Coverage of the Exeter problem set

This review groups recurring methods in Phillips Exeter Academy's Mathematics 3–4 (August 2025), using its 838 main exercises and the reference on printed pages 97–114. The numbers on the right point to representative source exercises.

Problems 2, 3, 5, 8, and 10 extend the original review. Problems 11–13 add three settings for spherical measurement, matrix transformations and self-similarity, and empirical growth models. Coverage is by major strand. Specialized applications such as refraction and the cycloid are outside this compact set.

01 **A cup cut from a circle** — Exeter 116, 160, 252
Sector length and area · cone geometry · cubic scaling

02 **Two possible survey markers** — Exeter 81, 145, 166, 168
Circle equations · sine and cosine rules · ambiguous triangles

03 **A light on a turning wheel** — Exeter 121, 165, 310, 704
Circular motion · phase shift · trig addition identity · polar direction

04 **A reflection in three dimensions** — Exeter 403, 476, 526
Dot products · plane equations · perpendicular projection

05 **Locating a signal** — Exeter 431, 679–680, 727, 788
Conic loci · eccentricity · asymptotes · distance equations

06 **Where the receiver goes** — Exeter 577, 627
Quadratic models · focus and directrix · geometric scaling

07 **Cooling with an unknown room temperature** — Exeter 481, 585, 696
Shifted exponential models · ratios of differences · logarithms

08 **Funding an annual award** — Exeter 637–640, 670, 740
Compound interest · arithmetic and geometric sums · present value

09 **Where the bikes settle** — Exeter 623–626
Transition probabilities · matrix multiplication · equilibrium

10 **A robot's seven moves** — Exeter 721–722, 769, 798
Permutations · binomial coefficients · conditional probability

11 **Two routes around a globe** — Exeter 181, 247, 254, 261, 289
Spherical coordinates · great circles · surface area and shell volume

12 **A triangular print** — Exeter 149, 237, 373–375, 517, 736
Matrix composition · inverses · determinants · fractals and limits

13 **How a colony spreads** — Exeter 330, 510–511, 551, 594, 745
Log–log regression · residuals · rates of change · logistic optimization

01 A cup cut from a circle

A sector of radius 15 cm and central angle 216° is rolled into a right circular cone without overlap. Its curved edge forms the open rim. Stand the cup tip down with its axis vertical.

a. Find the rim radius and the vertical height of the cup.

b. Find the cup's capacity and the area of paper in its curved side. Give exact values.

c. Water fills 8/27 of the capacity. How far below the rim is the water surface? Explain how the volume fraction determines a length.

CONCEPTS TESTED Exeter problems 116, 160, 252

Sector arc length determines the cone's radius. Similarity then converts a volume fraction into a depth.

a. Match the two edges

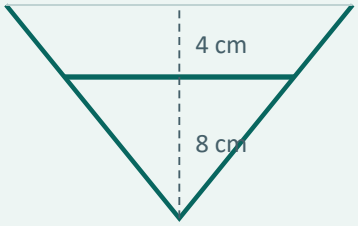
The sector spans 3/5 of a circle. Its arc length is $(3/5)(2\pi)(15) = 18\pi$ cm. Set $2\pi r = 18\pi$ to get **r = 9 cm**. The sector radius becomes the slant height, so the vertical height is $\sqrt{(15^2 - 9^2)}$ = **12 cm**.

b. Use the two shapes

The cone holds $(1/3)\pi(9^2)(12)$ = **324π cm³**. Its curved side uses exactly the paper in the sector, an area of $(3/5)\pi(15^2)$ = **135π cm²**.

c. Scale from the tip

The water forms a smaller, similar cone. If its height is k times the full height, its radius scales by k too, so its volume fraction is k^3. From $k^3 = 8/27$, k = 2/3. The water is 8 cm deep, so its surface is **4 cm below the rim**.



The waterline

An 8 cm water depth is two-thirds of the full height. Its volume is $(2/3)^3$ of the capacity.

02 Two possible survey markers

On a level map, station B is 20 km due east of station A. A marker M lies on a ray from A that rises 3 km north for every 4 km east. M is exactly 13 km from B.

a. Find both possible distances AM. Show why both put M on the specified ray. Write B's range circle and use the cosine rule to confirm your distance equation.

b. Find the area of triangle ABM for each possible location. Use the sine rule to explain the two possible angles at M.

c. From B, the direction to M is closer to due north than to due west. Which location fits? Justify your choice.

CONCEPTS TESTED Exeter problems 81, 145, 166, 168

A bearing defines a ray and a range defines a circle. Their intersections explain why two triangles fit the same survey.

a. Intersect the ray and the range circle

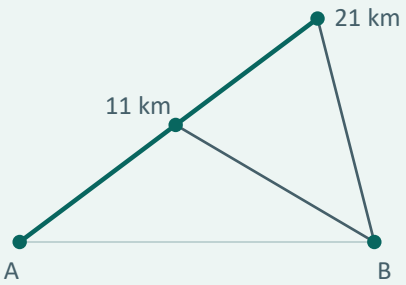
Set A = (0, 0), B = (20, 0). The ray's unit direction is (4/5, 3/5), so at distance s its coordinates are (4s/5, 3s/5). The range gives $(4s/5 - 20)^2 + (3s/5)^2 = 169$. This reduces to $s^2 - 32s + 231 = (s - 11)(s - 21) = 0$. Both **11 km and 21 km** are positive, so both points lie on the ray. The range circle is $(x - 20)^2 + y^2 = 169$. The cosine rule gives $13^2 = 20^2 + s^2 - 2(20)s(4/5)$, the same quadratic.

b. Use segment AB as a common base

The northward distances are 3(11)/5 = 6.6 km and 3(21)/5 = 12.6 km. The respective areas are ½(20)(6.6) = **66 km²** and ½(20)(12.6) = **126 km²**. By the sine rule, sin M / 20 = sin A / 13, so sin M = 12/13. Let α = arcsin(12/13) ≈ 67.38°. The other angle is 180° − α ≈ 112.62°. The 11 km triangle has the obtuse angle at M because $20^2 > 11^2 + 13^2$. The 21 km triangle has the acute angle there.

c. Read the bearing from B

The nearer point is 11.2 km west and 6.6 km north of B. The farther point is 3.2 km west and 12.6 km north. Only the latter has a larger northward component, so **AM = 21 km**.



Two triangles, one ray

Both slanted segments from B have length 13 km. The extra bearing distinguishes their endpoints.

03 A light on a turning wheel

A vertical wheel has radius 9 m and center (0, 11), with y measured in meters above the floor. A rim light starts at the rightmost point and turns counterclockwise once every 48 s at constant speed.

a. Write the coordinates of the light at time t seconds, using radians.

b. During the first revolution, when is the light exactly 15.5 m high? How far along the rim does it travel while above that height?

c. Give the 2 × 2 matrix that takes any displacement from the center to its value 12 s later. Check it on the starting displacement. A detector measures the light's signed displacement from the hub along the unit vector (4/5, 3/5). Use an angle identity to express its reading as one shifted cosine. When is its first maximum? Give the detector unit vector in polar coordinates.

CONCEPTS TESTED Exeter problems 121, 165, 310, 704

Circular motion produces a sinusoidal height. Its two threshold crossings determine both elapsed time and arc length.

a. Set the phase from the starting position

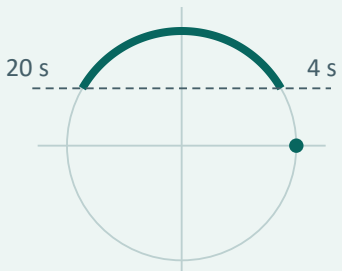
The angle is $\theta = (2\pi/48)t = \pi t/24$. At the rightmost point the horizontal displacement is 9 and the vertical displacement is zero. Thus **$x = 9\cos(\pi t/24)$, $y = 11 + 9\sin(\pi t/24)$**.

b. Find both crossings

At $y = 15.5$, $\sin\theta = (15.5 - 11)/9 = 1/2$. In one turn, $\theta = \pi/6$ or $5\pi/6$, giving **t = 4 s and 20 s**. Between these angles the light is higher. It sweeps $2\pi/3$ radians, so the arc length is $9(2\pi/3) =$ **6π m**.

c. Apply a quarter-turn

Twelve seconds is one quarter of the period. A counterclockwise quarter-turn maps (u, v) to (−v, u). Its matrix has **first row (0, −1) and second row (1, 0)**, acting on column vectors. It sends (9, 0) to (0, 9), placing the light at (0, 20). Set $\phi = \arctan(3/4)$ in quadrant I. The detector's dot product is $(9/5)(4\cos\theta + 3\sin\theta) = 9(\cos\theta\cos\phi + \sin\theta\sin\phi) =$ **$9\cos(\theta - \phi)$**. Its first maximum is 9 m at $\theta = \phi$, or **$t = 24\phi/\pi \approx 4.92$ s**. Its length is 1, so its polar coordinates are **$(1, \phi)$**.



The interval above 15.5 m

The highlighted arc runs from 30° to 150°. That is one-third of the circle and one-third of the 48 s period.

04 A reflection in three dimensions

A flat mirror extends through the points A = (2, 2, 1), B = (3, 4, 1), and C = (2, 4, 2). A point of light is at P = (7, 4, 6). Coordinates are in meters. Treat the mirror as an entire plane.

a. Find an equation of the mirror plane. Explain how you chose a normal vector.

b. Find the point H on the plane nearest to P and the distance PH.

c. Find the reflection of P in the mirror. Verify that the plane bisects the segment from P to its image at right angles.

CONCEPTS TESTED Exeter problems 403, 476, 526

Two nonparallel directions in the plane determine its normal direction. Moving along that normal gives the shortest path and the mirror image.

a. Build a normal from two directions

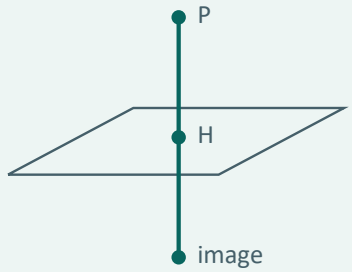
The plane contains AB = (1, 2, 0) and AC = (0, 2, 1). Both have dot product zero with n = (2, −1, 2). They are nonparallel, so n is normal to their plane. Using A gives $2(x - 2) - (y - 2) + 2(z - 1) = 0$, or **$2x - y + 2z = 4$**.

b. Follow the perpendicular to the mirror

On the line P − tn, the plane's left side is 22 − 9t since n · P = 22 and n · n = 9. Setting this equal to 4 gives t = 2. Thus **H = (3, 6, 2)** and PH = 2|n| = **6 m**. For any other point Q in the plane, PH is perpendicular to HQ, so $PQ^2 = PH^2 + HQ^2 > PH^2$.

c. Continue the same distance

The reflected point is 2H − P = **(−1, 8, −2)**. Its midpoint with P is (3, 6, 2), which satisfies the plane equation. The segment direction is (−8, 4, −8) = −4n, so it is perpendicular to the mirror.



The perpendicular through H

P and its image lie 6 m from the plane on opposite sides. Their midpoint H is the foot of the perpendicular.

05 Locating a signal

On a coordinate plane measured in kilometers, receivers L = (−6, 0) and R = (6, 0) measure their distances to a signal at point P in that plane. One report says PL + PR = 18. A second report says PL − PR = 8.

a. Identify the locus from the first report and write its equation in standard form.

b. Do the same for the second report. State which branch satisfies the signed difference. Find both conics' eccentricities and the hyperbola's asymptotes.

c. Use both reports to locate every possible P. Check the distances to the receivers.

CONCEPTS TESTED Exeter problems 431, 679–680, 727, 788

The sum and difference of two distances define confocal conics. Together they reduce two curves to two possible points.

a. Use the distance sum

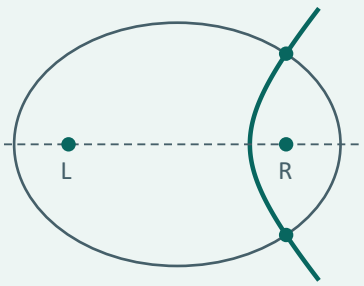
The two fixed receivers are the foci of an ellipse. Their half-separation is c = 6 and the sum is 2a = 18, so a = 9. Then $b^2 = a^2 - c^2 = 45$. The equation is $\mathbf{x^2/81 + y^2/45 = 1}$.

b. Use the signed difference

The absolute difference 2a = 8 gives a hyperbola with a = 4 and $b^2 = c^2 - a^2 = 20$. The report selects $\mathbf{x^2/16 - y^2/20 = 1, x \geq 4}$. P must be nearer to R, which selects the right branch. Eccentricity is c/a, hence **2/3** for the ellipse and **3/2** for the hyperbola. The asymptotes come from $x^2/16 - y^2/20 = 0$, giving $\mathbf{y = \pm(\sqrt{20}/4)x}$.

c. Recover the distances first

Adding and subtracting the reports gives PL = 13 and PR = 5. Subtract the squared distance equations to get $(x + 6)^2 - (x - 6)^2 = 169 - 25$, hence 24x = 144. At x = 6, PR = 5 gives **P = (6, 5) or (6, −5) km**. Each has distances $\sqrt{(12^2 + 5^2)} = 13$ and $\sqrt{(0^2 + 5^2)} = 5$.



Two reports, two intersections

The ellipse meets the right branch once above the receivers and once below. Reflection in the horizontal axis preserves both distances.

06 Where the receiver goes

A dish has a parabolic cross section 24 cm across and 4 cm deep. Put its vertex at (0, 0), let the axis point straight up, and measure x and y in centimeters. Signals arrive parallel to the axis and reflect to the focus.

a. Write an equation for the cross section and state the physical interval of x.

b. Find the focus and directrix. Show that a rim point is equally far from each. Where does the receiver sit relative to the rim?

c. A second dish has the same width and twice the depth. Where is its receiver relative to its own rim? Explain the change using the equation.

CONCEPTS TESTED Exeter problems 577, 627

A measured cross section fixes a quadratic model. The focus–directrix definition then predicts the receiver's position.

a. Fit the profile

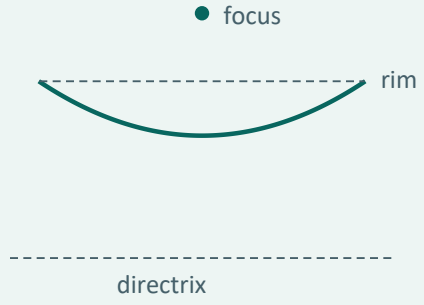
Symmetry and the vertex give $y = kx^2$. The rim includes (12, 4), so $k = 4/144 = 1/36$. The dish profile is **$y = x^2/36$, $-12 \le x \le 12$**.

b. Use equal distances

For focus (0, p) and directrix $y = -p$, equal distances give $x^2 + (y - p)^2 = (y + p)^2$, or $x^2 = 4py$. Here $4p = 36$, so the **focus is (0, 9) and the directrix is $y = -9$**. At (12, 4) the distances are $\sqrt{(12^2 + 5^2)} = 13$ and $4 + 9 = 13$ cm. The receiver is **5 cm above the rim**.

c. Double the depth at fixed width

Now (12, 8) is on the profile, so $y = x^2/18$ and $4p = 18$. The receiver is at (0, 4.5), **3.5 cm below the rim**. Doubling the quadratic coefficient halves the focal distance while the rim rises.



The shallower dish

The vertex sits halfway between the focus and the directrix. A 24 cm opening and 4 cm depth put the focus beyond the rim.

07 Cooling with an unknown room temperature

A ceramic tile cools in a room of constant temperature. Its recorded temperatures are 92°C at 0 min, 68°C at 6 min, and 52°C at 12 min. Assume its excess above room temperature decays exponentially, and treat the readings as exact.

a. Find the fraction of excess temperature retained every 6 min. Explain how the readings reveal it.

b. Find the room temperature and write a model T(t), with t in minutes.

c. When does the tile first reach 30°C? Give the answer to the nearest tenth of a minute. Does the model ever reach room temperature at a finite time?

CONCEPTS TESTED Exeter problems 481, 585, 696

Equal-time temperature drops reveal an exponential factor even when the baseline is unknown. Logarithms recover the time.

a. Compare consecutive drops

Write the temperatures as A + D, A + Dq, and A + Dq². Their successive drops are D(1 − q) and Dq(1 − q), so the ratio of the drops is q. From the readings, q = (68 − 52)/(92 − 68) = 16/24 = **2/3**.

b. Recover the baseline

The first drop gives D(1 − 2/3) = 24, so D = 72°C and A = 92 − 72 = **20°C**. Each 6 min contributes one factor of 2/3. The model is $\mathbf{T(t) = 20 + 72(2/3)^{t/6}}$.

c. Solve for the exponent

At 30°C the excess is 10°C. From $(2/3)^{t/6} = 10/72$, t = 6 ln(10/72)/ln(2/3) ≈ **29.2 min**. The excess is positive at every finite t and tends to zero as t grows. The model approaches 20°C from above.

Minutes	Tile (°C)	Excess (°C)
0	92	72
6	68	48
12	52	32

Subtract the room temperature

The excess column has a constant ratio of 2/3. The raw temperature column has a changing ratio.

08 Funding an annual award

A fund starts empty and earns 4% interest each year. Carry full precision and round reported balances to cents. At the end of each year, immediately after interest is credited, a sponsor deposits $360. Assume the rate stays fixed.

a. Give a recurrence and a closed formula for the balance just after the nth deposit.

b. After which deposit does the balance first reach at least $4,500? Check the adjacent years. Compare a zero-interest plan with year-end deposits of $300, $330, $360, and so on. When does it first meet the same target?

c. Return to the 4% annual rate. Under this plan there are no deposits after today. How much must be invested today to pay a $360 award at each year-end forever while preserving the initial balance? Explain.

CONCEPTS TESTED Exeter problems 637–640, 670, 740

The timing of equal payments determines a geometric sum. A logarithm gives the first qualifying whole year.

a. Track the age of each deposit

With $B_0 = 0$, $\mathbf{B_n = 1.04\,B_{n-1} + 360}$. After n deposits the oldest has earned interest n − 1 times and the newest has earned none. Sum $360[1 + 1.04 + \dots + 1.04^{n-1}]$ to get $\mathbf{B_n = 9{,}000(1.04^n - 1)}$.

b. Round to the first eligible year

The target requires $1.04^n \geq 1.5$, so $n \geq \ln(1.5)/\ln(1.04) \approx 10.34$. The **11th deposit** is the first. After 10 deposits the balance is $4,322.20. After 11 it is $4,855.09. The alternative deposits form an arithmetic sequence. Pair the first and last terms to obtain $S_n = n[600 + 30(n - 1)]/2$. The totals are $4,350 at year 10 and $4,950 at year 11, so it too first qualifies in **year 11**.

c. Balance interest and withdrawal

For the principal K to be the same after each award, $1.04K - 360 = K$. Thus $\mathbf{K = \$9{,}000}$. Each year it earns exactly $360, which funds that year's award. Equivalently, the discounted awards sum to $(360/1.04)/(1 - 1/1.04) = \$9{,}000$.

Year	Balance ($)
0	0.00
1	360.00
2	734.40

Check the timing

Only the first deposit has earned interest by the second year-end. Its balance grows to $374.40 before the next $360 arrives.

09 Where the bikes settle

A bike service has 320 bikes at two stations, East and West. Each day, a bike starting at East has a 75% chance of finishing at East. A bike starting at West has a 15% chance of finishing at East. Otherwise it finishes at West. Initially, 200 bikes are at East.

a. Write the daily transition matrix for the column vector (East, West) of expected counts. Find the expected counts after one day.

b. Find the equilibrium expected counts. Show how the difference between East's expected count and its equilibrium changes each day.

c. On which day is East's expected count first within one bike of equilibrium? Count the starting day as day 0.

CONCEPTS TESTED Exeter problems 623–626

A transition matrix gives a recurrence. Measuring the gap from equilibrium turns that recurrence into geometric decay.

a. Read the matrix by columns

The East-start column is (0.75, 0.25) and the West-start column is (0.15, 0.85). The **matrix rows are (0.75, 0.15) and (0.25, 0.85)**. Applying it to (200, 120) gives **(168, 152)** after one day.

b. Subtract the fixed point

If e is today's expected East count, tomorrow's is $0.75e + 0.15(320 - e) = 0.60e + 48$. The fixed point solves $e = 0.60e + 48$, so **East = 120 and West = 200**. Subtracting 120 on both sides gives $e_{n+1} - 120 = 0.60(e_n - 120)$. Thus the gap shrinks by a factor of 0.60 each day.

c. Test the one-bike boundary

The starting gap is 80 bikes, so after n days the gap is $80(0.60)^n$. At day 8 this is about 1.344 bikes, and at day 9 it is about 0.806 bike. The gap decreases, so **day 9** is the first day that qualifies.

Day	East	Gap
0	200	80
1	168	48
2	148.8	28.8

Compare the gaps

Fractional entries are expected counts. Each day's gap is 60% of the previous one, even though the station total changes by a different percentage.

10 A robot's seven moves

A robot starts at (0, 0). Each second it independently moves one unit east with probability 0.6 or one unit north with probability 0.4. It stops after seven moves.

a. How many paths end at (4, 3)? Which coefficient in the expansion of $(u + v)^7$ counts them, and why? Reconcile your count with the 7! permutations of seven individually labeled moves.

b. Find the probability that the robot ends at (4, 3).

c. Given that it ended at (4, 3), find the probability that it visited (1, 2). Explain why a ratio of path counts works with this biased robot.

CONCEPTS TESTED Exeter problems 721–722, 769, 798

A binomial coefficient counts paths. Given the endpoint, their equal probabilities turn a conditional question into counting.

a. Choose the eastward move positions

Choose 4 of the 7 positions to hold an east move. This gives 7!/(4!3!) = **35 paths**. In the expansion, choose u from four factors and v from three. These choices produce the same **coefficient 35 of u^4v^3**. Label the four east moves and three north moves individually. Each route represents 4! × 3! = 144 of the 7! labeled orders, so 7!/144 = 35.

b. Weight each path

Independence gives each such path probability $(0.6)^4(0.4)^3$. The 35 paths are disjoint, so the endpoint probability is $\mathbf{35(0.6)^4(0.4)^3 = 0.290304}$.

c. Count paths through the checkpoint

A visit to (1, 2) must occur at step 3. Choose the one east move among the first three moves (3 ways), then the one north move among the last four (4 ways). That gives 12 paths. All 35 endpoint paths have the same four east and three north moves, so their weights cancel. The conditional probability is **12/35**, about 34.3%.

Segment	East	North
To (1, 2)	1	2
To (4, 3)*	3	1

Split at the checkpoint

*The second segment starts at (1, 2). Any ordering of the first segment can be paired with any ordering of the second.

11 Two routes around a globe

A hollow spherical globe has outer radius 60 cm and inner radius 57 cm. Ports P and Q on its outer surface are at 45° north, at longitudes 0° and 90° east respectively. The origin is the center. The positive x- and y-axes meet the equator at 0° and 90° east respectively, and the positive z-axis points north.

a. Find the latitude circle's height above the equatorial plane and its radius. How long is the shorter route from P to Q along that parallel?

b. Write Cartesian coordinates for both ports. Use their dot product to find the shortest surface route and explain why it differs from the parallel route.

c. How much outer surface must be painted, and what volume of material forms the shell? Give exact values and units.

CONCEPTS TESTED Exeter problems 181, 247, 254, 261, 289

Latitude sets the radius of a horizontal slice. A 3D dot product gives the great-circle angle. Surface area and shell volume measure different parts of the model.

a. Slice horizontally

At latitude 45°, the height is 60 sin 45° = **30√2 cm**, and the horizontal radius is 60 cos 45° = **30√2 cm**. The longitude change is π/2 radians, so the parallel route has length **15π√2 cm**.

b. Measure the angle at the globe's center

Write k = 30√2. Then **P = (k, 0, k), Q = (0, k, k)**. Their dot product is $k^2 = 1800$, so cos θ = 1800/60² = 1/2 and θ = π/3. The short great-circle arc has length 60(π/3) = **20π cm**. Its plane passes through the sphere's center. The latitude plane is above the center, so its quarter-circle arc takes a longer route.

c. Distinguish surface from material

The outer area is $4\pi(60^2)$ = **14,400π cm²**. Subtracting the inner sphere's volume from the outer sphere's gives $(4\pi/3)(60^3 - 57^3)$ = **41,076π cm³**.

Compare routes on the same sphere

60 cm is the radius of the great circle. The latitude circle has radius 30√2 cm. Its swept angle is 90°, while the shorter great-circle route sweeps just 60° at the center.

12 A triangular print

A printer's master tile has corners (0, 0), (8, 0), and (0, 8) in centimeters. Its proof mode first shears (x, y) to (x + y, y), then halves both coordinates.

a. Give the composite matrix on column vectors, image vertices, and area scale factor. Check the determinant against the geometric area.

b. An ink dot appears at (3, 1) on the proof. Recover its original position with an inverse matrix. Why would a setting that flattened every y-coordinate to zero lose information?

c. In a separate print mode, stage 0 is the full master. At each step, join the side midpoints of every retained triangle and keep its three corner triangles. After n stages find the ink area, the chance a uniform random point of the master is inked, and the total boundary length including holes. What happens in the limit?

CONCEPTS TESTED Exeter problems 149, 237, 373–375, 517, 736

A composed matrix changes a tile's area and can be inverted. Repeating half-scale copies then links similarity, fractals, probability, and limits.

a. Compose in print order

Using row notation, the shear has rows (1, 1), (0, 1). Left-multiply by ½I to get **A with rows (½, ½), (0, ½)**. The images are **(0, 0), (4, 0), (4, 4)**. The determinant is ¼. The areas are 32 and 8 cm², confirming the **¼ area scale**.

b. Reverse the two equations

From Y = y/2 and X = (x + y)/2, get y = 2Y and x = 2X − 2Y. The inverse has **rows (2, –2), (0, 2)**, giving **(4, 2)** from (3, 1). Flattening maps a two-dimensional region onto a line. The resulting matrix has determinant zero and multiple original points give the same output.

c. Count copies before scaling

At stage n there are 3^n tiles with lengths scaled by $(½)^n$. They meet only at vertices. Ink area is **$32(3/4)^n$ cm²** and the probability is **$(3/4)^n$**. The original perimeter is 8(2 + √2) cm. All boundaries total **$8(2 + √2)(3/2)^n$ cm**. Area and probability tend to zero while these stage perimeters grow without bound.

Check one print stage

Each half-size copy has area 8 cm². Three copies give 24 cm² and cover 75% of the master. The total boundary grows because each triangular hole introduces three more edges.

13 How a colony spreads

A culture covers 4, 8, and 64 cm² at times 1, 2, and 4 hours in a dish with 100 cm² of usable space. A researcher fits a power law to these three readings and compares it with a model that accounts for crowding.

a. Fit $Y = mX + b$ by least squares, using $X = \log_2(t)$ and $Y = \log_2(A)$, where t and A are the numerical values of time in hours and area in cm². Report the three signed residuals, observed minus fitted, on this scale.

b. Recover the power law. Predict the area at 3 h, and use secant slopes from 2 to 2 + h to find the fitted model's instantaneous growth rate at 2 h. Is a prediction at 8 h credible?

c. The alternative hourly rule is $U_{n+1} = U_n + 0.5U_n(1 - U_n/100)$. Starting at 64 cm², predict the next value. Identify the positive equilibrium and maximize the one-hour gain on $0 \le U \le 100$.

CONCEPTS TESTED Exeter problems 330, 510–511, 551, 594, 745

Logarithms turn a power law into a straight-line fit. Secant slopes give a local growth rate, while a logistic model accounts for limited space.

a. Fit the transformed observations

The points are (0, 2), (1, 3), (2, 6). Their means are 1 and 11/3. The least-squares slope is $\Sigma(X - \bar{X})(Y - \bar{Y})/\Sigma(X - \bar{X})^2 = 4/2 = 2$. Thus $b = 11/3 - 2 = 5/3$. For **$Y = 2X + 5/3$**, the signed residuals are **1/3, −2/3, 1/3** in $\log_2$ units.

b. Turn a secant into a tangent

Exponentiating gives **$A(t) = kt^2$**, $k = 2^{5/3}$. At 3 h the interpolated area is $9k \approx$ **28.57 cm²**. The secant slope is $k[(2 + h)^2 - 4]/h = k(4 + h)$. As h tends to zero, the slope tends to **$4k \approx 12.70$ cm²/h**. At 8 h the extrapolation $64k \approx 203.19$ cm² exceeds the entire dish.

c. Optimize the space-limited gain

The next area is $64 + 0.5(64)(0.36) =$ **75.52 cm²**. For positive equilibrium the gain must be zero, giving **U = 100 cm²**. Completing the square gives gain $= 12.5 - (U - 50)^2/200$. Its maximum is **12.5 cm² in one hour at U = 50 cm²**.

What the residual measures

A signed $\log_2$ residual of +1/3 means the observation is 2 raised to the 1/3 power times the fitted area. The fit minimizes squared errors in $\log_2$ units.