a great circle must.

Exercise 316-1 · When data vectors align

Printed page 116 · Vectors and components / Line equations

Let $A = (-6, -4)$, $B = (3, 2)$, and $C = (6, 4)$.

(a) These points lie on a line through the origin. Find its slope.

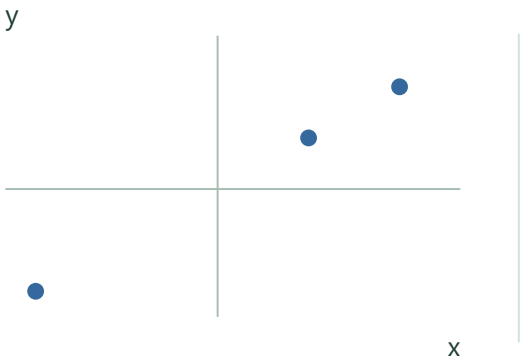
(b) Let $\mathbf{u}$ be the vector whose components are the x-coordinates of A, B, and C, and let $\mathbf{v}$ be the vector whose components are the y-coordinates of A, B, and C. Show that $\mathbf{v}$ is a positive scalar multiple of $\mathbf{u}$ (thus $\mathbf{u}$ and $\mathbf{v}$ point in the same direction).

(c) Explain why the scalar multiple in part (b) equals the slope you found in part (a).

(d) What would the vectors $\mathbf{u}$ and $\mathbf{v}$ have looked like if A, B, and C had not been collinear with the origin?

1. Pair the coordinates in order

Keep the same point order A,B,C in both vectors. The first component of u belongs with the first of v, and likewise for the two other component pairs.



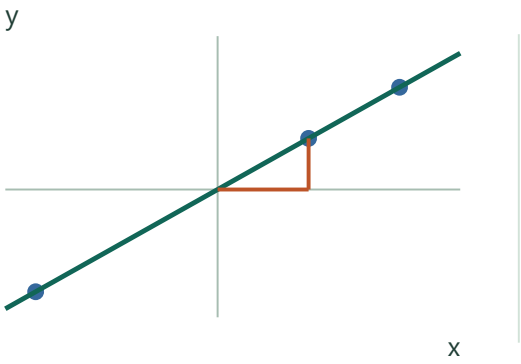
Three ordered pairs

Blue marks A, B and C.

Each dot keeps its x-coordinate paired with its own y-coordinate.

2. Read the slope as a multiplier

A line through the origin has equation y=mx. For the three data points the quotients −4/(−6), 2/3 and 4/6 are all 2/3. Thus multiplying every x-entry by 2/3 reproduces the corresponding y-entry.



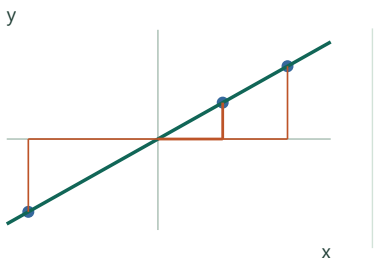
One slope triangle

run = 3; rise = 2

The rise/run triangle has slope 2/3 and the same line hits every dot.

3. Translate back to vector language

The component calculation gives v=(2/3)u. The positive sign preserves direction. If one component pair broke this rule, no one multiplier could work, so the data vectors would not be parallel through a common scalar with these x-values.



One multiplier

y = (2/3)x at all three.

1. (2/3)(−6,3,6) = (−12/3,6/3,12/3)
Multiply each component by the same scalar.

2. = (−4,2,4) = v
The positive multiple preserves the direction of u.

Every y-coordinate equals the same multiple of its x-coordinate, which also gives v=(2/3)u.

Check. Compare −4/(−6), 2/3, and 4/6. All three quotients are 2/3.

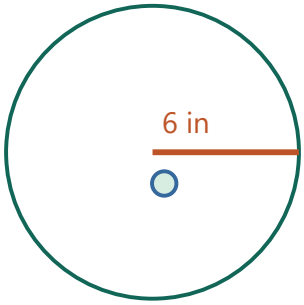
Exercise 319-1 · How many gumballs?

Printed page 116 · Packing / Volume / Measurement and estimation

A spherical globe, 12 inches in diameter, is filled with spherical gumballs, each having a 1-inch diameter. Estimate the number of gumballs in the globe, and explain your reasoning.

1. Compare one ball with the whole globe

The globe's radius is 6 inches and each gumball's radius is half an inch. The ratio of their radii is 12. Scaling all lengths by 12 multiplies volume by 12^3, so the globe holds 1728 gumball-volumes before allowing for air.



Cross section

R/r = 6/0.5 = 12

Volume ratio = 12^3

The radius ratio is a length comparison; cube it to compare volumes.

2. Estimate an occupied fraction

The volume ratio is $12^3=1728$. If the balls occupy a fraction f of the globe, then the count is $N=1728f$. Taking $f\approx0.60$ gives an estimate of roughly 1040 balls. The actual fraction depends on the arrangement and the gaps at the boundary.



An estimated fill fraction

$N \approx 1728 \times 0.60$

1. $N\cdot(4\pi/3)(\frac{1}{2})^3 = f\cdot(4\pi/3)6^3$
Only the occupied fraction of the globe volume is solid gum.

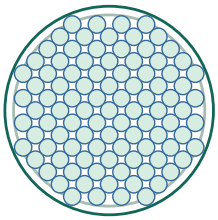
2. $N = f(6/(\frac{1}{2}))^3 = 1728f$
Cancel the common volume factor and divide by one ball's volume.

3. $N \approx 1728(0.60) = 1036.8$
Round the estimate, since the packing fraction is approximate.

Only an occupied fraction of the globe counts toward the volume of solid gumballs.

3. Give a realizable scale check

As a scale check, place centers at (i,j,k)/√2 inches for integer triples with even i+j+k. Distinct centers are at least 1 inch apart. Those with $i^2+j^2+k^2 \le 60$ lie within 5.5 inches of the globe center. Counting these triples gives 959 whole balls in one explicit arrangement.



One plane of the lattice

Blue disks are whole balls

cut through their centers.

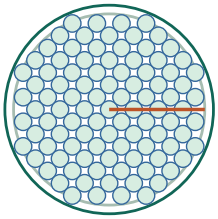
1. $(i^2+j^2+k^2)/2 \le (6-\frac{1}{2})^2 = 30.25$
A ball's center must stay at least one ball radius inside the globe.

2. $i^2+j^2+k^2 \le 60.5 \Rightarrow i^2+j^2+k^2 \le 60$
Multiply by two and use that the sum of integer squares is an integer.

A single coordinate layer illustrates the even-parity center rule; stacking all qualifying layers gives a realizable count.

4. Distinguish the estimate from a guarantee

The lattice spacing makes every pair of centers at least one diameter apart. Keeping all centers at most 5.5 inches from the middle leaves room for the final half-inch radius at the wall. This gives an explicit lower example. The occupied fraction of an actual poured batch depends on how it settles.



Center restriction

$R - r = 5.5$ in

The wall still clears each ball.

1. $i^2+j^2 \le 60-k^2$; i+j+k must be even
Fix an integer height k and count permitted pairs in that disk. For example, at k=7, $i^2+j^2 \le 11$ and i+j is odd.

2. k = 0,1,2,3,4,5,6,7: 97,88,89,80,69,52,37,16
For each integer i from −7 to 7, count j from −7 to 7 satisfying both tests.

3. N = 97+2(88+89+80+69+52+37+16) = 959
The seven positive-height layers have matching negative-height layers.

Shrink the allowable center radius by one gumball radius so an entire outer ball fits inside the globe.

Check. At f=0.60 the balls occupy about 543 of the globe's 905 cubic inches; the remaining volume is air.

Exercise 320-1 · Projecting data onto a line

Printed page 116 · Vector projections / Linear regression / The dot product

Let $A = (-3, -2)$, $B = (-1, -1)$, $C = (4, 3)$, $\mathbf{u} = [-3, -1, 4]$, and $\mathbf{v} = [-2, -1, 3]$.

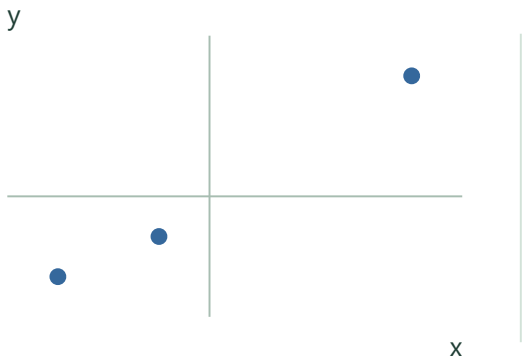
(a) Show that $\mathbf{u}$ and $\mathbf{v}$ point in different directions. Let $\mathbf{w}$ be the vector that results when $\mathbf{v}$ is projected onto $\mathbf{u}$. Show that $\mathbf{w}$ is approximately $[-2.19, -0.73, 2.92]$.

(b) Make a scatter plot. Verify that A, B, and C are not collinear. Notice that the x-coordinates of these points are the components of $\mathbf{u}$ and the y-coordinates are the components of $\mathbf{v}$.

(c) Verify that the points $A' = (-3, -2.19)$, $B' = (-1, -0.73)$, and $C' = (4, 2.92)$ lie on a line that goes through the origin. Notice that the y-coordinates of A', B', and C' appear as the components of $\mathbf{w}$, and that they are proportional to the components of $\mathbf{u}$.

1. Separate the two coordinate lists

The x-values form u=(−3,−1,4), the y-values form v=(−2,−1,3). Test whether v is already a multiple of u by comparing matching quotients. They give 2/3, 1 and 3/4, so the directions differ.



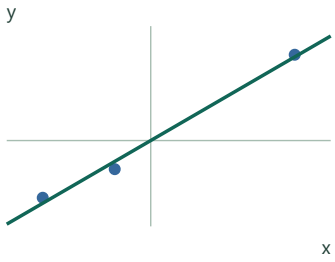
Input and output

The blue data do not all share one slope.

Place the three observations before fitting a line. Ratios from the origin differ.

2. Compute the scalar projection coefficient

To find the vector projection w=mu, require the leftover v−mu to be perpendicular to u. Expanding u·(v−mu)=0 gives m=(u·v)/(u·u)=(6+1+12)/(9+1+16)=19/26.



Fit through the origin

Slope m = 19/26

1. 0 = u·v − m(u·u)
Expand the perpendicularity condition for the residual arrow.

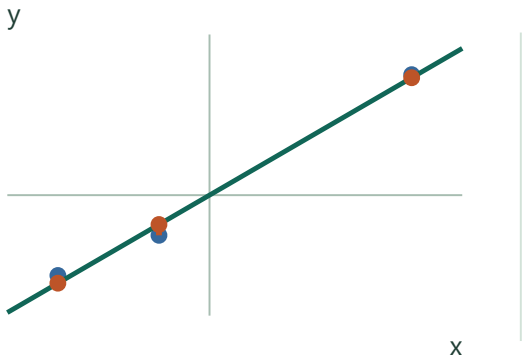
2. m(u·u) = u·v
Add m(u·u) to both sides.

3. m = (6+1+12)/(9+1+16) = 19/26
The denominator is positive because u is nonzero.

A single slope must account for all three points. Applying the perpendicular-residual condition determines the green line.

3. Read each projected point

Multiply each component of u by 19/26. The fitted heights are −57/26, −19/26 and 76/26. Placed over their original x-values, these three points lie exactly on y=(19/26)x. The printed points use the rounded heights −2.19, −0.73 and 2.92. Divide by their x-coordinates: (−2.19)/(−3)=(−0.73)/(−1)=2.92/4=0.73. Thus the printed points lie exactly on y=0.73x.



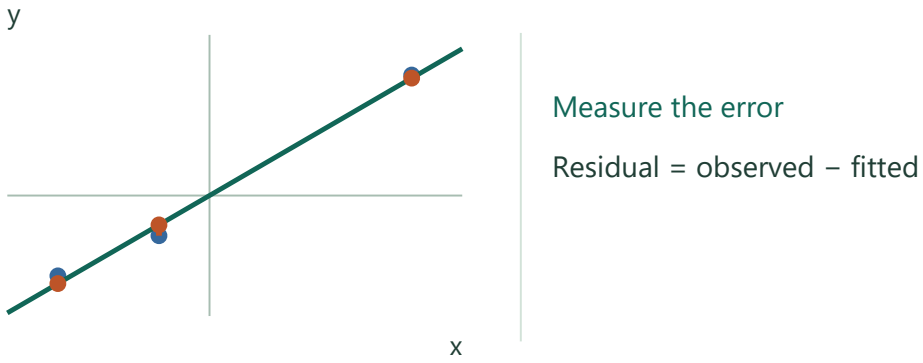
Move vertically

Orange = fitted values

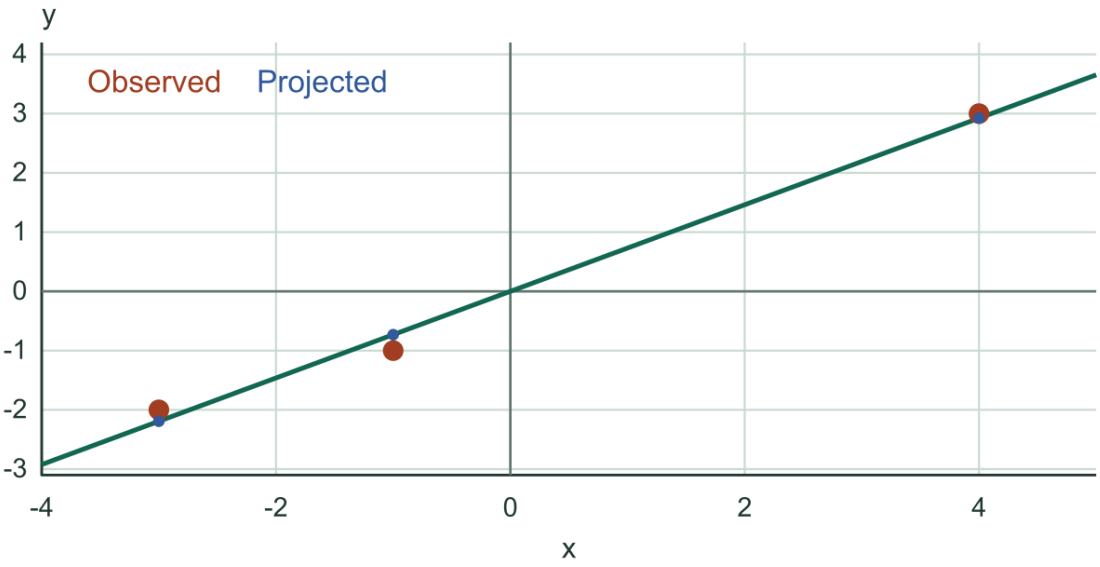
Move each data point vertically onto the line at its original x-coordinate.

4. Check the original scatter

The first original pair of points has slope 1/2, while the second pair has slope 4/5. Their differing slopes prove the original points are noncollinear. The short vertical marks in the worked drawing show how far each observed height lies from its fitted value.



Compare the discarded vertical offsets with the original scatter; a fitted line does not make the observed points collinear.



Check. The residual (5,−7,2)/26 has zero dot product with u.

Exercise 320-2 · Why regression uses squares

(Continuation)

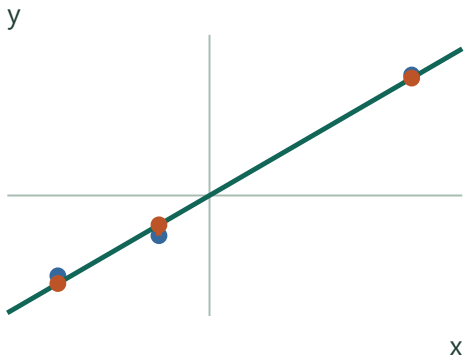
(a) You calculated $\mathbf{w}$ by first finding that it is m times as long as $\mathbf{u}$, where m is $\frac{\mathbf{u}\bullet\mathbf{v}}{\mathbf{u}\bullet\mathbf{u}}$. Notice that m is also the slope of the line through A', B', and C'. Now use a calculator to find an equation for the so-called *regression line* (or LinReg) for the data points A, B, and C. The slope should look familiar.

(b) Verify that the vector $\mathbf{r} = \mathbf{v} - \mathbf{w}$ is perpendicular to $\mathbf{u}$, then explain why this should have been expected. It is customary to call $\mathbf{r}$ a *residual vector*, because it is really just a list of residuals. Review the meaning of this data-analysis term if you need to.

(c) The regression line is sometimes called the *least-squares* line of best fit, because $\mathbf{w}$ was chosen to make $\mathbf{r}$ as short as possible. Explain this terminology. You will need to refer to the Pythagorean formula for calculating the length of a vector.

1. Read a residual as a vertical error

At each data point subtract the fitted height from the observed height. Stacking these differences produces a residual vector. Squaring its length adds the squares of the three vertical errors, which is exactly what least squares minimizes.



At each fixed x

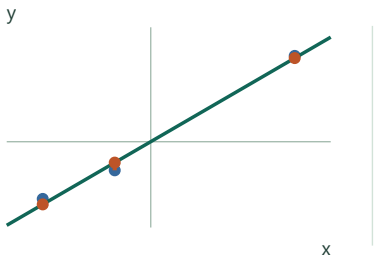
Orange segments span

observed and fitted y.

Hold each x fixed and look at its vertical discrepancy from the fitted height.

2. Recover the fitted line and the residual

Both original data vectors have mean zero, so the least-squares line has zero intercept. Its slope is 19/26. Observed minus fitted gives r=(5,−7,2)/26. Then u·r=(−15+7+8)/26=0.



Signed residuals

5/26, −7/26, 2/26

1. 26r = 26(−2,−1,3) − 19(−3,−1,4)
Multiply observed minus fitted by 26 to clear fractions.

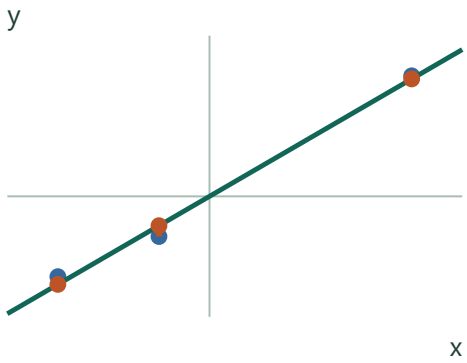
2. 26r = (−52+57,−26+19,78−76) = (5,−7,2)
Subtract matching coordinates.

3. 26u·r = (−3)5+(−1)(−7)+4·2 = 0
A zero dot product verifies orthogonality.

The three signed displacements form the residual vector.

3. Check both perpendicular directions

The centered x-values sum to zero. The residual entries sum to zero as well. Consequently both u and r are perpendicular to the constant vector (1,1,1), which represents a common vertical shift of all fitted heights.



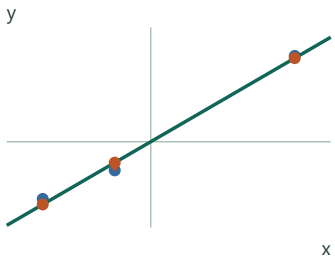
Zero sum

Positive and negative residuals balance.

Their sum is zero; adding a common intercept moves all predicted heights in the same direction.

4. Allow every slope and intercept

Allow any slope k and intercept b. The residual is $r+(19/26-k)u-b(1,1,1)$. The three directions are pairwise perpendicular: both r and u sum to zero, and $r\cdot u=0$. The sum of squared vertical errors is therefore $\|r\|^2+26(k-19/26)^2+3b^2$. It attains its unique minimum at $k=19/26$ and $b=0$.



Least-squares fit

A new slope or intercept adds squared error.

1. $v-ku-b1 = r+(19/26-k)u-b1$
Replace v by fitted plus residual; 1 denotes the all-ones vector.

2. $S = \|r\|^2+(19/26-k)^2\|u\|^2+b^2\|1\|^2$
All cross terms vanish because the three directions are pairwise perpendicular.

3. $S-\|r\|^2 = 26(19/26-k)^2+3b^2 \geq 0$
Both coefficients are positive, so equality requires both squares to vanish.

Changing either fitted coefficient adds a nonnegative square. The plotted line attains the minimum.

Check. The intercept normal equation also holds: $5-7+2=0$, so a vertical shift cannot improve the fit.

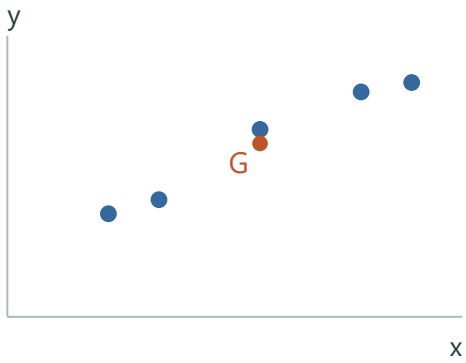
Exercise 326-1 · A regression line through the centroid

Printed page 116 · Linear regression / Least squares and residuals

Use a calculator to find the equation of the least-squares line (LinReg) for the five data points (2.0,3.2), (3.0,3.5), (5.0,5.0), (7.0,5.8), and (8.0,6.0). Let G be the *centroid* of these points — its x-coordinate is the average of the five given x-coordinates, and its y-coordinate is the average of the five given y-coordinates. Verify that G is on the least-squares line.

1. Center the x- and y-values

Add the five x-values and divide by five to get $\bar{x}=5$. Repeat for y to get $\bar{y}=4.7$. Their pair G is the centroid, the average location of the five points.



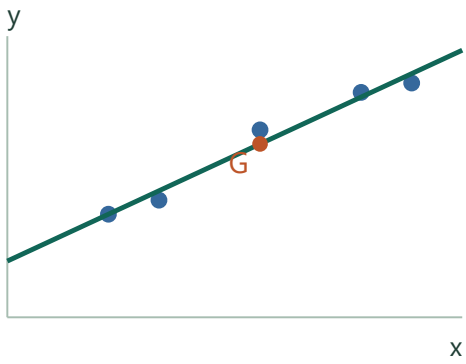
Centroid G

Mean point (5, 4.7)

The mean point is located by averaging coordinates separately, so it need not be one of the five observations.

2. Find the slope from centered products

Subtract the corresponding mean from every value. The x-deviations are $-3,-2,0,2,3$. Their squares sum to 26. The products of corresponding x- and y-deviations sum to 13, so the least-squares slope is $13/26=0.5$.



Centered contributions

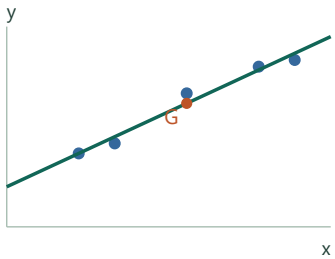
$\Sigma \Delta x \Delta y = 13$

$\Sigma (\Delta x)^2 = 26$

Centered horizontal and vertical offsets give the slope 13/26.

3. Choose and verify the intercept

The fitted average height must equal the average observed height. Solve $4.7=0.5\cdot 5+b$ to obtain $b=2.2$. Direct substitution of G into $y=0.5x+2.2$ verifies the claim.



Fitted line through G

$0.5(5) + 2.2 = 4.7$

1. $4.7 = 0.5(5)+b = 2.5+b$
Substitute the centroid coordinates and the fitted slope.

2. $b = 4.7-2.5 = 2.2$
Subtract 2.5 on each side.

Set the vertical position so the fitted value at the mean x equals the mean y.

Check. The five residuals are 0, −0.2, 0.3, 0.1, −0.2; they sum to zero and their dot product with the x-values is zero.

Exercise 341-1 · Diagonals of a cyclic quadrilateral

Printed page 117 · Cyclic quadrilaterals / The cosine rule / Ptolemy's theorem

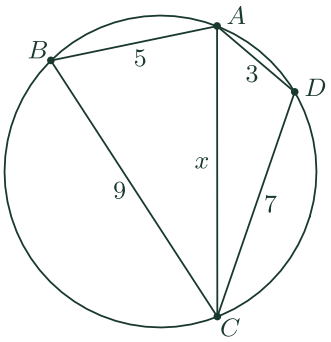
Quadrilateral $ABCD$ is inscribed in a circle, and the lengths of its sides are $AB = 5$, $BC = 9$, $CD = 7$, and $DA = 3$. Let x be the unknown length of diagonal AC.

(a) In terms of x, write an expression for $\cos B$.

(b) In terms of x, write an expression for $\cos D$.

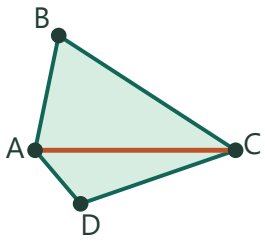
(c) A simple relationship holds between angles B and D. Use it to help you find the unknown length x.

(d) Find the length of diagonal BD.



1. Split the circle at diagonal AC

Label the vertices in order A,B,C,D as in the drawing. One triangle has sides 5 and 9 meeting at B. On the other side, sides 3 and 7 meet at D. Both triangles have the same unknown opposite side AC=x.



Shared side AC

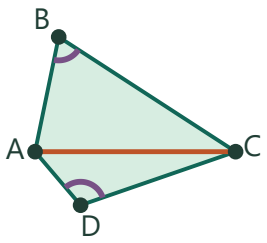
AB = 5, BC = 9

CD = 7, DA = 3

One diagonal splits the quadrilateral into two triangles with known pairs of sides.

2. Express both opposite angles

The cosine rule gives $x^2=5^2+9^2-2\cdot5\cdot9\cos B$ and $x^2=3^2+7^2-2\cdot3\cdot7\cos D$. Opposite angles of an inscribed quadrilateral sum to 180°, so $\cos D=-\cos B$.

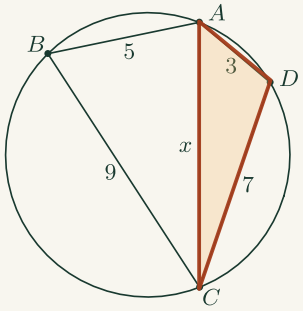


Angles B and D

$B + D = 180°$

$\cos D = -\cos B$

The opposite angles of a cyclic quadrilateral are supplementary, linking the two cosine-rule equations.

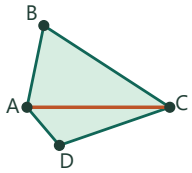


Triangle incident to D

Follow AD and DC through D. Both triangles use AC as their opposite side while the circle connects the two opposite angles.

3. Solve for AC

Rearranging the two cosine rules gives $(106-x^2)/90+(58-x^2)/42=0$. The first fraction is cos B and the second is cos D. Multiply by 90·42 and collect terms to obtain $9672=132x^2$. Taking the positive square root gives $AC=\sqrt{(806/11)}$.



Solve for AC

$AC^2 = 806/11$

1. $42(106-x^2)+90(58-x^2) = 0$
Multiply the cosine relation by both nonzero denominators.

2. $4452+5220 = (42+90)x^2$
Expand and move the squared terms to the right.

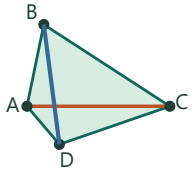
3. $x^2 = 9672/132 = 806/11$
Divide and reduce the fraction.

4. $x = \sqrt{(806/11)}$
The diagonal has positive length.

Using the same orange segment in both equations determines its length.

4. Recover the other diagonal

Ptolemy gives AC·BD=5·7+9·3=62, so BD=62√(11/806).



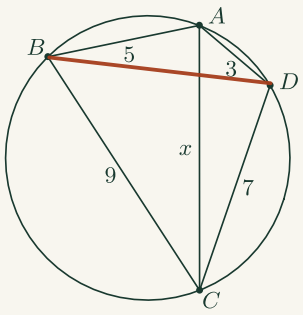
Second diagonal

AC × BD = 5·7 + 9·3

1. BD = (5·7+9·3)/AC = 62/√(806/11)
Divide Ptolemy's equation by the positive diagonal.

2. BD = 62√(11/806)
Taking the reciprocal of a positive square root exchanges the numerator and denominator inside it.

Draw the crossing diagonal. Ptolemy's theorem relates its length to AC by the products of opposite sides.



Second diagonal BD

This connection is a construction across the interior of the cyclic quadrilateral. It pairs with AC in the product-of-diagonals relation.

Check. Substituting x^2 into the cosines gives 4/11 and −4/11.

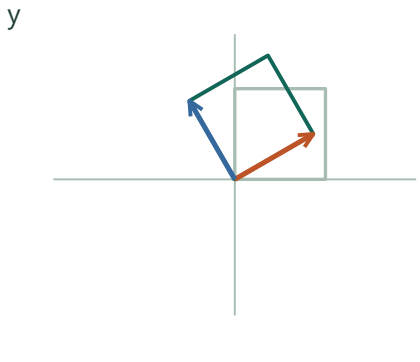
Exercise 351-1 · Which motions are linear?

Printed page 117 · Matrix transformations / Rotations / Reflections

You have used matrices to calculate the results of certain rotations and reflections. Which ones? Are translations calculated using matrices?

1. Construct a matrix from two arrows

The first and second columns are where the unit vectors (1,0) and (0,1) land. Under a counterclockwise rotation through θ they become (cos θ,sin θ) and (−sin θ,cos θ). This determines its action on every vector.



Rotation example: 30°

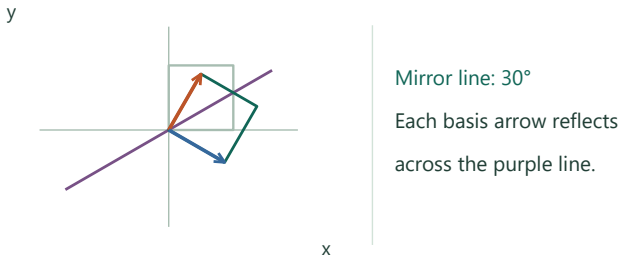
Colored arrows = columns

of the rotation matrix.

Rotate two basis directions. Their endpoints supply the matrix columns for every vector.

2. Reflect in a line through the origin

For a mirror making angle α with the x-axis, a direction angle φ becomes 2α−φ. Apply that to the two unit vectors above. The two images give the columns (cos 2α,sin 2α) and (sin 2α,−cos 2α).



1. φ = 0 ⇒ 2α−φ = 2α
Use the reflected direction of the first unit basis vector.

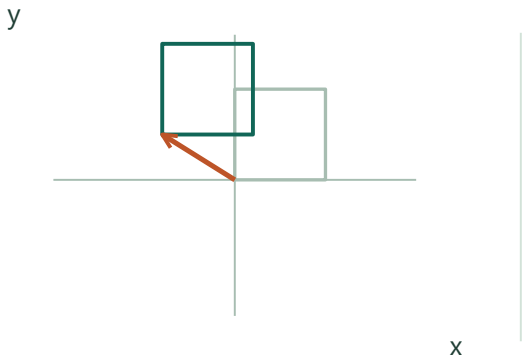
2. φ = π/2 ⇒ 2α−φ = 2α−π/2
The second basis vector initially points straight up.

3. (cos(2α−π/2),sin(2α−π/2)) = (sin 2α,−cos 2α)
Quarter-turn identities give the second matrix column.

Replace rotation by reflection about a line. The reflected basis directions change the second column's signs.

3. Test a translation at the origin

Any ordinary 2×2 matrix fixes zero, so it cannot perform a nonzero translation by itself. Add a vector after multiplying, or use homogeneous coordinates (x,y,1) with the translation in the third column.



Translate every vertex by the same arrow; the displaced origin witnesses the failure of linearity.

Check. At α=45° the reflection swaps x and y. A translation sends (0,0) to its displacement.

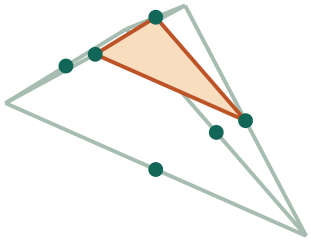
Exercise 363-1 · The solid left in the middle

Printed page 117 · Cross sections / Area and volume scaling / Volume

A regular pyramid with equilateral triangular faces is sliced by four cutting planes, each of which is parallel to a face of the pyramid and bisects the altitude drawn to that face. This dissects the pyramid into five pieces, four of which are smaller pyramids. Describe the fifth piece and determine its fraction of the original pyramid's volume.

1. Find the new vertices on the edges

A plane parallel to a face halfway up the solid cuts each edge from the opposite vertex halfway along its length. Repeat from each of the four vertices. The central piece therefore has the six original edge midpoints as its vertices.



Cut at the midpoints

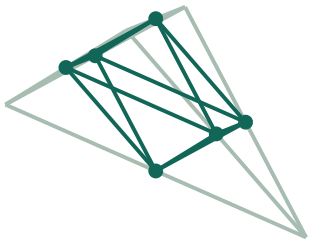
Triangle parallel to

the opposite face.

The cut across three incident edges joins three midpoints, making a smaller triangular face.

2. Count and classify the faces

Four equilateral faces of the middle piece are cuts. Four more are central triangles left on the original faces. All their side lengths equal half an original edge, so this is a regular octahedron.



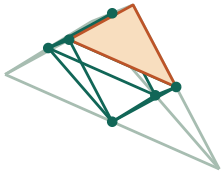
Central octahedron

6 vertices, 8 triangles

Connect midpoint pairs on every original face to expose the edge network of the central octahedron.

3. Subtract the four corners

Each corner has volume $(1/2)^3 = 1/8$ of the original. The remaining fraction is $1 - 4/8 = 1/2$.



Each corner: scale ½

Volume fraction = ⅛

4 corners remove ½.

1. $V_{corner}/V = (½)(½)(½) = ⅛$
Similarity halves each of three independent length dimensions.

2. $V_{middle}/V = 1 - 4(⅛) = ½$
Subtract four disjoint corner volumes.

Each corner is a half-scale tetrahedron. Remove four corner volumes of one eighth each.

Check. Four corner volumes plus the octahedron give $4/8 + 1/2 = 1$.

Omission note for appendix 391-1

A staggered arrangement fits 41 balls, and a general packing bound gives an upper limit of 44. We could not rule out an irregular arrangement of 42 balls. The exact capacity and whether the balls can move in a full box remain unverified.

Exercise 395-1 · Linearizing an inverse-square law

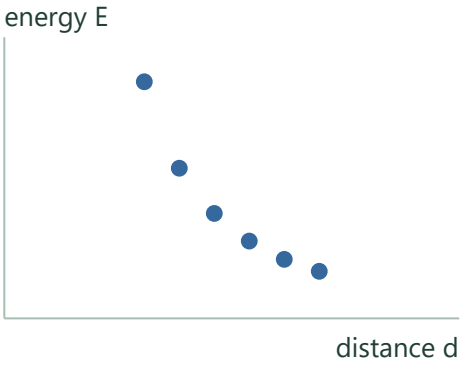
Printed page 117 · Data linearization / Linear regression / Power functions

The table at right contains experimental data. Each entry in the d-column stands for the distance from a photoelectric cell to a light source, and each entry in the E-column displays the amount of energy falling on the cell. A scientist suspects that the energy is closely related to the square of the distance. Create two new columns of transformed data, by calculating the *squares* of the d-entries and the *reciprocals* of the E-entries. Use the linear relationship between these variables to write down a simple equation that describes this example of an *inverse-square law*. Then predict the E-value that should correspond to a d-measurement of 36. Compare your equation with the theoretical prediction $E = 1444d^{-2}$.

d	E
12.0	10.1
15.0	6.41
18.0	4.48
21.0	3.30
24.0	2.52
27.0	2.01

1. Choose axes that test the scientist's idea

Write the suspected law $E=K/d^2$. Taking reciprocals gives $1/E=(1/K)d^2$. A plot of $y=1/E$ against $x=d^2$ should therefore be nearly straight and pass near zero.



Original readings

E falls as d increases.

Start from the six actual measurements before choosing a linearizing transformation.

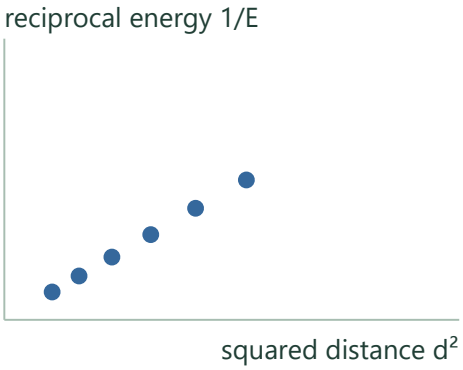
d	E
12.0	10.1
15.0	6.41
18.0	4.48
21.0	3.30
24.0	2.52
27.0	2.01

Measured energy E

Use the reciprocal of this column to test a straight-line relationship. The entries get smaller as distance increases.

2. Transform all six observations

For d=27,24,21,18,15,12 the squared distances are 729,576,441,324,225,144. Reciprocals of energy are approximately .49751,.39683,.30303,.22321,.15601,.09901.



Transform each point

$(d, E) \to (d^2, 1/E)$

Square each distance and invert each energy. The new points nearly lie on a line through the origin.

d	E
12.0	10.1
15.0	6.41
18.0	4.48
21.0	3.30
24.0	2.52
27.0	2.01

A matched measurement

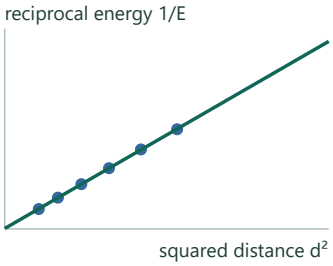
Read the last row across. Both transformations use data from this same distance–energy trial.

Original and transformed light measurements

Distance d	Energy E	d^2	1/E
12	10.1	144	0.09901
15	6.41	225	0.15601
18	4.48	324	0.22321
21	3.30	441	0.30303
24	2.52	576	0.39683
27	2.01	729	0.49751

3. Check the straight-line fit

Regress 1/E on d^2: $1/E \approx 0.000682013d^2 + 0.00202803$. At d=36 this gives E≈1.129. The nearly zero intercept supports a simple inverse-square law.



Unconstrained line

Intercept ≈ 0.00203

1. $\bar{x} = 406.5$, $\bar{y} \approx 0.2792664$
Average the six pairs $x=d^2$ and $y=1/E$.

2. $a = \Sigma[(x-\bar{x})(y-\bar{y})] / \Sigma(x-\bar{x})^2$
A least-squares line with an intercept uses centered data for its slope.

3. $a \approx 165.44380/242581.5 \approx 0.000682013$
Sum the six centered products and the six squared horizontal deviations.

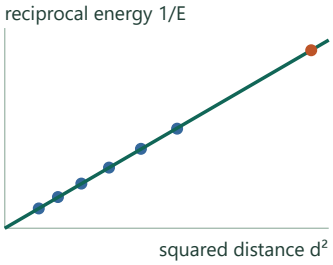
4. $b = \bar{y} - a\bar{x} \approx 0.00202803$
The fitted line passes through the mean point.

5. $E(36) \approx 1/(1296a+b) \approx 1.129$
Substitute 36^2 into the reciprocal-energy line before inverting.

Fit a slope and an intercept to the transformed scatter, leaving the intercept free.

4. Fit the inverse-square law and predict

To fit a pure inverse-square law, set $x=d^2$ and $y=1/E$ and fit $y=ax$ through the origin. Then $a=\Sigma(xy)/\Sigma(x^2) \approx 0.000686022$. Inverting gives $E \approx 1457.68/d^2$ and E≈1.125 at d=36.



Fit through the origin

d = 36 gives $d^2 = 1296$

Predicted E ≈ 1.125

1. $S(a) = \Sigma(y-ax)^2 = \Sigma y^2 - 2a\Sigma xy + a^2\Sigma x^2$
Expand each squared residual of the zero-intercept fit.

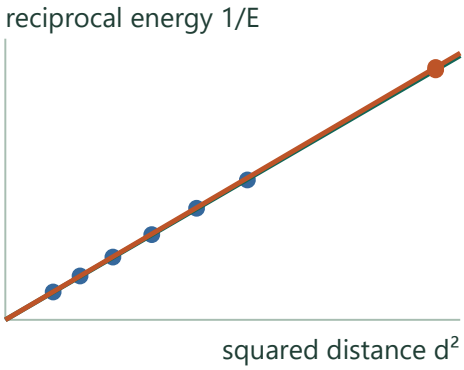
2. $S'(a) = -2\Sigma xy + 2a\Sigma x^2 = 0$
At the minimum the derivative vanishes; Σx^2 is positive.

3. $a = \Sigma xy/\Sigma x^2$; $E = 1/(ad^2)$
Solve for the slope, then undo the reciprocal transformation.

Require zero intercept for a pure inverse-square model; the highlighted prediction lies beyond the observations.

5. Compare with the theoretical model

The fitted coefficient is about 0.95% above the theoretical 1444. At d=36 the theoretical energy is 1444/1296≈1.114.



Compare theoretical slope

Orange: 1/1444

Fitted: 1/1457.68

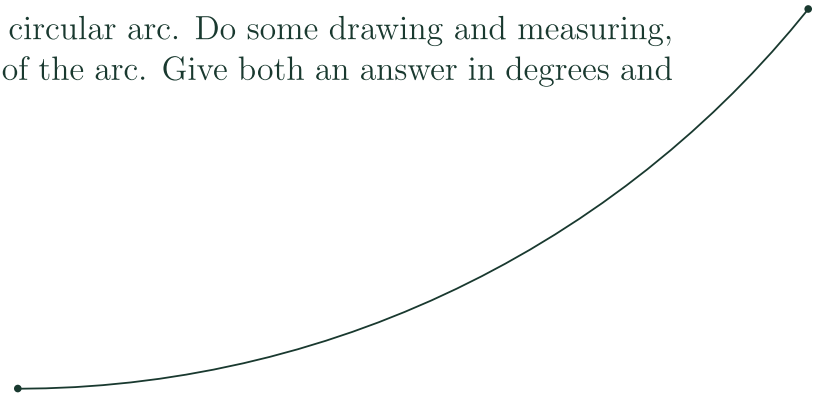
The theoretical and fitted lines nearly coincide. At the same d^2 their different reciprocal heights give slightly different energy predictions.

Check. Compute d^2E directly: it stays near 1444, e.g. 144·10.1=1454.4.

Exercise 406-1 · Measuring an arc without a marked center

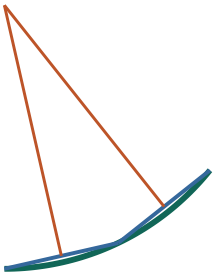
Printed page 117 · Arc length / Radian measure / Measurement and estimation

The figure at right shows a circular arc. Do some drawing and measuring, and then estimate the angular size of the arc. Give both an answer in degrees and an answer in radians. See page 95.



1. Identify a center without guessing it

Choose two chords of the same circle. The center is equidistant from both ends of each chord, so it lies on each perpendicular bisector. Their intersection locates the center, allowing the arc angle to be measured between radii.

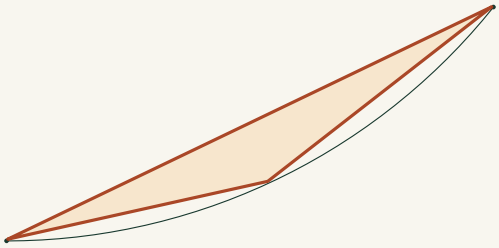


Perpendicular bisectors

Their intersection is

the estimated center.

Bisect two short chords perpendicularly; each bisector passes through the circle center.

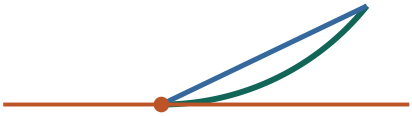


Two shorter chords

Choose an intermediate point on the arc and join it to each endpoint. Each perpendicular bisector passes through the center, which lies beyond this cropped drawing.

2. Use an independent tangent construction

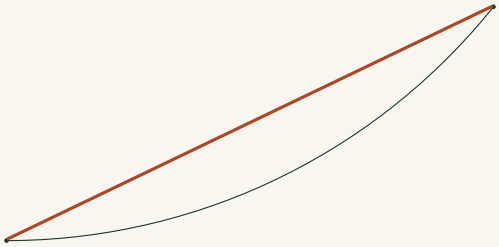
The enlarged source drawing shows a nearly horizontal tangent at its left endpoint. Join the endpoints by a chord. In one rendering the chord rises about 128 pixels and runs about 267 pixels. Their ratio is unchanged by uniform enlargement.



Horizontal tangent

Blue chord rises to the right.

Draw the chord between the endpoints to measure its direction relative to the left tangent.

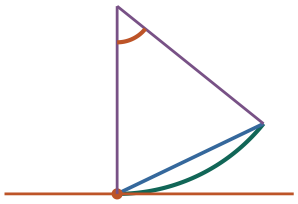


Full chord between endpoints

Connect the two visible endpoints by a straight segment. Its rise and run allow a direction angle to be measured independently of the rendering’s size.

3. Convert the measured chord angle

The chord–tangent angle is about arctan(128/267)=25.6°. The central angle is twice that: about 51° or 0.89 radian. Small changes in measurement change the last digit.



Central and tangent angles

Central angle ≈ 2 × 25.6°

1. α ≈ arctan(128/267) ≈ 25.6°
The rise and run measure the chord direction from the horizontal tangent.

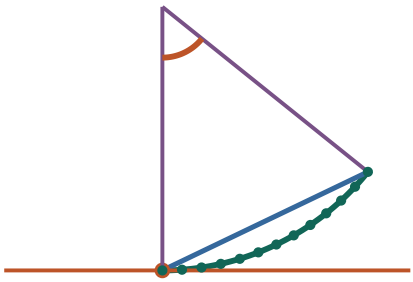
2. θ ≈ 2α ≈ 51.2°
The central angle subtending the same minor arc is twice the chord–tangent angle.

3. θ ≈ 51.2π/180 ≈ 0.89 rad
Convert degrees by multiplying by radians per degree.

Add the radii. The central angle subtending the chord is twice its chord–tangent angle.

4. Interpret the precision

The source is a drawing with finite line thickness. Slightly different endpoint or tangent measurements can change a decimal place. As an independent check, fit one circle to the 33 points along the rebuilt arc. The radii from the fitted center to its ends enclose about 51.24° (0.894 radian). Reporting about 51° reflects the precision of the printed diagram.

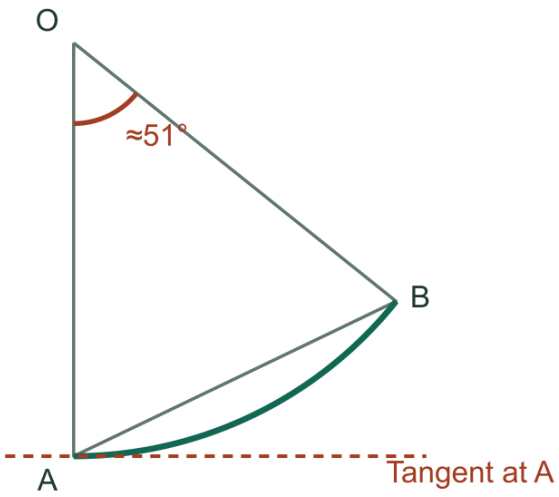


Sample points along the arc

Circle-fit estimate:

51.24° ≈ 0.894 rad

Several sample points constrain the fitted circle and provide a second estimate of the endpoint angle.



Check. The inferred radius is about $(267^2+128^2)/(2\cdot128)\approx342$ pixels. The chord formula $2\arcsin(\text{chord}/(2R))$ gives the same angle. As a separate measurement, fitting a circle to the visible arc pixels gives a radius of about 341 pixels and an angle of about 51.4° (0.90 rad).

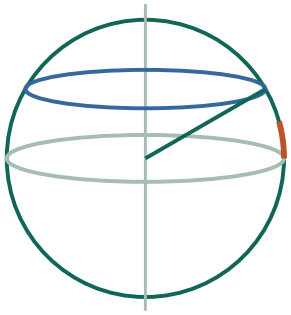
Exercise 406-2 · Why nautical miles were standardized

Printed page 117 · Spherical coordinates / Arc length / Unit conversion

The expression *one minute of longitude* does not define a distance, but the expression *one minute of latitude* does. Explain. Find this distance in miles, which was once called a *nautical mile*. What is the current definition of nautical mile, and why was the original definition changed?

1. Convert one angular minute

One degree is 60 angular minutes, and a full turn is 360 degrees. One minute is therefore 1/21600 of a turn. To turn that angle into miles, first identify the circle whose circumference is being divided.



Arc on a meridian

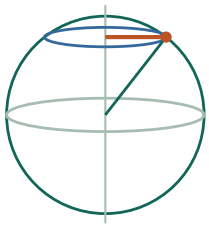
One minute = 1/21600

of a complete circle.

The marked portion of a meridian is enlarged for visibility. Divide a full turn into 360×60 equal angular increments.

2. Compare a meridian and a parallel

On a sphere a latitude minute along a meridian has length $2\pi R/21600$, about 1.15 statute miles. A longitude minute on a parallel has this length times cos(latitude), so it shrinks toward the poles.



Along a parallel

The same longitude angle

acts on radius R cos θ.

1. 1 arcminute = 1/(360·60) turn
Divide a whole rotation into degrees and then minutes.

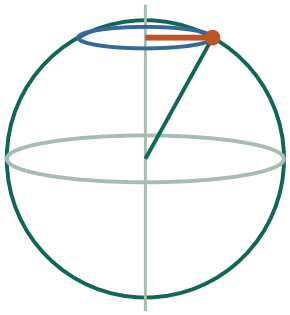
2. $s = 2\pi R/21600$
Multiply the meridian circumference by that fraction.

3. $s_{parallel} = (2\pi R/21600)\cos\varphi$
For longitude replace R by the parallel radius R cos φ.

Use the shorter blue parallel for a longitude change. Its radius varies with latitude.

3. Explain the standardized unit

Earth is slightly flattened; even meridian minutes vary with latitude. The international nautical mile is defined as exactly 1852 meters to provide one reproducible unit.



Fixed navigational unit

1 nautical mile = 1852 m

at every latitude.

A meter definition fixes the nautical mile even as the distance associated with an angular minute varies.

Check. 1852/1609.344≈1.15078 statute miles, close to the spherical estimate.

Exercise 412-1 · Changing Kepler's distance unit

Printed page 117 · Kepler's third law / Unit conversion

(Continuation) When distances are measured in astronomical units, Kepler's Third Law is $d^3 = t^2$. What would this equation look like if the distances were given in *miles*?

1. Convert the same orbit from AU to miles

Let d and m be the numerical values of the same orbital semimajor axis in AU and miles. Keep t in years. An orbit has positive semimajor axis and period. The unit conversion applies to length alone.

1 **d AU = m miles**
Both sides describe the same physical length.

2 **1 AU = K miles, K ≈ 92,955,807**
K is the positive number of miles per AU.

3 **m = Kd**
Replace each AU by K miles and compare the mile counts.

4 **d = m/K**
Divide by the nonzero conversion factor to recover the AU count.

2. Cube the converted distance

In the given solar-orbit model the coefficient 1 belongs to the AU-and-year system. Substituting the mile count changes that coefficient. Parentheses show which quantity is cubed.

1 **$d^3 = t^2$**
Start with the numerical distance in AU and period in years.

2 **$(m/K)^3 = t^2$**
Substitute m/K in place of the entire distance d.

3 **$m^3/K^3 = t^2$**
A cube multiplies three copies of both numerator and denominator.

4 **$m^3 = K^3t^2$**
Multiply both sides by K^3. This is the law for miles and years.

3. Check against a one-year orbit

Use Earth's reference orbit in this model to check the direction and power of the conversion.

1 **$t = 1 \Rightarrow m^3 = K^3 \cdot 1^2$**
A one-year period sets the right side equal to K cubed.

2 **$m = K \approx 92.96$ million miles**
Take the real cube root. One AU now has its expected mile count.

Check. At t=1 year the equation gives m=K miles, as required for Earth's orbit.

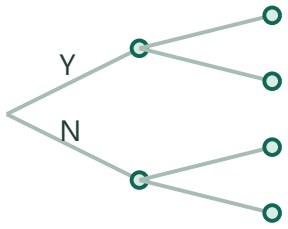
Exercise 450-1 · A dictionary in sixteen questions

Printed page 118 · Binary search / Exponent rules / Counting and permutations

Randy chooses a word in a dictionary and Andy tries to guess what it is, by asking questions to which Randy can answer only *yes* or *no*. There are 65000 words in the dictionary. Show that Andy can guess the word by asking at most 16 questions.

1. Interpret a yes/no answer

A single answer distinguishes two possible groups. Two answers can distinguish four groups. The doubling pattern suggests using the ordered dictionary to halve the candidate list with every question.



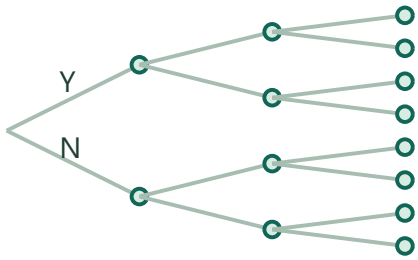
One answer, two branches

Two answers, four leaves.

A yes/no question creates two branches. Following an answer selects exactly one path.

2. Design the questions

Number the entries. Ask whether the secret entry lies in the first half of the remaining interval. Keep the indicated half and repeat. After k answers the largest possible interval has size $\lceil 65000/2^k \rceil$.



One branch survives

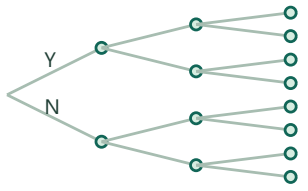
Each new question halves

the remaining interval.

Splitting the current interval as evenly as possible limits the greatest number of entries on the surviving branch.

3. Establish the worst-case guarantee

Since $2^{15}=32768<65000\le65536=2^{16}$, sixteen questions suffice in the worst case and fifteen cannot guarantee success.



15 questions: 32,768 leaves

16 questions: 65,536 leaves

65,000 entries fit.

1. $\lceil 65000/2^k \rceil \le 1 \Leftrightarrow 65000 \le 2^k$
At most one candidate may remain after k yes-or-no questions.

2. $32768 < 65000 \le 65536 \Rightarrow 15 < \log_2 65000 \le 16$
Bracket the target between adjacent powers of two.

3. $k_{min} = 16$
Question counts are whole numbers.

Compare the maximum number of leaves with 65,000. Fifteen cuts cannot identify every entry, while sixteen suffice.

Check. Alternatively label every entry with a distinct 16-bit code and ask for its bits.

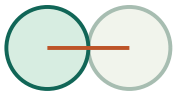
Exercise 477-1 · The gaps between pennies

Printed page 118 · Packing / Density

Imagine covering an unlimited plane surface with a single layer of pennies, arranged so that each penny touches six others tangentially. What percentage of the plane is covered?

1. Connect centers of touching pennies

Two equal tangent pennies of radius r have centers 2r apart. Draw the surrounding centers around one chosen center. They all lie on a circle of radius 2r.



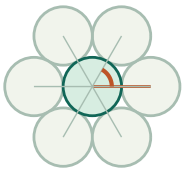
Two touching pennies

Center separation = 2r

Join the centers across the touching point. Each contributes one radius to the separation.

2. Explain the 60-degree pattern

The centers of six neighbors lie on a circle of radius 2r about each penny's center. Nonoverlap requires at least 60° between consecutive center directions. Six gaps use all 360°, so each gap is exactly 60°. Repeating this at adjacent centers produces the triangular lattice.



Six equal gaps

6 × 60° = 360°

1. neighbor spacing = 4r sin(θ/2) ≥ 2r
Bisect the triangle of center distances. Its equal sides are 2r; two neighbors cannot overlap.

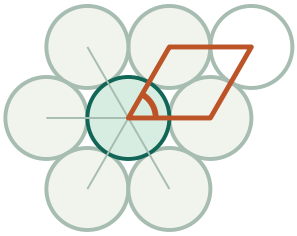
2. sin(θ/2) ≥ ½ ⇒ θ ≥ 60°
For the smaller separation use the increasing sine on 0° through 90°.

3. 6 × 60° = 360°
There is no surplus angle, so all six gaps equal 60°.

Six neighbors cannot lie any closer in angle without overlapping; the six gaps exhaust a turn.

3. Choose a repeating patch

Join a center to two adjacent neighbors and complete their parallelogram. Translating this patch along the two edge directions tiles the infinite pattern, so the covered fraction is the fraction in one patch. The corner sectors together account for one disk.



One repeating tile

One disk's worth of sectors

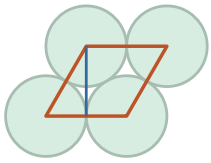
is inside a parallelogram.

A parallelogram of lattice directions repeats by translation, so one tile represents the entire packing.

4. Compute the fraction of the patch covered

Take two lattice vectors of length 2r at angle 60°. Their parallelogram has area $2\sqrt{3}\,r^2$. Each fundamental parallelogram accounts for one whole disk. Thus the covered fraction is $\pi/(2\sqrt{3}) \approx 90.69\%$.

Comparison lattice angle · 60°



Tile angle 60°

Height = 1.732r

Coverage = 90.69%

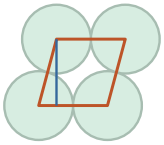
1. $A_{tile} = (2r)(2r \sin 60°) = 2\sqrt{3}\,r^2$
Multiply the base by the perpendicular height.

2. $A_{disk}/A_{tile} = \pi r^2/(2\sqrt{3}\,r^2) = \pi/(2\sqrt{3})$
The radius is positive, so cancel r^2.

Its perpendicular height is 2r sin 60°. Divide one disk area by the parallelogram area.

Compare settings · Comparison lattice angle

75°



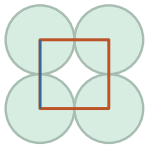
Tile angle 75°

Height = 1.932r

Coverage = 81.31%

For this comparison vary the lattice angle between 60° and 90°. The side stays 2r; its height is 2r sin(75°). The original six-neighbor packing is the 60° setting.

90°



Tile angle 90°

Height = 2.000r

Coverage = 78.54%

For this comparison vary the lattice angle between 60° and 90°. The side stays 2r; its height is 2r sin(90°). The original six-neighbor packing is the 60° setting.

Check. Within an equilateral triangle of centers the three 60° sectors total half a disk. Dividing by that triangle's area gives the same ratio.

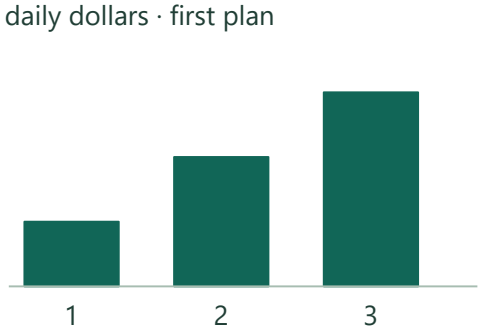
Exercise 477-2 · Linear pay versus doubling pennies

Printed page 118 · Arithmetic sequences and series / Geometric sequences and series

For the first 31 days of your new job, your boss offers you two salary options. The first option pays you \$1000 on the first day, \$2000 on the second day, \$3000 on the third day, and so on — in other words, $\$1000n$ on the nth day. The second option pays you one penny on the first day, two pennies on the second day, four pennies on the third day — the amount doubling from one day to the next. Which option do you prefer, and why?

1. List the first few payments

For the dollar plan the daily payments are \$1000, \$2000, \$3000 and so on. For the penny plan they are 1, 2, 4, 8 cents and so on. Each total requires adding every day's payment in the same currency.



Day 1 → day 3

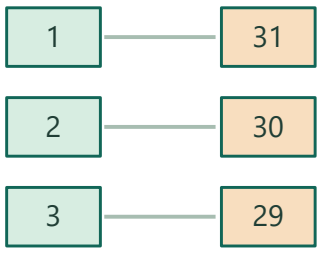
Dollars: 1000, 2000, 3000

Cents: 1, 2, 4

Keep the daily dollar amounts and the daily penny amounts in their own units initially.

2. Add the arithmetic schedule

Pair the first and last day numbers, 1+31=32. The 31 day numbers have average 16, so their sum is 31·16=496. Multiply by \$1000 to get \$496,000.



Pair from the ends

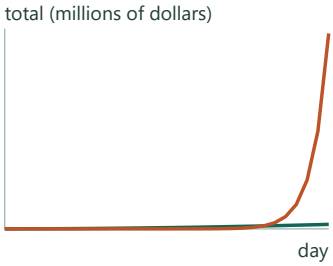
1 + 31 = 2 + 30 = 32

The middle day is 16.

Pairing opposite days gives a constant sum of 32 day numbers, which determines the arithmetic total.

3. Sum the doubling schedule

Let $S=1+2+4+\ldots+2^{30}$ cents. Doubling it gives the same intermediate powers, plus 2^{31} at the end and without the initial 1. Subtracting S yields $S=2^{31}-1$ cents, or $21,474,836.47. The penny option pays more across the 31 days.



Compare totals in dollars

Linear pay: $496,000

Pennies: $21,474,836.47

1. $2S = 2+4+\cdots+2^{31}$ cents
Doubling shifts each payment to the following power of two.

2. $2S-S = (2+\cdots+2^{31})-(1+\cdots+2^{30}) = 2^{31}-1$
Cancel the common powers from 2 through 2^{30}.

3. S/100 = 2147483647/100 dollars
Convert the total from cents to dollars before comparing plans.

Convert both cumulative payments to dollars; compounding the daily penny count eventually dominates the linear schedule.

Check. The final day alone pays 2^{30} cents, or $10,737,418.24, exceeding the entire first plan.

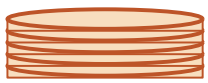
Exercise 477-3 · Will the last payment fit?

Printed page 118 · Volume / Measurement and estimation / Upper and lower bounds

(Continuation) You have chosen the second payment option, and on the thirty-first day your boss pays you the wages for that day — in pennies. You wonder whether all these coins are going to fit into your dormitory room, which measures 12 feet by 15 feet by 8 feet. Verify that a penny is 0.75 inch in diameter, and that seventeen of them make a stack that is one inch tall. Use this information to decide whether the pennies will all fit.

1. Select the payment for day 31

Day 1 contributes one penny. Doubling thirty times gives 2^{30} pennies for day 31 alone. A cumulative payment would count earlier days a second time for this question.



Day 31 alone

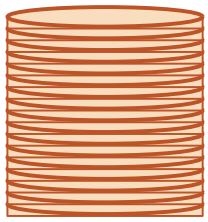
30 doublings of one penny

give 2^{30} pennies.

Count the last day's individual payment, after thirty doublings. The sketch represents a handful of the coins.

2. Check the dimensions of one penny

The U.S. Mint specifies a diameter of 0.750 in and thickness of 1.52 mm. Seventeen nominal pennies measure 25.84 mm, about 1.017 in. Treating the stack as one inch, as the question suggests, gives the following estimate.



Side view, enlarged

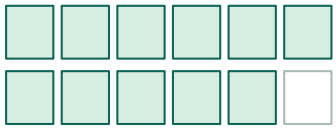
17 nominal thicknesses

≈ 1.017 inches

Stack the thin cylinders to turn a small thickness into an easier-to-check inch measurement.

3. Estimate the total solid volume

Model each cent as a cylinder of radius 0.375 in and height 1/17 in. The 2^{30} pennies on day 31 occupy $2^{30}\pi(0.375)^2/17$ cubic inches, about 16,148 ft³. Using the nominal thickness increases the estimate slightly.



Each tile: one room

Room volume = 1440 ft³

Penny volume ≈ 16,148 ft³

1. $V = 2^{30}\,\pi(0.375)^2(1/17)$ in³
Multiply one cylindrical penny's volume by the count on the last day.

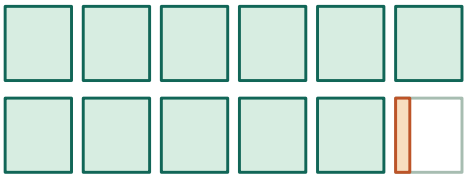
2. 1 ft³ = (12 in)³ = 1728 in³
Cube the linear conversion when converting volume.

3. $V \approx 2^{30}\pi(0.375)^2/(17\cdot1728)$ ft³ ≈ 16,148 ft³
Divide by cubic inches per cubic foot.

Divide the solid metal volume by a single room's volume. Eleven room-sized portions still do not exhaust it.

4. Compare the coins with the room

The room holds only 1440 ft³. Even eliminating all gaps would leave more than eleven rooms' worth.



Each tile: one room

Room volume = 1440 ft³

More than 11 rooms.

The partial final tile is additional solid metal. Even perfectly gap-free placement would already exceed one room.

Check. The number needed already exceeds the room's ideal volume capacity by a factor of about 11.2. Actual gaps reduce capacity further.

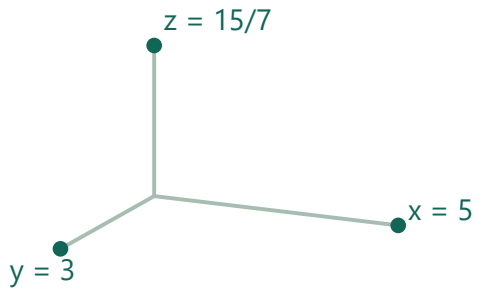
Exercise 502-1 · Tracing a plane on three walls

Printed page 118 · Planes in space / Line equations

Sketch the plane whose equation is $3x + 5y + 7z = 15$. Include in your sketch the three lines along which the plane intersects the xy-plane, the yz-plane, and the xz-plane. Notice that the equation $3x + 5y + 7z = 15$ can be rewritten as $z = \frac{15}{7} - \frac{3}{7}x - \frac{5}{7}y$. What are the meanings of the three fractions that appear in this equation?

1. Start at the intercepts

A point on the x-axis has y=z=0, so 3x=15 gives x=5. On the y-axis set x=z=0 and obtain y=3. On the z-axis set x=y=0 and obtain z=15/7. These three points anchor the sketch.



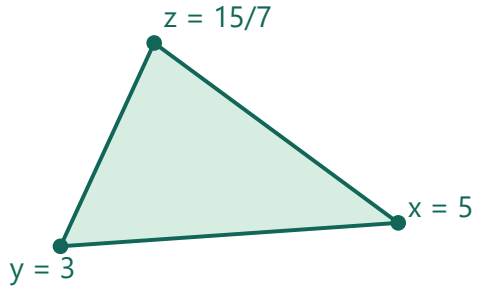
Set two coordinates to 0

Find one intercept per axis.

Each coordinate axis cuts the plane at its own intercept.

2. Trace the plane on each coordinate wall

Mark (5,0,0),(0,3,0),(0,0,15/7). Join each pair and extend the lines. In the xy, yz, xz planes the traces satisfy 3x+5y=15, 5y+7z=15, and 3x+7z=15 respectively.



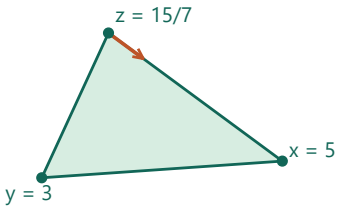
Three boundary traces

Each edge lies in one coordinate plane.

Join axis intercepts in pairs. Each joining edge is the trace where one coordinate is zero.

3. Interpret a coefficient by walking one unit

With y held fixed, an increase of 1 in x adds 3 to the left side of 3x+5y+7z=15. To keep the total fixed, 7z must fall by 3, so z falls by 3/7. The same balance with y increasing gives a fall of 5/7.



Move one unit in x

Hold y fixed. $\Delta z = -3/7$

1. $3\Delta x+5\Delta y+7\Delta z = 0$
Subtract the equations for two points on the same plane; the right sides cancel.

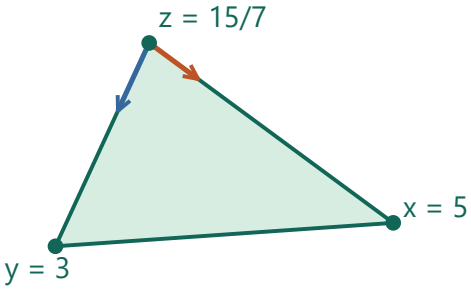
2. $\Delta y = 0, \Delta x = 1 \Rightarrow 3+7\Delta z = 0 \Rightarrow \Delta z = -3/7$
Hold y fixed and solve for the vertical step.

3. $\Delta x = 0, \Delta y = 1 \Rightarrow 5+7\Delta z = 0 \Rightarrow \Delta z = -5/7$
Repeat with x fixed.

A one-unit change in x requires a compensating change in height on the green plane.

4. Read the vertical intercept and slopes

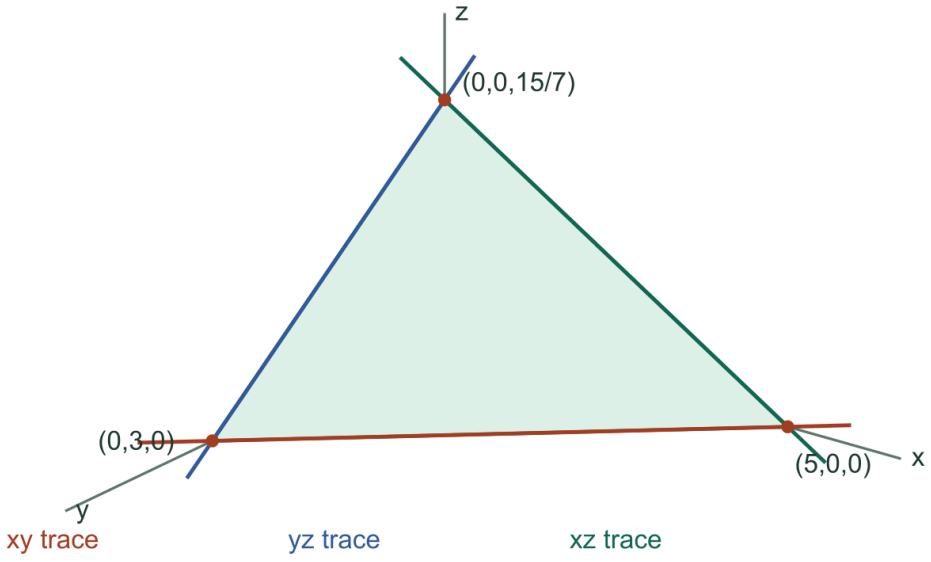
Solving gives $z=15/7-3x/7-5y/7$. The first term is the vertical intercept. At fixed y, one x-unit lowers z by 3/7; at fixed x, one y-unit lowers z by 5/7.



Move one unit in y

Hold x fixed. $\Delta z = -5/7$

Compare one-unit moves in two horizontal directions. Their different height changes are the slopes in the solved equation.



Check. The changes (1,0,−3/7) and (0,1,−5/7) both have zero dot product with the plane normal.

Exercise 561-1 · When is the last die removed?

Printed page 118 · Probability / Geometric sequences and series / Expected value

Forty-eight dice are rolled and each die that shows a "2" (a deuce) on top is removed. The remaining dice are rolled again and the deuces are removed. This procedure is repeated until all the dice are gone. How many rolls will be needed? Explain.

1. Follow one die first

A die survives a roll whenever one of its five non-deuce faces appears. Under independent fair rolls the chance of survival through n rolls is $(5/6)^n$.



One fair roll

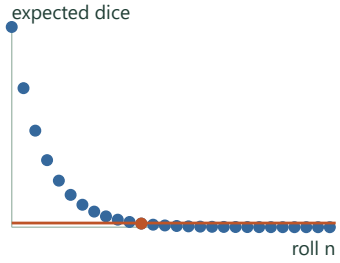
2 removes the die.

Five faces let it survive.

One of six outcomes removes the die; survival across independent rolls multiplies the five-sixths factor.

2. Distinguish a mean from completion

Adding the 48 survival probabilities gives $48(5/6)^n$ expected remaining dice. This falls below one on roll 22. A particular trial still has an integer count and can have one or more survivors at that time.



Mean number left

$48(5/6)^n$ drops below 1

at n = 22.

1. $48(5/6)^n < 1 \Rightarrow n \ln(5/6) < -\ln 48$
Divide by 48, then take the increasing natural logarithm.

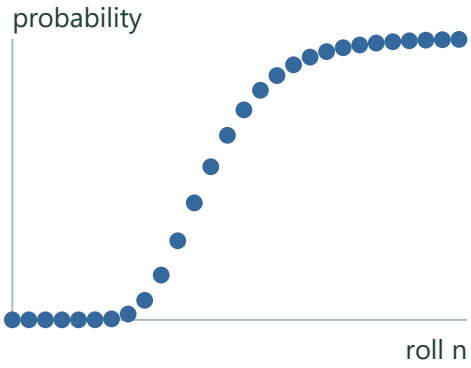
2. $n > -\ln 48 / \ln(5/6) \approx 21.23$
Dividing by the negative logarithm reverses the inequality.

3. $n_{min} = 22$
The number of completed rolls is an integer.

Now track an expected count across 48 dice. Falling below one expected die does not guarantee zero survivors.

3. Describe the last removal

For one die the chance of being removed by n is $1-(5/6)^n$. All 48 must be removed for the process to finish. Independence between dice gives $P(T \le n)=[1-(5/6)^n]^{48}$.



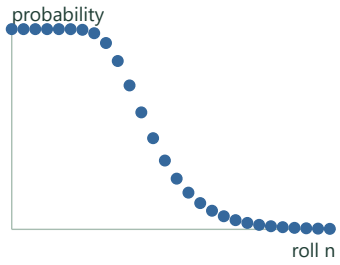
All 48 removed by n

Raise one-die removal probability to power 48.

Switch the vertical variable from expected count to the probability that every die is gone.

4. Find a typical scale for the wait

For a positive integer waiting time, add $P(T>n)$ for $n=0,1,2,\ldots$ to obtain its mean. Substitution gives about 24.96 rolls. Any specified finite n still gives positive survival probability, so there is no guaranteed finishing roll.



Still unfinished after n

The tail probabilities sum to about 24.96 rolls.

1. $P(T>n) = 1 - [1-(5/6)^n]^{48}$
Complement the probability that every die has already been removed.

2. $E[T] = \Sigma\, P(T>n)$, $n = 0,1,2,\ldots$
Each trial of length T contributes once for every integer n below T.

3. $E[T] \approx 24.96$
Sum the tail probabilities. Beyond n=150 the remaining sum is at most $48(5/6)^{151}/(1-5/6)$.

Take the complement to get the unfinished tail; adding that tail over all integer roll counts gives the expected finishing time.

Check. The distribution gives $P(T\le 0)=0$ and tends to 1. The experiment finishes almost surely, although arbitrarily long waits have positive probability.

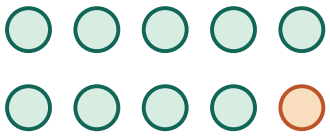
Exercise 574-1 · Splitting ten billion stones

Printed page 118 · Ratios and proportions / Scientific notation

A big pile of gravel contains 10^{10} stones. This pile is separated into two smaller piles by repeating the following process until the original pile is gone: Take ten stones from the original pile, then throw nine of them onto the left pile and one onto the right pile. When the original pile is gone, how many stones will be in the left pile? the right pile?

1. Track a single group of ten

Ten stones leave the original pile on every move. In each group, nine go to the left and one to the right. This preserves all ten stones and fixes the left-to-right ratio at 9:1.



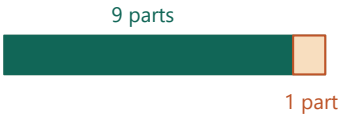
One group of ten

Nine green; one orange.

Color one group of ten by its two destinations.

2. Count the groups

Ten billion is 10^{10}. Dividing by ten stones per group gives 10^9 groups, with none left over.



10^{10} stones ÷ 10

One billion such groups.

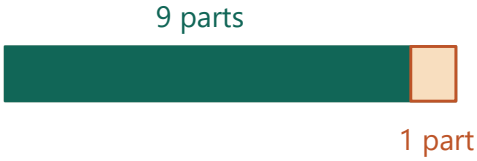
1. $G = 10^{10}/10^1 = 10^{(10-1)} = 10^9$
The exponent quotient counts complete ten-stone groups.

2. left = 9G, right = G, left+right = 10G
Each move contributes nine stones on one side and one on the other.

Repeat the same partition one billion times. The fraction going to either side stays fixed.

3. Convert groups back to stones

The left pile has $9 \cdot 10^9$ stones and the right pile has $1 \cdot 10^9$. Their sum is 10^{10}, the starting count. This is a finite process, even though the number of moves is large.



Left: 9×10^9

Right: 1×10^9

Both piles total 10^{10}.

Scale the two shares by the initial stone count and add them back as a check.

Check. The piles total 10^{10} and their sizes have the prescribed ratio 9:1.

Exercise 591-1 · Tuning an octave

Printed page 119 · Geometric sequences and series / Radicals and rational powers

When the note *middle C* is struck on a piano, it makes a string vibrate at 262 cycles per second. When the corresponding note one octave higher (denoted C′) is struck, it makes a string vibrate at 524 cycles per second (twice as fast as the first string). These two numbers form part of a geometric sequence of frequencies



$$\ldots, \mathrm{C}, \mathrm{C\#}, \mathrm{D}, \mathrm{D\#}, \mathrm{E}, \mathrm{F}, \mathrm{F\#}, \mathrm{G}, \mathrm{G\#}, \mathrm{A}, \mathrm{A\#}, \mathrm{B}, \mathrm{C}', \ldots$$

known as *equal-tempered tuning*. Given $\mathrm{C} = 262$ and $\mathrm{C}' = 524$, find the frequency of G, and the frequency of the note that is n steps above middle C. (For example, F is 5 steps above C. By the way, a musician would say "half-step" instead of "step.")

1. Read the keyboard by semitones

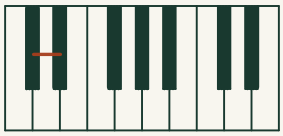
From one key to the immediately neighboring key is one semitone, counting black keys as well as white keys. The twelve semitones from C to the next C divide the octave into equal frequency ratios.



Adjacent pitches

Include the black keys.

Count all successive pitches, including black keys, to measure equal-tempered steps.



Black keys within the count

The first pair of black keys are C-sharp and D-sharp. Include black keys on the way from the starting C to a target pitch.

2. Find the multiplier

Let r be the factor for one semitone. After twelve steps the frequency is $262r^{12}$. Setting this equal to 524 gives $r^{12}=2$, so r is the positive twelfth root of 2.



One octave, 12 steps

$262 \times r^{12} = 524$ Hz

1. $262r^{12} = 524 \Rightarrow r^{12} = 524/262 = 2$
Divide by the positive starting frequency.

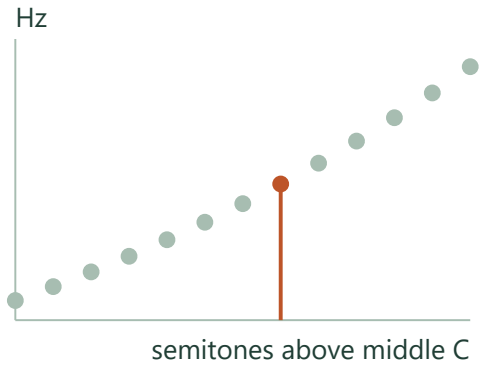
2. r = 2^(1/12)
Successive positive frequencies have a positive ratio, selecting the positive root.

Travel from C to the next C: twelve equal frequency ratios must multiply to two.

3. Find G and the general note

Count C♯, D, D♯, E, F, F♯, G: seven steps. G vibrates at 262·2^(7/12)≈392.56 Hz. Replacing seven with any step count n gives 262·2^(n/12) Hz.

Semitones from middle C · 7

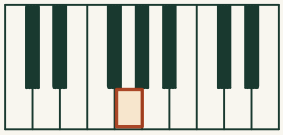


7 semitones

392.56 Hz

Multiplier: 2^(1/12)

Stop on G after seven semitones and take seven of those ratios.

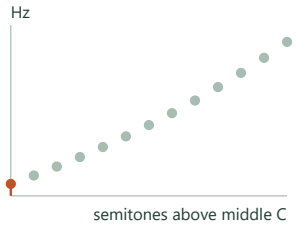


G in the first octave

This white key is G. Follow the ordered sequence of black and white keys from C and count the gaps between successive keys.

Compare settings · Semitones from middle C

0



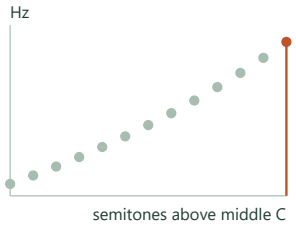
0 semitones

262.00 Hz

Multiplier: 2^(1/12)

After 0 semitones the frequency is 262 × 2^(0/12) ≈ 262.00 Hz.

12



12 semitones

524.00 Hz

Multiplier: 2^(1/12)

After 12 semitones the frequency is 262 × 2^(12/12) ≈ 524.00 Hz.

Check. Substitute n=0 and n=12 to recover both supplied frequencies.

Exercise 591-2 · The tempered fifth

Printed page 119 · Geometric sequences and series / Ratios and proportions

(Continuation) Without calculating any of the indicated frequencies, explain why the ratio G:C is the same as the ratio A:D, which is the same as the ratio B:E. Now calculate the ratio G:C and show that it is approximately equal to 3:2, the musical interval known as a *fifth*. The disagreement between the two ratios is why some violinists do not like to make music with pianists.

1. Compare the intervals before computing pitches

On the same keyboard, C to G, D to A, and E to B each span seven semitones. If the starting frequency is f, the higher one is fr^7. Dividing cancels f, so the ratio depends only on the number of steps.



A fifth spans 7 semitones

Example shown: C to G

C–G illustrates the same seven-semitone separation as D–A and E–B.

2. Calculate a tempered fifth

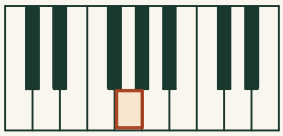
Since $r^{12}=2$, seven steps multiply frequency by $2^{(7/12)}\approx1.498307$. This explains why the three ratios are equal in equal temperament.



Same multiplier

$2^{(7/12)} \approx 1.498307$

The frequency of either starting note cancels when forming the upper-to-lower ratio.



G in the first octave · Figure from exercise 591-1

This white key is G. Follow the ordered sequence of black and white keys from C and count the gaps between successive keys.

3. Compare it to the pure fifth

Relative to a pure fifth of 1.5, the equal-tempered fifth is about $(1.5-1.498307)/1.5\approx0.113\%$ lower.



Pure ratio 3/2 = 1.5

Tempered fifth is about

0.113% lower.

1. fractional shortfall = $1-2^{(7/12)}/(3/2)$
Normalize the difference by the pure fifth's ratio.

2. $100[1-(2/3)2^{(7/12)}] \approx 0.113\%$
Multiply by 100 to express the relative change as a percent.

Compare that equal-tempered multiplier with 3/2; the small mismatch is a frequency-ratio difference.

Check. Twelve tempered fifths give exactly seven octaves, whereas $(3/2)^{12}$ differs slightly from 2^7.

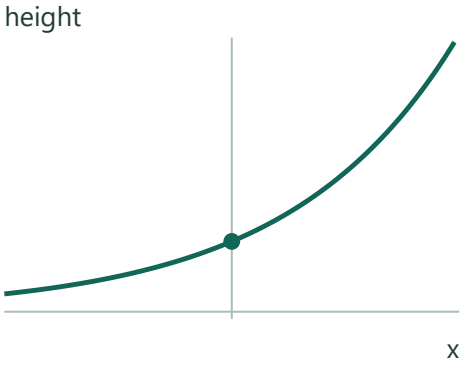
Exercise 609-1 · Slope of an exponential at zero

Printed page 119 · Exponential functions / Slopes and tangent lines / Rates of change

For any positive base b, the graph $y = b^x$ intersects the y-axis at $(0, 1)$. The slope m of the curve at this intersection depends on b, however. For example, you have probably already found that m is about 0.693 when $b = 2$. What is the (approximate) value of m when $b = 3$?

1. Locate the tangent being estimated

Every exponential y=bˣ goes through (0,1). The desired slope is the rise per horizontal unit in the immediate neighborhood of that point.



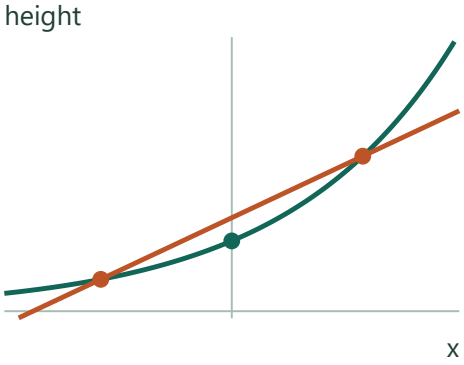
y = 3ˣ

All bases pass through (0,1).

Mark the intercept shared by all positive-base exponentials.

2. Balance samples on either side

Choose x=−h and x=h. Their outputs are 3^(−h) and 3ʰ. The horizontal distance is 2h, so the secant slope is [3ʰ−3^(−h)]/(2h). Smaller h places the samples closer to the requested tangent.



Symmetric secant

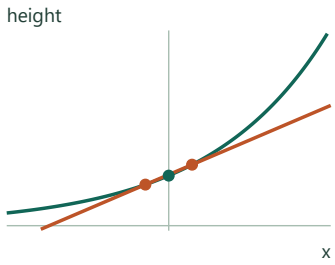
Samples at x = ±h

Run = 2h

Choose two points equally far left and right of zero. Their secant estimates the tangent near the intercept.

3. Evaluate and identify the limit

Use symmetric secant slope [3ʰ−3^(−h)]/(2h). At h=0.001 it gives about 1.098613. Its limit is ln 3≈1.098612.



Shrink the interval

The secant approaches slope ln 3 at (0,1).

1. $3^x = \exp(x \ln 3)$
Express the exponential using natural logarithms.

2. $d(3^x)/dx = \exp(x \ln 3) \ln 3$
The chain rule supplies the inner derivative.

3. $m(0) = \exp(0) \ln 3 = \ln 3$
At the marked intercept the exponential factor equals one.

Bring the two points closer. The visible interval is enlarged relative to the h=0.001 numerical calculation.

Check. Differentiate e^(x ln 3); at zero the exponential factor is 1.

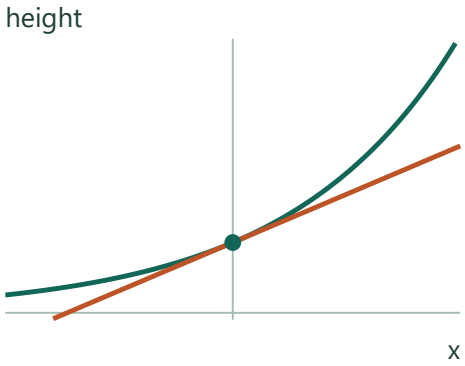
Exercise 609-2 · A table of intercept slopes

Printed page 119 · Exponential functions / Slopes and tangent lines / Logarithms

(Continuation) Make a table that includes (at least) the b-entries 1, 2, 3, 4, 6, 1/2, 1/3, and 1/4, and their corresponding m-entries. By the way, it is possible to save some work by writing your m-approximation formula in terms of b.

1. Use one expression for every base

Replace 3 in the symmetric secant formula by b. The limiting slope is ln b because b^x=e^(x ln b) and at x=0 its value is 1. For the constant function with b=1 the slope is zero.



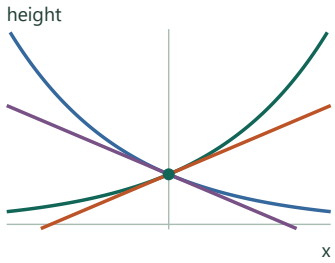
Example b = 3

Local slope = ln b

A tangent at (0,1) translates the base of an exponential into a slope.

2. Pair reciprocal bases

The graph of $(1/b)^x$ equals b^(−x), a reflection in the vertical axis. Corresponding tangent slopes at zero have opposite signs. Compute each positive slope once and use its negative for the reciprocal base.



Reciprocal bases

Reflect in x = 0

Slopes have opposite signs.

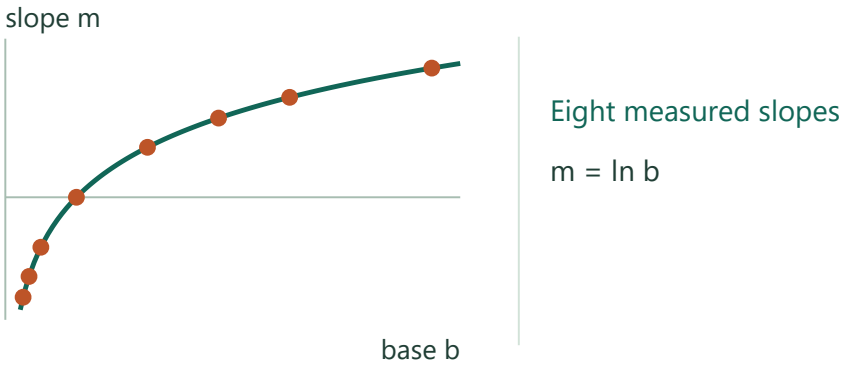
1. $\ln(1/b) = \ln 1 - \ln b$
Use the quotient law for b positive.

2. $\ln(1/b) = -\ln b$
Since ln 1 = 0, reciprocal bases give opposite tangent slopes.

Reflecting the exponential across the vertical axis replaces its base by the reciprocal and reverses the slope.

3. Fill the eight requested entries

For base b>0 use [bʰ−b^(−h)]/(2h) for small h. In the limit m=ln b. For bases 1,2,3,4,6,1/2,1/3,1/4, the slopes are respectively 0,0.693147,1.098612,1.386294,1.791759,−0.693147,−1.098612,−1.386294.



Place the eight bases and their slopes on new axes to read the whole table as a single curve.

Tangent slope at x = 0 for y = bˣ

Base b	Slope m = ln b
1	0
2	0.693147
3	1.098612
4	1.386294
6	1.791759
1/2	−0.693147
1/3	−1.098612
1/4	−1.386294

Check. Replacing b by 1/b reflects the graph in the vertical axis and changes the slope's sign.

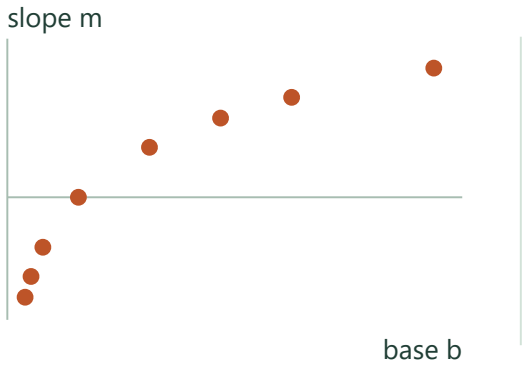
Exercise 609-3 · Straightening the slope table

Printed page 119 · Exponential functions / Data linearization / Logarithms

(Continuation) There are some familiar patterns in the table. Have you ever seen another table of values that exhibits this pattern? Make a scatter plot of the data. Can this nonlinear relationship be straightened?

1. Plot the raw variables

Use the base b for the horizontal coordinate and the tangent slope m for the vertical coordinate. Points with b below 1 fall below the horizontal axis. At b=1 the point is exactly (1,0).



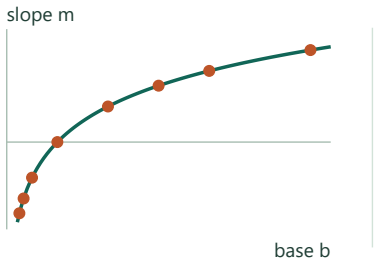
Plot table entries

Each dot is (b, m).

Keep the base as the horizontal variable and the tangent slope as the vertical one.

2. Recognize the logarithm in the scatter

Plot the ordered pairs from 609-2. The scatter follows a rising, concave-down logarithm passing through (1,0). Double the base and each slope rises by ln 2.



Fit the logarithmic shape

m increases by ln 2

whenever b doubles.

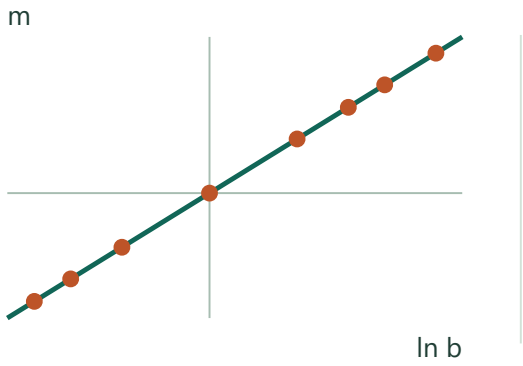
1. $m(2b)-m(b) = \ln(2b)-\ln b$
Evaluate the slope function at the doubled and original positive bases.

2. $m(2b)-m(b) = (\ln 2+\ln b)-\ln b = \ln 2$
The product law separates the constant increment.

Overlay the natural logarithm, which rises but flattens as the base increases.

3. Straighten the curve

Plot m against ln b to obtain a straight line of slope one through the origin. Equivalently, reverse the relation as $b=e^m$. A table of natural logarithms gives the same numbers directly.

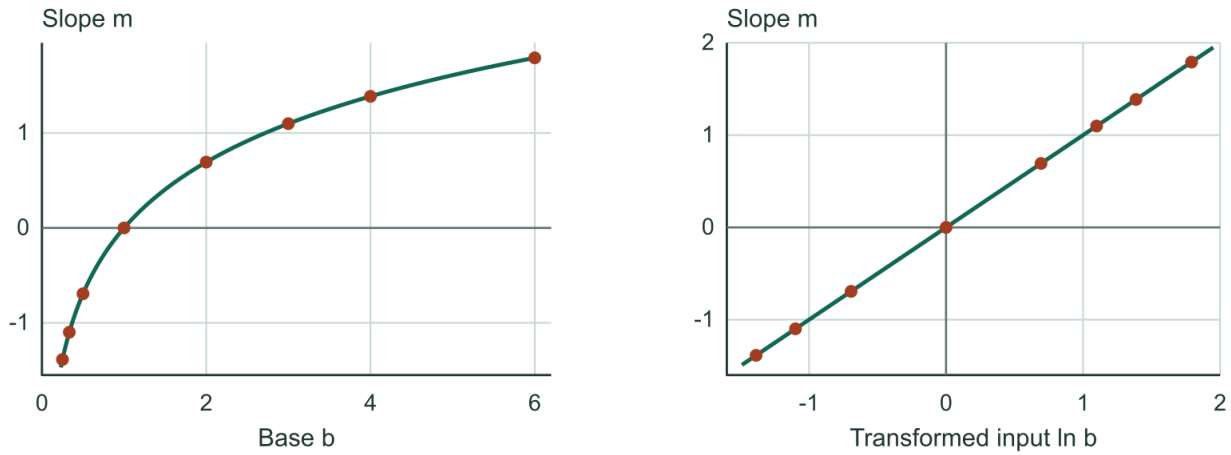


Change the horizontal scale

x = ln b; y = m

The dots lie on y = x.

Replace horizontal b by ln b without moving the vertical readings. The relation becomes a straight line.



Check. A product of bases adds their slopes: ln(2·3)=ln 2+ln 3, consistent with the row for 6.

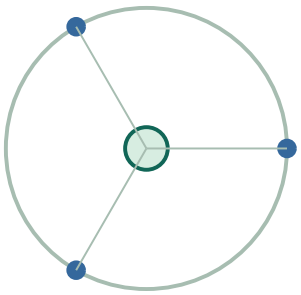
Exercise 612-1 · Three relay satellites

Printed page 119 · Sphere geometry / Spherical coordinates / Right-triangle trigonometry

(Continuation) Show that placing three satellites in one-day orbits (which are called *geosynchronous orbits*) enables communication between almost any two points on Earth.

1. Choose a configuration that can hold its position

Place three satellites equally around a circular orbit in the equatorial plane. They travel eastward in the same direction as Earth's rotation and make one revolution per sidereal day (one rotation relative to the stars). Each then stays over the same longitude as Earth rotates. Neighboring satellite longitudes differ by 120°.



Equatorial view

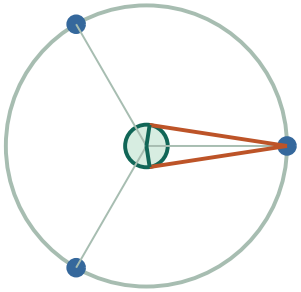
Three orbit positions

separated by 120°.

Space the three satellites equally on the same eastward equatorial orbit.

2. Determine one satellite's horizon

Use circular equatorial geostationary orbits, evenly spaced 120° apart, with relay links. At orbital radius about 6.6 Earth radii, a horizon ray is tangent to Earth. Its footprint half-angle is acos(R/r)≈81°.



Tangent sight lines

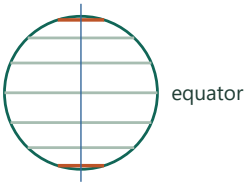
Orbit radius ≈ 6.6R

Horizon angle ≈ 81°

Draw sight lines tangent to the small Earth disk. Tangency sets the boundary of one satellite's footprint.

3. Test the gap between neighbors

On the equator every longitude is within 60° of a satellite. Overlapping footprints cover much of each hemisphere; signals can be relayed between satellites. At high latitudes coverage ends: at the longitude midpoint, cos(distance angle)=cos(latitude)cos 60°, so coverage is guaranteed to about ±72° latitude.



Worst longitude gap: 60°

Orange boundaries delimit guaranteed latitudes.

1. $\cos\delta = \cos\varphi\cos 60° = \tfrac{1}{2}\cos\varphi$
At the most distant longitude in each sector the nearest satellite is 60° away in longitude.

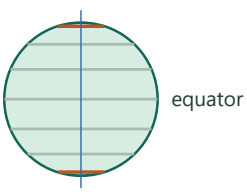
2. $\delta \le \arccos(R/r) \Rightarrow \tfrac{1}{2}\cos\varphi \ge R/r$
Cosine decreases as the central distance increases.

3. $\cos\varphi \ge 2R/r \Rightarrow -\arccos(2R/r) \le \varphi \le \arccos(2R/r)$
Cosine is even. This symmetric belt meets the horizon condition everywhere.

Measure latitude coverage at the longitude halfway between neighboring satellites, the largest possible longitude gap.

4. Measure how much of Earth is in the guaranteed belt

The guaranteed belt |latitude|≤φ has area $4\pi R^2 \sin\varphi$, where φ=acos(2R/r)≈72.4°. Its share of Earth's surface is sin φ≈95.3%. Any two points in this belt can communicate through one or more of the three satellites under the ideal line-of-sight model.



Guaranteed belt

Approximately ±72.4°

≈ 95.3% of surface area

1. $h = R\sin\varphi - (-R\sin\varphi) = 2R\sin\varphi$
The boundary parallels are equally far above and below the equatorial plane.

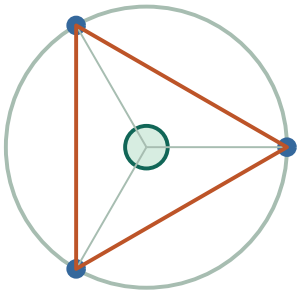
2. $A_{belt} = 2\pi Rh = 4\pi R^2\sin\varphi$
Apply the spherical zone area formula using its vertical height.

3. $A_{belt}/(4\pi R^2) = \sin\varphi \approx 0.953$
Divide by the total surface area.

Convert the limiting latitude to spherical belt area. The remaining caps account for the exception to global coverage.

5. Connect the relay path

Two sites under the same footprint can use the same satellite. With separate footprints they can use a link between satellites: the chord between adjacent satellites comes no closer than r cos 60° to the center, which exceeds R. At either pole the central angle to every subpoint is 90°, beyond the approximately 81° horizon.



Relay triangle

Each link clears Earth.

Polar caps can be missed.

Join the satellites in pairs. The sides clear the planet, so sites served by different satellites can exchange signals.

Check. Adjacent satellites see one another because their connecting chord stays at radius r cos 60°>R. A pole, 90° from all three subpoints, is outside every footprint.

Exercise 620-1 · Dollars per dollar per year

Printed page 119 · Dimensional analysis / Compounding

A simple example of a *rate* is the speed of a car, for example 55 *miles per hour*. Bankers also speak of interest *rates*, as in a savings account whose annual interest rate is 4 *percent*. What are the actual *units* for this rate? You hardly ever hear or see them expressed.

1. Give the percent sign a reference amount

Use $100 of principal for one year. The percentage compares the interest earned during that period with the principal that earned it.

1 **4% = 4/100 = 0.04**

Percent means per hundred.

2 **I = 0.04 × $100 = $4**

For the stated one-year return, 4% of the starting sum is interest.

3 **I/P = $4/$100 = 0.04**

Divide by the positive principal to obtain interest per dollar.

2. Identify the unit left after cancellation

For an annual simple rate r, divide the interest fraction by its time interval. P and the measured interval must both be nonzero to infer a rate this way.

1 **r = I/(P × time)**

The denominator contains the money invested and the time it earns interest.

2 **r = $4/($100 × 1 year)**

Use the observed interest, principal and time from the example.

3 **r = 0.04/year**

The dollar units cancel. The remaining unit is reciprocal years.

4 **0.04/year × $100 × 1 year = $4**

Multiplying by money and time restores dollars of interest.

3. Read an annual rate over longer times

The answer to the units question is dollars of interest per dollar of principal per year. A balance forecast also needs to specify whether later interest earns interest itself.

1 **r = 0.04/year ⇒ r × 1 year = 0.04**

One year converts the rate to the fraction earned in that year.

2 **$100 × (1 + 0.04) = $104**

Adding one year of interest produces this end-of-year balance.

3 **$104 × 1.04 = $108.16**

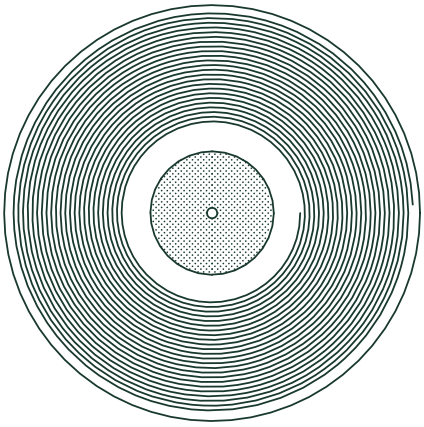
Only if that interest is reinvested at the same annual return does this second step follow.

Check. Multiplying rate by principal and one year gives dollars of interest: 0.04/year × $100 × 1 year=$4.

Exercise 630-1 · How long is a record groove?

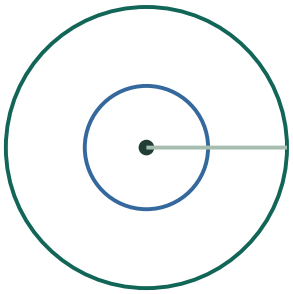
Printed page 119 · Arc length / Measurement and estimation / Angular and linear speed

A typical long-playing phonograph record (once known as an LP) plays for about 24 minutes at 33 1/3 revolutions per minute while a needle traces the long groove that spirals slowly in towards the center. The needle starts 5.7 inches from the center and finishes 2.5 inches from the center. Estimate the length of the groove.



1. Count all the revolutions

At 33⅓ revolutions per minute for 24 minutes the record turns 800 times. Each revolution traces a nearly circular piece of the groove. The outer and inner radii give the range of circumferences.



Outer radius 5.7 in

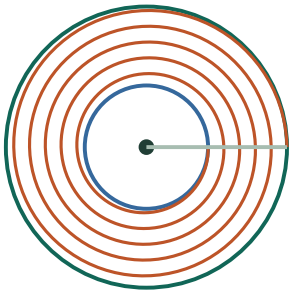
Inner radius 2.5 in

24 minutes → 800 turns

The groove sweeps one circumference per revolution, at gradually smaller radii.

2. Choose a model for spacing

Suppose the inward change in radius per turn is uniform. Between 5.7 and 2.5 inches the total radial change is 3.2 inches, or just 0.004 inch per turn. That small pitch justifies approximating each short spiral turn by a circle at its middle radius.

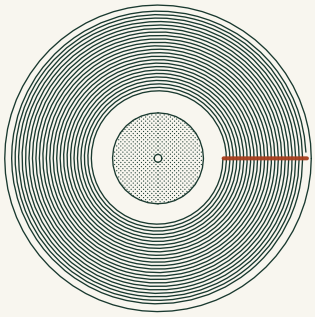


Even radial pitch

Radial step = 0.004 in

Spiral turns exaggerated.

Spread the turns apart in the illustration to make their inward drift visible. The actual pitch is much smaller.

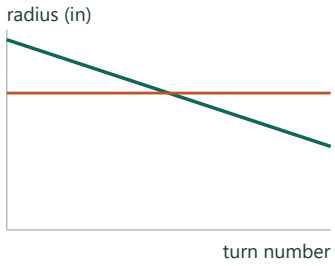


Radial width traversed

Compare the two endpoint radii along one ray. The difference is shared across many revolutions, making the inward movement per turn very small.

3. Add the changing circumferences

With equally spaced radii their mean is (5.7+2.5)/2=4.1 inches. The total is about 800·2π·4.1=6560π inches. Dividing by 12 gives about 1717 feet. The stylized rings in the printed picture illustrate the radial range.



Average radius: 4.1 in

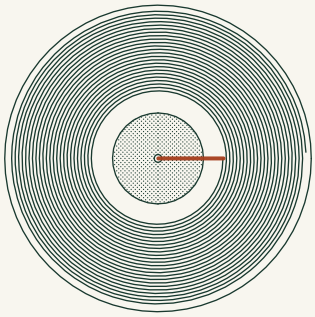
800 circumferences

≈ 1717 feet

1. C_mean = 2π(5.7+2.5)/2 = 8.2π in
Circumference is linear in radius, so average the endpoint radii for uniformly spaced turns.

2. L ≈ 800(8.2π)/12 ft ≈ 1717 ft
Sum 800 representative turns and convert inches to feet.

A linear change in radius has mean equal to the midpoint of the endpoint radii. Multiply its mean circumference by 800.



Inner radius of the groove

Measure from the same center to the inner edge of the grooved band. The innermost rings are visibly shorter per circuit.

Check. Dividing annular area $\pi(5.7^2-2.5^2)$ by the pitch 0.004 in gives the same estimate.

Exercise 635-1 · Energy on a logarithmic scale

Printed page 120 · Logarithms / Unit conversion / Mathematical modeling

Earthquakes can be classified by the amount of energy they release. Because of the large numbers involved, this is usually done logarithmically. The Richter scale is defined by the equation $R = 0.67\log(E) - 1.17$, where R is the rating and E is the energy carried by the seismic wave, measured in kilowatt-hours. (A kilowatt-hour is the energy consumed by ten 100-watt light bulbs in an hour).

(a) The 1989 earthquake in San Francisco was rated at 7.1. What amount of energy did this earthquake release? It could have sustained how many 100-watt light bulbs for a year?

(b) An earthquake rated at 8.1 releases more energy than an earthquake rated at 7.1. How many times more?

(c) Rewrite the defining equation so that E is expressed as a function of R.

(d) Adding 1 to any rating corresponds to multiplying the energy by what constant?

(e) Is it possible for a seismic wave to have a *negative* rating? What would that signify?

1. Part (c) · Isolate the logarithm first

Solving for energy once makes all the comparisons easier. E denotes the numerical energy in kilowatt-hours. The base-ten logarithm requires E > 0, so a real rating represents positive energy.

1 **R = 0.67 log₁₀ E − 1.17**
Start with the energy-rating model supplied in the question.

2 **R + 1.17 = 0.67 log₁₀ E**
Add 1.17 to both sides to undo the subtraction.

3 **(R + 1.17)/0.67 = log₁₀ E**
Divide both sides by the nonzero coefficient 0.67.

4 **E(R) = 10^((R + 1.17)/0.67)**
Raise 10 to both sides. Exponentiation inverts $\log_{10}$ and has positive output.

2. Part (a) · Recover the earthquake's energy

Insert the given rating into the inverse formula. Keep the division inside the exponent and round only after evaluating the power.

1 **E(7.1) = 10^((7.1 + 1.17)/0.67)**
Replace R by the earthquake's rating.

2 **E(7.1) = 10^(8.27/0.67)**
Add 7.1 and 1.17 before dividing.

3 **$E(7.1) \approx 2.2044 \times 10^{12}$ kWh**
Evaluate the power and restore the model's unit of energy.

3. Part (a) · Convert the energy to a bulb count

Assume the bulbs draw 100 W continuously for a 365-day year, and all the stated energy is available to them. Dividing by the annual energy of one bulb counts how many can be powered.

1 **100 W = 0.1 kW**
Divide by 1,000 watts per kilowatt.

2 **365 days × 24 h/day = 8,760 h**
Express a year in the time unit used in kilowatt-hours.

3 **0.1 kW × 8,760 h = 876 kWh**
Energy equals power times time for one continuously lit bulb.

4 **$E(7.1)/876 \approx 2.52 \times 10^{9}$ bulbs**
Both energies use kWh. About 2.52 billion bulbs could run for a year.

4. Parts (b) and (d) · Compare two ratings

A one-unit increase means taking a ratio of two positive energies. Denote the ratio by Q. The additive offset cancels, which explains why the multiplier is independent of the starting rating.

1 **$Q = E(R + 1)/E(R)$**
Use larger energy divided by smaller energy. The denominator is positive.

2 **$Q = 10^{((R + 2.17)/0.67)} / 10^{((R + 1.17)/0.67)}$**
Apply the energy formula at each rating.

3 **$Q = 10^{((R + 2.17 - R - 1.17)/0.67)}$**
Dividing powers with the same base subtracts their exponents.

4 **$Q = 10^{(1/0.67)} \approx 31.084$**
Since 8.1 = 7.1 + 1, its earthquake has about 31.1 times as much energy.

5. Part (e) · Find the boundary for a negative rating

A negative rating is possible. It describes an energy below the scale's zero-rating reference. The energy itself stays positive.

1 **$R < 0 \iff 0.67 \log_{10} E - 1.17 < 0$**
Replace R by the model expression, retaining E > 0.

2 **$\log_{10} E < 1.17/0.67$**
Add 1.17 and divide by positive 0.67, preserving the inequality.

3 **$0 < E < 10^{(1.17/0.67)} \approx 55.75$ kWh**
Base ten is increasing. At the exact upper boundary, the rating is zero.

4 **$E = 1$ kWh $\Rightarrow$ $R = -1.17$**
A concrete example works because $\log_{10} 1 = 0$.

Check. Taking the energy ratio first cancels the constant offset. At rating zero the inverse formula gives exactly the threshold between positive and negative ratings.

Exercise 656-1 · Adding sound intensities

Printed page 120 · Logarithm rules / Mathematical modeling

The human eardrum responds to a very wide range of loudness, tolerating intensities as great as $10^{12} I_0$, where I_0 is the intensity of a barely audible sound. The loudness of a sound whose intensity is I is said to be $10 \cdot \log(I \div I_0)$ *decibels*. On this scale, threshhold sounds $I = I_0$ have decibel level 0, while the loudest tolerable sounds are rated at 120. Intensity is often measured in watts per square meter.

(a) A loud stereo produces sound that is a hundred million times as intense as a threshhold sound. What is the decibel level of such sound?

(b) Suppose that two loud stereos are simultaneously playing the same music, side-by-side. What is the decibel level of their combined sound?

(c) The decibel level of a buzzing mosquito is 40. What is the decibel level of a gang of one thousand identical buzzing mosquitos?

1. Part (a) · Convert one stereo to decibels

Let I_0 be the positive reference intensity and I the intensity at the listener. Both have units of watts per square meter, so their ratio is dimensionless. A finite decibel level requires $I > 0$.

1 **$J = I/I_0 > 0$**

A real logarithm requires a positive argument.

2 **$I = 100{,}000{,}000\, I_0 \Rightarrow J = 10^8$**

Divide the stated intensity by the same reference intensity.

3 **$L = 10 \log_{10}(10^8) = 10 \times 8 = 80$ dB**

A base-ten logarithm returns the exponent on ten.

2. Part (b) · Add equal intensity contributions

At a location receiving equal intensities from the two stereos, the usual intensity-addition model applies if interference averages out. First add the physical intensities, then convert the total to decibels.

1 $I_1 + I_2 = 10^8\, I_0 + 10^8\, I_0 = 2 \times 10^8\, I_0$

Equal contributions double the intensity under this assumption.

2 $L = 10 \log_{10}(2 \times 10^8)$

Divide the combined intensity by I_0 inside the logarithm.

3 $L = 10(\log_{10} 2 + 8)$

Apply the product rule to two positive factors.

4 $L \approx 80 + 3.01 = 83.01$ dB

Doubling intensity adds about three decibels.

3. Part (b) · Check what "same music" specifies

If identical signals arrive perfectly in phase with equal sound-pressure amplitudes, the pressures add coherently. The question gives no arrival phase or listening position, so the combined level depends on the physical assumption.

1 $p + p = 2p$

In phase, identical pressure signals reinforce each other at the listener.

2 $I_{total}/I_{single} = (2p)^2/p^2 = 4$

Intensity scales with mean-square pressure. Use nonzero pressure amplitude p.

3 $L = 10 \log_{10}(4 \times 10^8) \approx 86.02$ dB

Four times one stereo's intensity adds about 6.02 dB to its 80 dB level.

4. Part (c) · Convert the mosquito level back to intensity

Treat the thousand insects as independent sources with equal individual intensities at the listener. The stated 40 dB is a logarithmic level. Recover each intensity before multiplying by the number of sources.

1 **$40 = 10 \log_{10}(I/I_0) \Rightarrow 4 = \log_{10}(I/I_0)$**
Divide both sides by ten.

2 **$I/I_0 = 10^4$**
Exponentiate base ten to undo the logarithm.

3 **$I_{total}/I_0 = 1{,}000 \times 10^4 = 10^7$**
Add the thousand independent contributions, all using the same reference.

4 **$L_{total} = 10 \log_{10}(10^7) = 70$ dB**
The thousandfold intensity increase adds 30 dB.

Check. Multiplying intensity by 1000 adds 30 dB; doubling adds only about 3 dB.

Exercise 680-1 · When doubling stops working

Printed page 121 · Sequences and recursion / Logistic growth

Spreading rumors. It is tempting to think that rumors spread like populations grow. In other words, the more who know a rumor, the more who will learn of it during the next hour. This does not take diminishing opportunity into account, however: The more who know a rumor, the *fewer* there are to tell it to. A simple quadratic model for this phenomenon is obtained by assuming that the number who learn of a rumor each hour is proportional to the number who already know it, and *also* proportional to the number who do not. For example, suppose that three PEA students know a certain rumor at 7 am, that there are 1000 students in all, and that any student will try to pass a rumor along to one more student each hour. This suggests the model $S_n = S_{n-1} + \dfrac{1000 - S_{n-1}}{1000} S_{n-1}$, where S_n is the number of students who know the rumor n hours after 7 am. It is given that $S_0 = 3$. The fraction in the recursive equation represents that portion of the student body that does not know the rumor. Calculate S_1, S_2, and S_3. Your answers will suggest the equation $S_n = S_{n-1} + S_{n-1}$, which describes 100% growth. Explain why this model becomes less realistic as n increases.

1. Estimate who hears new information

Suppose S of the 1000 students know the rumor at the start of an hour. The model gives a random listener probability (1000−S)/1000 of being uninformed. Each of the S tellers makes one contact.



3 of 1000 initially informed

One contact per teller

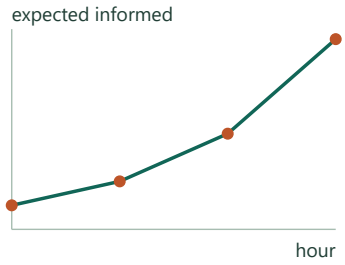
Chance of a new listener

= (1000 − S)/1000

Separate the initially informed students from possible new contacts.

2. Add expected new listeners

From a current count S, the model adds S(1−S/1000), giving next count $2S-S^2/1000$. Starting at 3 gives 5.991, 11.9461, and 23.7495. Fractional counts represent model averages.



Early hours

3 → 5.991 → 11.946

→ 23.750

1. $S' = S+S(1-S/1000)$
Add the model's new listeners to those already informed.

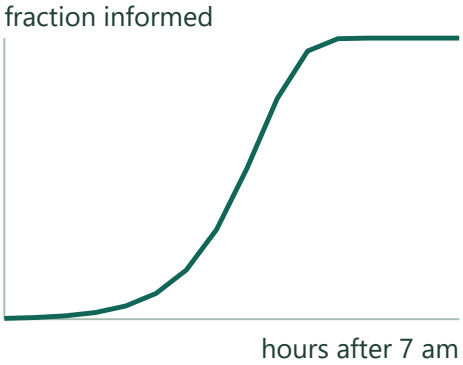
2. $S' = S+S-S^2/1000 = 2S-S^2/1000$
Distribute the product.

3. $S = 3 \Rightarrow S' = 6-9/1000 = 5.991$
Use the starting count to get the next model value.

Mark the first few expected counts after applying the recurrence once per hour.

3. Explain the changing growth rate

Early growth nearly doubles because most contacts are new. Later contacts increasingly reach informed students and the increase slows.



Early: nearly doubling

Later: fewer new listeners.

Over the full day the curve flattens as most tellers reach students who already know.

Check. At S=1000 the increase is zero. Pure doubling would eventually predict more informed students than exist.

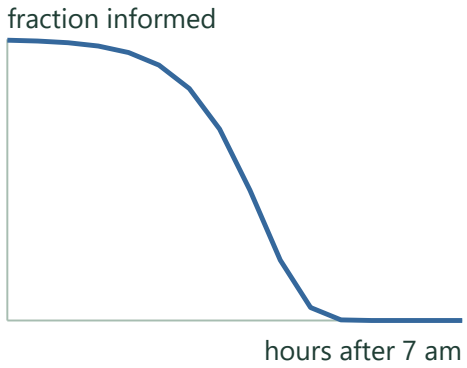
Exercise 680-2 · The uninformed fraction squares itself

Printed page 121 · Sequences and recursion / Logistic growth / Rates of change

(Continuation) It is convenient to apply this equation in a different form, in which S_n is replaced by $1000P_n$. In other words, P_n is the fractional part $S_n/1000$ of the student body that knows the rumor after n hours. After these replacements are made, the recursion simplifies to $P_n = P_{n-1} + (1 - P_{n-1})\,P_{n-1}$, and the initial condition is $P_0 = 0.003$.
(a) Use the recursion and the initial condition to calculate what percentage of the student body knows the rumor by 5 pm. What percentage know by ten o'clock check-in?
(b) Plot the ordered pairs (n, P_n) for $n = 0, 1, 2, \ldots, 15$.
(c) Which hourly increase $P_n - P_{n-1}$ is the greatest?
(d) If each student told *two* students the rumor, how would this affect the recursion?

1. Work with the fraction still unaware

Write P for the informed fraction. From P′=P+P(1−P), subtract from 1. The uninformed fraction becomes $(1-P)^2$ after one hour. Each hour therefore doubles the exponent on the initial unaware fraction 0.997.



Blue: still uninformed

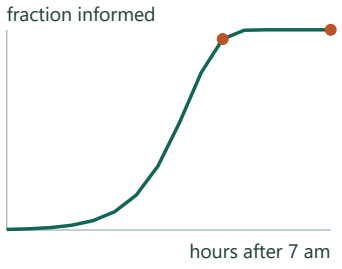
Each hour the fraction

is squared.

The height of the blue curve is the remaining uninformed fraction. Repeated squaring drives it toward zero.

2. Evaluate the two requested times

With P=S/1000, P′=2P−P², hence $1-P' = (1-P)^2$. Thus $P_n = 1 - 0.997^{(2^n)}$. At 5 pm n=10 gives 95.3885%; at 10 pm n=15 the result is 100(1−0.997^32768)%, extremely close to 100%.



5 pm = hour 10

10 pm = hour 15

1. $Q_0 = 1-3/1000 = 0.997$, $Q_{n+1} = Q_n^2$
Track the uninformed fraction instead of the informed fraction.

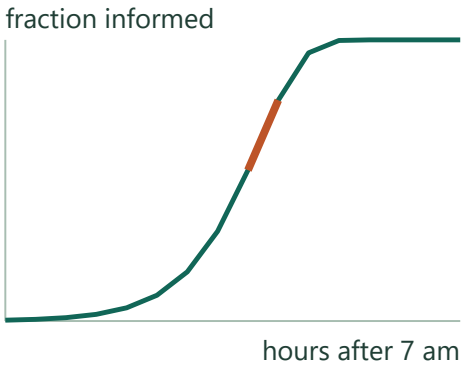
2. $Q_1 = Q_0^2$, $Q_2 = (Q_0^2)^2 = Q_0^4$
Each update doubles the previous exponent.

3. $P_n = 1-Q_n = 1-0.997^{(2^n)}$
After n squarings the exponent is 2^n; take the complement.

Switch to the informed fraction, one minus the blue height, to read the two evening times.

3. Plot the individual hours

For hours 0 through 15 the plotted percentages rounded to tenths are 0.3,0.6,1.2,2.4,4.7,9.2,17.5,31.9,53.7,78.5,95.4,99.8,100.0,100.0,100.0,100.0. The largest gain is hour 8 to 9 (3–4 pm), about 24.87 percentage points.



Largest increase

Hour 8 → 9

≈ 24.87 percentage points

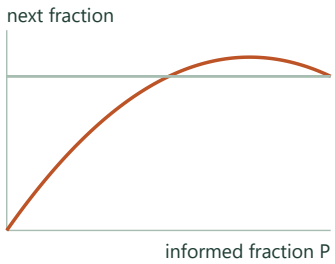
Compare successive vertical gains: the segment between hours eight and nine is steepest in the discrete model.

Rumor model by hour after 7 am, rounded to 0.1 percentage point

Hour	Informed (%)	Hour	Informed (%)
0	0.3	8	53.7
1	0.6	9	78.5
2	1.2	10	95.4
3	2.4	11	99.8
4	4.7	12	100.0
5	9.2	13	100.0
6	17.5	14	100.0
7	31.9	15	100.0

4. Test a two-contact rule

With two contacts, doubling the gain gives the formal rule $P' = P + 2P(1-P)$. It can exceed 1; accounting for duplicate contacts is necessary for a bounded model.



Two-contact shortcut

The orange prediction can exceed 1.

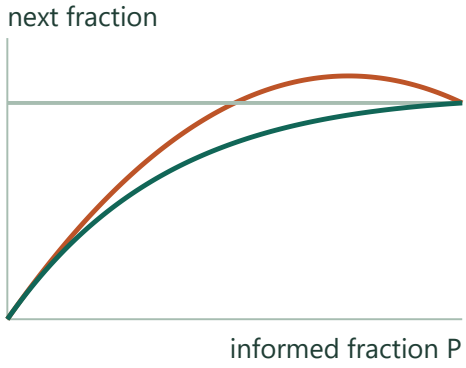
1. $P' - 1 = 3P - 2P^2 - 1 = (1-P)(2P-1)$
Subtract one from the two-contact rule and factor.

2. $\frac{1}{2} < P < 1 \Rightarrow P' - 1 > 0$
Both factors are positive on this interval, so the rule overcounts students.

Change to next-versus-current axes. A naive second contact doubles the gain and can predict more than the whole school.

5. Count duplicate contacts explicitly

For one explicit model with duplicate contacts, suppose each of the 2S calls independently selects one of the 1000 students uniformly, with replacement. A particular uninformed student then misses all calls with probability (999/1000)^(2S). The conditional expected new count is (1000−S)[1−(999/1000)^(2S)], always at most the number still uninformed.

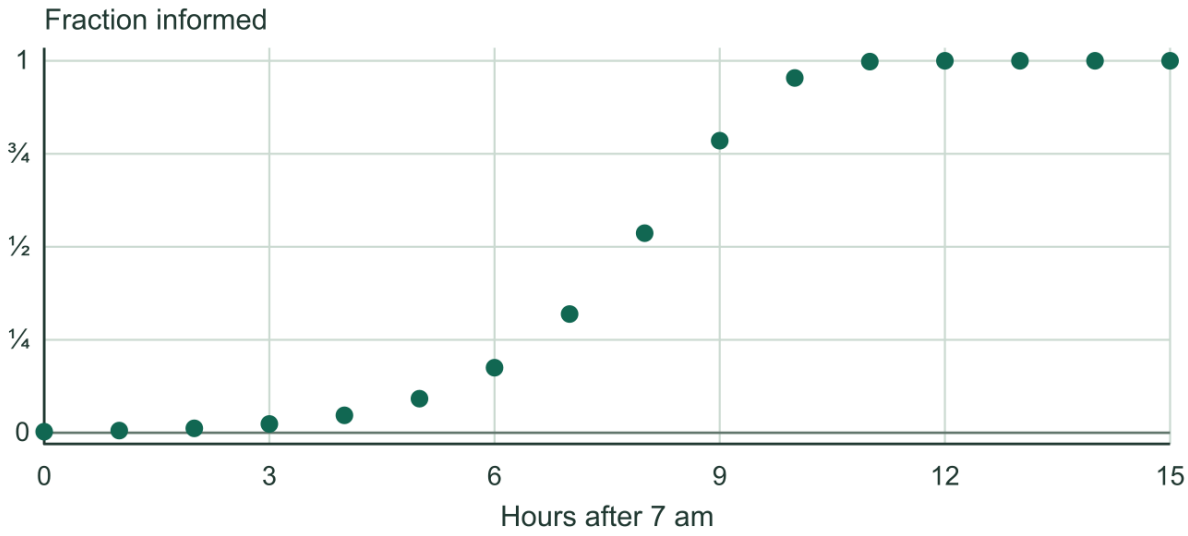


Account for repeated calls

Green: each listener is

counted at most once.

For independent calls with replacement the green calculation removes duplicate listeners. The updated fraction stays bounded by one.



Check. Differences equal $P_n(1-P_n)$, largest when the starting fraction lies closest to 1/2. Of the sampled values $P_8 \approx .537$ is closest.

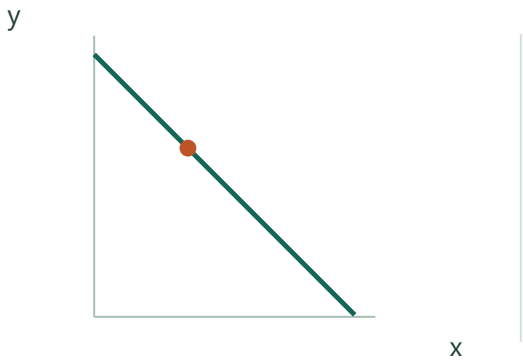
Exercise 728-1 · The limit on a line

Printed page 121 · Transition matrices / Geometric sequences and series / Limits and convergence

(Continuation) Show that the distances between consecutive points form a geometric sequence. What is the limiting position of this sequence of points? Suggestion: vectors might prove easier than the Pythagorean Theorem.

1. Use a preserved quantity

The transition matrix has column sums 1, so it preserves the sum of the two coordinates. Starting from a sum of 14 means every vector head lies on x+y=14. This turns the two-dimensional iteration into one variable.



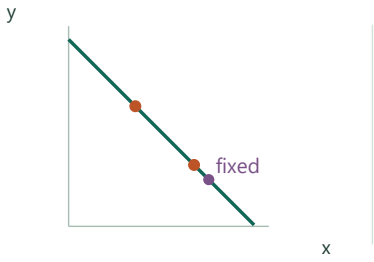
Invariant x + y = 14

Start at (5,9).

A column sum of one fixes the coordinate sum; the initial head sits on the indicated line.

2. Find the fixed point

The column sums preserve x+y=14. Write y=14−x to obtain x′=.2x+8.4. At equilibrium x=10.5 and y=3.5. Subtracting equilibrium shows each displacement scales by .2.



Fixed point (10.5, 3.5)

x′ = 0.2x + 8.4

1. x′ = 0.8x+0.6(14−x) = 0.2x+8.4
Eliminate y using the invariant total.

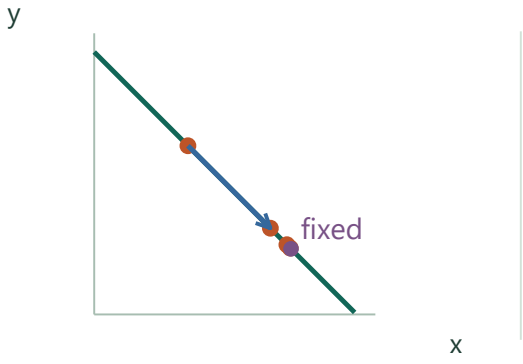
2. x* = 0.2x*+8.4 ⇒ 0.8x* = 8.4 ⇒ x* = 10.5
At equilibrium the new and old first coordinates coincide.

3. x′−10.5 = 0.2x−2.1 = 0.2(x−10.5)
Subtract the fixed point and factor.

Mark its fixed point and the first iterate. The remaining horizontal gap is one fifth of the preceding gap.

3. Follow the changes between consecutive terms

If x′=0.2x+8.4, applying the rule to two successive inputs multiplies their difference by 0.2. The second coordinate changes by the negative of the first, since the sum is fixed. Consequently each step vector, and hence its length, shrinks by this same positive factor.



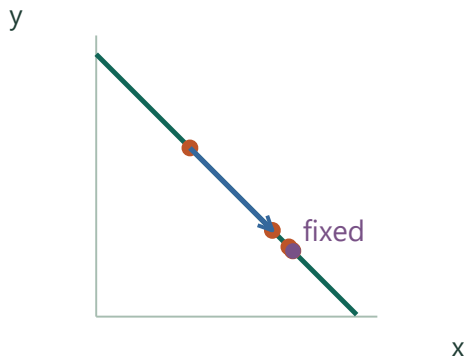
Steps shrink by 0.2

Every step lies on the line.

The arrows between successive heads point along the same line and shrink geometrically.

4. Measure the first step and the limiting head

The first step is (4.4,−4.4), so the successive distances are 4.4√2·(.2)^n for n=0,1,…; the limiting head is (10.5,3.5).



First step (4.4, −4.4)

Length 4.4√2; then each

length is multiplied by 0.2.

The shrinking arrows converge to the marked head while their lengths tend to zero.

Check. Multiplication by the original matrix fixes (10.5,3.5) and sends the direction (1,−1) to .2(1,−1).

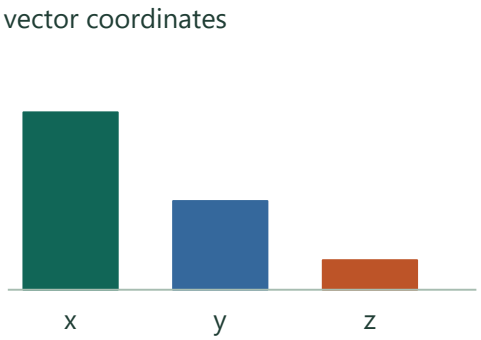
Exercise 736-1 · A matrix geometric sequence

Printed page 121 · Transition matrices / Sequences and recursion / Steady-state distributions

Choose a 3-by-3 matrix $\mathbf{M}$ of positive numbers so that the sum of the entries in each column is 1. Choose a column matrix $\mathbf{v}$ whose three positive components sum to 100.
(a) Why is $\mathbf{v}$, $\mathbf{Mv}$, $\mathbf{M}^2\mathbf{v}$, … called a *geometric* sequence? Calculate some of its terms.
(b) If these vectors are drawn with their tails at the origin, the heads define a sequence of coplanar points. Explain why.
(c) What seems to be happening to its terms? Calculate $\mathbf{M}^n$ for some large exponent n.

1. Choose a matrix with the required column totals

Place 0.6 on each diagonal and 0.2 in each remaining position. Every entry is positive and 0.6+0.2+0.2=1 in every column. Pick (60,30,10) as a starting column whose total is 100.



Multiplications: 0

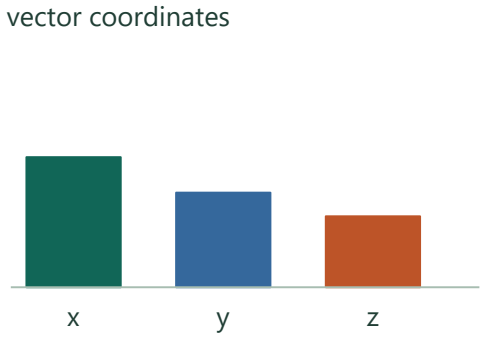
60.00, 30.00, 10.00

Sum remains 100.

Start the three bars at the coordinates of a vector whose total is 100.

2. Multiply to get initial terms

Example: take M with diagonal .6 and all off-diagonals .2, starting at v=(60,30,10). Each term is the preceding one multiplied by the same matrix. First terms are (44,32,24), then (37.6,32.8,29.6).



Multiplications: 1

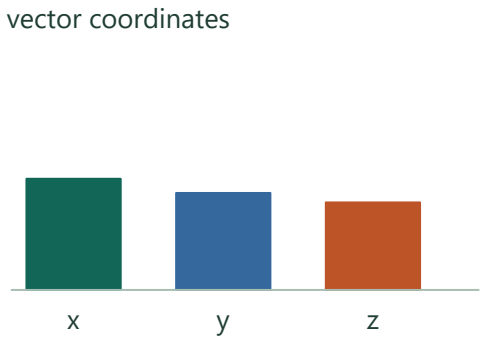
44.00, 32.00, 24.00

Sum remains 100.

A matrix multiplication redistributes the total; each coordinate moves toward the common mean.

3. Separate the mean from the deviation

Let J replace all three coordinates by their average. Then $J^2=J$, and $(I-J)$ keeps only the deviation from the mean. Multiplication by M keeps the mean unchanged and scales every deviation by 0.4. After n multiplications the deviation is multiplied by 0.4^n.



Multiplications: 2

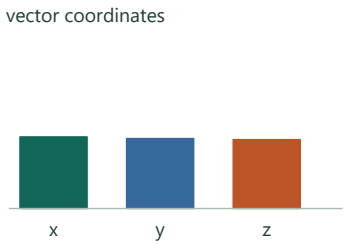
37.60, 32.80, 29.60

Sum remains 100.

Compare the deviations from 100/3 after another multiplication: only 40% of each prior deviation remains.

4. Interpret the plane and the matrix power

The coordinate sum stays 100 because column sums are 1. The heads lie in plane x+y+z=100. Put J equal to the matrix with every entry 1/3. Then $M=J+.4(I-J)$ and $M^n=J+.4^n(I-J)$. The limit vector is (100/3,100/3,100/3).



Multiplications: 4

34.02, 33.25, 32.74

Sum remains 100.

1. $J(I-J) = J-J^2 = 0$
Averaging twice gives the same mean; the mean of the deviations is zero.

2. $(I-J)^2 = I-2J+J^2 = I-J$
The projection onto deviations is also unchanged when repeated.

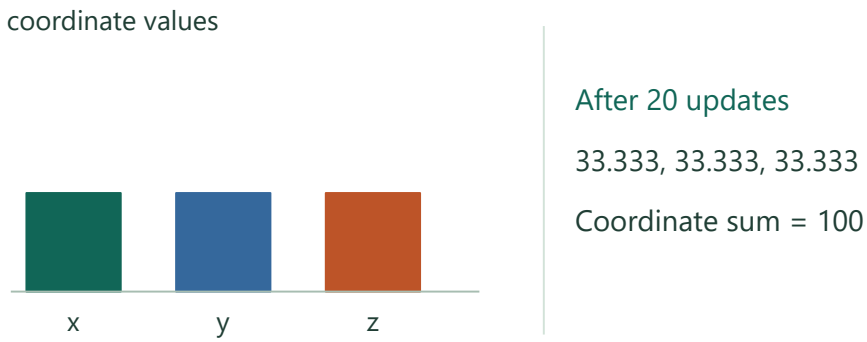
3. $M^n = J+0.4^n(I-J)$, $n \geq 1$
In powers of the sum, all mixed products vanish and the remaining projections are idempotent.

The endpoints remain in the constant-sum plane. The three coordinates approach equal heights.

5. Calculate a large power

At n=20 the diagonal entries are $1/3+(2/3)(0.4)^{20}$ and off-diagonal entries are $1/3-(1/3)(0.4)^{20}$. Both differ from 1/3 by less than 10^{-8}, so each coordinate approaches the original mean 100/3.

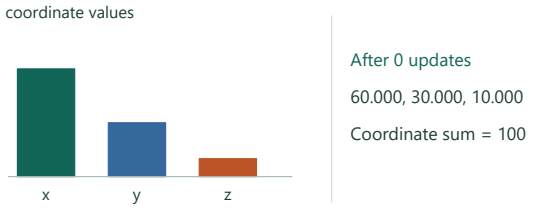
Matrix multiplications · 20



At the twentieth power the remaining differences are invisible on this scale, as predicted by the factor 0.4^{20}.

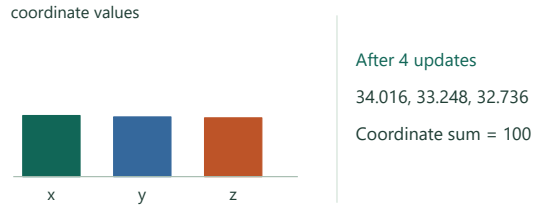
Compare settings · Matrix multiplications

0



After 0 multiplications the vector is (60.000, 30.000, 10.000). The common mean remains 100/3.

4



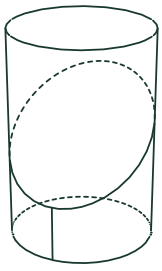
After 4 multiplications the vector is (34.016, 33.248, 32.736). The common mean remains 100/3.

Check. At n=20 each entry differs from 1/3 by less than 10^{-8}. The limit vector is fixed by M. Specifically the diagonal entries of M^{20} are $1/3+(2/3)(0.4)^{20}$ and the off-diagonal entries are $1/3-(1/3)(0.4)^{20}$.

Exercise 744-1 · Unwrapping a candle

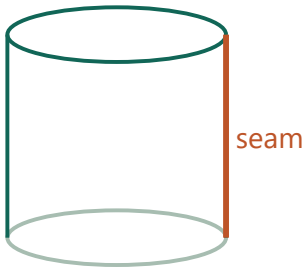
Printed page 121 · Sinusoidal functions / Nets and surface development / Cross sections

Jackie wraps a sheet of paper tightly around a wax candle whose diameter is two inches, then cuts through them both with a sharp knife, making a 45-degree angle with the candle's axis. After unrolling the paper and laying it flat, Jackie sees the wavy curve formed by the cut edge, and wonders whether it can be described mathematically. Sketch this curve, then show that it is sinusoidal. It is very helpful to use radian measure to describe angles in this problem.



1. Choose a seam and a height origin

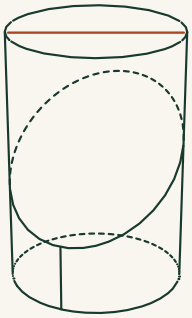
Let the seam be where the slanted cut is highest, and let z_0 be the height where the cutting plane meets the candle's vertical axis. The diameter is 2 inches, so the cylinder radius is 1 inch.



Straight seam

Radius is one inch.

Choose the uphill side as the seam so the unrolled curve starts at its maximum height.

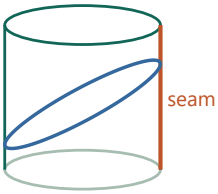


Diameter of the circular top

The top is drawn as an ellipse in perspective. The horizontal span represents a true diameter, so the center-to-rim distance is half this length.

2. Intersect the plane and the circular wall

Choose the horizontal x-axis in the steepest uphill direction. The 45° cut plane has equation $z=z_0+x$. On the cylinder $x=\cos\theta$, $y=\sin\theta$, so $z=z_0+\cos\theta$.



Sloping cut

$z - z_0 = x = \cos\theta$

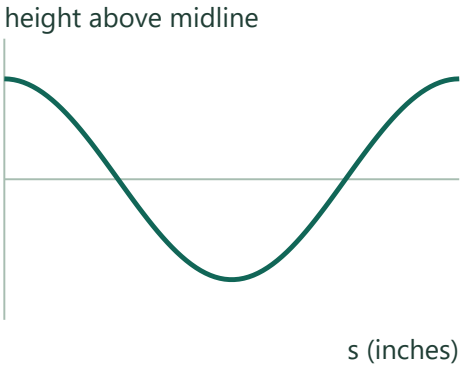
1. $(z-z_0)/x = \tan 45° = 1$ $(x \neq 0)$
Take rise over horizontal run along the steepest direction.

2. $z = z_0+x = z_0+\cos\theta$
Multiply by x and substitute the rim's x-coordinate. The equation also holds at x=0.

Intersect the sloping cut plane with the cylindrical wall. The vertical coordinate varies with horizontal x.

3. Unroll the horizontal distance

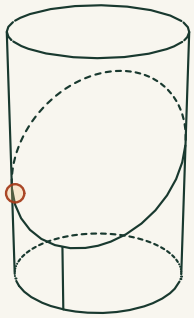
After unrolling, horizontal arc length $s = r\theta = \theta$ inches. The graph $z = z_0 + \cos s$ is one cosine wave, with amplitude 1 inch and wavelength 2π inches. Where the seam is placed changes the phase.



Horizontal arc length s

One turn: 2π inches

Cut the seam and flatten the wall. Arc length now plays the role of the horizontal coordinate.

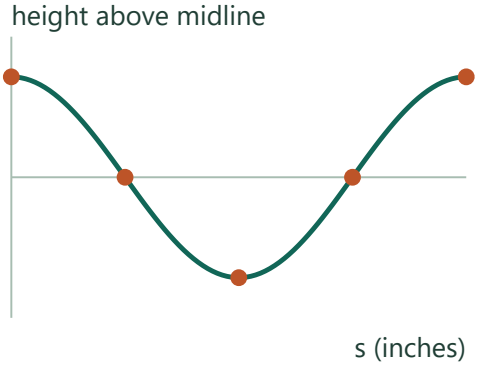


One height on the circular wall

A point on this cut has both an angular position around the axis and a vertical coordinate. Opening the sheet preserves the latter and straightens the horizontal distance around the candle.

4. Check the cut at four positions

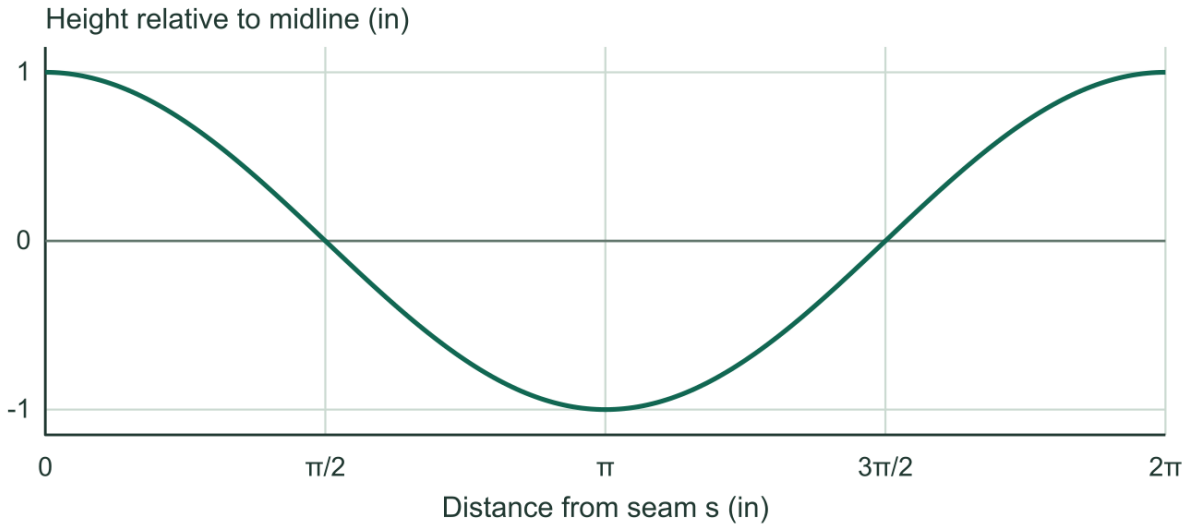
Starting at the seam, the cut is at height $z_0 + 1$. A quarter turn later it is at z_0, halfway around it is at $z_0 - 1$, and after a full turn it returns to its starting height. These positions explain the crest, crossings and trough in the unrolled graph.



Quarter-turn checkpoints

$+1,\ 0,\ -1,\ 0,\ +1$ in

At quarter-turn intervals the elevation relative to z_0 is alternately extreme or zero.



Check. Opposite points have heights z_0+1 and z_0-1, a difference of one diameter as the incline predicts.

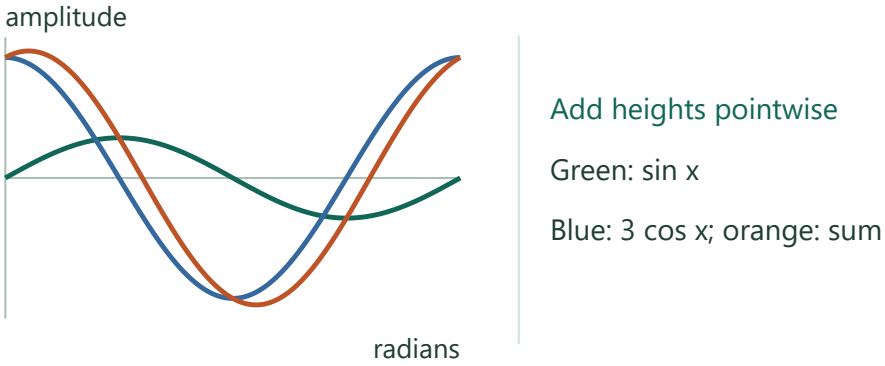
Exercise 745-1 · One wave from two

Printed page 122 · Sinusoidal functions / Angle addition formulas / Phase shifts

Show that the curve $y = \sin x + 3\cos x$ is sinusoidal, by writing it in an equivalent form $y = a\sin(x+p)$.

1. Expand the target form

Using angle addition, a sin(x+p)=a cos p·sin x+a sin p·cos x. For the coefficients to match the given expression for all x, require a cos p=1 and a sin p=3.



Add heights pointwise

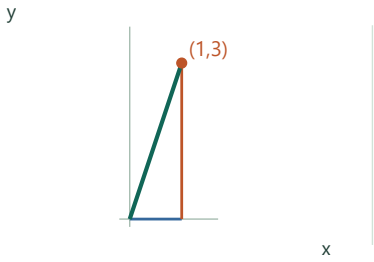
Green: sin x

Blue: 3 cos x; orange: sum

Add the two component heights at each x to obtain the orange signal.

2. Recover the amplitude

Square and add the two equations. The identity $\cos^2 p+\sin^2 p=1$ leaves $a^2=1^2+3^2=10$. Choose the positive amplitude $\sqrt{10}$.



Coefficient triangle

Legs 1 and 3

Hypotenuse $a = \sqrt{10}$

1. $(a\cos p)^2+(a\sin p)^2 = 1^2+3^2$
Square the two matched coefficients, then add.

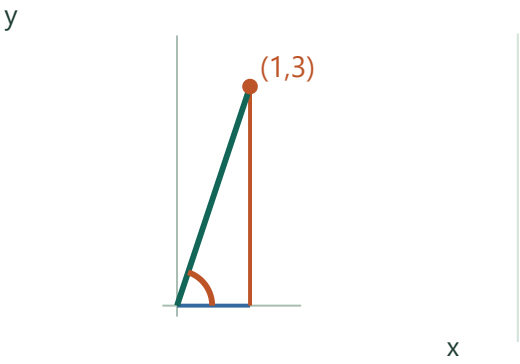
2. $a^2(\cos^2 p+\sin^2 p) = 10 \Rightarrow a^2 = 10$
Factor out a^2 and use the unit-circle identity.

3. $a = \sqrt{10}$
Choose the standard positive amplitude.

Treat the two coefficients as perpendicular legs. Their hypotenuse determines the amplitude.

3. Locate the phase

Divide the second coefficient equation by the first to obtain $\tan p=3$. Both sine and cosine coefficients are positive, placing p in quadrant I. Hence $y=\sqrt{10}\sin(x+\arctan 3)$.



Choose a first-quadrant p

$\tan p = 3$

$y = \sqrt{10}\sin(x + p)$

Locate the phase angle in the coefficient triangle, using the positive signs to choose its quadrant.

Check. At $x=0$ both sides are 3; at a quarter-turn both are 1. Matching the two coefficients proves equality for every x.

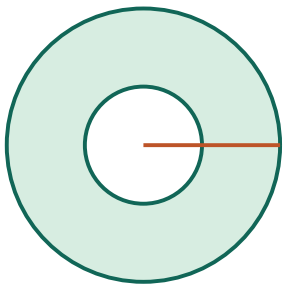
Exercise 745-2 · Unrolling a paper spool

Printed page 122 · Ring and shell geometry / Measurement and estimation

A long strip of paper, whose thickness is 0.002 inch, is rolled tightly on a cylinder whose radius is 0.75 inch. This produces a cylinder whose radius is 1.75 inches. Estimate the length of the paper strip. What assumptions did you make?

1. Look down the coil's axis

A tight roll forms an annulus. The area of its material is the outer disk area minus the core area. The paper's width along the cylinder axis is the same everywhere and cancels from the volume comparison.



Annular cross section

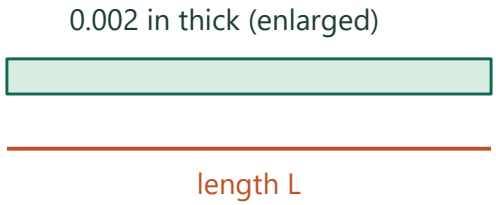
Outer radius 1.75 in

Inner radius 0.75 in

The hole contains no paper, so subtract its disk area from the outer disk area.

2. Picture the paper straightened

Unrolled without stretching, the same cross section becomes a long rectangle with thickness 0.002 inch and unknown length L. Equate its area $0.002L$ to $\pi(1.75^2-0.75^2)$.



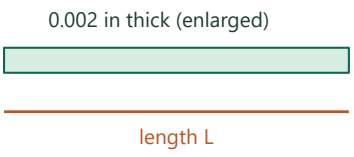
Unrolled cross section

Area = 0.002L

Flattening the paper keeps its material area while turning the annulus into a thin rectangle.

3. Solve with units and assumptions

Compare cross-sectional areas. The paper occupies $\pi(1.75^2-0.75^2)=2.5\pi$ square inches. Straightened, its cross section is $0.002L$. Thus $L=1250\pi$ in$\approx$327.25 ft. Assume uniform thickness, tight layers, and negligible gaps or compression.



Match the two areas

$0.002L = 2.5\pi$ in^2

$L = 1250\pi$ in $\approx$ 327 ft

1. $1.75^2-0.75^2 = (1.75-0.75)(1.75+0.75) = 2.5$
Factor the difference of squares.

2. $L = 2.5\pi/0.002 = 1250\pi$ in
Divide the unrolled area by its nonzero thickness.

3. $L = 1250\pi/12$ ft ≈ 327.25 ft
Convert inches of length to feet.

Match the two cross sections and divide the area by the paper thickness to recover a length.

Check. The one-inch radial build corresponds to 500 layers. At mean radius 1.25 inches their total circumference is $500\cdot 2\pi\cdot 1.25$.

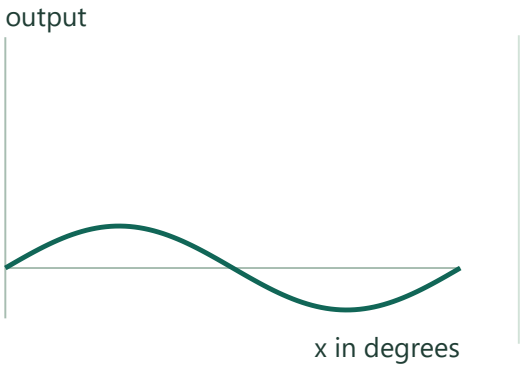
Exercise 775-1 · Three transformations of one oscillation

Printed page 122 · Graph transformations / Periodic functions / Domain and range

What do the functions $f(x) = \log(2 + \sin 60x)$, $g(x) = 2^{\sin 60x}$, and $h(x) = |2 + 3\sin 60x|$ have in common? How do they differ?

1. Follow one sine wave

Write u=sin(60x). Mark five equally spaced positions across one cycle and label their u-values 0, 1, 0, −1, 0. The three functions all read this same input. In degrees a full cycle takes 360/60=6 units of x. In radians it takes 2π/60=π/30.



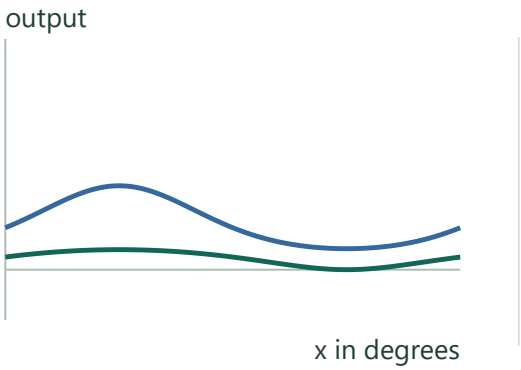
u = sin(60x)

Period in degree mode: 6

First graph the sine input over one period before applying an outer operation.

2. Transform the vertical scale

For the logarithm, 2+u stays between 1 and 3, so log is always defined and increases from 0 to $\log_{10}3$. With the exponential, u becomes an exponent: $2^{-1}=1/2$, $2^{0}=1$ and $2^{1}=2$. Both functions increase as u increases and have smooth curves.



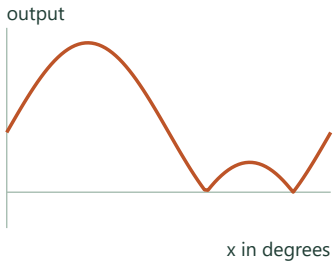
Green: $\log_{10}(2 + u)$

Blue: 2^{u}

The logarithm and the exponential change heights monotonically while keeping the same phase of maxima.

3. Locate the fold

Before taking the absolute value, 2+3u runs from −1 to 5. Where u drops below −2/3 the graph would be below zero. Reflect that part upward. The crossings at u=−2/3 become corners and the old trough at u=−1 becomes a small peak of height 1.



Orange: |2 + 3u|

The negative lobe folds up.

1. 2+3u < 0 ⇔ 3u < −2 ⇔ u < −2/3
Subtract two and divide by positive three to find the interval being folded.

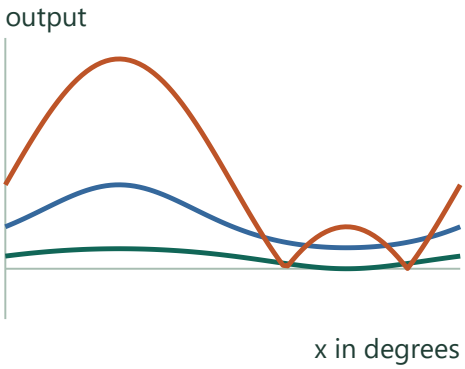
2. u < −2/3 ⇒ y = −(2+3u)
The absolute value reverses a negative height.

3. u ≥ −2/3 ⇒ y = 2+3u
On the other part of the wave the height stays unchanged.

Taking an absolute value reflects the negative piece of 2+3u upward, introducing corners.

4. Compare the graphs

All three have the same domain, period and highest-point locations. Only the folded curve has two zeros per cycle and a secondary peak. Sketch each against the same five positions, adding the two fold points to the last graph.



One period, three outputs

Two corners in the

absolute-value graph.

Align the three curves at the same x. Only the orange curve hits zero twice per cycle.

Check. At u=0 the values are $\log_{10}2$, 1, and 2. All peak when u=1; the third reaches its minimum at a different phase.

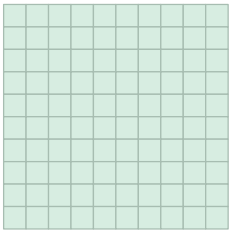
Exercise 783-1 · Layers of paint

Printed page 122 · Counting and permutations / Three-dimensional geometry

A large wooden cube is formed by gluing together 1000 small congruent cubes, and then it is painted red. After the paint is dry, the large cube is taken apart into small cubes again. How many of these small cubes have paint on three of their faces? on exactly two of their faces? on exactly one of their faces? on none of their faces?

1. Mark where paint can reach

Ten cubes fit along each edge because $10^3=1000$. A small cube has one painted face for each coordinate that lies on an outside layer. A corner lies on three such layers, an ordinary edge cube on two, and a face-interior cube on one.



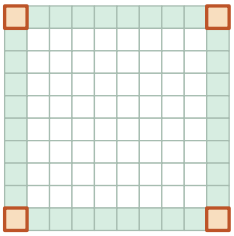
One exterior face

10 × 10 small squares

Look straight at one painted face. A cube's position relative to the outer boundaries determines how many faces receive paint.

2. Count corners and open edges

There are eight corners. Each of the twelve edges has ten positions, but its two endpoints are already counted. Thus exactly two faces are painted on $12\times8=96$ cubes.



Four corners on this face

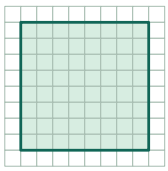
Each edge also has

8 noncorner positions.

Separate whole-cube corners from the open interiors of edges to avoid counting an endpoint twice.

3. Peel away the outer shell

On each face, removing its boundary leaves an 8-by-8 array. Six faces give 384 cubes with one painted face. Removing one layer on both sides of every coordinate leaves an 8-by-8-by-8 core, so 512 cubes have no paint.



Face interior: 8 × 8

The fully unpainted core

has a third factor of 8.

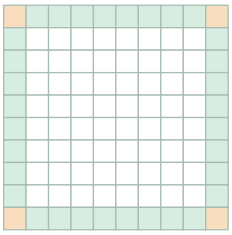
1. $N_1 = 6(10-2)^2 = 6\cdot64 = 384$
For exactly one painted face, exclude both outer positions in each of the other two directions.

2. $N_0 = (10-2)^3 = 512$
An unpainted cube must be interior along all three axes.

Remove the outside border of each face, then think one layer inward on all six sides to locate the unpainted core.

4. Check in two ways

The four groups contain 8+96+384+512=1000 cubes. Counting painted squares instead gives 3×8+2×96+384=600, the same as six 10-by-10 faces.



Check painted area

3×8 + 2×96 + 384

= 600 face squares

Count painted unit squares again, allowing a corner cube to contribute three and an edge cube two.

Check. 8+96+384+512=1000. The painted-face count $3{\cdot}8+2{\cdot}96+384=600$ equals $6{\cdot}10^2$.

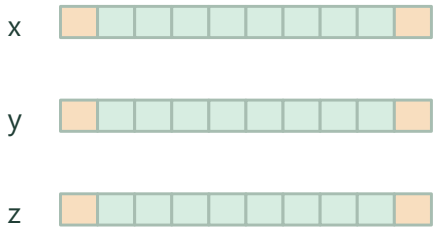
Exercise 783-2 · A binomial that counts painted faces

Printed page 122 · Binomial coefficients / Counting and permutations

(Continuation) Here is a curiosity: Expand the binomial power $(a+b)^3$ and then replace a by 2 and replace b by 8.

1. Choose a coordinate three times

In each direction distinguish the two exterior positions from the eight interior positions. Write a for the first count and b for the second. Choosing positions independently along three directions gives (a+b)(a+b)(a+b).



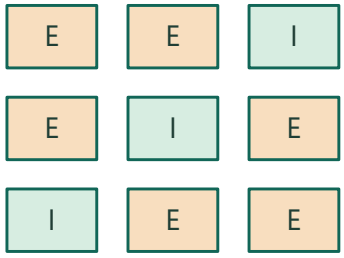
Each coordinate

2 ends, 8 interior slots

The orange fraction of each coordinate strip consists of its two outside positions; the eight other positions are interior.

2. Collect choices by type

Selecting a three times gives a^3. Selecting a twice and b once gives $3a^2b$, since the b can be chosen in three positions. Similarly one a gives $3ab^2$, and selecting b three times gives b^3.



E = exterior position

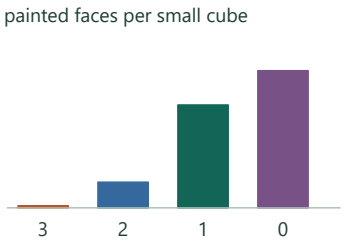
I = interior position

Three choices for one I.

Choosing which coordinate directions reach an end partitions the cube into four disjoint groups.

3. Substitute the cube counts

With a=2, b=8, the four terms are 8, 96, 384 and 512. They count cubes with three, two, one and zero painted faces in that order. Their sum recovers 10^3.



$(2 + 8)^3 = 1000$

8 + 96 + 384 + 512

1. $3a^2b = 3{\cdot}2^2{\cdot}8 = 96$
Choose the one interior direction and count two boundary coordinates.

2. $3ab^2 = 3{\cdot}2{\cdot}8^2 = 384$
Choose the one boundary direction and count two interior coordinates.

Evaluate the four terms of the binomial expansion to recover the counts in the preceding painting problem.

Check. Independent choices of position in the three directions give $(2+8)^3=10^3$.

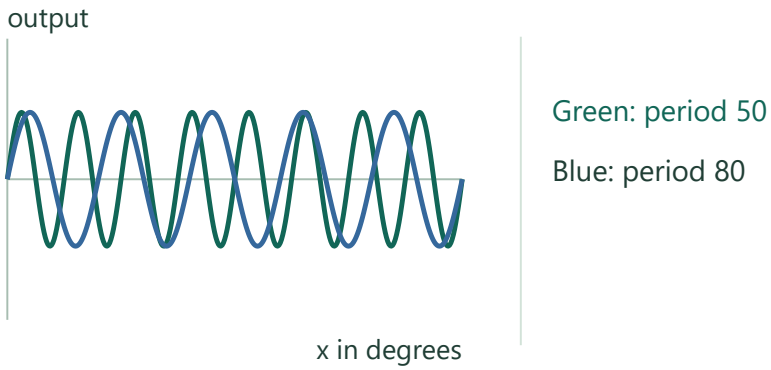
Exercise 788-1 · Beating periods

Printed page 122 · Periodic functions / Amplitude and period

Find periods for $f(x) = \sin(7.2x)$, $g(x) = \sin(4.5x)$, and $h(x) = f(x) + g(x)$.

1. Find each separate repeat

A sine repeats after its angle increases by 360° in degree mode. For a horizontal shift T, solve 7.2T=360 and 4.5T=360. The two periods are 50 and 80 units of x.



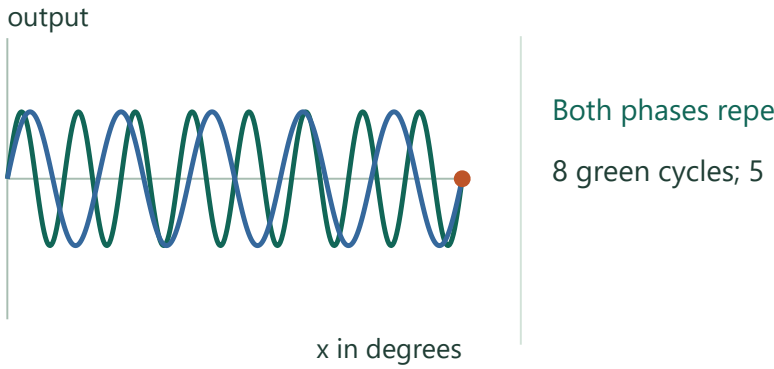
Green: period 50

Blue: period 80

Plot the component sines on the same interval. The faster wave fits more cycles.

2. Make both waves start over

A common shift satisfies T=50m=80n with positive integers m,n. Divide by ten to get 5m=8n. Since 5 and 8 are coprime, the first match is m=8, n=5, hence T=400.



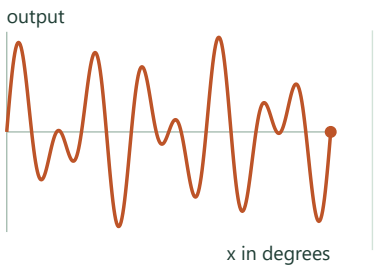
Both phases repeat at 400

8 green cycles; 5 blue

The phases next meet at their starting values and directions after eight and five complete revolutions.

3. Check that the sum has no shorter period

Shifting a sine by T replaces it by a linear combination of sine and cosine at its own frequency. The two distinct frequencies are independent (over a common cycle, their products integrate to zero). For the sum to be identical for every x, each frequency must therefore return to its own original phase.



Sum of distinct frequencies

Both phases must repeat.

1. $\sin(\omega(x+T)) = \sin(\omega x)\cos(\omega T)+\cos(\omega x)\sin(\omega T)$
Expand the translated component by angle addition.

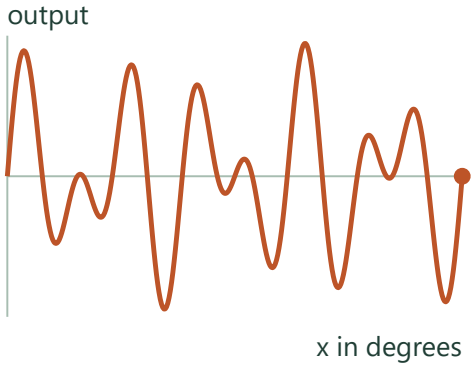
2. $\cos(\omega T) = 1$, $\sin(\omega T) = 0$
Independence of the sine and cosine components at each distinct frequency forces their original coefficients.

3. $\omega T = 360° \cdot k$
A positive period must return each degree-mode phase to a complete turn.

Add the ordinates at each x; both constituent frequencies must return to their starting phase for the entire function to repeat.

4. Convert to radians

Replace the 360-degree angle increment by 2π. This gives 5π/18 and 4π/9. Eight repeats of the first, or five of the second, give the sum's period 20π/9.



Same full cycle in radians

400° = 20π/9 radians

Read the same horizontal interval using radians. All periods are rescaled by π/180.

Check. Dividing the frequencies by their common factor 0.9 yields coprime integers 8 and 5.

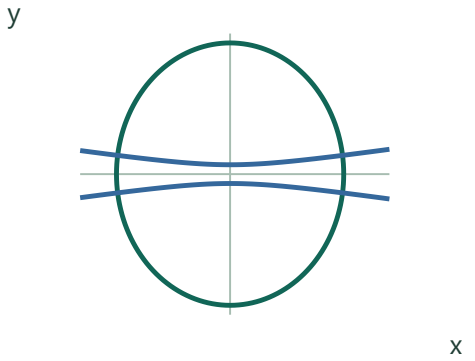
Exercise 795-1 · Orthogonal confocal conics

Printed page 122 · Ellipses / Hyperbolas / Orthogonality

(Continuation) Sketch the hyperbola $48y^2 - x^2 = 48$ and the ellipse $4x^2 + 3y^2 = 588$. Verify that they are confocal and share point P. The two tangent lines to the curves at point P are related. Sketch the tangent lines and explain the relationship. Note that you have encountered the equations of *both* tangent lines already.

1. Put both curves on one sketch

Divide the hyperbola by 48 and the ellipse by 588. The hyperbola $y^2 - x^2/48 = 1$ opens up and down from $(0, \pm 1)$, approaching $y = \pm x/\sqrt{48}$. The ellipse $x^2/147 + y^2/196 = 1$ meets the axes at $(\pm\sqrt{147}, 0)$ and $(0, \pm 14)$.



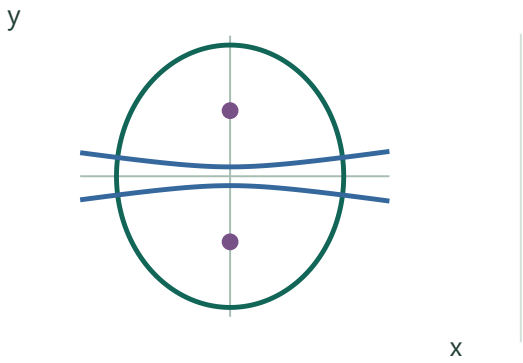
Ellipse and hyperbola

Both have a vertical axis.

Draw both curves on equal coordinate scales so their tangent angles can be compared.

2. Compare focal distances

The hyperbola uses $c^2=a^2+b^2=1+48$; the ellipse uses $c^2=a^2-b^2=196-147$. Both have a vertical focal axis and $c=7$, giving the same two foci $(0,\pm7)$.



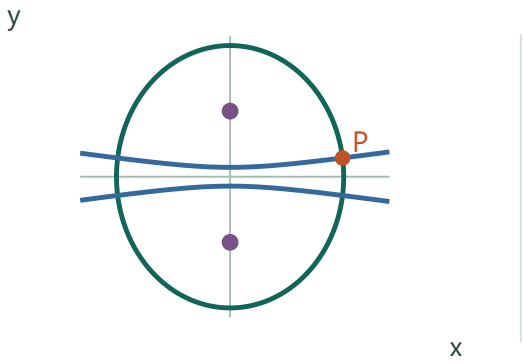
Same focal pair

(0, 7) and (0, −7)

Mark the two shared foci. The different signs in the two focal-distance formulas give the same distance 7.

3. Mark the shared point

At $P=(12,2)$, $48\cdot4-144=48$ and $4\cdot144+3\cdot4=588$. This locates P on the upper hyperbola branch and the upper-right portion of the ellipse.



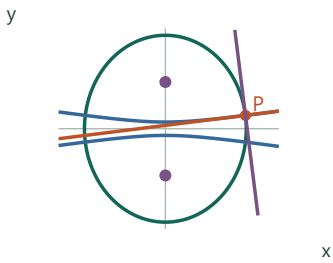
Shared point P

(12, 2) lies on both.

Place P on both curves by testing its coordinates in each equation.

4. Draw and compare the tangents

Implicit differentiation gives $96y\cdot y'-2x=0$ and $8x+6y\cdot y'=0$. At P the slopes are $1/8$ and -8. Use $y-2=m(x-12)$ to obtain $8y-x=4$ and $8x+y=98$. Their slope product -1 shows that the two tangents meet at right angles.



Tangents through P

Slopes ⅛ and −8

Product = −1

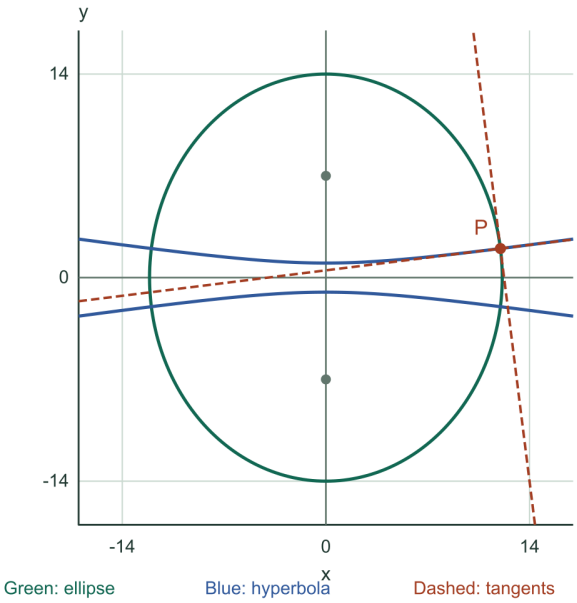
1. $96yy' = 2x \Rightarrow y' = x/(48y) = 12/(48\cdot2) = 1/8$
Differentiate the hyperbola and divide by 96y, which is nonzero at P.

2. $6yy' = -8x \Rightarrow y' = -4x/(3y) = -8$
Apply the same steps to the ellipse.

3. $8(y-2) = x-12 \Rightarrow 8y-x = 4$
Clear the hyperbola tangent's denominator.

4. $y-2 = -8(x-12) \Rightarrow 8x+y = 98$
Expand the ellipse tangent and collect constants.

Draw each tangent at P. Their slopes are negative reciprocals, making the intersection perpendicular.



Check. The slopes multiply to -1. Substituting the point gives $48\cdot4-144=48$ and $4\cdot144+3\cdot4=588$.

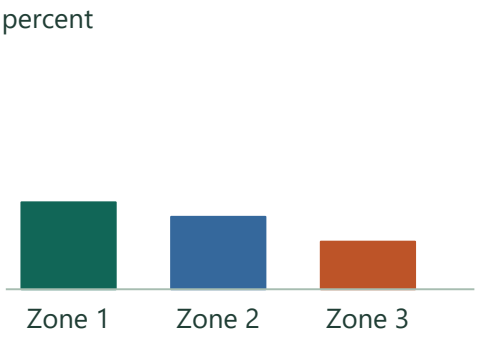
Exercise 796-1 · Where a taxi fleet settles

Printed page 122 · Transition matrices / Steady-state distributions

The Metropolis Cab Company, which has a fleet of 1000 taxis, has divided the city into three zones, with one office in each zone. The matrix $\begin{bmatrix} 0.42 & 0.33 & 0.12 \\ 0.35 & 0.48 & 0.31 \\ 0.23 & 0.19 & 0.57 \end{bmatrix}$ shows the distribution of destinations from each zone. For example, 23% of the pickups in zone 1 are discharged in zone 3. After discharging a fare, an MCC taxi waits in that zone for a new fare. On the basis of this long-term data, what do you expect that the current distribution of MCC taxis is?

1. Read one column as one starting zone

For a pickup in zone 1, read down the first column: 42% finish in zone 1, 35% in zone 2, 23% in zone 3. The entries sum to 100%. Let x,y,z be the fractions in the three zones immediately after trips.



From zone 1

Each bar is a possible

destination probability.

Read down the first matrix column to see three mutually exclusive destinations of a zone-1 trip.

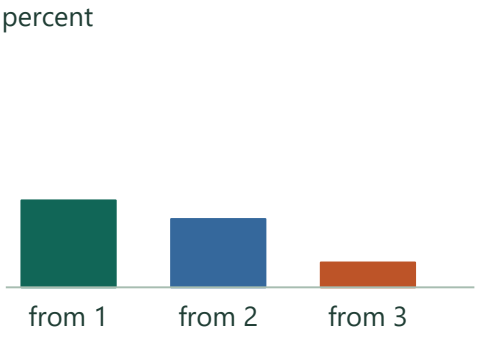
$$\begin{bmatrix} 0.42 & 0.33 & 0.12 \\ 0.35 & 0.48 & 0.31 \\ 0.23 & 0.19 & 0.57 \end{bmatrix}$$

Pickups from zone one

This first column partitions all trips that begin in zone 1. Read downward to see how one origin contributes to three possible destinations.

2. Balance arrivals with the current fractions

At equilibrium the next distribution equals the current one. The first row gives x=.42x+.33y+.12z; the second gives y=.35x+.48y+.31z. The fractions must also satisfy x+y+z=1.



To zone 1

Multiply by x, y, z

before adding arrivals.

Switch from an origin column to one destination row. Each shown entry weights the share arriving from its own origin.

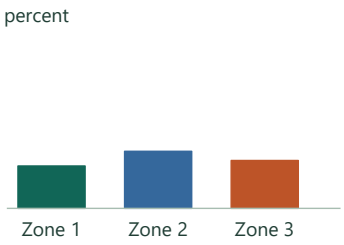
$$\begin{bmatrix} 0.42 & 0.33 & 0.12 \\ 0.35 & 0.48 & 0.31 \\ 0.23 & 0.19 & 0.57 \end{bmatrix}$$

Arrivals into zone two

Read the middle row across. Each entry tells how one of the three origin populations contributes to the next zone-2 population.

3. Eliminate the final unknown

Replace z by 1−x−y. The two equations become .70x−.21y=.12 and −.04x+.83y=.31. Solving gives approximately x=.287635, y=.387356, z=.325009. Multiplying by 1000 gives about 288, 387 and 325 cabs at trip endpoints.



Endpoint shares

≈ 288, 387, 325 per 1000

1. 70x−21y = 12, −4x+83y = 31
Use z=1−x−y in each balance equation and multiply by 100.

2. 4(70x−21y)+70(−4x+83y) = 48+2170
Multiply the equations by 4 and 70 so x cancels.

3. 5726y = 2218 ⇒ y = 1109/2863
Collect y terms and divide.

4. x = (12+21y)/70, z = 1−x−y
Back-substitute and recover the third share from the total.

Solve for the distribution that reproduces all three endpoint counts after one more trip.

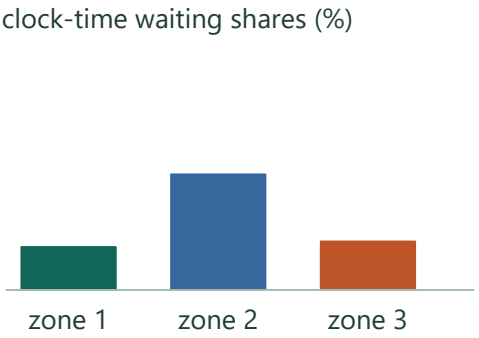
Long-run distribution immediately after trips

Destination zone	Share of trip endpoints (%)	Cabs per 1,000 (rounded)
1	28.7635	288
2	38.7356	387
3	32.5009	325

4. Distinguish an endpoint count from a clock-time count

A longer wait makes a cab more likely to be seen during a random-time survey. Among waiting cabs the shares are proportional to xw_1, yw_2, zw_3, where w is the mean wait in each zone. Equal waits recover the endpoint shares. Locating the entire fleet at an arbitrary time also requires information about taxis in transit.

Mean zone 2 wait relative to others · 2×



Zone 2 wait factor: 2

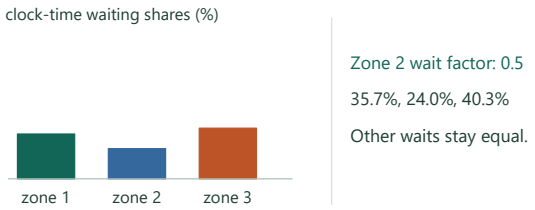
20.7%, 55.8%, 23.4%

Other waits stay equal.

For a clock-time sample, double the example waiting time in zone 2 and renormalize. Its share among waiting cabs increases.

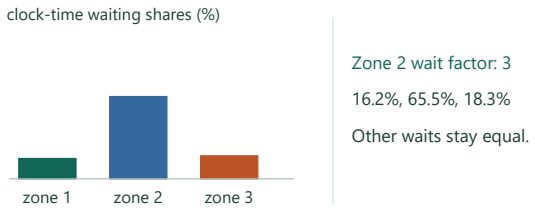
Compare settings · Mean zone 2 wait relative to others

0.5×



Sampling only waiting taxis at a random time gives zone 2 about 24.02% of observations; the fraction of trip endpoints is still about 38.74%.

3×



Sampling only waiting taxis at a random time gives zone 2 about 65.48% of observations; the fraction of trip endpoints is still about 38.74%.

Check. Repeated multiplication of any initial distribution by the positive column-stochastic matrix converges to these fractions; they sum to one and reproduce themselves on multiplication.

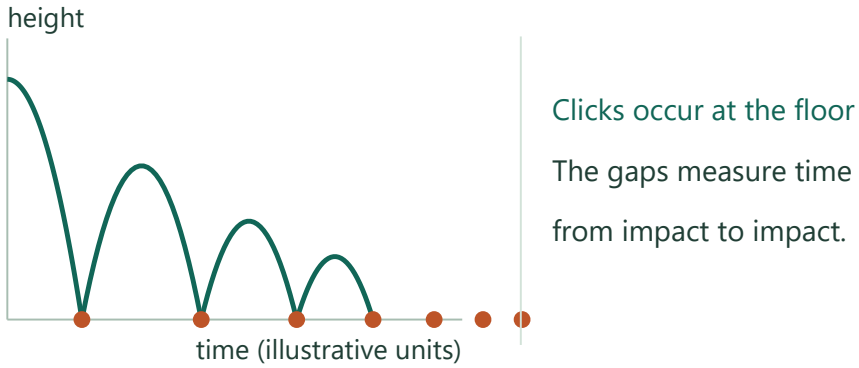
Exercise 803-1 · The rhythm of a bouncing ball

Printed page 122 · Geometric sequences and series / Time and distance

When you listen to the sound of a bouncing ping-pong ball that has been dropped onto a cement floor, what mathematical pattern do you hear?

1. Listen to the gaps

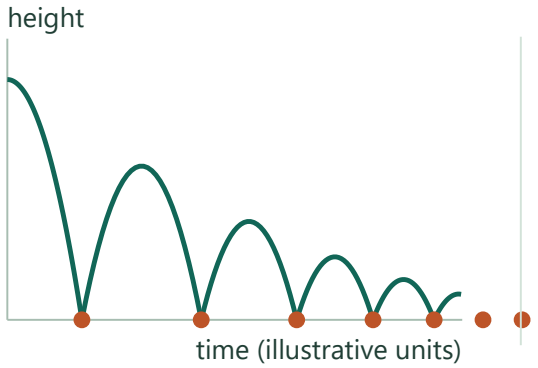
Each impact makes a click. The silence until the next click is the duration of one up-and-down flight. The gaps become progressively shorter as the bounce loses energy.



Place a mark at each impact on a time axis. Closer marks are clicks heard closer together.

2. Turn a rebound speed into a time

Ignoring air resistance, a launch at speed v takes v/g seconds to decelerate to zero and another v/g to return to the floor. A click-to-click interval is therefore 2v/g.



One whole flight

Rise time = fall time

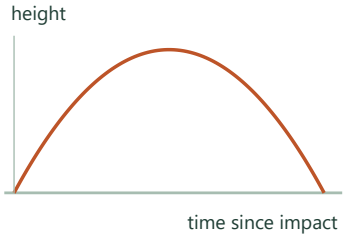
Interval = 2v/g

The arch from one click to the next contains an ascent and a descent of equal duration.

3. Predict the rhythm

If each bounce retains a fraction e of the preceding speed, each interval is e times the preceding interval. The gaps form a decreasing geometric sequence. If the given fraction is a height ratio r, the speed ratio is √r because maximum height is $v^2/(2g)$. The sound of a real ball only follows this pattern approximately.

Completed rebounds after first flight · 0



Flight 1

Height fraction = 1.000

Duration fraction = 1.000

1. $h_2/h_1 = [v_2^2/(2g)]/[v_1^2/(2g)] = (v_2/v_1)^2$
Compare peak heights under the same gravity.

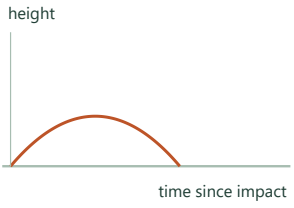
2. $v_2/v_1 = \sqrt{(h_2/h_1)}$
Both launch speeds are positive, so take the positive root.

3. $T_2/T_1 = (2v_2/g)/(2v_1/g) = v_2/v_1$
The same ratio governs successive gaps between impacts.

In this example bounce height scales by 0.64, while the width of each time interval scales by its square root, 0.8.

Compare settings · Completed rebounds after first flight

2



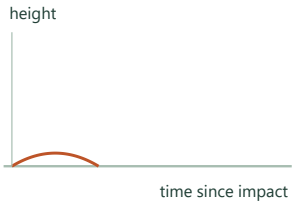
Flight 3

Height fraction = 0.410

Duration fraction = 0.640

With 64% of peak height retained per bounce, flight 3 reaches 0.410 times the initial flight's height and lasts 0.640 times as long.

5



Flight 6

Height fraction = 0.107

Duration fraction = 0.328

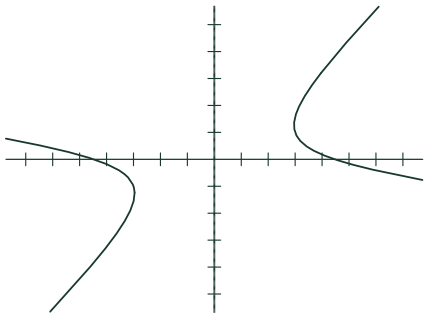
With 64% of peak height retained per bounce, flight 6 reaches 0.107 times the initial flight's height and lasts 0.328 times as long.

Check. Ballistic height is proportional to v^2 while flight time is proportional to v. This gives the same square-root relation.

Exercise 812-1 · Finding axes hidden by an xy term

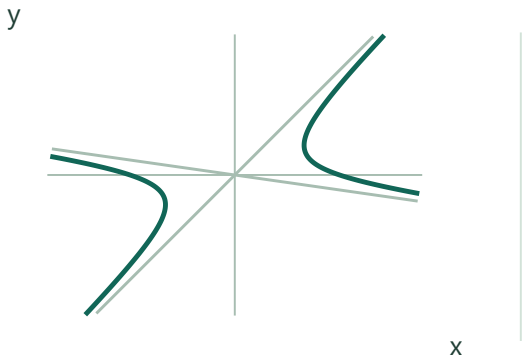
Printed page 122 · Hyperbolas / Coordinate transformations / Asymptotes

The figure at right shows part of the graph of $x^2 + 6xy - 7y^2 = 20$. Identify the asymptotes of this hyperbola, then use symmetry to help you find coordinates for the vertices, which are the points closest to the origin.



1. Identify the two limiting directions

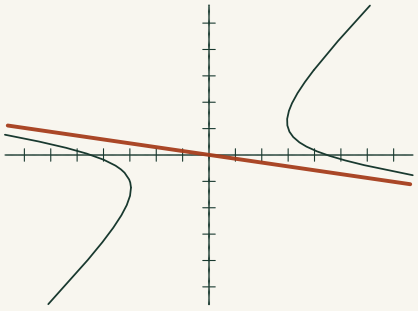
Factor $x^2+6xy-7y^2=(x-y)(x+7y)$. On the branches the product is 20. When either factor grows without bound the other must approach zero, so the asymptotes are the lines $y=x$ and $y=-x/7$ through the origin.



Asymptote lines

$y = x$ and $y = -x/7$

The two factor lines give the directions the branches approach.

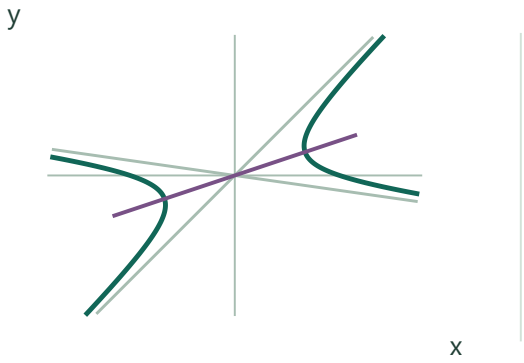


Shallow negative-slope asymptote

This second line passes through the origin and decreases slightly to the right. Each outer branch approaches one end of each asymptote.

2. Locate the symmetry axis

The lines have direction vectors (1,1) and (7,−1). Normalize them before adding to bisect the smaller angle. In common units their sum is proportional to (5,5)+(7,−1)=(12,4), pointing along (3,1). Its perpendicular points along (−1,3).



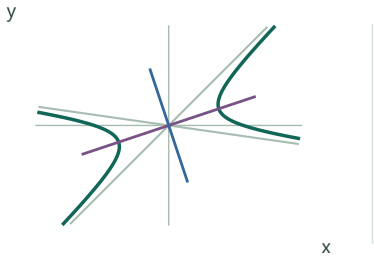
Bisect their directions

Transverse direction (3,1)

Bisect the angle between normalized asymptote directions to identify the transverse direction.

3. Measure along the rotated axes

Project the position onto the two unit vectors: $U=(3x+y)/\sqrt{10}$ and $V=(-x+3y)/\sqrt{10}$. Expand $2U^2-8V^2$ to recover $x^2+6xy-7y^2$. In these coordinates the curve satisfies $U^2=10+4V^2$.



Use unit axes U, V

$2U^2 - 8V^2 = 20$

1. $10(2U^2-8V^2) = 2(3x+y)^2-8(-x+3y)^2$
Substitute both projections, clearing their common squared denominator.

2. $= 18x^2+12xy+2y^2-8x^2+48xy-72y^2$
Expand the two squares and distribute the coefficients.

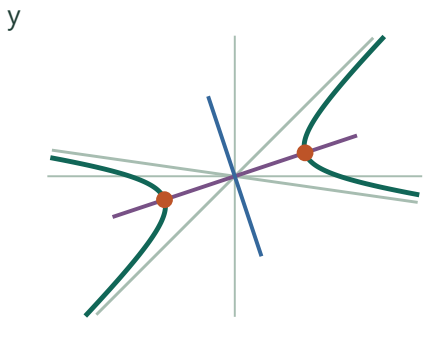
3. $= 10(x^2+6xy-7y^2) = 200$
Collect like terms and apply the original level equation.

4. $2U^2-8V^2 = 20 \Rightarrow U^2 = 10+4V^2$
Divide by ten, add $8V^2$ and divide by two.

Taking coordinates along that direction and its perpendicular removes the xy term.

4. Prove these points are the closest

The squared distance to the origin is $U^2+V^2=10+5V^2$. Its least possible value is 10, attained precisely when $V=0$. Then $U=\pm\sqrt{10}$, and moving those distances along $(3,1)/\sqrt{10}$ gives the two vertices $(3,1)$ and $(-3,-1)$.

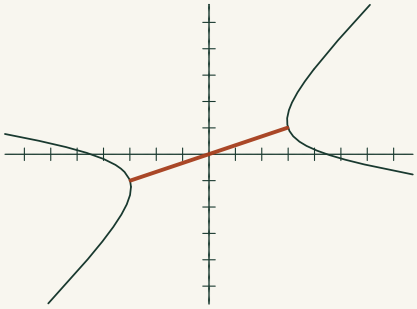


Nearest branch points

V = 0

Vertices ±(3, 1)

Slide toward the origin on either branch. The squared distance reaches its minimum exactly on the transverse axis.



Closest point on each branch

The two symmetric tips lie on the same line through the origin. Distance to the origin is measured radially, so a tip need not be the leftmost or rightmost point of a tilted branch.

Check. The squared radius is $U^2+V^2=10+5V^2$, uniquely minimized at $V=0$. Either vertex gives 20 in the original equation.

Exercise 812-2 · A rotated hyperbola in parameters

Printed page 122 · Hyperbolas / Rotations / Parametric equations

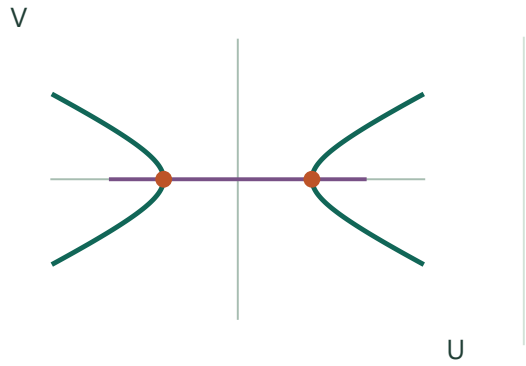
(Continuation) A similar approach to the one used in #460 can be used to find a parametric description of this hyperbola.

(a) Calculated the distance between the vertices and the distance between the foci.

(b) Find parametric equations and sketch the graph using a graphing tool.

1. Read the lengths in rotated coordinates

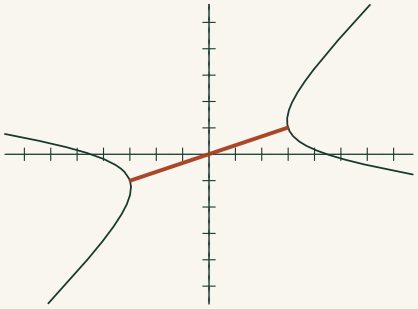
Divide $2U^2-8V^2=20$ by 20. In the standard hyperbola $U^2/10-V^2/(5/2)=1$, the semiaxes satisfy $a^2=10$ and $b^2=5/2$. The two vertices are $2a=2\sqrt{10}$ apart.



Rotated U,V plane

Vertices are $2\sqrt{10}$ apart

Use the axis-aligned form; the points on the U-axis are the vertices.

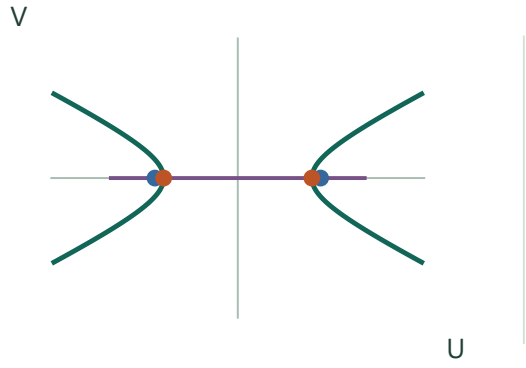


Closest point on each branch · Figure from exercise 812-1

The two symmetric tips lie on the same line through the origin. Distance to the origin is measured radially, so a tip need not be the leftmost or rightmost point of a tilted branch.

2. Find the foci

For a hyperbola $c^2=a^2+b^2$, so $c^2=25/2$. Each focus is c units from the center along the U-axis, and the distance from one focus to the other is $2c=5\sqrt{2}$.



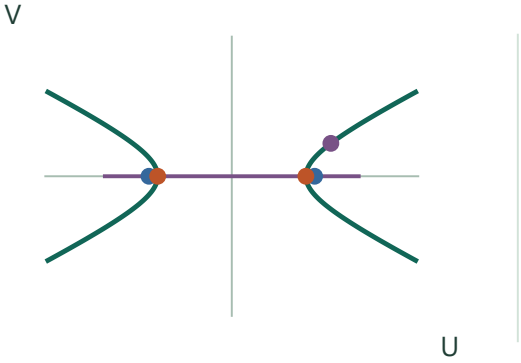
Foci lie past vertices

Foci are $5\sqrt{2}$ apart

Locate the foci a distance c from the center, beyond the vertices.

3. Make the equation true for every parameter

Use $\cosh^2 t - \sinh^2 t = 1$. Set $U = \pm\sqrt{10}\cosh t$ and $V = \sqrt{(5/2)}\sinh t$. The two signs select opposite branches, and varying t through all real values traverses each branch.



Move along one branch

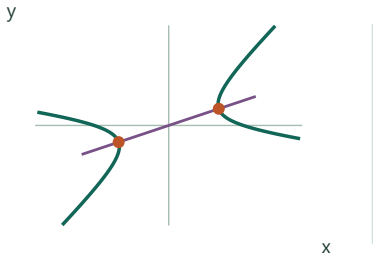
$U = \sqrt{10}\cosh t$

$V = \sqrt{(5/2)}\sinh t$

A parameter moves along a branch without leaving the curve because $\cosh^2 t - \sinh^2 t = 1$.

4. Rotate the plotted points back

Use $x = (3U - V)/\sqrt{10}$ and $y = (U + 3V)/\sqrt{10}$. For a graphing tool without hyperbolic functions enter $U = \sqrt{10}\sec t$ and $V = \sqrt{(5/2)}\tan t$ instead, skipping the points where cosine vanishes. At $V = 0$ your plot should pass through $(3,1)$ and $(-3,-1)$; away from them its arms approach $y = x$ and $y = -x/7$.



Rotate points back

Vertices $\pm(3,1)$

Branches tilt with U.

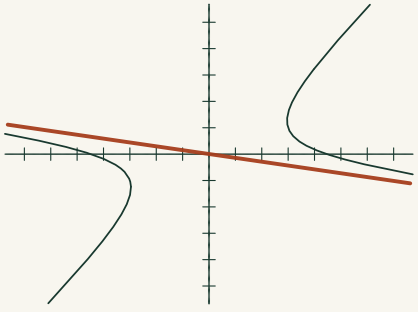
1. $\sqrt{10}\,U = 3x + y$, $\sqrt{10}\,V = -x + 3y$
Start with projections onto the two unit axes.

2. $\sqrt{10}(3U - V) = 10x$
Triple the first equation and subtract the second to eliminate y.

3. $\sqrt{10}(U + 3V) = 10y$
Add the first equation to three times the second to eliminate x.

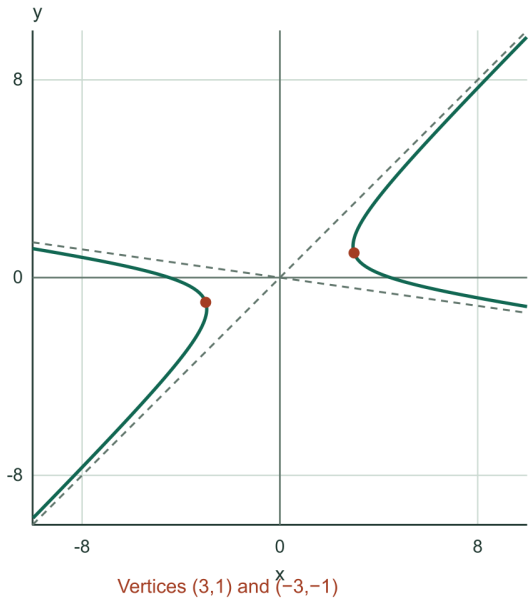
4. $x = (3U - V)/\sqrt{10}$, $y = (U + 3V)/\sqrt{10}$
Divide by ten, using $\sqrt{10}/10 = 1/\sqrt{10}$.

Rotate the whole set of points and axes together to recover the tilted hyperbola in the original xy-plane.



Shallow negative-slope asymptote · Figure from exercise 812-1

This second line passes through the origin and decreases slightly to the right. Each outer branch approaches one end of each asymptote.



Check. At t=0 the hyperbolic parametrizations give the two vertices. The identity $\cosh^2 t - \sinh^2 t = 1$ verifies every point.

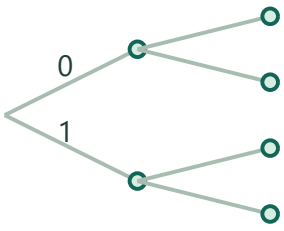
Exercise 812-3 · Counting a byte

Printed page 123 · Binary representations / Counting and permutations

The smallest unit of information is called a *bit*. A sequence of eight bits is called a *byte*. A bit represents two possible values (say 0 or 1). How many different values can a byte represent?

1. Build a short bit string

One bit has the two possibilities 0 and 1. Attach a second bit to each possibility: 00, 01, 10, 11. A leading zero still occupies a position.



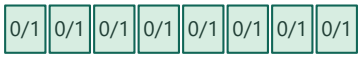
Two positions

00, 01, 10, 11

Append a 0 or a 1 to each first-bit choice to obtain four leaves.

2. Double for every new position

Either possible new bit can follow every previous string, so the count doubles each time. Eight independent positions give 2^8=256 patterns. As unsigned integers they encode 0 through 255 inclusive.



Eight distinct positions

$2 \times 2 \times \ldots \times 2 = 256$

An initial zero counts.

1. $N_8 = 2N_7 = 2(2^7) = 256$
Appending either bit doubles the number of seven-bit strings.

2. largest unsigned value = 256−1 = 255
The contiguous list of 256 nonnegative integers starts with zero.

Give each of the eight slots its own binary choice. The number of completed strings doubles for every extra slot.

Check. As unsigned binary integers these run from 0 through 255 inclusive.

Exercise 816-1 · Largest distance difference on a line

Printed page 123 · Reflections / Optimization / Inequalities

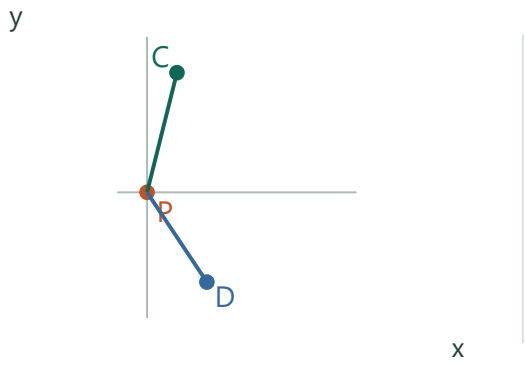
Consider the two points $D = (2, -3)$ and $C = (1, 4)$. Point $P = (p, 0)$ is on the x-axis.

(a) Write a formula for the difference of the lengths PC and PD.

(b) Show that this difference is a maximum when $p = 5$.

1. Write the two distances

From P=(p,0) to C=(1,4), the horizontal and vertical differences give PC=√[(p−1)²+16]. Similarly PD=√[(p−2)²+9]. Subtract in the stated order PC−PD.



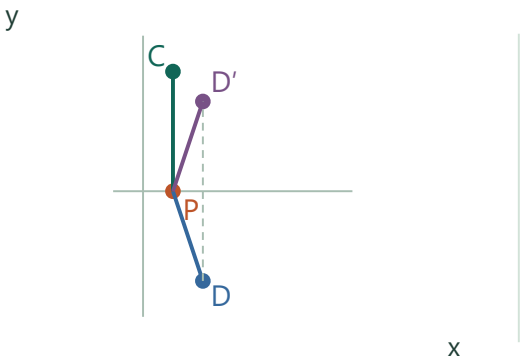
Compare PC and PD

P lies on the x-axis.

Compare two distances from the same movable point P.

2. Put both points on the same side

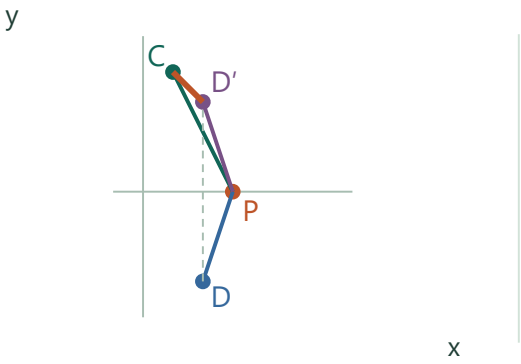
Reflect D across the x-axis to D′=(2,3). Since P lies on the mirror, PD=PD′. The difference has become PC−PD′, with both fixed points above the line.



Reflecting D across the axis replaces PD with a segment of equal length ending at D′.

3. Bound every possible difference

In triangle CPD′ the path from C to D′ to P is at least as long as CP. Subtract PD′ to obtain PC−PD′≤CD′=√2. Equality occurs exactly when D′ lies on the segment from C to P.

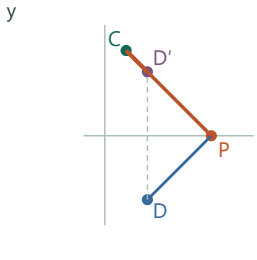


Draw CD′ to close a triangle; its fixed length bounds the distance difference.

4. Find where equality is attained

The line through (1,4) and (2,3) is y=5−x. Extend it beyond D′ to y=0, obtaining P=(5,0). Here PC=4√2 and PD=3√2, so the upper bound really is reached.

Position of P on the x-axis · 5



Equality requires alignment

p = 5

PC − PD = 1.414

1. m = (3−4)/(2−1) = −1
Compute the slope through C and the reflected point D′.

2. y−4 = −(x−1) ⇒ y = 5−x
Use C as the point on the line.

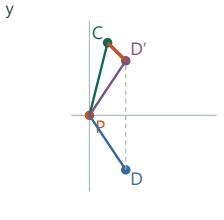
3. 0 = 5−p ⇒ p = 5
Set y to zero to locate the intersection with the mirror.

4. PC−PD = √32−√18 = 4√2−3√2 = √2
The aligned point attains the bound proved in the preceding step.

Move P to the line through C and D′. At p=5 the reflected point lies between C and P, so the bound is attained.

Compare settings · Position of P on the x-axis

0



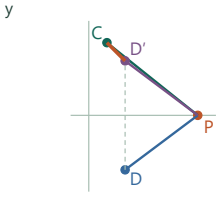
Equality requires alignment

p = 0

PC − PD = 0.518

At p = 0, PC − PD ≈ 0.5176. The reflected path still bends; the difference is below √2.

6



Equality requires alignment

p = 6

PC − PD = 1.403

At p = 6, PC − PD ≈ 1.4031. The reflected path still bends; the difference is below √2.

Check. At p=5 the two lengths are 4√2 and 3√2; their difference attains the bound.

Exercise 816-2 · Equal angles at the reflecting line

Printed page 123 · Reflections / Angle bisectors / Distance and loci

(Continuation) Consider the reflection of D over the x-axis onto D'.

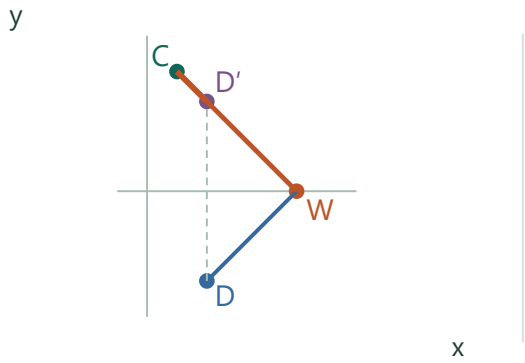
(a) The line CD' crosses the x-axis at point W, find the coordinates of W.

(b) How does W relate to P?

(c) What can you say about the relationship between the angle that line CW makes with the x-axis and the angle that the line DW makes with the x-axis?

1. Construct the crossing

Reflect (2,−3) upward to D′=(2,3). From C=(1,4) to D′ the rise is −1 over a run of 1. Extend the line y=5−x until y=0: W=(5,0).



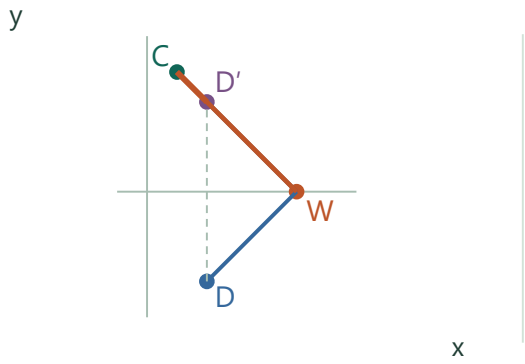
Line CD′: y = 5 − x

Crossing W = (5,0)

Extend the line through C and reflected D′ to intersect the axis at W.

2. Connect W to the maximum

In the previous exercise equality required that C,D′,P lie on one straight segment, in that order. W is the unique point of the x-axis on this extension, so W is that maximizing P.



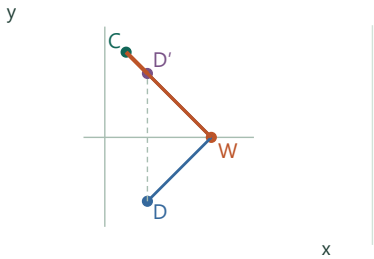
W attains the maximum

D′ lies between C and W.

At that crossing the broken path via D′ straightens, giving equality in the preceding bound.

3. Compare the two angles

From W toward C the direction is (−4,4); toward D it is (−3,−3). Each has equal horizontal and vertical component magnitudes. The two lines therefore make equal acute angles of 45° with the x-axis on opposite sides.



Equal acute angles

CW and DW each make

45° with the x-axis.

1. tan α = 4/4 = 1, tan β = 3/3 = 1
Use the magnitudes of vertical and horizontal displacements to measure the two acute angles.

2. α = β = arctan(1) = 45°
The reflected segments make equal angles with the mirror although their vertical components point opposite ways.

Reflect WD′ across the axis to recover WD. The acute angles above and below the axis agree.

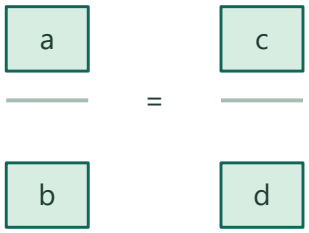
Check. The direction vectors from W to C and D are (−4,4) and (−3,−3); their horizontal and vertical magnitudes match in each case.

Exercise 819-1 · How many proportions work?

A *proportion* is created by filling the four blank spaces of ____ : ____ = ____ : ____ with numbers. Given four distinct numbers a, b, c, and d, how many ways are there of using all four numbers to fill the blanks? Assuming that at least one of these permutations does produce a correct proportion, what is the probability that a *random* permutation of a, b, c, and d will produce a correct proportion?

1. Count the four-slot arrangements

There are 4 choices for the first slot, then 3, 2 and 1, making 24 equally likely permutations. A proportion a:b=c:d is true exactly when its defined ratios give ad=bc. With four distinct entries, a single zero would make only one cross-product zero, so a successful set has no zero.



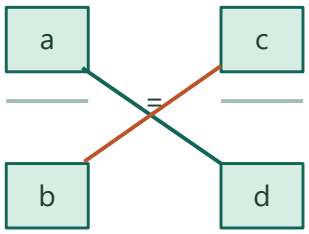
Arrange four entries

4! = 24 permutations

The four boxes distinguish the slots of a proportion; even swapping two labels usually changes the outcome.

2. Count the arrangements of a successful pairing

There are three ways to partition four entries into two unordered pairs. For any pairing with equal products, choose which pair occupies the two outer slots (2 choices), then order the entries of each pair (2 choices each). Each successful pairing supplies eight arrangements.



Equal cross products

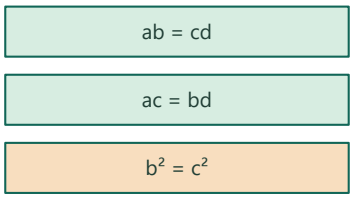
Two orders per pair,

two sides for the pairs.

The crossed diagonals specify the two product pairs. One successful pairing accounts for eight slot arrangements.

3. Decide whether a second pairing can work

If ab=cd and ac=bd, division gives b/c=c/b. Distinctness then forces c=−b, and substitution gives d=−a. A third pairing would require $a^2=b^2$, contradicting four distinct entries.



1. (ab)/(ac) = (cd)/(bd) ⇒ b/c = c/b
Divide the two successful pair equalities. All four factors must be nonzero.

2. $b^2 = c^2$ ⇒ (b−c)(b+c) = 0
Clear denominators and factor the difference.

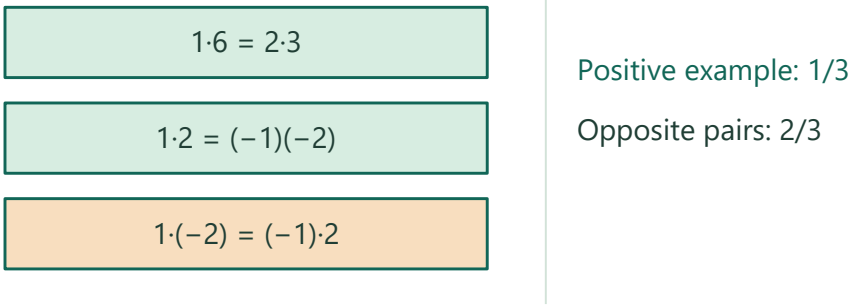
3. c = −b, ab = (−b)d ⇒ d = −a
Four distinct entries exclude c=b; then cancel b.

4. ad = bc ⇒ $-a^2 = -b^2$ ⇒ a = ±b
A third successful pairing would repeat an entry, contradicting distinctness.

Compare two successful product equations. Dividing them forces the two relevant entries to have equal squares.

4. Give the probabilities and examples

With positive numbers exactly one pairing can work, so the conditional probability is 8/24=1/3; {1,2,3,6} illustrates this. The question allows signed numbers. An opposite-pair set such as {1,−1,2,−2} has two successful pairings, giving 16/24=2/3. All other successful sets have probability 1/3.



Use separate examples to see why positive data allow one successful pairing while signed data may allow two.

Check. For {1,2,3,6} only 1·6=2·3 works. For {1,−1,2,−2} both 1·2=(−1)(−2) and 1·(−2)=(−1)·2 work. Two simultaneous product equalities force opposite-number pairs on division.

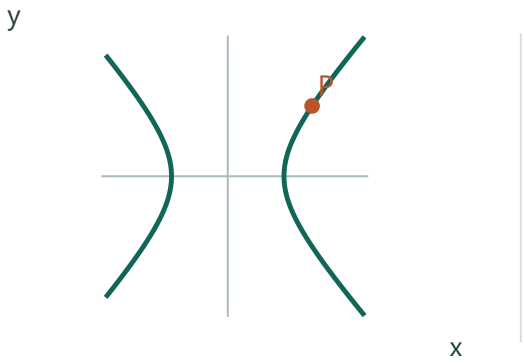
Exercise 831-1 · A tangent to a horizontal hyperbola

Printed page 123 · Hyperbolas / Tangent lines / Slopes and tangent lines

The point $P = (6, 5)$ is on the hyperbola $5x^2 - 4y^2 = 80$. Verify this and make a sketch. Then find an equation for the line that intersects the hyperbola tangentially at P.

1. Locate the branch and the point

Divide by 80 to obtain $x^2/16-y^2/20=1$. The vertices are $(\pm4,0)$, and the asymptotes are $y=\pm(\sqrt{20}/4)x$. Mark $P=(6,5)$ on the upper part of the right branch: $5\cdot36-4\cdot25=80$.



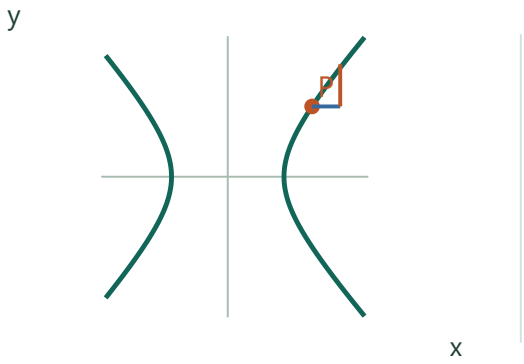
Right branch

$5(6^2) - 4(5^2) = 80$

Mark the verified point on the right-hand branch.

2. Find the tangent direction

Along the curve y changes with x. Differentiate both sides to get $10x-8y\cdot y'=0$, so $y'=5x/(4y)$. At (6,5) the slope is 3/2.



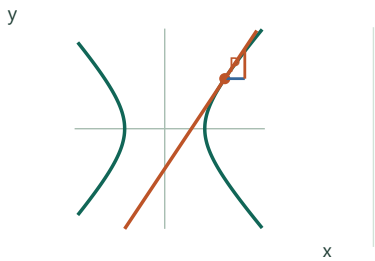
Local slope

Run 2, rise 3

At P the implicit derivative fixes the rise/run ratio, shown by the two legs.

3. Draw and verify the tangent

From P rise 3 for a run of 2. The line is $y-5=(3/2)(x-6)$, or $3x-2y=8$. Replacing y by $(3x-8)/2$ in the hyperbola gives $-4(x-6)^2=0$, so the line has a double intersection at P.



Tangent through (6,5)

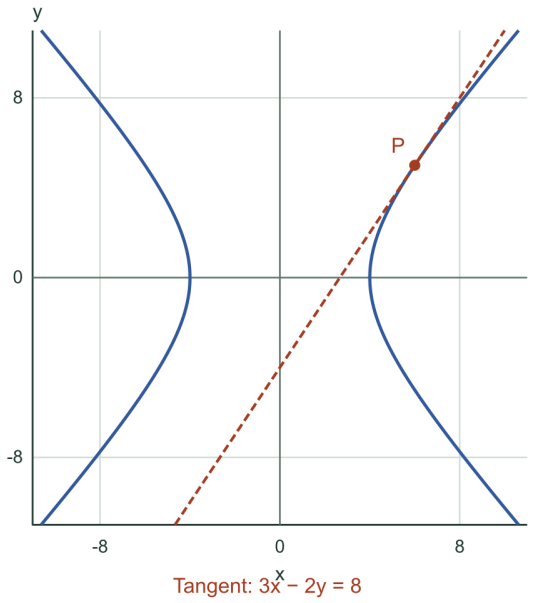
$3x - 2y = 8$

1. $y = (3x-8)/2 \Rightarrow 5x^2-(3x-8)^2 = 80$
Substitute the candidate tangent into the original equation.

2. $5x^2-9x^2+48x-64-80 = 0$
Expand the square and subtract 80.

3. $-4(x^2-12x+36) = -4(x-6)^2 = 0$
Factor. The repeated solution is x=6, hence y=5.

Extend that slope through P. Substitution gives one repeated intersection.



Check. Substitute y=(3x−8)/2 into the curve: the intersection reduces to $(x-6)^2=0$, a double root.

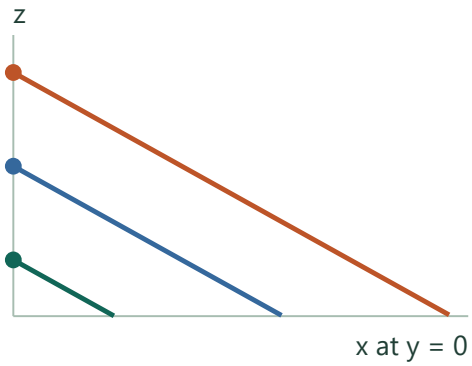
Exercise 834-1 · Three equally spaced planes

Printed page 123 · Planes in space / Distance and loci

Figure out as much as you can about the planes $2x+3y+6z=12$, $2x+3y+6z=32$, and $2x+3y+6z=52$. Justify your conclusions.

1. Read the perpendicular direction

The expression 2x+3y+6z is the dot product of (x,y,z) with n=(2,3,6). Moving within a level plane leaves this product constant, so n is perpendicular to all three planes. Different constants give three disjoint parallel planes.



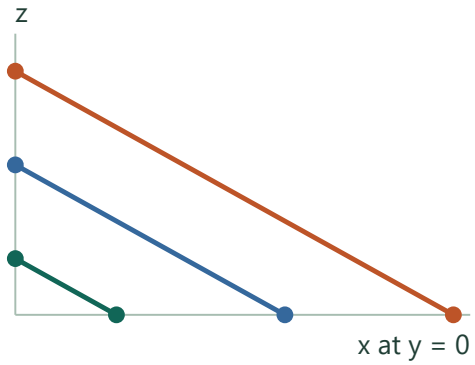
Parallel section lines

2x + 6z = k at y = 0

Slicing the three planes with y=0 exposes equal slopes and different intercepts.

2. Sketch each plane using intercepts

For level k, setting two variables to zero in turn gives (k/2,0,0), (0,k/3,0), (0,0,k/6). For example level 12 cuts the axes at 6, 4 and 2. Scaling these values by 32/12 or 52/12 locates the intercepts of the other planes.



Intercepts on x and z

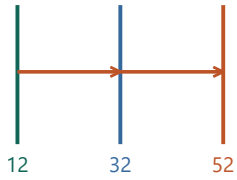
x = k/2; z = k/6

On the y-axis: y = k/3

Locate each plane's intercepts by setting the other coordinates to zero.

3. Convert changes of level into distances

The normal length is √(4+9+36)=7. One unit of travel along n/7 increases the dot product by 7. Adjacent levels differ by 20, giving separation 20/7; the outer levels differ by 40, giving 40/7. The level-32 plane is exactly halfway between the others.



Along the unit normal

Each 20-unit level change

takes 20/7 length units.

1. Δk = n·(s n/7) = (s/7)(n·n)
Move a distance s along the positive unit normal n/7.

2. Δk = (s/7)49 = 7s ⇒ s = Δk/7
Adjacent levels give 20/7; the outer levels give 40/7.

Look along the normal (2,3,6). Its length 7 converts differences of level to perpendicular separation.

Check. The middle plane has constant (12+52)/2=32 and bisects the space between the outer planes.

Exercise 834-2 · Pricing a lottery annuity due

Printed page 123 · Annuities / Present and future value / Compounding

A lottery winner is given two payment options: Receive 131 million dollars in 25 yearly installments of equal size, the first payable immediately, or receive a single immediate payment of 70.3 million dollars. Assuming that these plans are of equal value to the state lottery system, what interest rate is the state getting on its investments?

1. Put the installments on a timeline

Each installment is 131/25=5.24 million dollars. Place the first at time 0 and the last at time 24. The first payment already has its present value; later payments must be discounted.



25 installments

The first occurs today.

Align the 25 payments at times 0 through 24 so the immediate check is counted once.

2. Discount all 25 payments

Let r be the annual rate. An amount A due j years later is worth $A/(1+r)^j$ now. Set the sum of these 25 values equal to 70.3 million. The sum $5.24[1+(1+r)^{-1}+\cdots+(1+r)^{-24}]$ is a finite geometric series.



Example discount rate 6%

Later equal checks have smaller values today.

1. $q = 1/(1+r)$, $S = 1+q+\cdots+q^{24}$
Use a single discount factor for each year; the first payment is immediate.

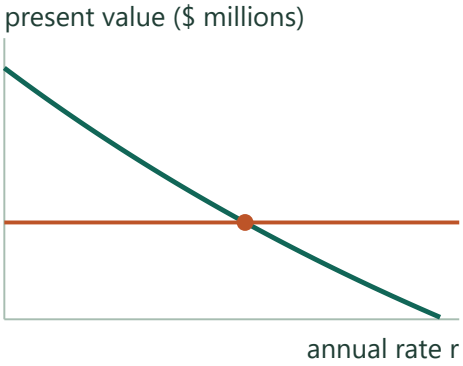
2. $S-qS = 1-q^{25}$
Subtract the shifted sum so intermediate powers cancel.

3. $5.24(1-q^{25})/(1-q) = 70.3$
For $r \neq 0$, divide by $1-q$. At r=0 the undiscounted total is 131 million.

Discount every check from its own date. At a positive rate the present-value bars decrease even though the promised checks are equal.

3. Determine the rate and check uniqueness

A calculator can bisect the rate interval: at 6% the sum exceeds 70.3, while at 7% it is below 70.3. Keep the half containing the crossing. The result is r≈.061168, or 6.12%. Each future payment's present value falls as r rises, so there is exactly one crossing.



Solve PV = 70.3 million

Unique crossing

r ≈ 6.12%

Plot the sum of the discounted checks against the unknown rate. Its monotone crossing fixes the rate.

Check. At zero interest the value is 131 million, and at r≈.061168 the 25 separately discounted payments total 70.30 million.

Exercise 838-1 · Digits in a historic prime

Printed page 123 · Logarithms / Digits and place value / Prime numbers

As of 11 December 2003, the largest integer known to be prime was $2^{20996011}-1$. It was found as part of the GIMPS (Great Internet Mersenne Prime Search), which harnessed the power of 211000 personal computers during the eight-year project. It was reported in the *Boston Globe* that this new prime is a 6320430-digit number. Make calculations to confirm this statement. Elsewhere in the article, it was also reported that it would take a book of about 1500 pages to print all the digits. Make calculations to show that this is also a plausible statement. The title of the article is "Largest prime number discovered." Does this make sense? Explain. Does this prime still hold the record?

1. Locate the leading decimal place

Set n = 20,996,011. Count the digits of 2^n first. A positive integer with k digits lies between the smallest k-digit number and the next power of ten. The logarithm is available because 2^n is positive.

1 **$10^{(k-1)} \le 2^n < 10^k$**
These are the lower and upper place-value boundaries for k digits.

2 **$k - 1 \le \log_{10}(2^n) < k$**
The increasing logarithm preserves both inequalities.

3 **$\log_{10}(2^n) = n \log_{10} 2 \approx 6{,}320{,}429.100291$**
Move the positive integer exponent outside the logarithm.

4 **$k = 6{,}320{,}429 + 1 = 6{,}320{,}430$**
The integer part of the logarithm gives the leading-place exponent.

2. Show that subtracting one preserves the count

Subtracting one can shorten a numeral, as 1000 becomes 999. Here the lower place-value boundary has a factor of 5, while 2^n has only the prime factor 2. This makes the lower inequality strict before subtraction.

1 **h = 6,320,429, $10^h < 2^n < 10^{(h+1)}$**
The counts locate the interval, and its lower power of ten cannot equal 2^n.

2 **$10^h + 1 \le 2^n$**
Both quantities are integers, so a strict increase is at least one.

3 **$10^h \le 2^n - 1 < 10^{(h+1)}$**
Subtract one. The result still lies inside the same digit-count interval.

4 **digits of $2^n - 1 = h + 1 = 6{,}320{,}430$**
The decimal count reported in the article follows.

3. Test a concrete page layout

The article asks whether about 1,500 pages is plausible. Estimate the digits required per page, then test an explicit arrangement. The final page may be partly filled.

1 **6,320,430 ÷ 1,500 = 4,213.62 digits/page**
This is the average needed if every page shares the load.

2 **70 rows × 60 digits/row = 4,200 digits/page**
Choose an illustrative layout close to that average.

3 **1,504 × 4,200 = 6,316,800 digits**
These many full pages leave 3,630 digits.

4 **1,504 + 1 = 1,505 pages**
One partial page holds the remainder. That is about 1,500 pages.

4. Explain why a largest prime cannot exist

Consider any proposed finite list containing all primes. Build a number that leaves remainder one upon division by each listed prime. It is enough to find a new prime divisor. The constructed number itself may be composite.

1 **$P = p_1 \times p_2 \times \ldots \times p_n$**
Form the product of the proposed list of primes.

2 **$M = P + 1 > 1$**
Every integer greater than one has a prime divisor. Call one such divisor q.

3 **M leaves remainder 1 on division by each p_i**
Each p_i divides P exactly, so none divides P + 1.

4 **q differs from every p_i**
The list omitted a prime, contradicting its claim to contain them all.

5. Distinguish a discovery record from a largest prime

A headline can describe the largest prime known on a specific date. The 2003 number has since been surpassed. For example, GIMPS announced a larger verified prime in October 2024.

1 **136,279,841 > 20,996,011**
Compare the exponent in the 2024 discovery with the one from 2003.

2 **2^136279841 − 1 > 2^20996011 − 1**
Powers of two strictly increase with the exponent. Subtracting one preserves the order.

Check. The later exponent is much larger, so even without a current record table this disproves a continuing record for the 2003 number.

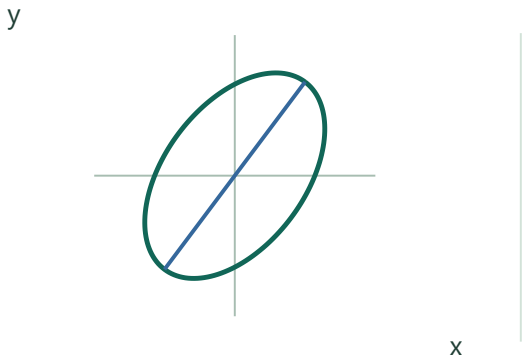
Exercise 838-2 · Undoing a rotation of an ellipse

Printed page 124 · Ellipses / Rotations / Coordinate transformations

The graph of $481x^2 - 384xy + 369y^2 = 5625$ is an ellipse, but the presence of the xy term in the equation prevents us from recognizing the curve. One approach to this problem is to use a rotation of coordinates, as follows:
(a) Replace each x in the equation by $0.6X - 0.8Y$ and each y by $0.8X + 0.6Y$. Verify that the coefficient of XY in the resulting equation is 0.
(b) Find the dimensions of the ellipse that is represented by the XY equation.
(c) The coordinates (X, Y) were obtained by applying a clockwise rotation to (x, y). Calculate the size of this acute angle.
(d) In the xy-coordinate system, the graph of the given equation is an ellipse whose major symmetry axis is a line of positive slope through the origin. Use the preceding information to find the slope of this line and coordinates for the vertices of the ellipse. Sketch the curve.

1. Rotate and expand the mixed term

Substitute the two supplied expressions. From x^2 the XY coefficient is $2(.6)(-.8)$; from xy it is $.6^2 - .8^2$; from y^2 it is $2(.8)(.6)$. With the original multipliers, $481(-.96) - 384(-.28) + 369(.96) = 0$.



Tilted major axis

Rotate axes to remove XY.

The oblique blue direction indicates a coordinate system in which the ellipse has no mixed term.

2. Collect the pure squares

The X^2 coefficient is $481(.36) - 384(.48) + 369(.64) = 225$. For Y^2 it is $481(.64) + 384(.48) + 369(.36) = 625$. Divide $225X^2 + 625Y^2 = 5625$ by 5625 to get $X^2/25 + Y^2/9 = 1$. Its full axis lengths are 10 and 6.



In X,Y coordinates

Semiaxes = 5 and 3

1. $225X^2 + 625Y^2 = 5625$
Collect the surviving squared terms after the mixed coefficient cancels.

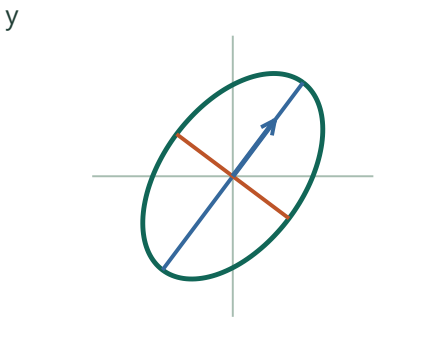
2. $(225/5625)X^2 + (625/5625)Y^2 = 1$
Divide all three terms by the positive level.

3. $X^2/25 + Y^2/9 = 1 \Rightarrow a = 5, b = 3$
Take positive square roots for the two semiaxis lengths.

In the rotated system, read both semiaxes directly at the intercepts.

3. Recover the angle

The positive X-axis in the original plane points along (.6,.8), whose angle is arctan(.8/.6)=arctan(4/3)≈53.13°. The coordinate transformation from (x,y) to (X,Y) rotates the coordinate pair clockwise by this angle.



Back in the xy-plane

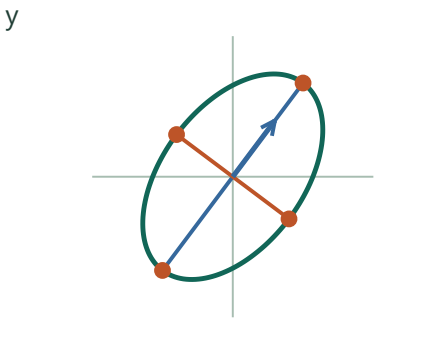
Positive X points along

(0.6, 0.8).

Return to the original axes; the longer axis has direction (0.6,0.8).

4. Place the vertices and sketch

For X=±5, Y=0 the original coordinates are ±(3,4), so the major axis slope is 4/3. For X=0, Y=±3 they are ±(−2.4,1.8). Draw a smooth centered ellipse through the four endpoints with these perpendicular axes.

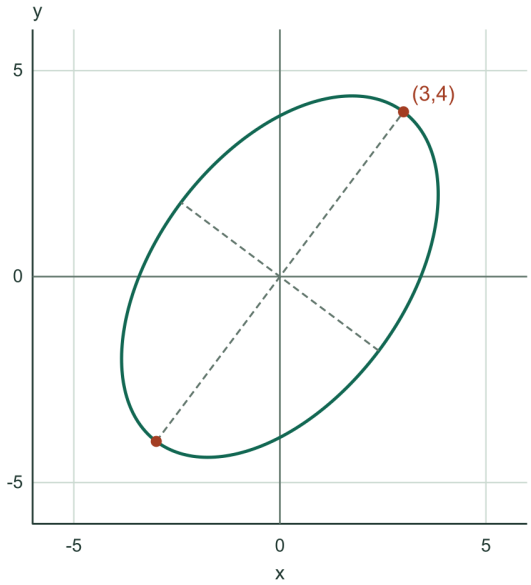


Major endpoints ±(3,4)

Minor endpoints

±(−2.4, 1.8)

Rotate the four intercepts back. The distances from the center stay 5 and 3.



Check. At (3,4), 481·9−384·12+369·16=5625. Rotations preserve lengths, so the major vertices remain five units from the origin.

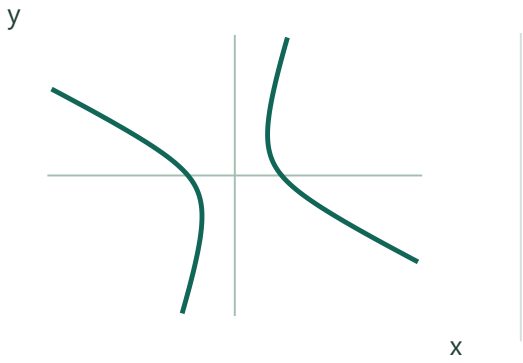
Exercise 838-3 · Which bisector is the transverse axis?

Printed page 124 · Hyperbolas / Rotations / Coordinate transformations

The graph of $3x^2 + 5xy - 2y^2 = 12$ is a hyperbola, whose major symmetry axis is the line $y = (\tan 22.5)x$. Make calculations that confirm this statement.

1. Represent the cross term

A conic with a mixed term is tilted. For a rotation through θ use x=X cosθ−Y sinθ and y=X sinθ+Y cosθ. The XY coefficient becomes (−5)sin(2θ)+5cos(2θ), collecting the contributions from $3x^2$, 5xy and $-2y^2$.



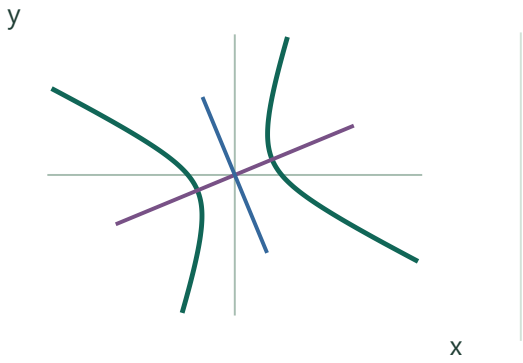
Mixed xy coefficient

The hyperbola is tilted.

Graph the conic and observe its tilted branches in the original coordinate plane.

2. Choose the rotation

Making that coefficient zero requires tan(2θ)=1. Choose the acute solution 2θ=45°, or θ=22.5°. The rotated axes are symmetric directions, but we must still decide which one contains the vertices.



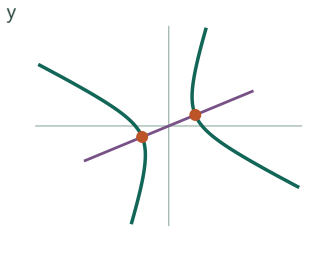
Candidate symmetry axes

2θ = 45°; θ = 22.5°

The angle that cancels the mixed term supplies two perpendicular symmetry axes.

3. Select the transverse axis

The coefficients of X^2 and Y^2 become $(1+5\sqrt{2})/2$ and $(1-5\sqrt{2})/2$. Only the first is positive. Since the right side equals positive 12, the hyperbola crosses the X-axis, and its vertices lie there. In the original coordinates this line has slope $\tan 22.5° = \sqrt{2}-1$, confirming the claim.



Only one axis meets it

The positive coefficient puts vertices on X.

1. $A = 3\cos^2\theta + 5\sin\theta\cos\theta - 2\sin^2\theta$
Collect X^2 after substituting the rotated coordinates.

2. $A = [1+5\cos(2\theta)+5\sin(2\theta)]/2$
Use the double-angle identities.

3. $2\theta = 45° \Rightarrow A = (1+5\sqrt{2})/2$
Each trigonometric term has value $\sqrt{2}/2$.

4. $B = 1-A = (1-5\sqrt{2})/2$
The two new squared coefficients sum to $3-2=1$. Only A is positive.

Test which axis contains real intersections; its coefficient must have the same sign as the constant.

Check. The axis slope is $\tan 22.5° = \sqrt{2}-1$. Substituting its direction into the symmetric matrix with rows $(3,2.5),(2.5,-2)$ gives a parallel vector.

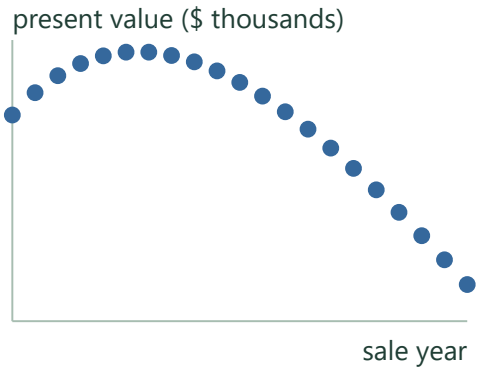
Exercise 838-4 · When should an asset be sold?

Printed page 124 · Present and future value / Optimization

Eugene has an object worth \$75000, which is expected to increase in value by \$5000 per year. Furthermore, Eugene anticipates that money invested today will earn 5% interest annually for the foreseeable future.
(a) If Eugene sells the object a year from now, what is its *present value*?
(b) What is the present value of the object if the sale occurs two years from now? twenty years from now?
(c) In how many years should Eugene sell the object to maximize its worth?

1. Compare every offer in today's dollars

After n years the selling price is $75{,}000+5{,}000n$. Money invested today would grow by the factor 1.05^n. Divide the future price by that factor to obtain the present value V_n. After 1, 2 and 20 years it is \$76,190.48, \$77,097.51 and \$65,955.66 respectively.



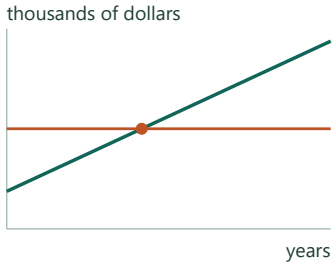
Discount to today

Whole-year sale dates.

Convert different future sale dates into a common present-value unit before comparing them.

2. Test whether another year helps

To compare V_{n+1} with V_n, multiply both by the positive denominator 1.05^{n+1}. Waiting helps exactly when the extra \$5,000 beats the interest .05(75,000+5,000n). This means n<5.



Green: interest forgone

Orange: annual \$5k gain

They meet at n = 5.

1. $V_{n+1} > V_n \Leftrightarrow 80000+5000n > 1.05(75000+5000n)$
Multiply by the positive common denominator.

2. $5000 > 3750+250n \Rightarrow 1250 > 250n$
Cancel the original sale price and collect the interest terms.

3. $n < 5$
Divide by 250. Equality at n=5 makes the neighboring years tie.

Compare one year's extra appreciation to the interest that year's sale proceeds could earn.

3. Locate the maximum

At n=5 the price is \$100,000 and one more year's \$5,000 appreciation matches the 5% return. After that the same increase is too small. Years 5 and 6 both maximize the whole-year value, about \$78,352.62.

Whole years until sale · 5



Year 5

Sale: \$100,000

Present value: \$78,352.62

Keep only whole-year dates. The two adjacent points have the same greatest height.

Compare settings · Whole years until sale

0



Year 0

Sale: \$75,000

Present value: \$75,000.00

Selling in year 0 yields \$75,000 then, equivalent to \$75,000.00 today. Compare years 5 and 6.

20



Year 20

Sale: \$175,000

Present value: \$65,955.66

Selling in year 20 yields \$175,000 then, equivalent to \$65,955.66 today. Compare years 5 and 6.

4. Allow a fractional selling date

If appreciation is linear even within a year and discounting uses 1.05^t, maximize $V(t)=(75{,}000+5{,}000t)/1.05^t$. Its logarithmic derivative is $1/(15+t)-\ln(1.05)$; it changes from positive to negative at $t=1/\ln(1.05)-15\approx5.50$ years.



Continuous-time model

One interior maximum

at about 5.50 years.

1. ln V(t) = ln 5000 + ln(15+t) − t ln 1.05
Factor the sale price and take logs of positive quantities.

2. (ln V)′ = 1/(15+t) − ln 1.05
Differentiate each term.

3. t* = 1/ln 1.05 − 15
Set the derivative to zero. Its sign changes from positive to negative as t increases.

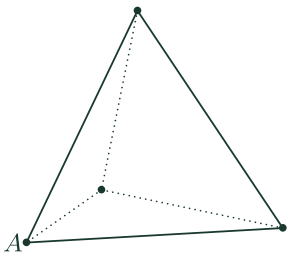
Under continuous appreciation and discounting the peak moves between those yearly points.

Check. From year 5 to 6 the price rises from $100,000 to $105,000, exactly one year's discount growth. Afterward the same $5000 is a smaller percentage.

Exercise 838-5 · A random walk on four vertices

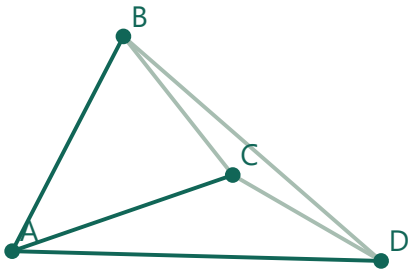
Printed page 124 · Random walks / Transition matrices / Probability

Let A, B, C, and D be the vertices of a regular tetrahedron, each of whose edges is 1 meter long. A bug starts at vertex A and crawls along the edges of the tetrahedron. At each vertex, it randomly chooses which of the three available edges it will follow next, each of the three being equally likely. What is the probability that the bug will find itself at vertex A after it has crawled 7 meters?



1. Translate distance into moves

Each edge is 1 meter, so 7 meters means seven choices of edge. At every corner the bug can go to any of the other three vertices. There is no edge that leaves it at its current vertex.



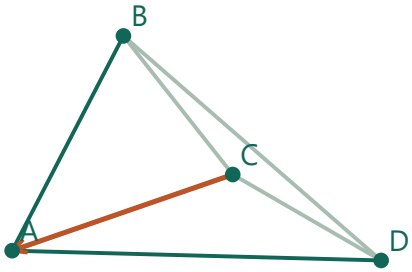
One edge = one move

Three choices at each tip.

Every pair of vertices is joined by a one-meter edge, so distance becomes a discrete step count.

2. Group the states

Let p_n be the probability of being at A after n moves. A bug at A must leave; a bug away from A has exactly one of its three edges pointing back to A. Thus $p_{n+1}=(1-p_n)/3$, with $p_0=1$.

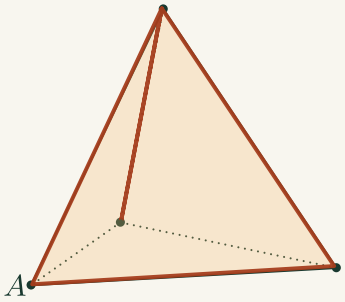


Group all other vertices

At B, C or D one of three

edges returns to A.

From any non-A vertex exactly one of the three destinations is A. This gives the one-variable recurrence.

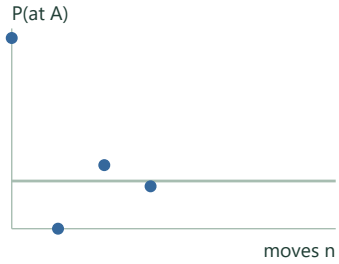


Three exits from a corner

From the upper corner trace edges to A, to the back corner shown with dotted lines, and to the far-right corner. Only one of the three choices arrives at A.

3. See the oscillation

The first few values are $p_1=0$, $p_2=1/3$, $p_3=2/9$. A fixed probability p satisfies $3p=1-p$, giving $p=1/4$. Subtract that value from the recurrence: $p_{n+1}-1/4=-(p_n-1/4)/3$.



Alternate about ¼

Each deviation reverses

and shrinks by a factor 3.

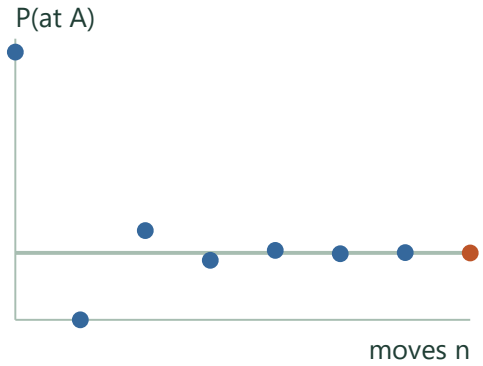
1. $p_{n+1}-¼ = (1-p_n)/3 - ¼$
Subtract the fixed probability from each new term.

2. $= (4-4p_n-3)/12 = -(p_n-¼)/3$
Use a common denominator and factor the deviation from equilibrium.

The plotted probabilities alternate around the equilibrium after each application of the recurrence.

4. Evaluate the seventh move

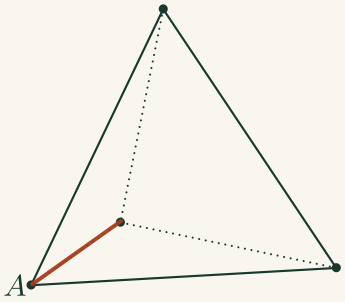
Start with deviation $p_0 - 1/4 = 3/4$ and multiply by $-1/3$ seven times. This gives $p_7 = 1/4 + (3/4)(-1/3)^7 = 182/729$, about 0.249657. Of the 3^7 equally likely seven-edge walks, 546 end at A.



After seven moves

182/729 ≈ 0.249657

Follow seven moves; the remaining deviation from one quarter is small but nonzero.



A return edge

The dotted connection from the interior-looking corner also ends at A. Dotted edges are hidden by the perspective drawing but remain available to the bug.

Check. There are $3^7 = 2187$ equally likely paths; counting returns by the same adjacency graph gives 546 ending at A.

Exercise 838-6 · An amortization experiment

Printed page 125 · Loan amortization / Compounding / Spreadsheet modeling

Repaying loans, part III. Jordan is in the market for a house. He is looking to buy a \$250 000 home. To buy the house, Jordan will make a down payment of \$50 000 and take out a \$200 000 mortgage to finance the purchase. He will pay off the mortgage in 360 equal monthly payments.

(a) Jordan is shopping around for a loan and sees various banks advertising a range of interest rates. Using annual interest rates of 3%, 3.5%, 4%, ... 6%, determine the monthly payment needed to pay off the mortgage (banks divide the annual interest rate by 12 to find the monthly rate). Also, for each interest rate calculate the total amount paid during the lifetime of the loan. What is the effect of the relatively small changes to the interest rate?

(b) Jordan qualifies for a 30-year fixed rate mortgage at 4% annual interest. Create a spreadsheet with the following columns: month (1-360), remaining loan balance, interest paid this month, principal paid this month. Make one scatter plot with month on the horizontal axis; interest paid this month and principal paid this month should both be on the vertical axis. After which month does over half your payment go to principal? Would this remain the same if the annual interest rate was 9%?

(c) Many financial advisers encourage clients to pay an additional \$100 above their monthly mortgage payment. Investigate the effect of Jordan adding \$100 above his monthly payment on his 4% loan.

(d) Suppose Jordan had an option of a 15-year fixed mortgage versus a 30-year fixed mortgage. Compare the effects on the monthly payment and total payments of each loan.

1. Start with what Jordan borrows

Subtract the \$50,000 down payment from the \$250,000 purchase price. The initial balance is \$200,000. If the quoted annual rate is R, the monthly rate is i=R/12 and the 30-year term has N=360 payments.



\$50k down + \$200k borrowed

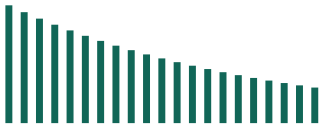
Price = \$250,000

Finance the remainder.

Partition the purchase price into the down payment and the principal that must be financed.

2. Obtain a fixed payment

Discount the monthly payments M from months 1 through N and set their sum equal to the borrowed amount. The finite geometric sum gives 200000=M[1−(1+i)^−N]/i. Solve this equality for M before substituting each rate.



Sum 360 discounted checks

Bars sample the schedule at 18-month intervals.

1. $q = (1+i)^{-1}$, $P/M = q+q^2+\cdots+q^N$
The first installment is one month in the future.

2. (1−q)P/M = q−q^(N+1)
Subtract the shifted geometric sum.

3. P/M = q(1−q^N)/(1−q) = [1−(1+i)^(−N)]/i
Use q/(1−q)=1/i, valid for the quoted positive rates.

4. M = Pi/[1−(1+i)^(−N)]
Multiply by M and divide by the positive discount factor difference.

Discount the equal monthly payments back to their common starting date; their total must equal the principal.

3. Calculate and compare the seven rate scenarios

For monthly rate i and N payments, equate principal to a discounted annuity: M=200000i/[1−(1+i)^(−N)]. At N=360 the ordered pairs (monthly payment, total paid) in dollars are: 3% (843.21,303554.90); 3.5% (898.09,323312.18); 4% (954.83,343739.01); 4.5% (1013.37,364813.42); 5% (1073.64,386511.57); 5.5% (1135.58,408808.08); 6% (1199.10,431676.38). Round only reported amounts.



Keep 360 months fixed

3%: $843.21 per month

6%: $1,199.10 per month

Hold principal and term constant and vary the annual rate, changing the monthly payment.

Payments on $200,000 over 360 months, excluding the down payment

Annual rate	Monthly payment ($)	Total of payments ($)
3%	843.21	303,554.90
3.5%	898.09	323,312.18
4%	954.83	343,739.01
4.5%	1,013.37	364,813.42
5%	1,073.64	386,511.57
5.5%	1,135.58	408,808.08
6%	1,199.10	431,676.38

4. Build the first spreadsheet row

At 4%, first-month interest is 200000(.04/12)=$666.67. Of the approximately $954.83 payment, $288.16 reduces principal. The next row begins with the preceding ending balance. Carry full precision through the calculations and round the displayed dollars.



Interest first; the rest pays principal.

Month 1 at 4%

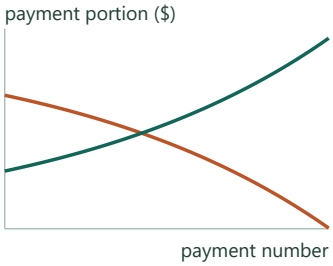
Interest: $666.67

Principal: $288.16

Split the first payment into interest on the opening balance and principal reduction.

5. Continue the rows and find the crossover

In each monthly row interest equals prior balance times i; principal equals M minus interest; ending balance equals prior balance minus principal. Plot both payment portions against month. Principal first exceeds half in payment 153 at 4%, or payment 269 at 9%.



Orange: monthly interest

Green: monthly principal

They add to a fixed total.

1. $P_j = M/(1+i)^{(N-j+1)}$
The principal portion grows by factor 1+i monthly; the final payment's principal is M/(1+i).

2. $P_j > M/2 \Leftrightarrow (1+i)^{(N-j+1)} < 2$
Cancel the positive payment and invert the positive factors.

3. $j > N+1-\ln(2)/\ln(1+i)$
Take logs and rearrange; ln(1+i)>0. Choose the first integer strictly above the threshold.

Carry the smaller balance forward. The interest portion falls as the principal portion rises.

Six rows from the 4% amortization schedule

Month	Ending balance ($)	Interest ($)	Principal ($)
1	199,711.84	666.67	288.16
2	199,422.71	665.71	289.12
152	143,086.05	478.54	476.29
153	142,608.17	476.95	477.88
359	951.66	6.33	948.50
360	0.00	3.17	951.66

6. Interpret the scatter plot

The interest series falls because its rate applies to a shrinking balance. The principal series rises by the same amount because the regular payment is fixed. The first majority-principal payments are 153 at 4% and 269 at 9%, just after months 152 and 268 respectively.

Payment number at 4% · 153



Orange: interest Green: principal

Payment 153 of 360

Interest: $476.95

Principal: $477.88

Mark the first month where principal exceeds half the payment. The crossover depends on the rate.

Compare settings · Payment number at 4%

1



Orange: interest Green: principal

Payment 1 of 360

Interest: $666.67

Principal: $288.16

The fixed payment is approximately $954.83. At payment 1, principal receives 30.18% of it.

360



Orange: interest Green: principal

Payment 360 of 360

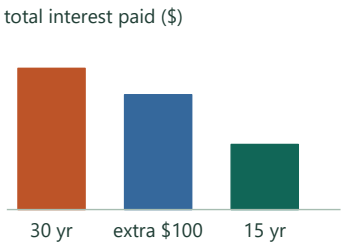
Interest: $3.17

Principal: $951.66

The fixed payment is approximately $954.83. At payment 360, principal receives 99.67% of it.

7. Test the extra payment and shorter term

Paying an extra $100 each month at 4% pays the loan off with a smaller final payment in month 301, saving about $26,854.93 in interest. A 15-year 4% loan costs $1479.38 monthly, total $266287.65, versus $954.83 and $343739.01 over 30 years. Totals exclude the down payment and fees.



Early payoff

Extra $100 reduces balance before later interest accrues.

1. $B_n = (1+i)B_{n-1} - A$, $B_0 = P$, $A = M + 100$
Accrue one month's interest, then subtract the increased payment.

2. $B_n = P(1+i)^n - A[(1+i)^n - 1]/i$
Iterate the recurrence and sum the geometric series of payment factors.

3. $B_n \le 0 \Leftrightarrow n \ge -\ln(1 - Pi/A)/\ln(1+i)$
Isolate the growth factor and take logs. Here $A > Pi$, so $1 - Pi/A$ is positive.

4. final payment = $(1+i)B_{300}$
Round the payoff time up to a whole month and pay only the remaining amount due in month 301.

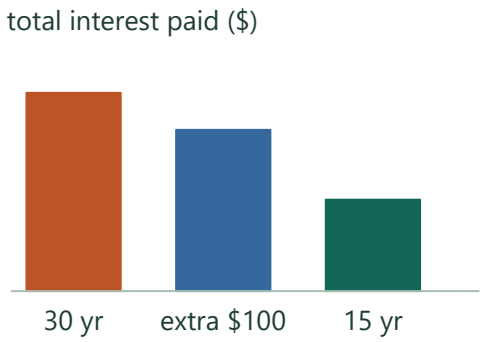
Compare the interest cost after sending an extra $100 each month to principal.

Borrowing $200,000 at 4% with three repayment plans

Payment plan	Payoff month	Total paid ($)	Interest paid ($)
30-year schedule	360	343,739.01	143,739.01
30-year + $100 monthly	301	316,884.08	116,884.08
15-year schedule	180	266,287.65	66,287.65

8. Compare the costs of the alternatives

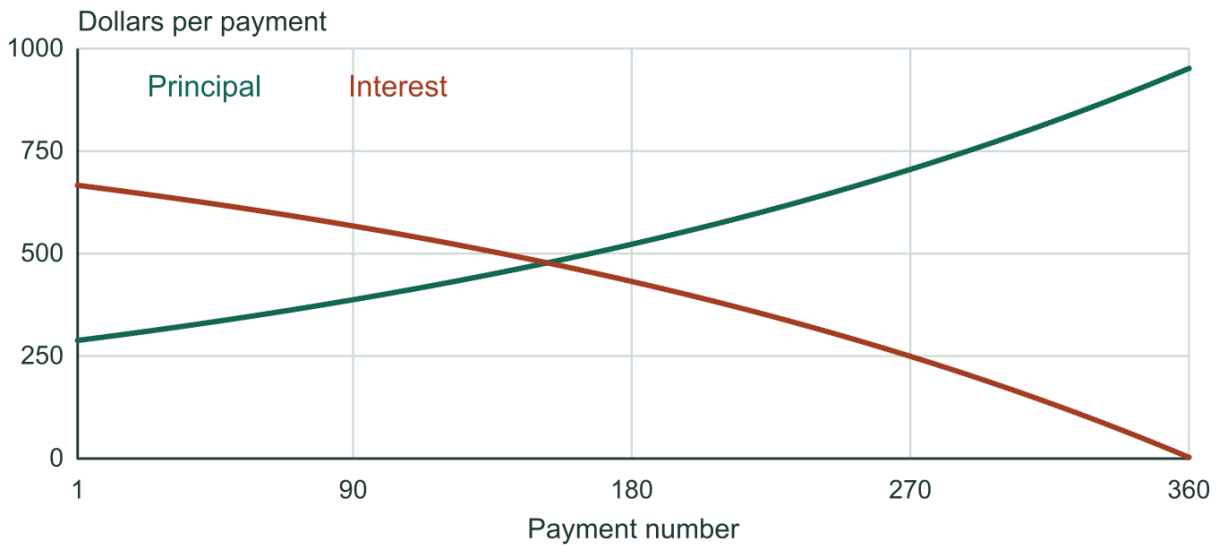
At 3% versus 6%, doubling the quoted rate adds about $356 per month and $128,121 in total payments. At 4%, switching from 30 years to 15 raises the monthly payment about $524.55 but saves about $77,451.36 in interest. The accompanying workbook plots the two payment components for every month.



Same $200k principal

A shorter term cuts the total interest paid.

Hold the amount borrowed fixed and shorten the term to 15 years. The higher monthly payment buys a smaller lifetime interest bill.



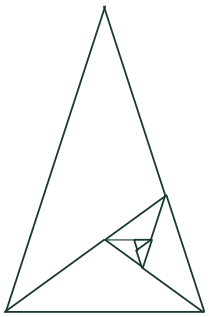
Open the editable mortgage workbook and monthly schedules.

Check. Each row's principal and interest sum to that payment. Summing all principal reductions gives $200,000, and iterating the balances to zero reproduces the annuity formula.

Exercise 838-7 · An endless chain of similar triangles

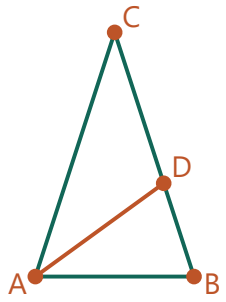
Printed page 125 · Similarity and scaling / Geometric sequences and series / The golden ratio

Draw a triangle that has two 72-degree angles, and two 2-inch sides. Bisect one of the 72-degree angles. This creates two isosceles triangles, one of which is similar to the triangle you started with. The process can thus be repeated forever, each time applied to a triangle with two 72-degree angles. Make a list of the lengths of the *longest sides* of this sequence of triangles. What kind of numerical sequence do you discover?



1. Name the sides on the drawing

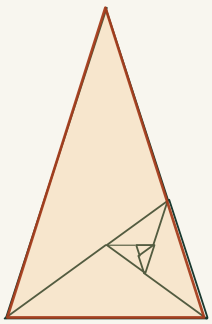
Let AB=b be the short base and AC=BC=L=2 the equal sides. The base angles A and B are both 72°, so the apex angle C is 36°. Bisect angle A and name the point on BC where that line arrives D.



Base angles: 72°

AD bisects angle A.

The orange bisector divides the 72° base angle into two 36° angles.

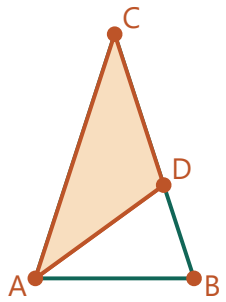


Two equal original sides

The long sides meet at the narrow top angle. The two lower corners have equal angles, so comparing the original triangle to a smaller copy tests similarity.

2. Identify the equal lengths

In triangle ABD the angles are 36°,72°,72°, so AD=AB=b. In triangle ACD both A and C measure 36°, so CD=AD=b. Consequently BD=BC−CD=L−b.



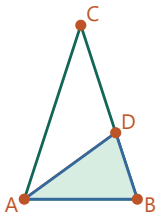
Equal-angle pairs

AD = AB = CD = b

Identify the equal opposite sides in the two smaller triangles.

3. Match the smaller similar triangle

Triangle ABD has the same three angles as ABC. Its short side BD corresponds to AB; a long side b corresponds to a long side L. Thus (L−b)/b=b/L. Writing q=b/L gives $1-q=q^2$.



Smaller copy ABD

Short side BD = L − b

b/L = (L − b)/b

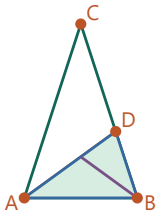
1. $(L-b)/b = b/L \Rightarrow L^2-Lb = b^2$
Cross-multiply the proportional sides; both are positive.

2. $1-b/L = (b/L)^2 \Rightarrow q^2+q-1 = 0$
Divide by L^2 and use q=b/L.

Highlight ABD and match its apex and base angles with the large triangle before forming the side ratio.

4. Repeat the construction

The positive solution $q=(\sqrt{5}-1)/2\approx.618$ is the scale factor at every repetition. The longest sides form a decreasing geometric sequence, in inches: 2, 2q, $2q^2$, $2q^3$,... . The first three are about 2, 1.236 and .764.



One more bisector

Each long side is q times

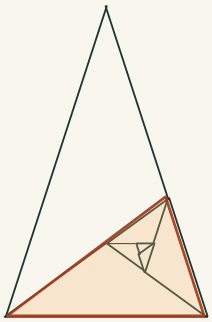
the previous long side.

1. $q = [-1\pm\sqrt{(1+4)}]/2$
Apply the quadratic formula to $q^2+q-1=0$.

2. $q = (\sqrt{5}-1)/2$
Discard the negative solution because the scale is a ratio of lengths.

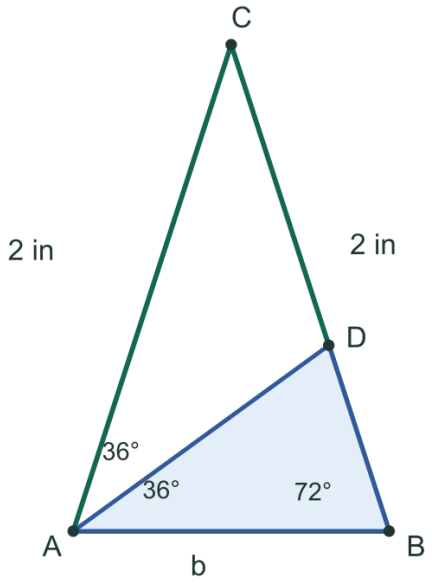
3. $2q^2 = 2(1-q) = 3-\sqrt{5} \approx 0.764$
Use the similarity relation once more to compute the third longest side.

Bisect a base angle of the smaller copy. The repeated construction rescales the longest side by the same q each time.



Next similar triangle ABD

The base and first bisector are two sides of the smaller triangle. Its third side is only a portion of the original long side, and the pattern can repeat inside it.



Check. The ratio satisfies $q^2+q=1$. Numerically the first lengths are 2,1.2361,0.7639,...; the second and third sum to the first.