Cross-concept practice

Show the reasoning that connects each part. Use exact values unless a question asks for rounding.

01 A cup cut from a circle

A sector of radius 15 cm and central angle 216° is rolled into a right circular cone without overlap. Its curved edge forms the open rim. Stand the cup tip down with its axis vertical.

a. Find the rim radius and the vertical height of the cup.

b. Find the cup's capacity and the area of paper in its curved side. Give exact values.

c. Water fills 8/27 of the capacity. How far below the rim is the water surface? Explain how the volume fraction determines a length.

02 Two possible survey markers

On a level map, station B is 20 km due east of station A. A marker M lies on a ray from A that rises 3 km north for every 4 km east. M is exactly 13 km from B.

a. Find both possible distances AM. Show why both put M on the specified ray. Write B's range circle and use the cosine rule to confirm your distance equation.

b. Find the area of triangle ABM for each possible location. Use the sine rule to explain the two possible angles at M.

c. From B, the direction to M is closer to due north than to due west. Which location fits? Justify your choice.

03 A light on a turning wheel

A vertical wheel has radius 9 m and center (0, 11), with y measured in meters above the floor. A rim light starts at the rightmost point and turns counterclockwise once every 48 s at constant speed.

a. Write the coordinates of the light at time t seconds, using radians.

b. During the first revolution, when is the light exactly 15.5 m high? How far along the rim does it travel while above that height?

c. Give the 2 × 2 matrix that takes any displacement from the center to its value 12 s later. Check it on the starting displacement. A detector measures the light's signed displacement from the hub along the unit vector (4/5, 3/5). Use an angle identity to express its reading as one shifted cosine. When is its first maximum? Give the detector unit vector in polar coordinates.

04 A reflection in three dimensions

A flat mirror extends through the points A = (2, 2, 1), B = (3, 4, 1), and C = (2, 4, 2). A point of light is at P = (7, 4, 6). Coordinates are in meters. Treat the mirror as an entire plane.

a. Find an equation of the mirror plane. Explain how you chose a normal vector.

b. Find the point H on the plane nearest to P and the distance PH.

c. Find the reflection of P in the mirror. Verify that the plane bisects the segment from P to its image at right angles.

05 Locating a signal

On a coordinate plane measured in kilometers, receivers L = (−6, 0) and R = (6, 0) measure their distances to a signal at point P in that plane. One report says PL + PR = 18. A second report says PL − PR = 8.

a. Identify the locus from the first report and write its equation in standard form.

b. Do the same for the second report. State which branch satisfies the signed difference. Find both conics' eccentricities and the hyperbola's asymptotes.

c. Use both reports to locate every possible P. Check the distances to the receivers.

06 Where the receiver goes

A dish has a parabolic cross section 24 cm across and 4 cm deep. Put its vertex at (0, 0), let the axis point straight up, and measure x and y in centimeters. Signals arrive parallel to the axis and reflect to the focus.

a. Write an equation for the cross section and state the physical interval of x.

b. Find the focus and directrix. Show that a rim point is equally far from each. Where does the receiver sit relative to the rim?

c. A second dish has the same width and twice the depth. Where is its receiver relative to its own rim? Explain the change using the equation.

07 Cooling with an unknown room temperature

A ceramic tile cools in a room of constant temperature. Its recorded temperatures are 92°C at 0 min, 68°C at 6 min, and 52°C at 12 min. Assume its excess above room temperature decays exponentially, and treat the readings as exact.

a. Find the fraction of excess temperature retained every 6 min. Explain how the readings reveal it.

b. Find the room temperature and write a model T(t), with t in minutes.

c. When does the tile first reach 30°C? Give the answer to the nearest tenth of a minute. Does the model ever reach room temperature at a finite time?

08 Funding an annual award

A fund starts empty and earns 4% interest each year. Carry full precision and round reported balances to cents. At the end of each year, immediately after interest is credited, a sponsor deposits $360. Assume the rate stays fixed.

a. Give a recurrence and a closed formula for the balance just after the nth deposit.

b. After which deposit does the balance first reach at least $4,500? Check the adjacent years. Compare a zero-interest plan with year-end deposits of $300, $330, $360, and so on. When does it first meet the same target?

c. Return to the 4% annual rate. Under this plan there are no deposits after today. How much must be invested today to pay a $360 award at each year-end forever while preserving the initial balance? Explain.

09 Where the bikes settle

A bike service has 320 bikes at two stations, East and West. Each day, a bike starting at East has a 75% chance of finishing at East. A bike starting at West has a 15% chance of finishing at East. Otherwise it finishes at West. Initially, 200 bikes are at East.

a. Write the daily transition matrix for the column vector (East, West) of expected counts. Find the expected counts after one day.

b. Find the equilibrium expected counts. Show how the difference between East's expected count and its equilibrium changes each day.

c. On which day is East's expected count first within one bike of equilibrium? Count the starting day as day 0.

10 A robot's seven moves

A robot starts at (0, 0). Each second it independently moves one unit east with probability 0.6 or one unit north with probability 0.4. It stops after seven moves.

a. How many paths end at (4, 3)? Which coefficient in the expansion of $(u + v)^7$ counts them, and why? Reconcile your count with the 7! permutations of seven individually labeled moves.

b. Find the probability that the robot ends at (4, 3).

c. Given that it ended at (4, 3), find the probability that it visited (1, 2). Explain why a ratio of path counts works with this biased robot.

11 Two routes around a globe

A hollow spherical globe has outer radius 60 cm and inner radius 57 cm. Ports P and Q on its outer surface are at 45° north, at longitudes 0° and 90° east respectively. The origin is the center. The positive x- and y-axes meet the equator at 0° and 90° east respectively, and the positive z-axis points north.

a. Find the latitude circle’s height above the equatorial plane and its radius. How long is the shorter route from P to Q along that parallel?

b. Write Cartesian coordinates for both ports. Use their dot product to find the shortest surface route and explain why it differs from the parallel route.

c. How much outer surface must be painted, and what volume of material forms the shell? Give exact values and units.

12 A triangular print

A printer’s master tile has corners (0, 0), (8, 0), and (0, 8) in centimeters. Its proof mode first shears (x, y) to (x + y, y), then halves both coordinates.

a. Give the composite matrix on column vectors, image vertices, and area scale factor. Check the determinant against the geometric area.

b. An ink dot appears at (3, 1) on the proof. Recover its original position with an inverse matrix. Why would a setting that flattened every y-coordinate to zero lose information?

c. In a separate print mode, stage 0 is the full master. At each step, join the side midpoints of every retained triangle and keep its three corner triangles. After n stages find the ink area, the chance a uniform random point of the master is inked, and the total boundary length including holes. What happens in the limit?

13 How a colony spreads

A culture covers 4, 8, and 64 cm² at times 1, 2, and 4 hours in a dish with 100 cm² of usable space. A researcher fits a power law to these three readings and compares it with a model that accounts for crowding.

a. Fit $Y = mX + b$ by least squares, using $X = \log_2(t)$ and $Y = \log_2(A)$, where t and A are the numerical values of time in hours and area in cm². Report the three signed residuals, observed minus fitted, on this scale.

b. Recover the power law. Predict the area at 3 h, and use secant slopes from 2 to 2 + h to find the fitted model’s instantaneous growth rate at 2 h. Is a prediction at 8 h credible?

c. The alternative hourly rule is $U_{n+1} = U_n + 0.5U_n(1 - U_n/100)$. Starting at 64 cm², predict the next value. Identify the positive equilibrium and maximize the one-hour gain on $0 \le U \le 100$.